Approximation by Simple Poles – Part II: System Level Synthesis Beyond Finite Impulse Response

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Abstract—**In Part I, a novel Galerkin-type method for finite dimensional approximations of transfer functions in Hardy space was developed based on approximation by simple poles. In Part II, this approximation is applied to system level synthesis, a recent approach based on a clever reparameterization, to develop a new technique for optimal control design. To solve system level synthesis problems, prior work relies on finite impulse response approximations that lead to deadbeat control, and that can experience infeasibility and increased suboptimality, especially in systems with large separation of time scales. The new design method does not result in deadbeat control, is convex and tractable, always feasible, can incorporate prior knowledge, and works well for systems with large separation of time scales. Suboptimality bounds with convergence rate depending on the geometry of the pole selection are provided. An example demonstrates superior performance of the method.**

Index Terms—**System level synthesis, optimal control, H_∞ control, optimization**

I. Introduction

In Part I [1] the approximation by a finite collection of transfer functions with simple poles was studied as a Galerkin-type method for approximating transfer functions in Hardy space. The present paper applies this simple pole approximation (SPA) to optimal design of linear feedback controllers. A powerful approach for solving optimal control problems involves not directly optimizing over the controller, but rather over an affine function of a closed-loop transfer function (which depends implicitly on the controller), and then recovering the optimal controller that realizes this closed-loop behavior afterwards. Examples include sensitivity minimization [2], the Youla parameterization [3], Q-parameterization [4], input-ouput parameterization (IOP) [5], and system level synthesis (SLS) [6], [7]. For the present paper, we focus on the closed-loop system responses for state feedback controllers, and so restrict our attention to SLS rather than sensitivity minimization (which minimizes a different closed-loop transfer function), Youla or Q-parameterization (which do not directly parameterize using the closed-loop transfer function) or IOP (which focuses on output feedback).

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis is valuable for applications and has a long history (see, e.g., [8]). However, it remains challenging to solve efficiently as methods for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis alone do not readily yield optimal solutions to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis. The SLS reparameterization for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis results in a convex but infinite dimensional optimization problem. In order to solve it, prior work [9] has approximated that the closed-loop responses are finite impulse responses (FIR) in order to arrive at a tractable finite dimensional optimization problem. However, this results in deadbeat control (DBC), which often experiences poorly damped oscillations between discrete sampling times that can even persist in steady state, as well as lack of robustness to model uncertainty and parameter variations because of the high control gains required to reach the origin in finite time [10]. We denote SLS with the FIR approximation by DBC for the remainder of the paper.

With DBC, the number of poles in the closed-loop transfer functions is equal to the length of the FIR, potentially resulting in large numbers of poles that can lead to high computational complexity for the control design, lack of robustness in the resulting controller, and implementation challenges in practice [11, Chapter 19]. This is especially problematic when the optimal solution has a long settling time, such as in systems with large separation of time scales, where short sampling times are needed to capture the fast dynamics, which are also coupled with much slower dynamics. This leads to closed-loop impulse responses settling only after a large number of time steps. In addition, FIR closed-loop responses have all poles at the origin, which results in infeasibility in case of stable but uncontrollable poles in the plant. To resolve this, DBC introduces a slack variable enabling constraint violation, which leads to additional suboptimality [9]. Furthermore, in this case DBC leads to a quasi-convex problem, requiring an iterative approach such as golden section search to solve rather than a single convex optimization [9]. This approach then requires inversion of a transfer function with order equal to the length of the FIR, which is potentially large and may be numerically unstable, to recover the optimal closed-loop transfer functions and resulting controller [9].

The present paper combines SLS with SPA [1] to develop a new control method which addresses these limitations. This approach is not FIR, so it does not suffer from the drawbacks of deadbeat control. Moreover, the number of poles is independent of the settling time of the optimal closed-loop responses, and therefore SPA even works well for systems with large separation of timescales. It results in a convex and tractable optimization for the design, avoiding the need for iterative methods, does not need to invert transfer functions to recover the optimal closed-loop solution, requires only a small number of poles, guarantees feasibility for stabilizable

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systems without introducing slack variables, and additional suboptimality resulting from these can be avoided. Finally, if prior information is known about the optimal solution, such as the locations of some of the optimal poles (e.g., for model matching [12], model reference control [13], design based on the internal model principle [14], expensive control [15, Theorem 3.12(b)], etc.), then these can be incorporated directly into the design for improved performance.

