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## 2.4 PRIORITY QUEUES

---

- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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# Priority queue

---

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.

Queue. Remove the item least recently added.

Randomized queue. Remove a random item.

Priority queue. Remove the **largest** (or **smallest**) item.

<i>operation</i>	<i>argument</i>	<i>return value</i>
<i>insert</i>	P	
<i>insert</i>	Q	
<i>insert</i>	E	
<i>remove max</i>		Q
<i>insert</i>	X	
<i>insert</i>	A	
<i>insert</i>	M	
<i>remove max</i>		X
<i>insert</i>	P	
<i>insert</i>	L	
<i>insert</i>	E	
<i>remove max</i>		P

# Priority queue API

Requirement. Generic items are Comparable.

Key must be Comparable (bounded type parameter)	
public class MaxPQ<Key extends Comparable<Key>>	
MaxPQ()	<i>create an empty priority queue</i>
MaxPQ(Key[] a)	<i>create a priority queue with given keys</i>
void insert(Key v)	<i>insert a key into the priority queue</i>
Key delMax()	<i>return and remove the largest key</i>
boolean isEmpty()	<i>is the priority queue empty?</i>
Key max()	<i>return the largest key</i>
int size()	<i>number of entries in the priority queue</i>

# Priority queue applications

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- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Number theory. [sum of powers]
- Artificial intelligence. [A\* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

# Priority queue client example

**Challenge.** Find the largest  $M$  items in a stream of  $N$  items.

- Fraud detection: isolate \$\$ transactions.
- File maintenance: find biggest files or directories.

$N$  huge,  $M$  large

**Constraint.** Not enough memory to store  $N$  items.

```
% more tinyBatch.txt
Turing      6/17/1990   644.08
vonNeumann  3/26/2002   4121.85
Dijkstra    8/22/2007   2678.40
vonNeumann  1/11/1999   4409.74
Dijkstra    11/18/1995   837.42
Hoare       5/10/1993   3229.27
vonNeumann  2/12/1994   4732.35
Hoare       8/18/1992   4381.21
Turing      1/11/2002   66.10
Thompson    2/27/2000   4747.08
Turing      2/11/1991   2156.86
Hoare       8/12/2003   1025.70
vonNeumann  10/13/1993  2520.97
Dijkstra    9/10/2000   708.95
Turing      10/12/1993  3532.36
Hoare       2/10/2005   4050.20
```

```
% java TopM 5 < tinyBatch.txt
Thompson    2/27/2000   4747.08
vonNeumann  2/12/1994   4732.35
vonNeumann  1/11/1999   4409.74
Hoare       8/18/1992   4381.21
vonNeumann  3/26/2002   4121.85
```

sort key

# Priority queue client example

**Challenge.** Find the largest  $M$  items in a stream of  $N$  items.

- Fraud detection: isolate \$\$ transactions.
- File maintenance: find biggest files or directories.

$N$  huge,  $M$  large

**Constraint.** Not enough memory to store  $N$  items.

```
use a min-oriented pq      MinPQ<Transaction> pq = new MinPQ<Transaction>();  
                          ↑  
while (StdIn.hasNextLine())  
{  
    String line = StdIn.readLine();  
    Transaction item = new Transaction(line);  
    pq.insert(item);  
    if (pq.size() > M) ← pq contains  
        pq.delMin();      largest M items  
    }  
                          ↑  
Transaction data  
type is Comparable  
(ordered by $$)
```

# Priority queue client example

---

**Challenge.** Find the largest  $M$  items in a stream of  $N$  items.

**order of growth of finding the largest  $M$  in a stream of  $N$  items**

implementation	time	space
sort	$N \log N$	$N$
elementary PQ	$M N$	$M$
binary heap	$N \log M$	$M$
best in theory	$N$	$M$

# Priority queue: unordered and ordered array implementation

<i>operation</i>	<i>argument</i>	<i>return value</i>	<i>size</i>	<i>contents (unordered)</i>	<i>contents (ordered)</i>
<i>insert</i>	P		1	P	P
<i>insert</i>	Q		2	P Q	P Q
<i>insert</i>	E		3	P Q E	E P Q
<i>remove max</i>		Q	2	P E	E P
<i>insert</i>	X		3	P E X	E P X
<i>insert</i>	A		4	P E X A	A E P X
<i>insert</i>	M		5	P E X A M	A E M P X
<i>remove max</i>		X	4	P E M A	A E M P
<i>insert</i>	P		5	P E M A P	A E M P P
<i>insert</i>	L		6	P E M A P L	A E L M P P
<i>insert</i>	E		7	P E M A P L E	A E E L M P P
<i>remove max</i>		P	6	E M A P L E	A E E L M P

A sequence of operations on a priority queue

# Priority queue: unordered array implementation

```
public class UnorderedMaxPQ<Key extends Comparable<Key>>
{
    private Key[] pq;      // pq[i] = ith element on pq
    private int N;         // number of elements on pq

    public UnorderedMaxPQ(int capacity)
    {   pq = (Key[]) new Comparable[capacity]; }

    public boolean isEmpty()
    {   return N == 0; }

    public void insert(Key x)
    {   pq[N++] = x; }

    public Key delMax()
    {
        int max = 0;
        for (int i = 1; i < N; i++)
            if (less(max, i)) max = i;
        exch(max, N-1);
        return pq[--N];
    }
}
```

no generic  
array creation

less() and exch()  
similar to sorting methods

null out entry  
to prevent loitering

# Priority queue elementary implementations

---

Challenge. Implement **all** operations efficiently.

**order of growth of running time for priority queue with N items**

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	$\log N$	$\log N$	$\log N$

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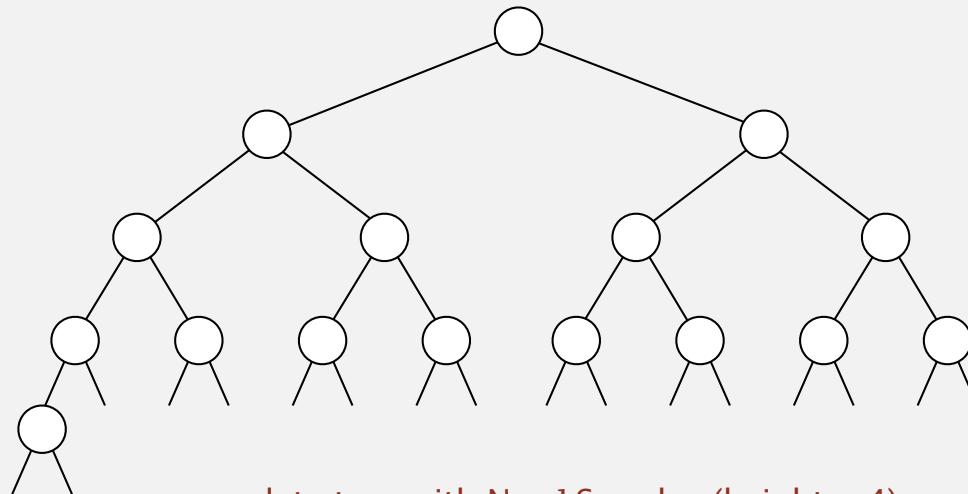
- ▶ *API and elementary implementations*
- ▶ ***binary heaps***
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Complete binary tree

---

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



Property. Height of complete tree with  $N$  nodes is  $\lfloor \lg N \rfloor$ .

Pf. Height only increases when  $N$  is a power of 2.

# A complete binary tree in nature

---



# Binary heap representations

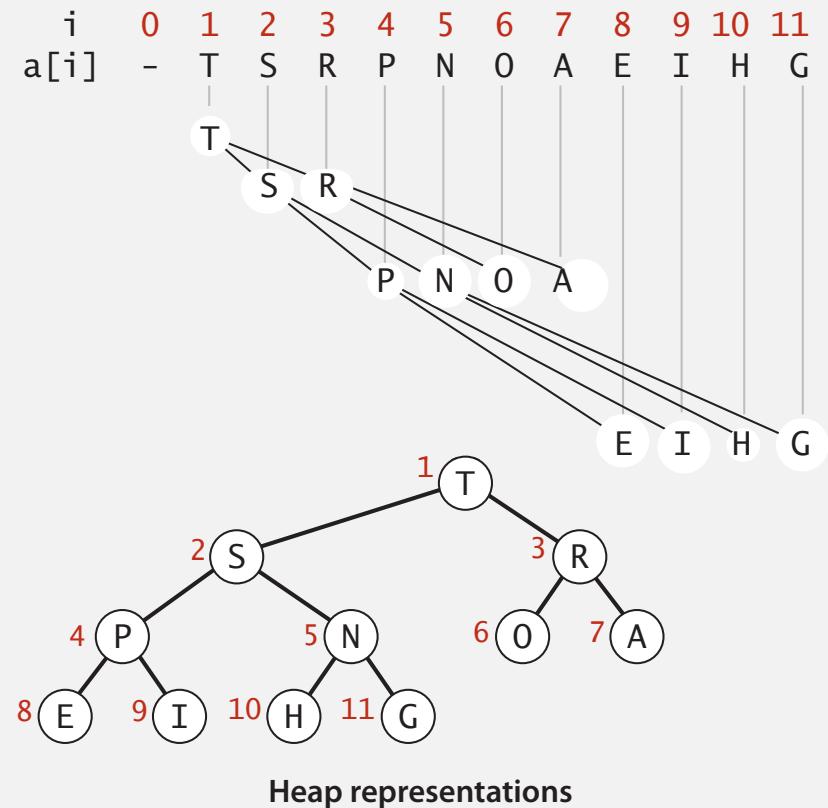
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in **level** order.
- No explicit links needed!



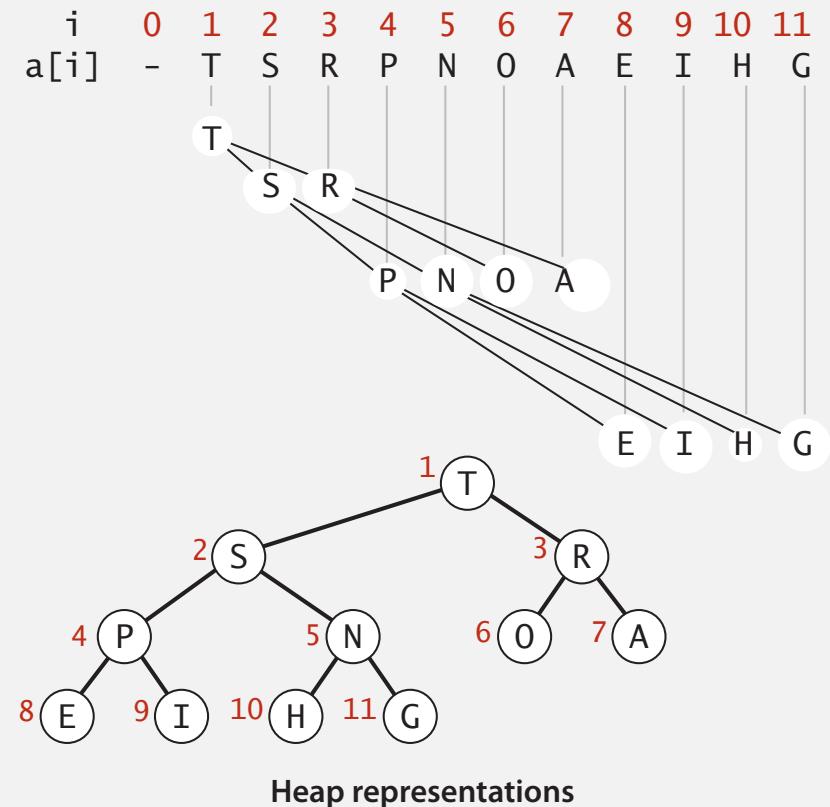
# Binary heap properties

---

Proposition. Largest key is  $a[1]$ , which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at  $k$  is at  $k/2$ .
- Children of node at  $k$  are at  $2k$  and  $2k+1$ .