A suboptimality certificate is provided which shows the convergence rate of SPA to the ground-truth optimal solution based on the geometry of the pole selection. Unlike a similar certificate for DBC, this does not require a long enough time horizon for the optimal impulse response to decay to be valid, and its convergence rate does not depend on this decay rate. This certificate is then specialized to a particular pole selection based on an Archimedes spiral as in [1, Theorem 4]. An example shows superior performance of SPA over DBC, and is fully reproducible with all code publicly available [16].

The paper is organized as follows. Section II provides preliminaries and problem setup, Section III provides the SPA method and suboptimality certificates, Section IV shows an illustrative example, Section V gives the proofs, and Section VI offers concluding remarks.

II. Preliminaries

We use the same notation as in Part I [1], and refer the reader to the preliminaries and main results sections there for further details. Recall also Assumptions A1-A5 from Part I.

Consider the following LTI system in discrete time

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \hat{B}w(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^p$, $w(k) \in \mathbb{R}^q$, and $y(k) \in \mathbb{R}^m$ are the state, controller input, disturbance input, and performance output vectors at time step k, respectively. Let σ be the stable plant poles (i.e., the stable eigenvalues of A). It will be useful to introduce the following related system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

where $v(k) \in \mathbb{R}^n$ and the other signals are defined analogously to (1). Consider a linear (possibly dynamic) state feedback control law of the form $u(z) = K(z)x(z)$ where $K \in \mathcal{RH}_\infty$, and let $T_{\text{desired}}(z)$ be some desired closed-loop transfer function for model reference or model matching control (note that we can set $T_{\text{desired}}(z) = 0$ if desired). For any signals $a(z)$ and $b(z)$, let $T_{a\to b}(z)$ denote the closed-loop transfer function from $a(z)$ to $b(z)$. For any transfer function Φ, let $\mathcal{I}(\Phi)$ and $\mathcal{C}(\Phi)$ denote its impulse response and convolution (i.e. causal Toeplitz) operators, respectively (see [1] for more details). The goal is to choose a controller $K(z)$ that is a solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ [11], [17] optimal control problem given by

$$\begin{aligned} \min_{K(z)} \; & ||T_{w\to y}(z) - T_{\text{desired}}(z)||_{\mathcal{H}_2} \\ & + \lambda\, ||T_{w\to y}(z) - T_{\text{desired}}(z)||_{\mathcal{H}_\infty} \\ \text{s.t.} \quad & T_{v\to x}(z), T_{v\to u}(z) \in \frac{1}{z}\mathcal{RH}_\infty, \end{aligned} \tag{2}$$

where $\lambda \in [0, \infty]$ is constant. As $T_{w\to y}(z)$ is nonconvex in $K(z)$, (2) is known to be a challenging problem. We make the following feasibility assumption:

(A6) A solution to (2) *exists, i.e.,* (A, B) *is stabilizable, and the optimal closed-loop transfer functions are rational (hence they have finitely many poles).*

While one can construct pathological examples where this assumption does not hold (e.g., a controllable SISO system with $y = x$ and $T_{\text{desired}}(z) = e^{\frac{1}{z}}$), in the standard mixed $\mathcal{H}_2/\mathcal{H}_\infty$ setting Assumption A6 is satisfied automatically [18].

By Assumption A6 there exists an optimal solution $(T^*_{v\to x}, T^*_{v\to u})$ to (2). As $T^*_{v\to x}, T^*_{v\to u} \in \frac{1}{z}\mathcal{RH}_\infty$, we can write their partial fraction decomposition as

$$\begin{aligned} T^*_{v\to u}(z) &= \sum\nolimits_{q\in\mathcal{Q}} \sum\nolimits_{j=1}^{m^*_q} H^*_{(q,j)} \frac{1}{(z-q)^j} \\ T^*_{v\to x}(z) &= \sum\nolimits_{q\in\hat{\mathcal{Q}}} \sum\nolimits_{j=1}^{\hat{m}_q} G^*_{(q,j)} \frac{1}{(z-q)^j} \end{aligned} \tag{3}$$

where $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ are finite sets of stable poles closed under complex conjugation, $H^*_{(q,j)}$ and $G^*_{(q,j)}$ are coefficient matrices, and m^*_q and $\hat{m}_q$ are the multiplicities of the pole q in $T^*_{v\to u}$ and $T^*_{v\to x}$, respectively. It will be shown (in the proof of Lemma 2) that the following relationship between the poles $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ holds: $\hat{\mathcal{Q}} \subset \mathcal{Q} \cup \sigma$. Thus, each pole of $T^*_{v\to x}$ must be a pole of at least one of $T^*_{v\to u}$ and the plant.

A recent approach was proposed to solve problem (2) for the special case, where $y = \left[(Qx)^\intercal \; (Ru)^\intercal \right]^\intercal$ for constant matrices Q and R, $T_{\text{desired}}(z) = 0$, and $\hat{B} = I$. This approach is known as system level synthesis (SLS) [9], and the key idea is to reparameterize the control design in terms of the closed-loop transfer functions $\Phi_x(z) = T_{v\to x}(z)$ and $\Phi_u(z) = T_{v\to u}(z)$. This transforms (2) into an infinite dimensional convex optimization problem at the price of the additional affine constraint

$$(zI - A)\Phi_x - B\Phi_u = I \tag{4}$$

After solving (2) subject to (4), a controller that yields the optimal closed-loop responses can be recovered via $K(z) = \Phi_u(z)\Phi_x^{-1}(z)$, and realizations of $K(z)$ exist which do not require transfer function inversion.

A. Finite Impulse Response Approximation

To obtain a tractable optimization problem, in [9] the FIR approximation is made for the closed-loop transfer functions Φ_x and Φ_u, i.e., $\Phi_x(z) = \sum_{i=1}^T G_i z^{-i}$ and $\Phi_u(z) = \sum_{i=1}^T H_i z^{-i}$ where G_i and H_i are coefficient matrices and T is the length of the FIR, resulting in DBC. For an uncontrollable plant it is not feasible to achieve FIR closed-loop transfer functions, so to maintain feasibility DBC introduces a slack variable V that allows (4) to be violated. The objective then becomes non-convex, so DBC uses a quasi-convex upper bound of the objective [9]. The true (i.e., realized) closed-loop responses are then given by $T_{v\to x}(z) = \Phi_x(z)\left(I + \frac{V}{z^T}\right)^{-1}$ and $T_{v\to u}(z) = \Phi_u(z)\left(I + \frac{V}{z^T}\right)^{-1}$. Let J^* be the optimal cost of (2), $J(T)$ the optimal cost of DBC with an FIR of length T, and $C_*, \rho_* > 0$ such that $||\mathcal{I}(T^*_{v\to x})(k)||_2 \le C_*\rho_*^k$ for all $k \ge 0$. Then for T sufficiently large such that $C_*\rho_*^T < 1$, DBC is feasible and satisfies the following suboptimality bound [9, Theorem 4.7] for some $c > 0$:

$$\frac{J(T) - J^*}{J^*} \leq \frac{C_* \rho_*^T}{1 - C_* \rho_*^T} \left(1 + \frac{\lambda c}{1 - \rho_*^T}\right). \tag{5}$$

When ρ_* is small (i.e., the optimal closed-loop response is slow), $C_* \rho_*^T < 1$ may require large T, the convergence rate of $C_* \rho_*^T$ in (5) is slow, and the term $\frac{1}{1 - C_* \rho_*^T}$ (which arises from the slack variable V) will further slow convergence.

III. Main Results

A. Simple Pole Approximation (SPA) Control Design

To introduce our new method, we begin by reformulating (2) using the SLS reparameterization, which results in the following convex but infinite dimensional optimization problem which is a strict generalization of the formulation in [9]:

Recall that σ are the stable poles of the plant, where each $q \in \sigma$ has multiplicity m_q, and $\mathcal{P}$ represents a selection of poles within the unit disk [1]. To obtain a tractable optimization problem, we approximate Φ_x and Φ_u using $\mathcal{P}$ and σ by

$$\begin{aligned} \Phi_u(z) &= \sum\nolimits_{p \in \mathcal{P}} H_p \frac{1}{z - p} \\ \Phi_x(z) &= \sum\nolimits_{p \in \mathcal{P}} G_p \frac{1}{z - p} + \sum\nolimits_{q \in \sigma} \sum\nolimits_{i=1}^{m_q + 1} G_{(q,i)} \frac{1}{(z - q)^i} \end{aligned} \tag{6}$$

where H_p, G_p, and $G_{(q,i)}$ are coefficient matrices. We refer to this as the simple pole approximation (SPA) since all of the poles of Φ_u are simple. Lemma 1 shows that SPA is always feasible for (A, B) stabilizable, and its proof explains the asymmetry in the poles of Φ_x, Φ_u due to the plant poles.