# Promotion in a heap

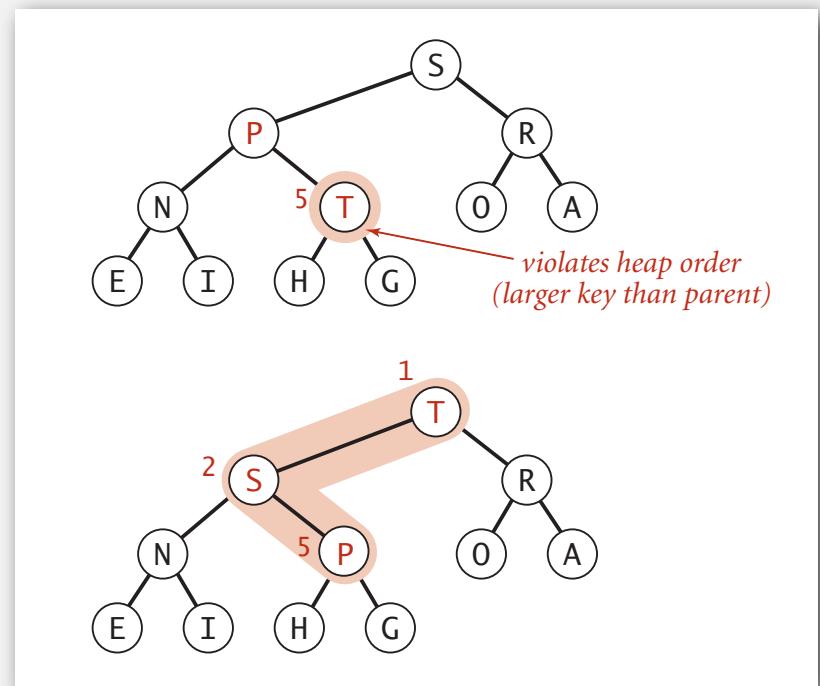
Scenario. Child's key becomes **larger** key than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
    while (k > 1 && less(k/2, k))
    {
        exch(k, k/2);
        k = k/2;
    }
}
```

parent of node at k is at k/2



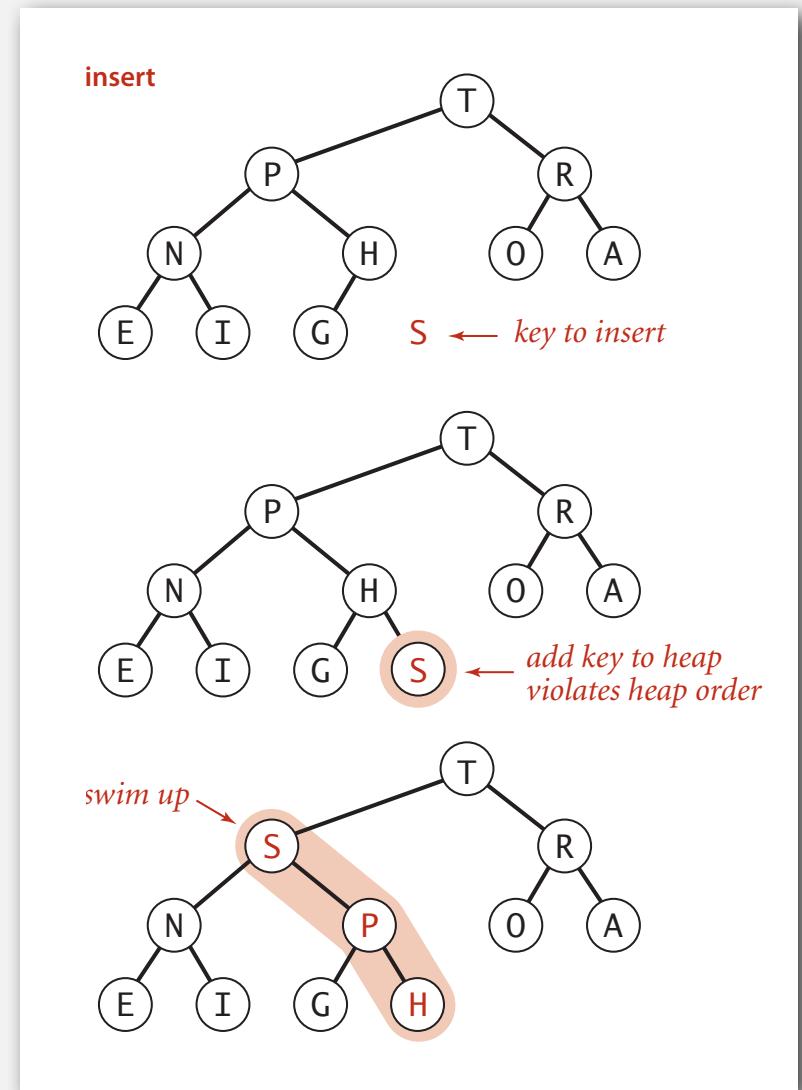
Peter principle. Node promoted to level of incompetence.

# Insertion in a heap

**Insert.** Add node at end, then swim it up.

**Cost.** At most  $1 + \lg N$  compares.

```
public void insert(Key x)
{
    pq[++N] = x;
    swim(N);
}
```



# Demotion in a heap

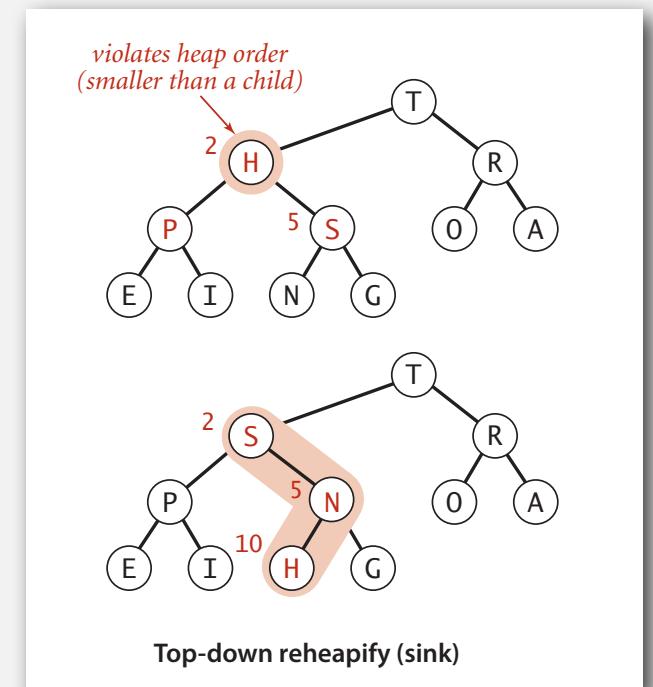
Scenario. Parent's key becomes **smaller** than one (or both) of its children's.

To eliminate the violation:

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

why not smaller child?

```
private void sink(int k)
{
    while (2*k <= N)           children of node at k
    {                           are 2k and 2k+1
        int j = 2*k;
        if (j < N && less(j, j+1)) j++;
        if (!less(k, j)) break;
        exch(k, j);
        k = j;
    }
}
```



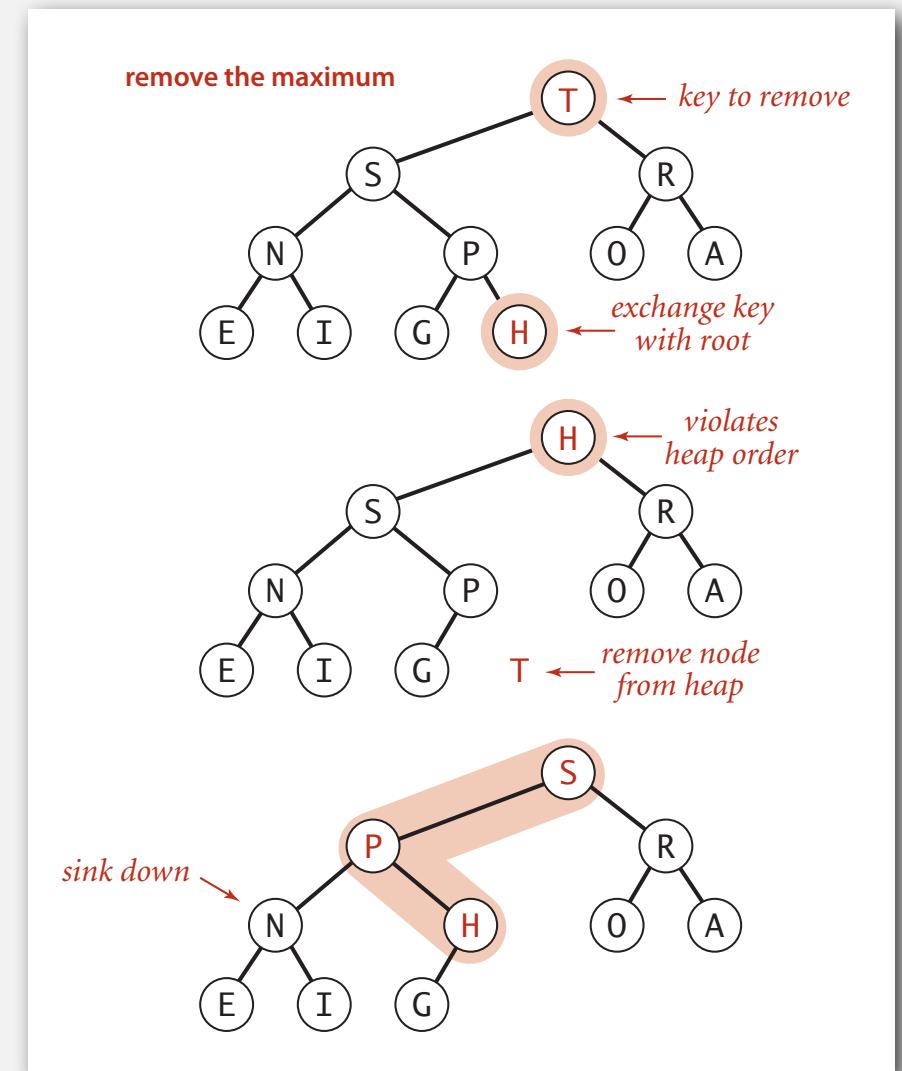
Power struggle. Better subordinate promoted.

# Delete the maximum in a heap

Delete max. Exchange root with node at end, then sink it down.

Cost. At most  $2 \lg N$  compares.

```
public Key delMax()
{
    Key max = pq[1];
    exch(1, N--);
    sink(1);
    pq[N+1] = null; ← prevent loitering
    return max;
}
```



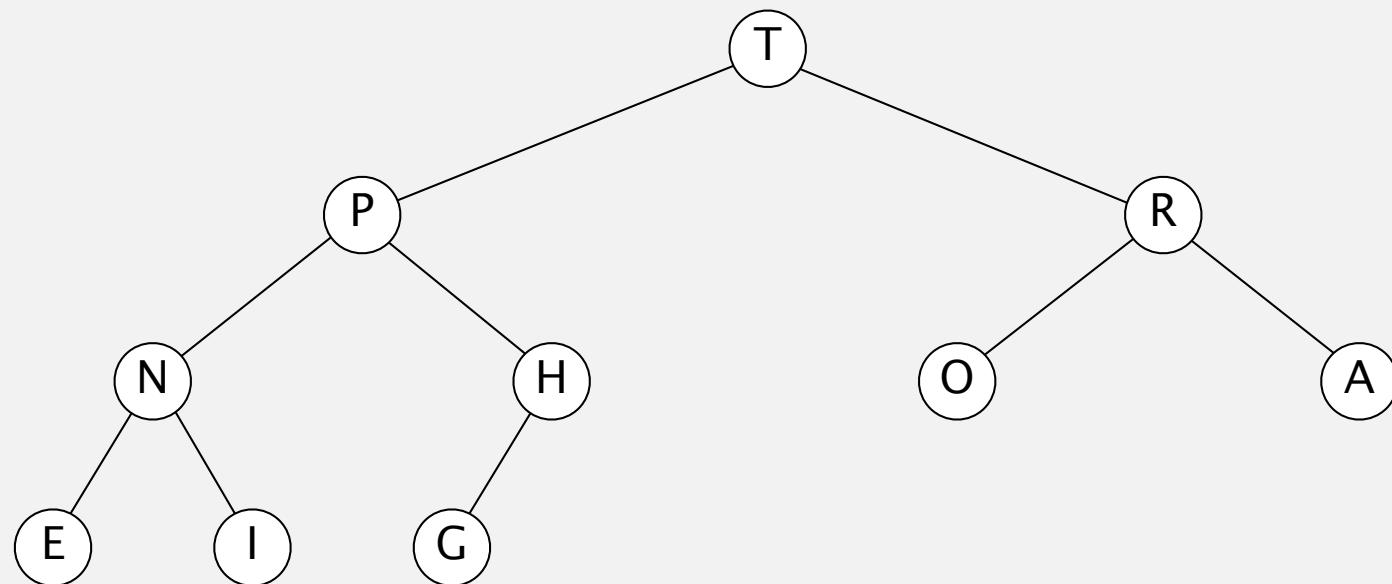
# Binary heap demo

---

**Insert.** Add node at end, then swim it up.

**Remove the maximum.** Exchange root with node at end, then sink it down.

heap ordered



T	P	R	N	H	O	A	E	I	G
---	---	---	---	---	---	---	---	---	---

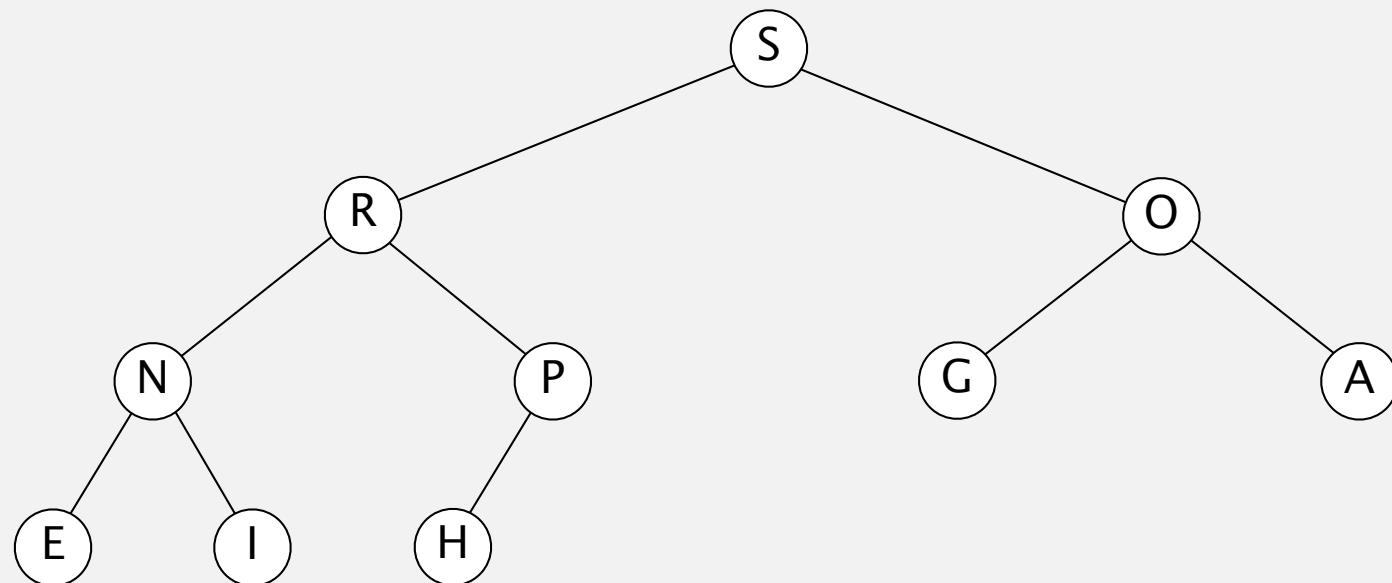
# Binary heap demo

---

**Insert.** Add node at end, then swim it up.

**Remove the maximum.** Exchange root with node at end, then sink it down.

heap ordered



# Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
    private Key[] pq;
    private int N;

    public MaxPQ(int capacity)
    {   pq = (Key[]) new Comparable[capacity+1]; }

    public boolean isEmpty()
    {   return N == 0;   }
    public void insert(Key key)
    public Key delMax()
    {   /* see previous code */ }

    private void swim(int k)
    private void sink(int k)
    {   /* see previous code */ }

    private boolean less(int i, int j)
    {   return pq[i].compareTo(pq[j]) < 0;   }
    private void exch(int i, int j)
    {   Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;   }

}
```

fixed capacity  
(for simplicity)

PQ ops

heap helper functions

array helper functions

# Priority queues implementation cost summary

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	$\log N$	$\log N$	1
d-ary heap	$\log_d N$	$d \log_d N$	1
Fibonacci	1	$\log N$ †	1
impossible	1	1	1

← why impossible?

† amortized

# Binary heap considerations

---

## Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

## Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to log N  
amortized time per op  
(how to make worst case?)

## Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

## Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement with `sink()` and `swim()` [stay tuned]

# Immutability: implementing in Java

**Data type.** Set of values and operations on those values.

**Immutable data type.** Can't change the data type value once created.

```
public final class Vector {           ← can't override instance methods
    private final int N;
    private final double[] data;        ← all instance variables private and final

    public Vector(double[] data) {
        this.N = data.length;
        this.data = new double[N];
        for (int i = 0; i < N; i++)      ← defensive copy of mutable
            this.data[i] = data[i];       ← instance variables
    }

    ...
}
```

instance methods don't change  
instance variables

**Immutable.** String, Integer, Double, Color, Vector, Transaction, Point2D.

**Mutable.** StringBuilder, Stack, Counter, Java array.

# Immutability: properties

---

**Data type.** Set of values and operations on those values.

**Immutable data type.** Can't change the data type value once created.

**Advantages.**

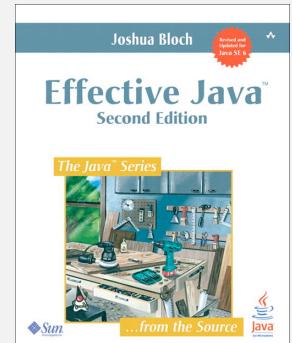
- Simplifies debugging.
- Safer in presence of hostile code.
- Simplifies concurrent programming.
- Safe to use as key in priority queue or symbol table.



**Disadvantage.** Must create new object for each data type value.

*“Classes should be immutable unless there's a very good reason to make them mutable.... If a class cannot be made immutable, you should still limit its mutability as much as possible.”*

— Joshua Bloch (Java architect)



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- ▶ *heapsort*
- ▶ *event-driven simulation*

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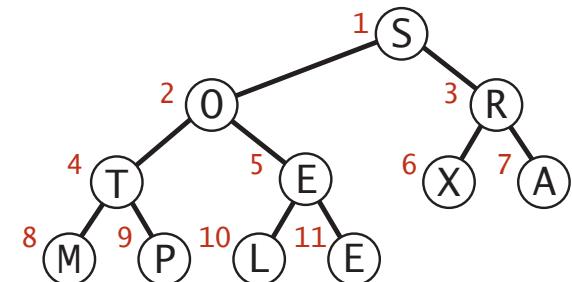
- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Heapsort

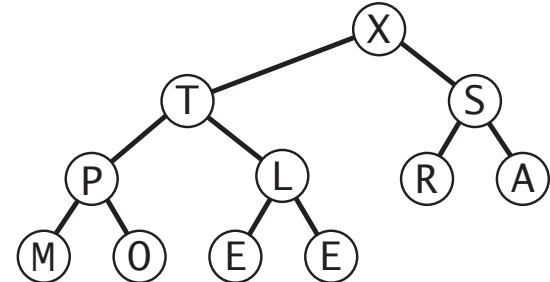
Basic plan for in-place sort.

- Create max-heap with all  $N$  keys.
- Repeatedly remove the maximum key.

start with array of keys  
in arbitrary order



build a max-heap  
(in place)



sorted result  
(in place)

1 A  
2 E  
3 E  
4 L  
5 M  
6 O  
7 P  
8 R  
9 S  
10 T  
11 X

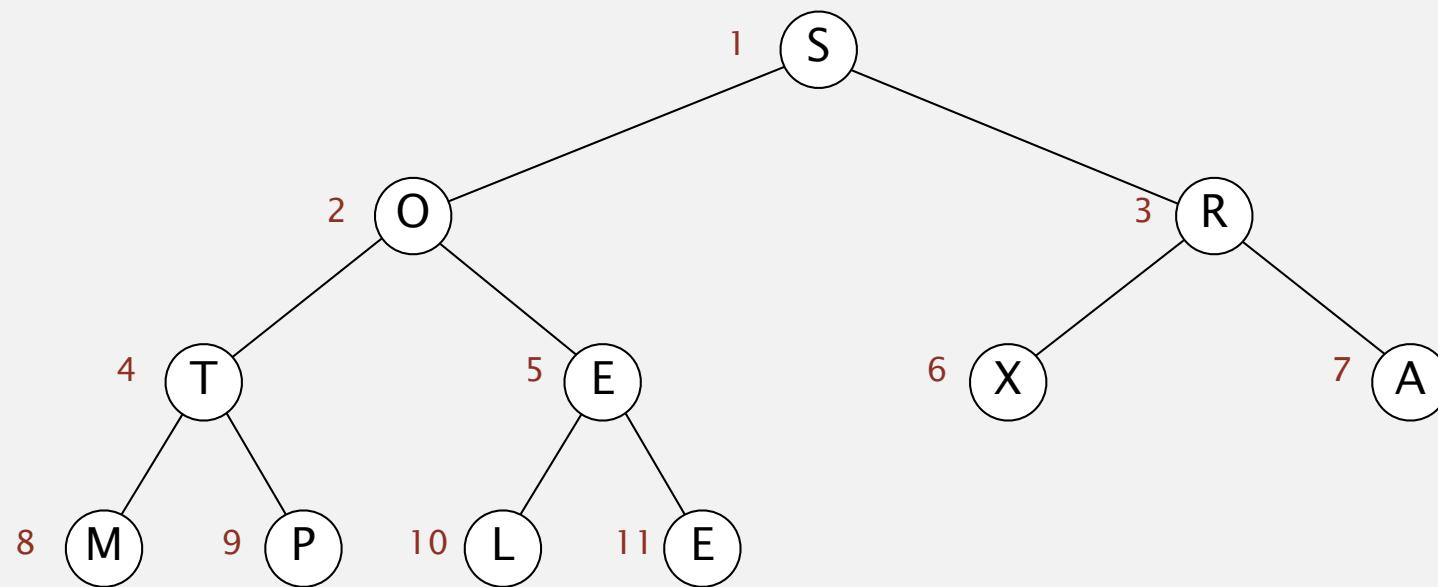
# Heapsort demo

Heap construction. Build max heap using bottom-up method.