Lemma 1. *Under Assumption A6, the SPA of* (6) *yields a feasible solution for* (2) *with the SLS constraint* (4)*.*

Although it is possible to select any poles $\mathcal{P} \subset \mathbb{D}$ for the SPA method, we provide several recommendations that often lead to improved performance. First, we suggest to include the stable poles of the plant σ in $\mathcal{P}$ to allow the design to cancel out any controllable modes of the plant for which it is advantageous to do so. In addition, for any poles of the optimal solution which are known a priori (see Section I), including these in $\mathcal{P}$ can lead to a dramatic improvement in performance. For the remaining poles, the Archimedes spiral is a natural choice as it provides an approximately even pole selection over $\mathbb{D}$ and converges at the rate $(|\mathcal{P}| + 2)^{-1/2}$ [1].

For any $q \in \sigma$, let $\tilde{m}_q = 1$ if $q \in \mathcal{P}$ and $\tilde{m}_q = 0$ otherwise. Then the SPA of (6) applied to (2) subject to the SLS constraint (4) results in the following optimal control design problem, consisting of the objective

$$\begin{aligned} \min_{H_p, G_p, G_{(q,i)}} & \left|\left| \mathcal{I}(C \Phi_x \hat{B}) + \mathcal{I}(D \Phi_u \hat{B}) - \mathcal{I}(T_{\text{desired}}) \right|\right|_F \\ & + \lambda \left|\left| \mathcal{C}(C \Phi_x \hat{B}) + \mathcal{C}(D \Phi_u \hat{B}) - \mathcal{C}(T_{\text{desired}}) \right|\right|_2, \end{aligned} \tag{7}$$

subject to the following SLS constraints (whose form given below is derived in the proof of Lemma 2):

$$\begin{aligned} G_{(q,2)} + (qI - A) G_{(q,1)} - B H_q = 0, &\quad \forall\, q \in \sigma \cap \mathcal{P} \\ (pI - A) G_p - B H_p = 0, &\quad \forall\, p \in \mathcal{P} - \sigma \\ G_{(q,i+1)} + (qI - A) G_{(q,i)} = 0, &\quad \forall\, q \in \sigma, \\ &\quad i \in \{1 + \tilde{m}_q, ..., m_q\} \\ (qI - A) G_{(q, m_q + \tilde{m}_q)} = 0, &\quad \forall\, q \in \sigma \\ \sum\nolimits_{p \in \mathcal{P} - \sigma} G_p + \sum\nolimits_{q \in \sigma} G_{(q,1)} = I & \end{aligned} \tag{8}$$

and the impulse responses

$$\begin{aligned} \mathcal{I}(\Phi_u)(k) &= \sum_{p \in \mathcal{P}} p^{k-1} H_p \\ \mathcal{I}(\Phi_x)(k) &= \sum_{p \in \mathcal{P} - \sigma} p^{k-1} G_p + \sum_{q \in \sigma} \sum_{i=1}^{m_q + 1} p^{k-i} \binom{k-1}{i-1} G_{(q,i)}. \end{aligned} \tag{9}$$

As the poles of Φ_x, Φ_u lie in $\sigma \cup \mathcal{P}$, which are stable poles, $\Phi_x, \Phi_u \in \frac{1}{z} \mathcal{RH}_\infty$. It is straighforward to see that in (8)-(9) the SLS constraints and the impulse responses are affine and linear, respectively, in the coefficients H_p, G_p, and $G_{(q,i)}$. As the impulse responses $\mathcal{I}$ and convolution operators $\mathcal{C}$ appearing in the objective (7) are linear in the impulse responses of Φ_x and Φ_u, this implies that the terms inside the norms $||\cdot||_F$ and $||\cdot||_2$ are affine in the coefficients H_p, G_p, and $G_{(q,i)}$. Therefore, since $||\cdot||_F$ and $||\cdot||_2$ are convex, the SPA control design (7)-(9) is convex.