we assume array entries are indexed 1 to N

array in arbitrary order



S	O	R	T	E	X	A	M	P	L	E
1	2	3	4	5	6	7	8	9	10	11

# Heapsort demo

---

**Sortdown.** Repeatedly delete the largest remaining item.

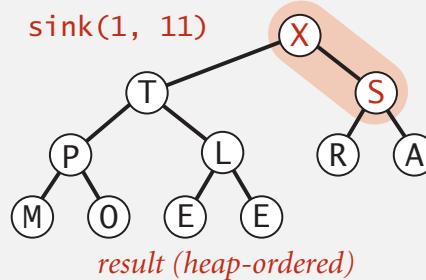
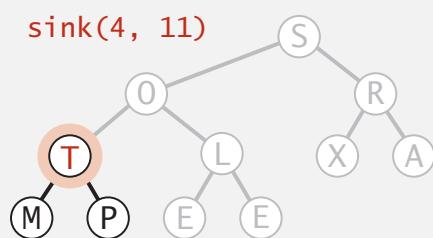
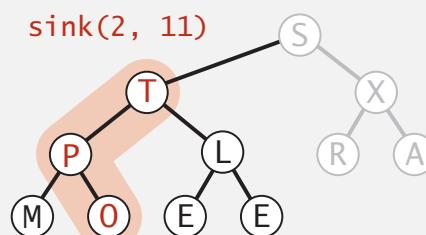
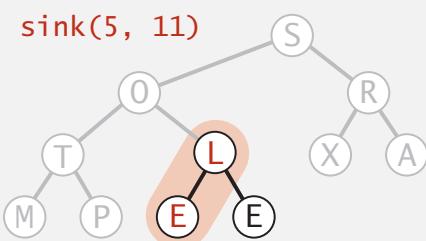
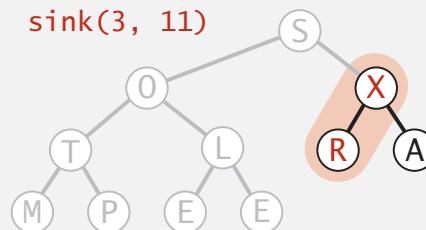
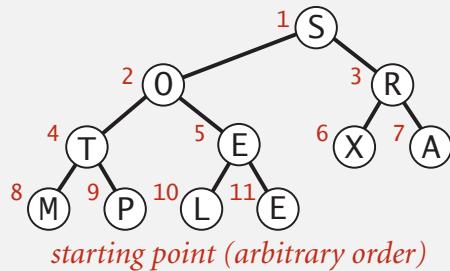
array in sorted order



# Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
    sink(a, k, N);
```

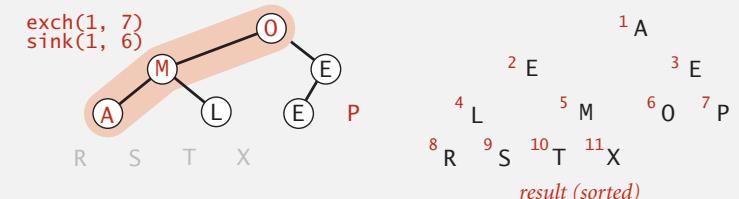
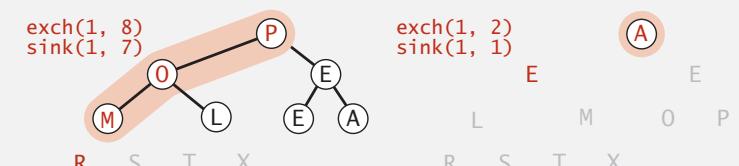
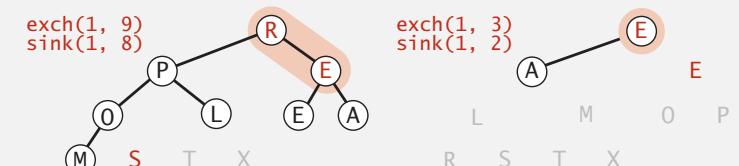
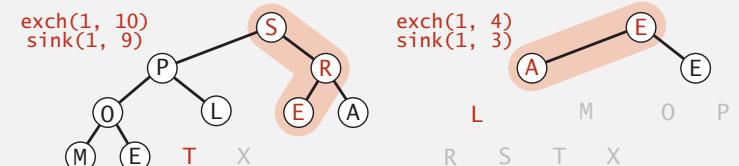
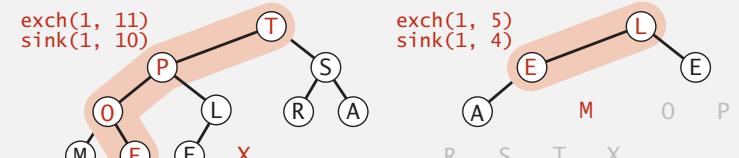
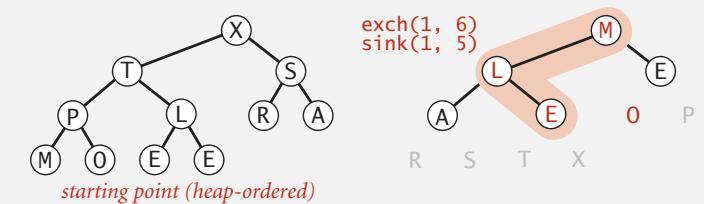


# Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
    exch(a, 1, N--);
    sink(a, 1, N);
}
```



# Heapsort: Java implementation

```
public class Heap
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int k = N/2; k >= 1; k--)
            sink(a, k, N);
        while (N > 1)
        {
            exch(a, 1, N);
            sink(a, 1, --N);
        }
    }

    private static void sink(Comparable[] a, int k, int N)
    { /* as before */ }

    private static boolean less(Comparable[] a, int i, int j)
    { /* as before */ }

    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

but convert from  
1-based indexing to  
0-base indexing

# Heapsort: trace

---

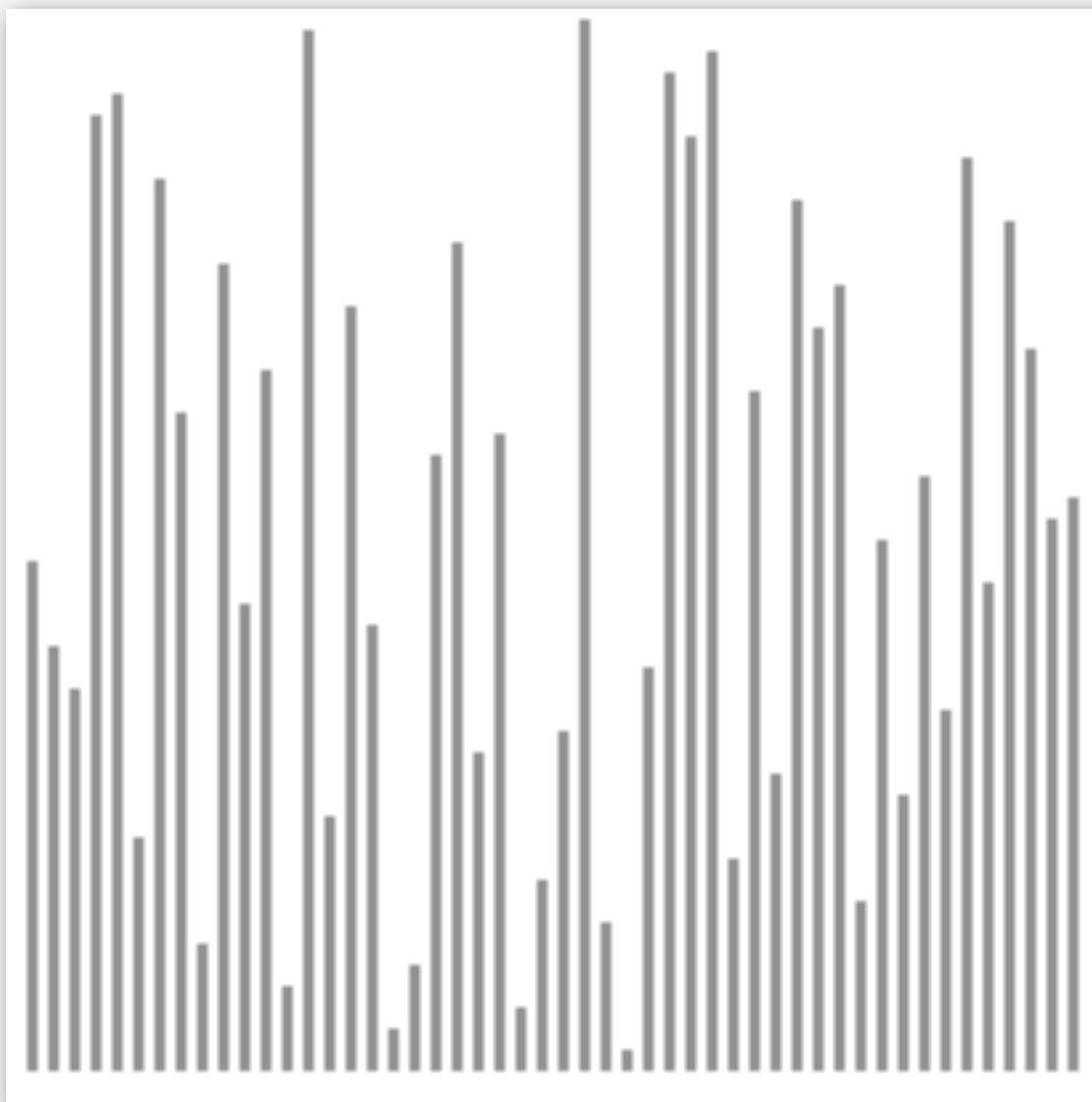
N	k	a[i]											
		0	1	2	3	4	5	6	7	8	9	10	11
<i>initial values</i>		S	O	R	T	E	X	A	M	P	L	E	
11	5	S	O	R	T	L	X	A	M	P	E	E	
11	4	S	O	R	T	L	X	A	M	P	E	E	
11	3	S	O	X	T	L	R	A	M	P	E	E	
11	2	S	T	X	P	L	R	A	M	O	E	E	
11	1	X	T	S	P	L	R	A	M	O	E	E	
<i>heap-ordered</i>		X	T	S	P	L	R	A	M	O	E	E	
10	1	T	P	S	O	L	R	A	M	E	E	X	
9	1	S	P	R	O	L	E	A	M	E	T	X	
8	1	R	P	E	O	L	E	A	M	S	T	X	
7	1	P	O	E	M	L	E	A	R	S	T	X	
6	1	O	M	E	A	L	E	P	R	S	T	X	
5	1	M	L	E	A	E	O	P	R	S	T	X	
4	1	L	E	E	A	M	O	P	R	S	T	X	
3	1	E	A	E	L	M	O	P	R	S	T	X	
2	1	E	A	E	L	M	O	P	R	S	T	X	
1	1	A	E	E	L	M	O	P	R	S	T	X	
<i>sorted result</i>		A	E	E	L	M	O	P	R	S	T	X	

Heapsort trace (array contents just after each sink)

# Heapsort animation

---

50 random items



- ▲ algorithm position
- in order
- not in order

<http://www.sorting-algorithms.com/heap-sort>

## Heapsort: mathematical analysis

---

Proposition. Heap construction uses  $\leq 2N$  compares and exchanges.

Proposition. Heapsort uses  $\leq 2N \lg N$  compares and exchanges.

Significance. In-place sorting algorithm with  $N \log N$  worst-case.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ←  $N \log N$  worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, **but**:

- Inner loop longer than quicksort's.
- Makes poor use of cache memory.
- Not stable.

# Sorting algorithms: summary

---

	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	$N$ exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
quick	x		$N^2 / 2$	$2N \ln N$	$N \lg N$	$N \log N$ probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	$2N \ln N$	N	improves quicksort in presence of duplicate keys
merge		x	$N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee, stable
heap	x		$2N \lg N$	$2N \lg N$	$N \lg N$	$N \log N$ guarantee, in-place
???	x	x	$N \lg N$	$N \lg N$	$N \lg N$	holy sorting grail

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.4 PRIORITY QUEUES

---

- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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## 2.4 PRIORITY QUEUES

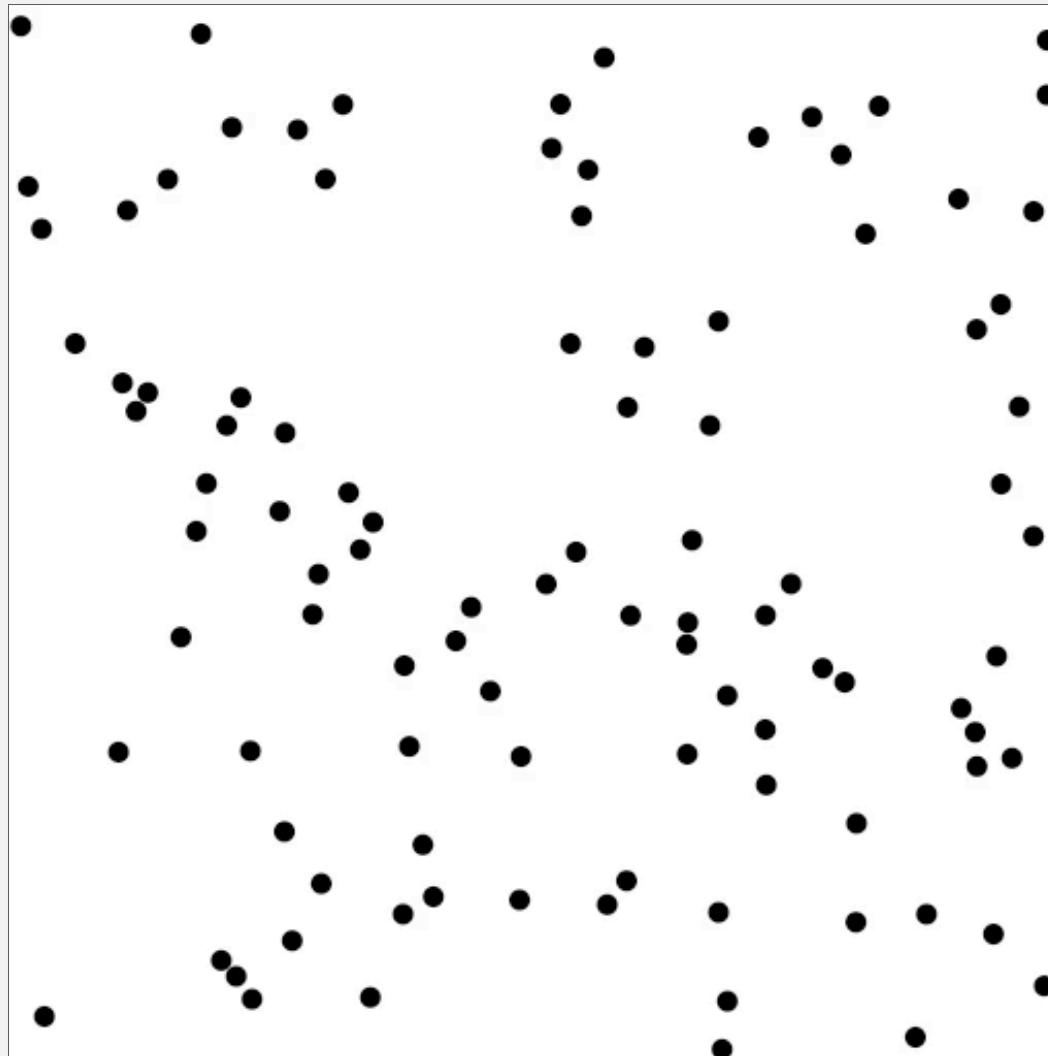
---

- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Molecular dynamics simulation of hard discs

---

**Goal.** Simulate the motion of  $N$  moving particles that behave according to the laws of elastic collision.



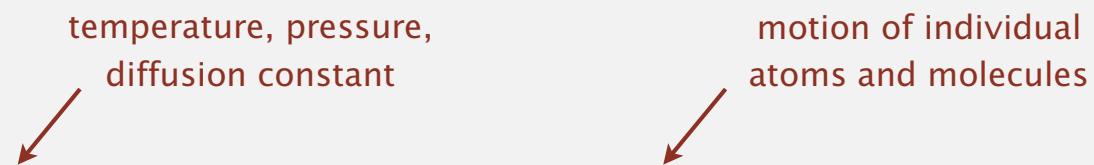
# Molecular dynamics simulation of hard discs

---

**Goal.** Simulate the motion of  $N$  moving particles that behave according to the laws of elastic collision.

## Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.



**Significance.** Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

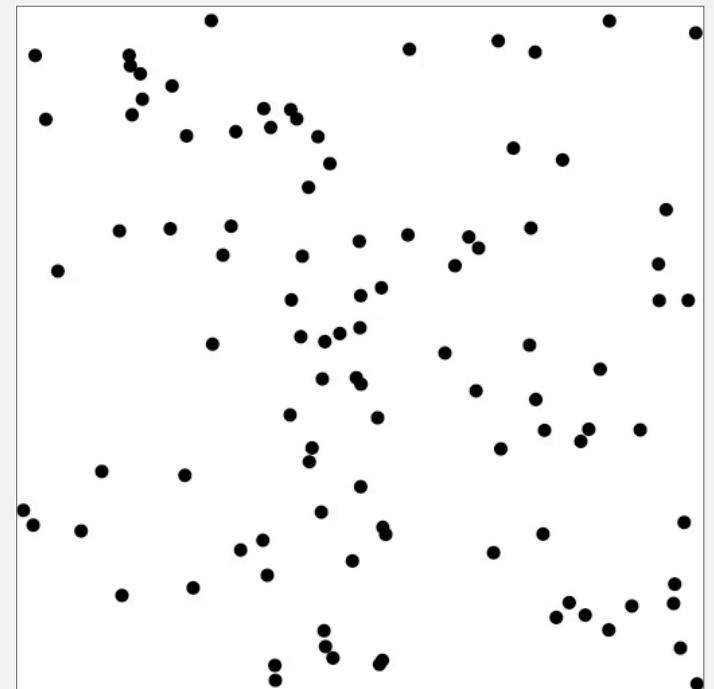
# Warmup: bouncing balls

Time-driven simulation.  $N$  bouncing balls in the unit square.

```
public class BouncingBalls
{
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        Ball[] balls = new Ball[N];
        for (int i = 0; i < N; i++)
            balls[i] = new Ball();
        while(true)
        {
            StdDraw.clear();
            for (int i = 0; i < N; i++)
            {
                balls[i].move(0.5);
                balls[i].draw();
            }
            StdDraw.show(50);
        }
    }
}
```

main simulation loop

```
% java BouncingBalls 100
```



## Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;          // position
    private double vx, vy;          // velocity
    private final double radius;    // radius
    public Ball(...)
    { /* initialize position and velocity */ }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }

    public void draw()
    { StdDraw.filledCircle(rx, ry, radius); }
}
```

check for collision with walls



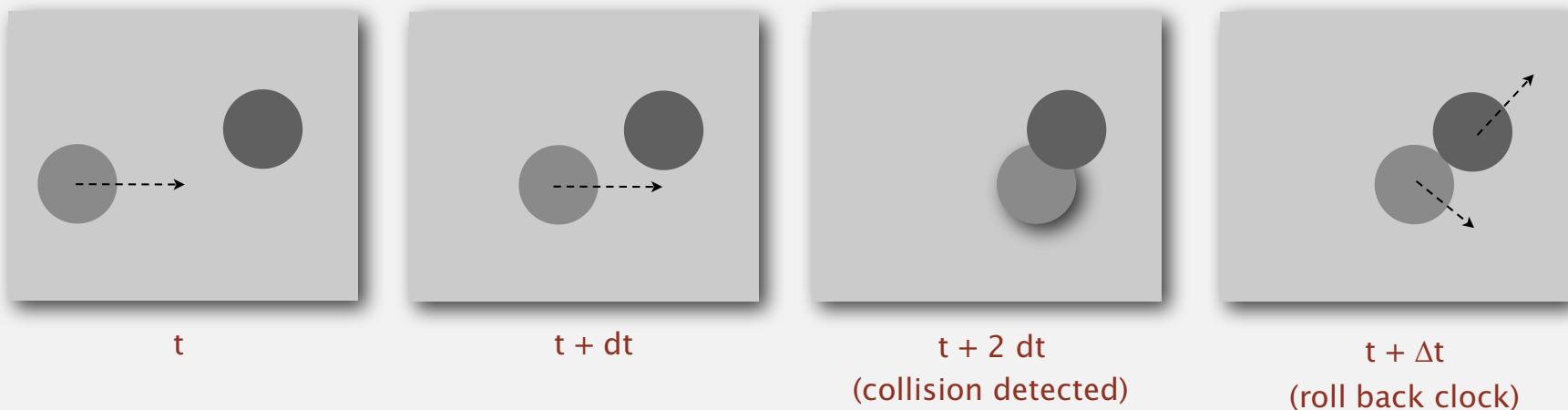
Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: which object does the check? too many checks?

# Time-driven simulation

---

- Discretize time in quanta of size  $dt$ .
- Update the position of each particle after every  $dt$  units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.