As representations of $\mathcal{I}$ and $\mathcal{C}$ would require matrices of infinite size, in order to evaluate the norms $||\cdot||_F$ and $||\cdot||_2$ in the objective (7) in practice, we introduce a finite $T > 0$ and replace all instances of $\mathcal{I}$ and $\mathcal{C}$ in (7) by $\mathcal{I}_T$ and $\mathcal{C}_T$, respectively (see [1] for the notation). Then these norms become the standard Frobenius and spectral matrix norms, so (7)-(9) can be formulated as a tractable semidefinite program (SDP), and as a quadratic program (QP) in the special case of $\mathcal{H}_2$ design (i.e., $\lambda = 0$). As the dimension of (7)-(9) is independent of the time horizon T, by iteratively increasing T (i.e., $T \to 2T$) the Frobenius and spectral norms in the objective will approximate the true $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively, to within any arbitrary numerical tolerance without increasing the dimension of the underlying optimization.

As feasibility is guaranteed for SPA by Lemma 1 and (7)-(9) is convex, unlike with DBC there is no need to introduce a slack variable or use iterative quasi-convex optimization methods for SPA. Instead, SPA can be solved with a single convex optimization (a SDP or QP), and then the closed-loop responses are given by $T_{v \to x}(z) = \Phi_x(z)$ and $T_{v \to u}(z) = \Phi_u(z)$, which do not require inverting transfer functions like DBC does. Furthermore, note that the poles in $\mathcal{P}$ can be chosen to lie anywhere within the open unit disk, so this method does not result in FIR closed-loop transfer functions and, hence, avoids deadbeat control.

B. Suboptimality Bounds

Recall that $d(z, \mathcal{P})$ is the distance from z to $\mathcal{P}$, that $\max_{z \in \mathcal{Q}} d(z, \mathcal{P})$ measures the geometric approximation error between approximating poles $\mathcal{P}$ and optimal poles $\mathcal{Q}$, and $D(\mathcal{P}) = \max_{z \in \mathbb{D}} d(z, \mathcal{P})$ measures the worst case geometric approximation error (for unknown $\mathcal{Q}$). In addition, $r \in (0,1)$ is such that $\mathcal{P} \subset \overline{B}_r$, and δ is a measure of the minimum distance between each approximating pole in $\mathcal{P}$ and σ (see [1] for further details). Also, recall Assumptions A1-A5 from Part I [1]. Our main theoretical result shows that the relative error of the SPA method decays at least linearly with $D(\mathcal{P})$.

Theorem 1 (General Suboptimality Bound)**.** *Let J^* denote the optimal cost of* (2)*, and let $J(\mathcal{P})$ denote the optimal cost of* (7)-(9) *for any choice of $\mathcal{P}$. Suppose Assumption A6 is met, and $\mathcal{P}$ satisfies Assumptions A1-A5. Then there exists a constant $\hat{K} = \hat{K}(\mathcal{Q}, G^*_{(q,j)}, H^*_{(q,j)}, r, \delta) > 0$ such that*

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P}). \tag{10}$$

While the DBC suboptimality bound in (5) only holds for T sufficiently large such that $||\mathcal{I}(T^*_{v\to x})(T)||_2 \leq C_* \rho_*^T < 1$, the SPA bound in (10) does not have this requirement. The constant term in the DBC bound is expressed in terms of the $\mathcal{H}_\infty$ norm of the optimal controller, whereas $\hat{K}$ from Theorem 1 depends on the partial fraction decomposition of the optimal closed-loop responses. Furthermore, the DBC bound includes a multiplicative term $\frac{1}{1 - C_* \rho_*^T}$ resulting from the slack variable, whereas the SPA bound has no such term because it does not need a slack variable. Finally, the convergence for the DBC bound depends on the rate of decay of the optimal closed-loop impulse response, whereas the SPA bound convergence depends on the distance between $\mathcal{P}$ and the optimal closed-loop poles. Therefore, SPA is preferable when the optimal impulse response takes long to decay, such as in stabilizable systems with large separation of time scales. However, DBC may be preferable when the optimal responses decay fast, or when many poles are desired, since its convergence rate approaches exponential as the number of poles approaches infinity. In addition, if some optimal poles can be included in $\mathcal{P}$ due to prior knowledge (see Section III-A), this will typically have the effect of decreasing both $D(\mathcal{P})$ and $\hat{K}$ in (10), significantly reducing the relative error of SPA. In contrast, it is not clear how such prior knowledge could be included with DBC to reduce its relative error.