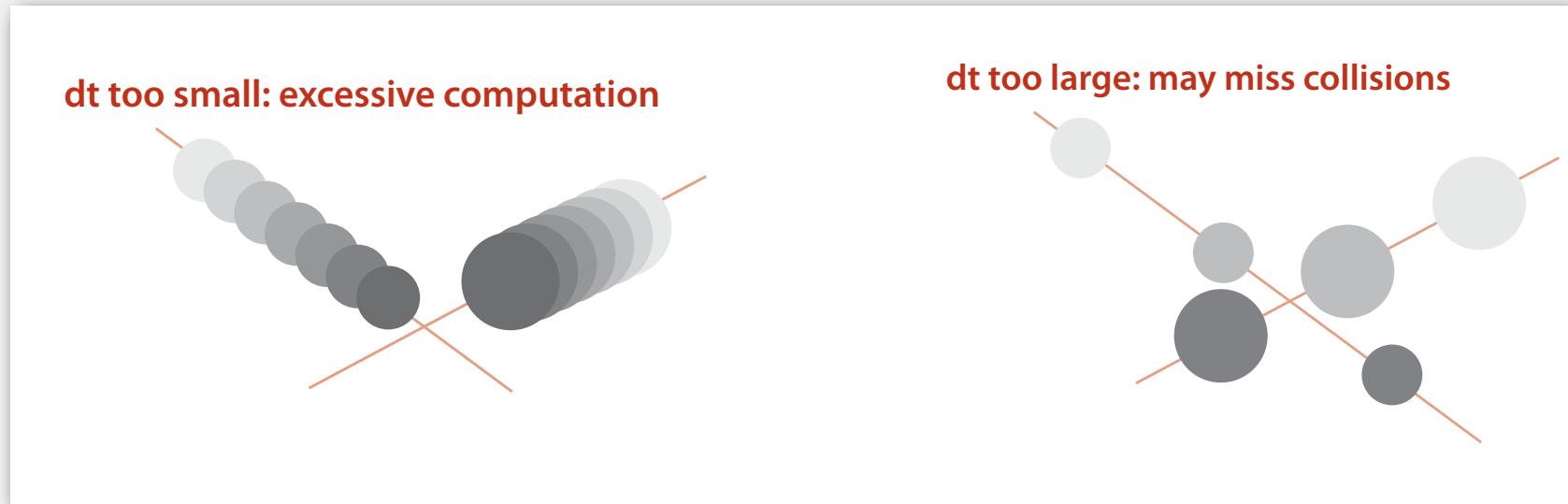


# Time-driven simulation

---

## Main drawbacks.

- $\sim N^2 / 2$  overlap checks per time quantum.
- Simulation is too slow if  $dt$  is very small.
- May miss collisions if  $dt$  is too large.  
(if colliding particles fail to overlap when we are looking)



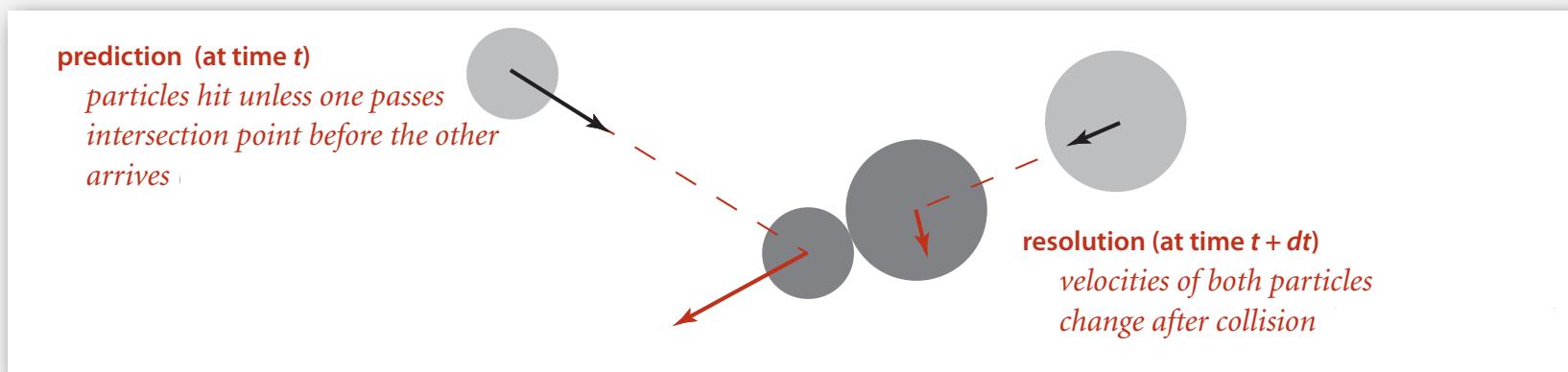
# Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Remove the min = get next collision.

**Collision prediction.** Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

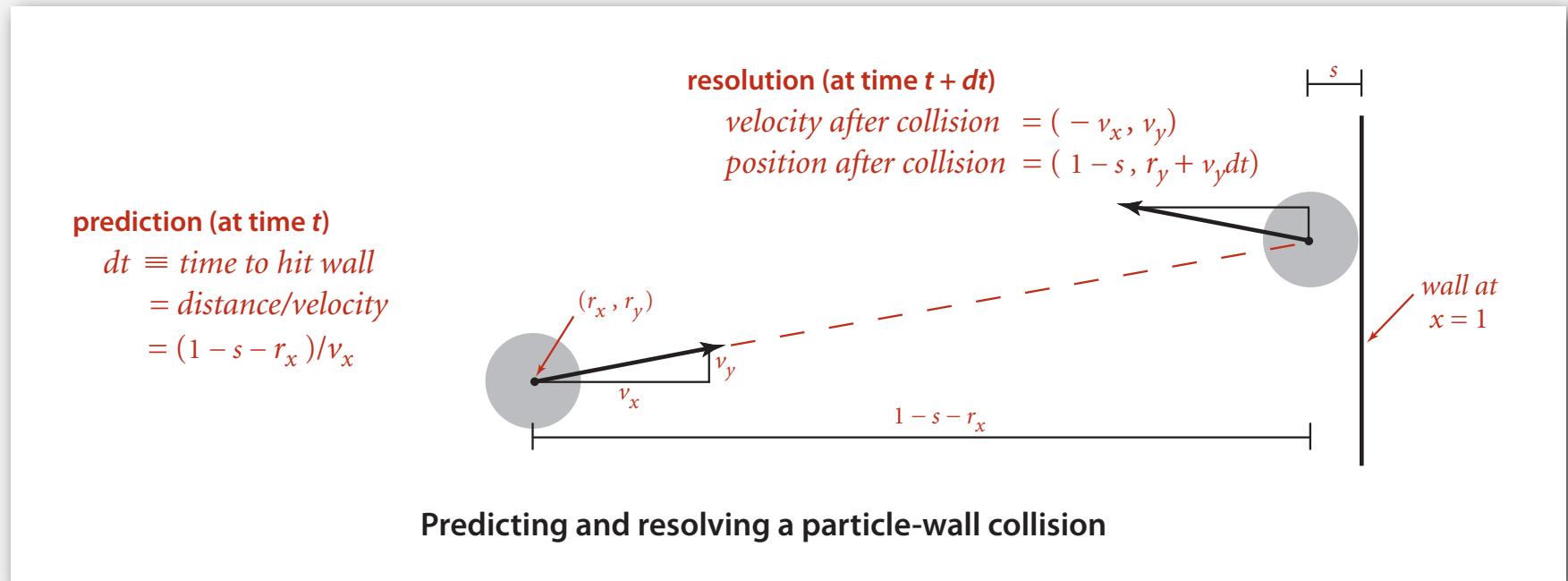
**Collision resolution.** If collision occurs, update colliding particle(s) according to laws of elastic collisions.



# Particle-wall collision

## Collision prediction and resolution.

- Particle of radius  $s$  at position  $(rx, ry)$ .
- Particle moving in unit box with velocity  $(vx, vy)$ .
- Will it collide with a vertical wall? If so, when?

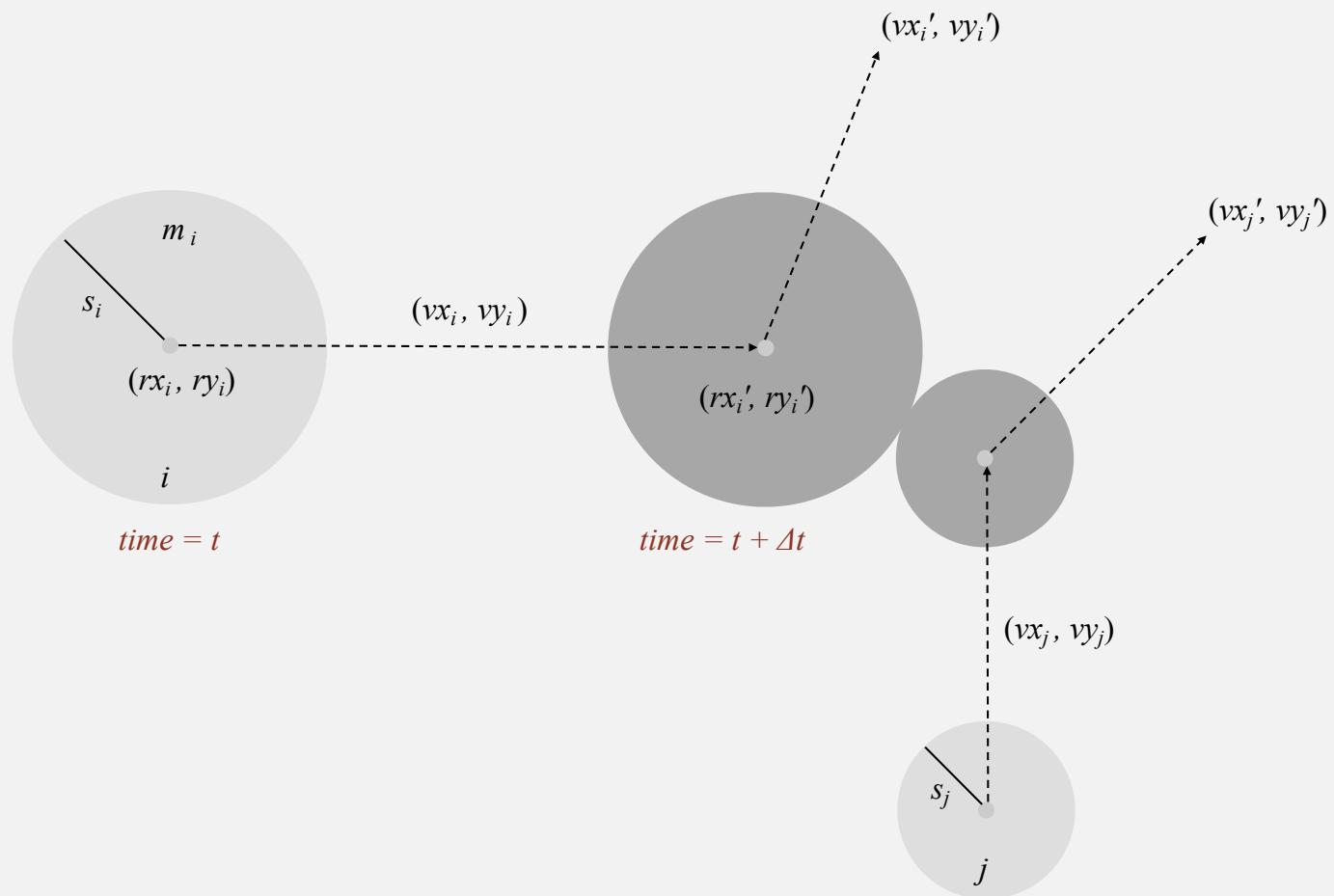


# Particle-particle collision prediction

---

## Collision prediction.

- Particle  $i$ : radius  $s_i$ , position  $(rx_i, ry_i)$ , velocity  $(vx_i, vy_i)$ .
- Particle  $j$ : radius  $s_j$ , position  $(rx_j, ry_j)$ , velocity  $(vx_j, vy_j)$ .
- Will particles  $i$  and  $j$  collide? If so, when?