Corollary 1 shows that, for the Archimedes spiral pole selection in [1], the relative error of SPA converges to zero at the rate $(|\mathcal{P}| + 2)^{-1/2}$ since $|\mathcal{P}_n| = 2n - 2$ for each $n > 0$.

Corollary 1 (Spiral Suboptimality Bound)**.** *For each even integer $n > 0$, let $\mathcal{P}_n$ denote the selection of $(2n-2)$ poles given by p_k for $k \in [-(n-1), n-1]$, where*

$$\theta_k = 2\sqrt{\pi k},\ r_k = \sqrt{\frac{k}{n}},\ p_k = (r_k, \theta_k),\ p_{-k} = (r_k, -\theta_k).$$

Then there exists a constant $\hat{K} = \hat{K}(\mathcal{Q}, G^*_{(q,j)}, H^*_{(q,j)}) > 0$ and $N > 0$ such that $n \geq N$ implies

$$\frac{J(\mathcal{P}_n) - J^*}{J^*} \leq \frac{\hat{K}}{\sqrt{n}}. \tag{11}$$

Note that N in Corollary 1 only needs to be chosen to ensure that $D(\mathcal{P}_N) < 1$, $|\mathcal{P}_N| \geq m_{\max}$, and that $\delta > 0$ for $\mathcal{P}_N$, and so is typically satisfied in practice with small N (see the remark following [1, Theorem 3]).

IV. Numerical Example

To compare DBC and SPA, we consider the example of using a power converter to provide frequency and voltage control services to the power grid, which arises naturally as a result of interfacing renewable generation to the grid [19]. This example served as the motivation to develop the SPA method, because of the inadequate performance of DBC resulting from the large separation of time scales in power systems containing power converter interfaced devices [20]. Let w represent the frequency and voltage magnitude at the connection point, and let y represent the power output of the converter. Then this can be formulated in the form of (2) with matrices given by

$$A = \begin{bmatrix} 0.988 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.995 & 0 \\ 0 & 0 & 0 & 0 & 0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.005 & 0 \\ 0 & 0.1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} -0.0001 & 0 \\ 0 & 1 \\ 0.0066 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 0.829 & -0.428 & 1.02 & 0 \\ 0 & 0.428 & 0.829 & 0 & -1.02 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{\text{desired}}(z) = \begin{bmatrix} T^{\hat{y}}_{\text{desired}}(z) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-5.3e-5}{z-0.999} & 0 \\ 0 & \frac{-0.1}{z-0.999} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For ease of comparison to the ground-truth optimal solution we choose $\lambda = 0$, and we note that an infinite impulse response method for SLS exists for this special case [21], but we emphasize that similar results to those shown here hold for $\lambda \neq 0$ (although the exact ground-truth optimal solution is difficult to obtain). With $\lambda = 0$, the objective is to minimize

$$\left|\left| \begin{bmatrix} T_{w\to\hat{y}}(z) - T^{\hat{y}}_{\text{desired}}(z) \\ 0.01\ T_{w\to u}(z) \end{bmatrix} \right|\right|_{\mathcal{H}_2}$$

where $\hat{y} = Cx$. Let $t \in \mathbb{R}$ and $k \in \mathbb{Z}$ denote continuous and discrete time, respectively, with a sample time of $h = 1$ ms chosen to avoid aliasing from the fast converter dynamics.

To solve (2), for DBC we use golden section search as suggested in [9, p. 380], which involves solving SDPs iteratively to find Φ_x and Φ_u. Then, $T_{w\to y}$ can be recovered from Φ_x and Φ_u by inverting $\left(I + \frac{V}{z^T}\right)$ (see Section II). For SPA, we let $\mathcal{P}$ consist of the stable poles of the plant and T_{desired}, and select the remaining poles from the Archimedes spiral as in [1, Theorem 4]. Then, solving (2) using SPA only requires a single SDP, and then $T_{w\to y}$ is given by a linear transformation of Φ_x and Φ_u, so no transfer function inversion is necessary. To solve the SDPs in each case, Matlab was used with YALMIP and the solver MOSEK. This control design implementation is available online [16].

The DBC and SPA control design approaches are run for varying numbers of poles. For DBC, the problem is infeasible for 29 or less poles, converges in 18 (golden section) iterations for 30 poles, and converges in 7 iterations for 300 poles. As DBC includes constraint violations, Φ_x and Φ_u are not equal to $T_{v\to x}$ and $T_{v\to u}$ for DBC, and recovery of $T_{v\to x}$ and $T_{v\to u}$ for DBC requires inverting $\left(I + \frac{V}{z^T}\right)$ (see Section II), but for large numbers of poles (i.e., large T) this leads to out of memory and numerical errors. Therefore, the figures show only the result of using Φ_x and Φ_u in place of the true system responses $T_{v\to x}$

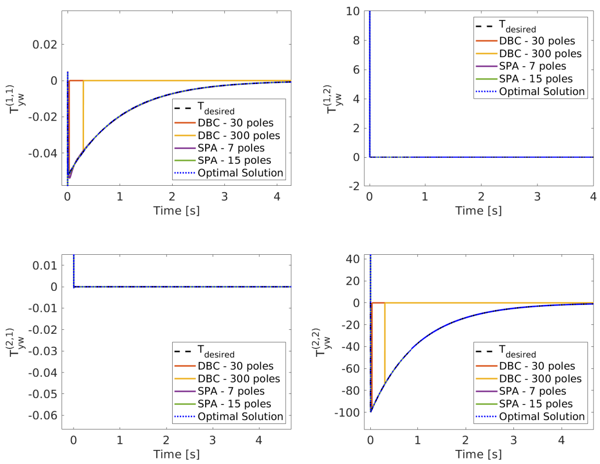

Fig. 1. The impulse responses of control designs for SLS with the FIR approximation (DBC) and simple pole approximation (SPA) as a function of the number of closed-loop poles. The desired transfer function T_{desired} and the ground-truth optimal solution are also shown.

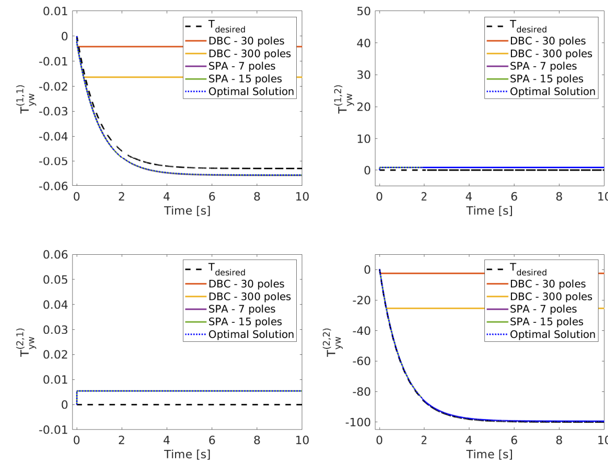

Fig. 2. Step responses of control designs for SLS with the FIR approximation (DBC) and simple pole approximation (SPA) as a function of the number of closed-loop poles. The desired transfer function T_{desired} and the ground-truth optimal solution are also shown.

and $T_{v\to u}$ for DBC, so the true DBC results are actually worse than the DBC results shown in these figures. The SPA method is feasible for any number of poles, and requires only one SDP for each number of poles. It is run for 7 and 15 poles, and the true system responses are easily recovered.

The impulse responses for the solutions of DBC, SPA, and the optimal and desired transfer functions are shown in Fig. 1. For DBC, the impulse response is close to the optimal impulse response only for the first 30 ms or 300 ms for 30 and 300 poles, respectively, after which the impulse response becomes zero (an undesirable but inevitable feature of DBC). However, the optimal impulse response takes several seconds to decay, so overall the matching is very poor for DBC, with 300 poles only slightly better than with 30 poles. In contrast, for SPA the impulse response shows a small initial mismatch during the first few milliseconds, but after this the matching is very close, with the 15 pole solution having slightly better matching than the 7 pole case. From the impulse responses it is clear that SPA is much closer to the optimal solution, and with orders of magnitude fewer poles.

The step responses for the solutions of DBC, SPA, and the optimal and desired transfer functions are shown in Fig. 2. For DBC, the step responses deviate greatly from the optimal step response, with the 300 pole solution closer than with 30 poles. With SPA the step responses are very close to the optimal step response, with the 15 pole solution having slightly better matching during the initial transient than the 7 pole case. From the step responses it is clear that SPA results in much closer matching with the optimal transfer function than DBC, and with far fewer poles.