# Particle-particle collision prediction

## Collision prediction.

- Particle  $i$ : radius  $s_i$ , position  $(rx_i, ry_i)$ , velocity  $(vx_i, vy_i)$ .
- Particle  $j$ : radius  $s_j$ , position  $(rx_j, ry_j)$ , velocity  $(vx_j, vy_j)$ .
- Will particles  $i$  and  $j$  collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0 \\ \infty & \text{if } d < 0 \\ -\frac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v) (\Delta r \cdot \Delta r - \sigma^2) \quad \sigma = \sigma_i + \sigma_j$$

$$\Delta v = (\Delta vx, \Delta vy) = (vx_i - vx_j, vy_i - vy_j)$$

$$\Delta r = (\Delta rx, \Delta ry) = (rx_i - rx_j, ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$

$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$

$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

**Important note:** This is high-school physics, so we won't be testing you on it!

# Particle-particle collision resolution

**Collision resolution.** When two particles collide, how does velocity change?

$$\begin{aligned} vx_i' &= vx_i + Jx / m_i \\ vy_i' &= vy_i + Jy / m_i \\ vx_j' &= vx_j - Jx / m_j \\ vy_j' &= vy_j - Jy / m_j \end{aligned}$$

Newton's second law  
(momentum form)

$$Jx = \frac{J \Delta rx}{\sigma}, \quad Jy = \frac{J \Delta ry}{\sigma}, \quad J = \frac{2m_i m_j (\Delta v \cdot \Delta r)}{\sigma(m_i + m_j)}$$

impulse due to normal force

(conservation of energy, conservation of momentum)

**Important note:** This is high-school physics, so we won't be testing you on it!

# Particle data type skeleton

```
public class Particle
{
    private double rx, ry;          // position
    private double vx, vy;          // velocity
    private final double radius;    // radius
    private final double mass;      // mass
    private int count;              // number of collisions

    public Particle(...) { }

    public void move(double dt) { }
    public void draw() { }

    public double timeToHit(Particle that) { }
    public double timeToHitVerticalWall() { }
    public double timeToHitHorizontalWall() { }

    public void bounceOff(Particle that) { }
    public void bounceOffVerticalWall() { }
    public void bounceOffHorizontalWall() { }

}
```

predict collision  
with particle or wall

resolve collision  
with particle or wall

# Particle-particle collision and resolution implementation

```
public double timeToHit(Particle that)
{
    if (this == that) return INFINITY;
    double dx = that.rx - this.rx, dy = that.ry - this.ry;
    double dvx = that.vx - this.vx; dvy = that.vy - this.vy;
    double dvdr = dx*dvx + dy*dvy;
    if( dvdr > 0) return INFINITY; ← no collision
    double dvdv = dvx*dvx + dvy*dvy;
    double drdr = dx*dx + dy*dy;
    double sigma = this.radius + that.radius;
    double d = (dvdr*dvdr) - dvdv * (drdr - sigma*sigma);
    if (d < 0) return INFINITY;
    return -(dvdr + Math.sqrt(d)) / dvdv;
}
```

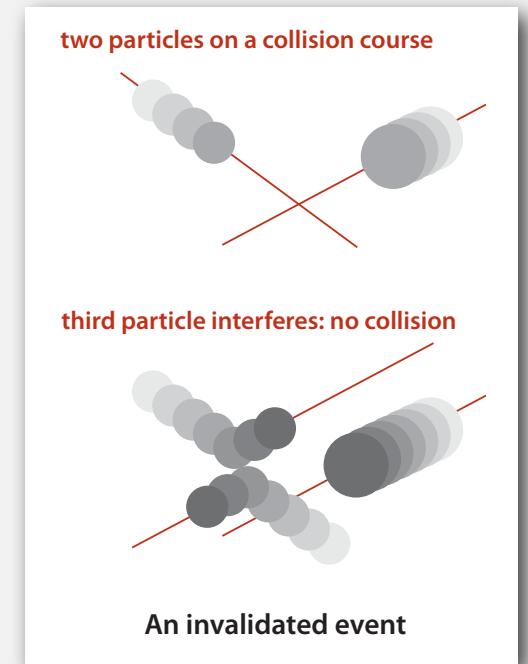
```
public void bounceOff(Particle that)
{
    double dx = that.rx - this.rx, dy = that.ry - this.ry;
    double dvx = that.vx - this.vx, dvy = that.vy - this.vy;
    double dvdr = dx*dvx + dy*dvy;
    double dist = this.radius + that.radius;
    double J = 2 * this.mass * that.mass * dvdr / ((this.mass + that.mass) * dist);
    double Jx = J * dx / dist;
    double Jy = J * dy / dist;
    this.vx += Jx / this.mass;
    this.vy += Jy / this.mass;
    that.vx -= Jx / that.mass;
    that.vy -= Jy / that.mass;
    this.count++;
    that.count++;     Important note: This is high-school physics, so we won't be testing you on it!
}
```

# Collision system: event-driven simulation main loop

## Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

“potential” since collision may not happen if some other collision intervenes



## Main loop.

- Delete the impending event from PQ (min priority =  $t$ ).
- If the event has been invalidated, ignore it.
- Advance all particles to time  $t$ , on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

# Event data type

## Conventions.

- Neither particle null  $\Rightarrow$  particle-particle collision.
- One particle null  $\Rightarrow$  particle-wall collision.
- Both particles null  $\Rightarrow$  redraw event.

```
private class Event implements Comparable<Event>
{
    private double time;          // time of event
    private Particle a, b;        // particles involved in event
    private int countA, countB;   // collision counts for a and b

    public Event(double t, Particle a, Particle b) { }           ← create event

    public int compareTo(Event that)
    {   return this.time - that.time;   }                         ← ordered by time

    public boolean isValid()
    {   }
}
```

invalid if intervening collision

## Collision system implementation: skeleton

```
public class CollisionSystem
{
    private MinPQ<Event> pq;           // the priority queue
    private double t = 0.0;              // simulation clock time
    private Particle[] particles;       // the array of particles

    public CollisionSystem(Particle[] particles) { }

    private void predict(Particle a)      add to PQ all particle-wall and particle-
    {                                     -particle collisions involving this particle
        if (a == null) return;
        for (int i = 0; i < N; i++)
        {
            double dt = a.timeToHit(particles[i]);
            pq.insert(new Event(t + dt, a, particles[i]));
        }
        pq.insert(new Event(t + a.timeToHitVerticalWall() , a, null));
        pq.insert(new Event(t + a.timeToHitHorizontalWall(), null, a));
    }

    private void redraw() { }

    public void simulate() { /* see next slide */ }
}
```

# Collision system implementation: main event-driven simulation loop

```
public void simulate()
{
    pq = new MinPQ<Event>();
    for(int i = 0; i < N; i++) predict(particles[i]);
    pq.insert(new Event(0, null, null));
```

initialize PQ with collision events and redraw event

```
while(!pq.isEmpty())
{
    Event event = pq.delMin();
    if(!event.isValid()) continue;
    Particle a = event.a;
    Particle b = event.b;
```

get next event

```
for(int i = 0; i < N;
     particles[i].move(event.time - t);
    t = event.time;
```

update positions and time

```
if      (a != null && b != null) a.bounceOff(b);
else if (a != null && b == null) a.bounceOffVerticalWall()
else if (a == null && b != null) b.bounceOffHorizontalWall();
else if (a == null && b == null) redraw();
```

process event

```
predict(a);
predict(b);
```

predict new events based on changes

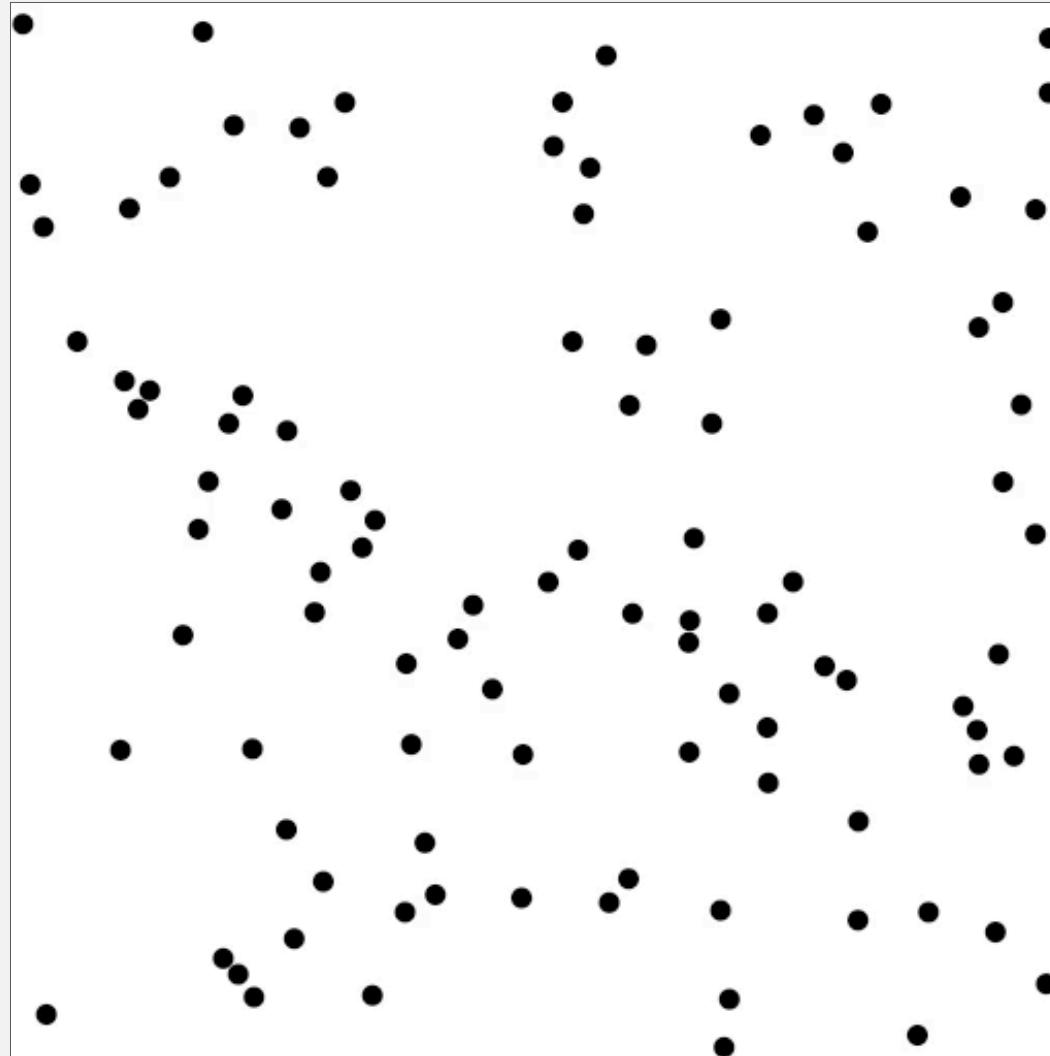
}

}

# Particle collision simulation example 1

---

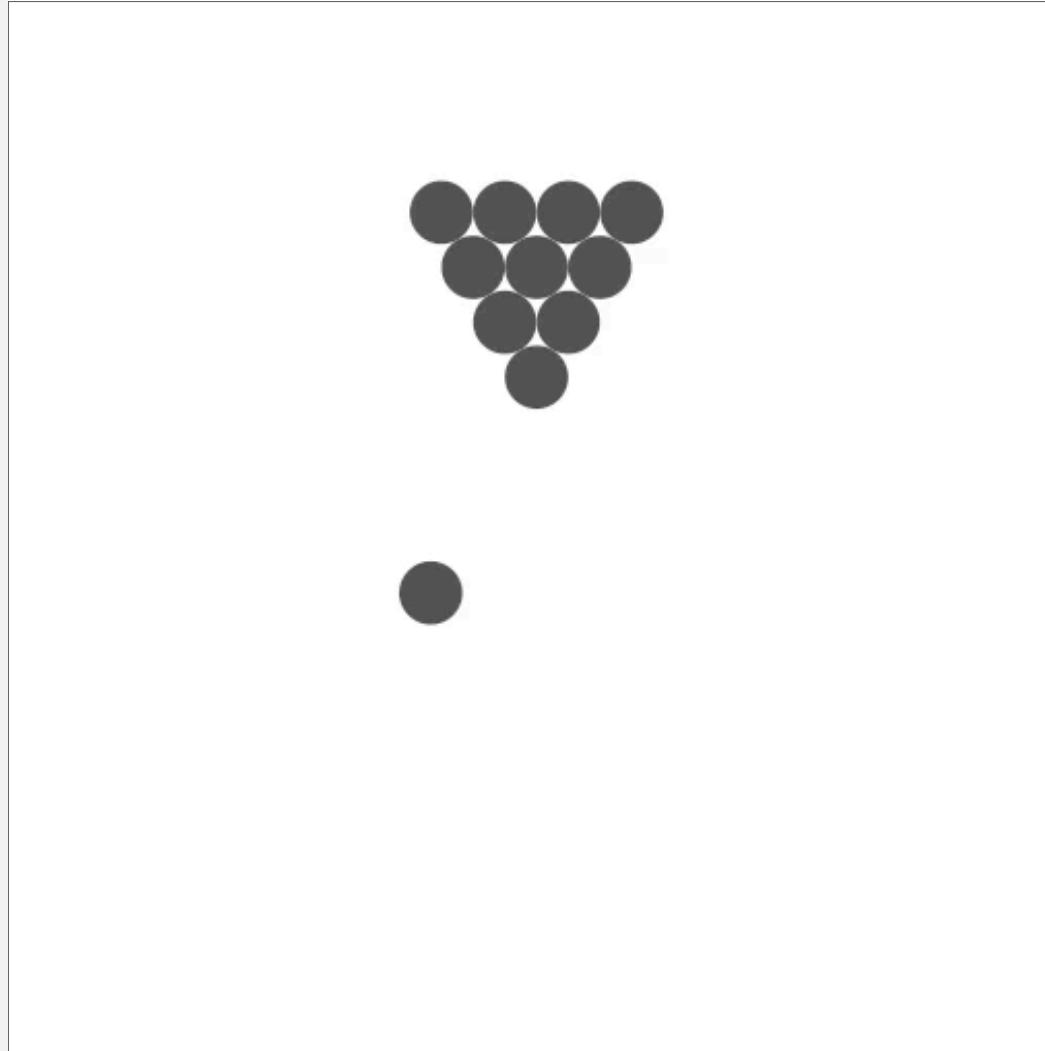
```
% java CollisionSystem 100
```



## Particle collision simulation example 2

---

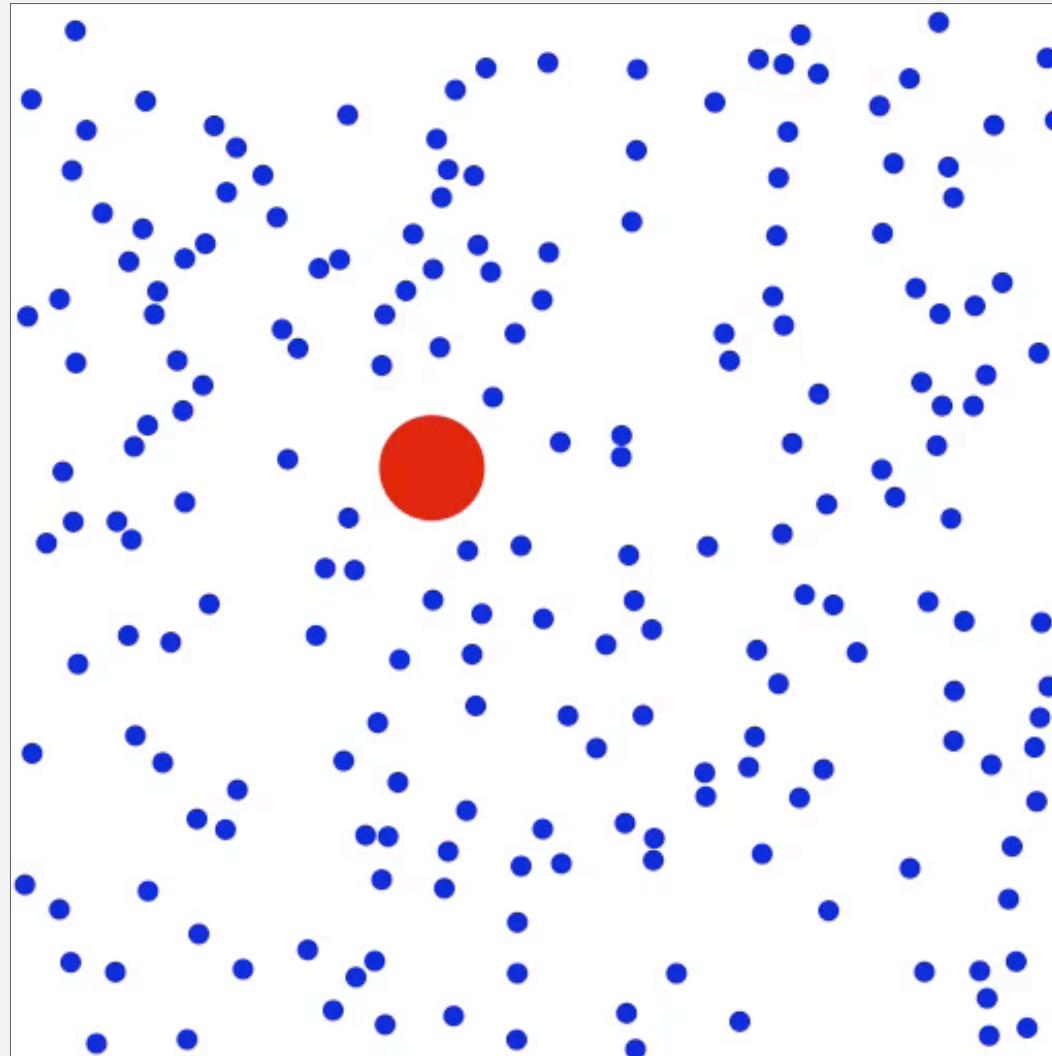
```
% java CollisionSystem < billiards.txt
```



## Particle collision simulation example 3

---

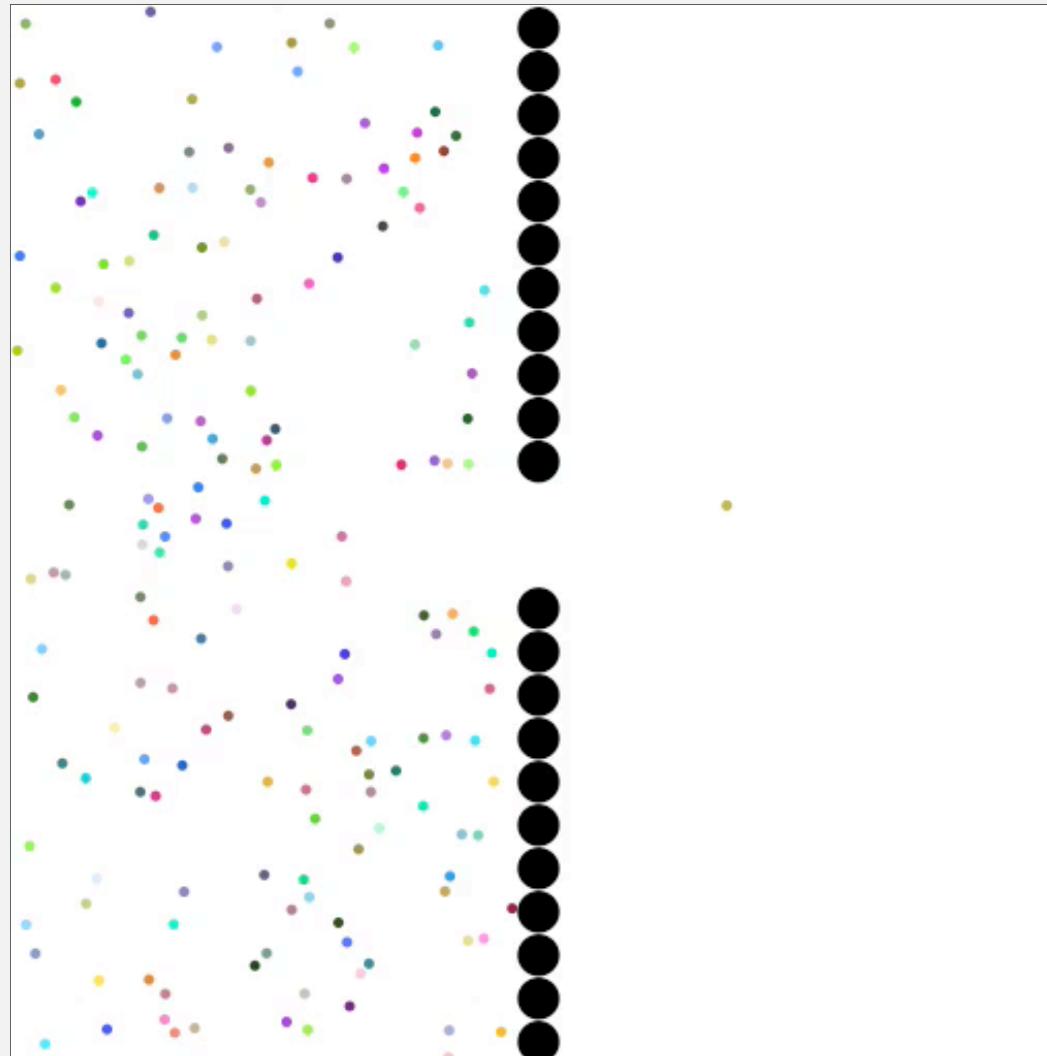
```
% java CollisionSystem < brownian.txt
```



# Particle collision simulation example 4

---

```
% java CollisionSystem < diffusion.txt
```



# Algorithms

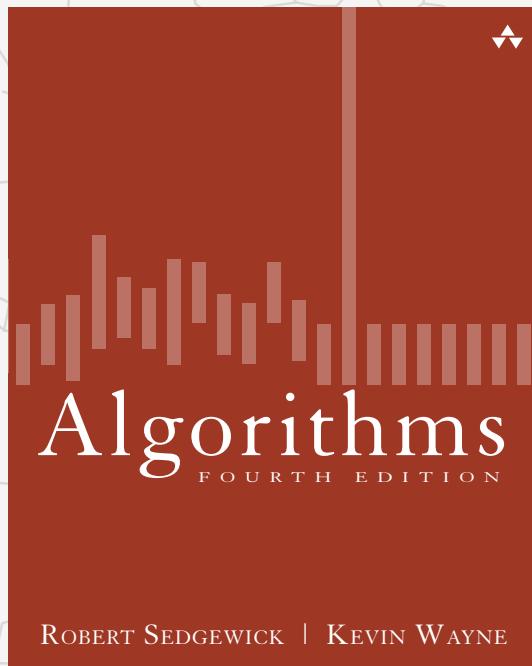
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- ▶ *heapsort*
- ▶ *event-driven simulation*



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