V. Proofs

Proof of Lemma 1. Working in coordinates where A is in Jordan normal form, we may have $A = \operatorname{diag}(A_u, A_c)$ and $B = \begin{bmatrix} B_u^\intercal & B_c^\intercal \end{bmatrix}^\intercal$ where (A_c, B_c) is controllable and (A_u, B_u) is not. As (A_c, B_c) is controllable, there exists K_c such that the eigenvalues of $A_c + B_c K_c$ lie in $\mathcal{P}$. Let $K = \begin{bmatrix} 0 & K_c \end{bmatrix}$. Then $A + BK = \begin{bmatrix} A_u & B_u K_c \\ 0 & A_c + B_c K_c \end{bmatrix}$ so $\Phi_x(z) = \begin{bmatrix} (zI - A_u)^{-1} & \star \\ 0 & (zI - (A_c + B_c K_c))^{-1} \end{bmatrix}$ and $\Phi_u(z) = K\Phi_x(z) = \begin{bmatrix} 0 & K_c(zI - (A_c + B_c K_c))^{-1} \end{bmatrix}$. Thus, since (A, B) is stabilizable, the poles of Φ_x lie in $\mathcal{P} \cup \sigma$ and contain the uncontrollable plant poles, while the poles of Φ_u lie in $\mathcal{P}$. So, $\Phi_x, \Phi_u \in \frac{1}{z}\mathcal{RH}_\infty$ and satisfy (4). ☐

The key technical result required to prove Theorem 1 is Lemma 2, which extends the approximation error bounds of [1, Theorem 1] to bound the error between a feasible solution (Φ_u, Φ_x) of (7)-(9) and the optimal solution (Φ_u^*, Φ_x^*) of (2).

Lemma 2. *Under the conditions of Theorem 1, let (Φ_x^*, Φ_u^*) denote the optimal solution to* (2). *Then there exist $\Phi_x, \Phi_u \in \frac{1}{z}\mathcal{RH}_\infty$ which are a feasible solution to* (7)-(9)*, and constants $K_\infty^u, K_2^u, K_\infty^x, K_2^x > 0$, such that*

$$||\Phi_u - \Phi_u^*||_{\mathcal{H}_\infty} \le K_\infty^u D(\mathcal{P}), \quad ||\Phi_u - \Phi_u^*||_{\mathcal{H}_2} \le K_2^u D(\mathcal{P}) \tag{12}$$

$$||\Phi_x - \Phi_x^*||_{\mathcal{H}_\infty} \le K_\infty^x D(\mathcal{P}), \quad ||\Phi_x - \Phi_x^*||_{\mathcal{H}_2} \le K_2^x D(\mathcal{P}). \tag{13}$$

Before proving Lemma 2, we will require several technical results as given in the next few lemmas and corollaries. Lemma 3 bounds the distance between two simple transfer functions in terms of the distance between their poles.

Lemma 3. *Let k be any integer, m a positive integer, and $z \in \overline{\mathbb{D}}$. Let $q, p_1, ..., p_m \in \mathbb{D}$, and let $\hat{d}(q) = \max_i |p_i - q|$. If $z \in \partial\mathbb{D}$, let $\delta = d(\partial\mathbb{D}, \{p_i\}_{i=1}^m) > 0$ and $\eta = d(q, \partial\mathbb{D}) > 0$; if not, suppose $d(z, \{p_i\}_{i=1}^m) \ge \delta > 0$ and $d(z, q) \ge \eta > 0$. Then there exists $K > 0$ such that*

$$\left| \frac{(z-q)^k}{\prod_{i=1}^m (z - p_i)} - (z-q)^{k-m} \right| \le K\hat{d}(q).$$

Proof of Lemma 3. Let $\mathcal{P} = \{p_1, ..., p_m\}$. We compute

$$\left|\frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)} - (z-q)^{k-m}\right| = \frac{\left|(z-q)^m - \prod_{i=1}^{m}(z-p_i)\right|}{\left|(z-q)^{m-k}\prod_{i=1}^{m}(z-p_i)\right|}.$$

Noting that the proofs of [1, Eqs. 10, 12] are still valid for $z \in \overline{\mathbb{D}}$ (i.e. $|z| \le 1$), applying them here we have that

$$\left|(z-q)^m - \prod_{i=1}^{m}(z-p_i)\right| \le ((|q|+2)^m - (|q|+1)^m)\,\hat{d}(q).$$

Furthermore,

$$\left|(z-q)^{m-k}\prod_{i=1}^{m}(z-p_i)\right| \ge d(z,q)^{m-k}d(z,\mathcal{P})^m \ge \eta^{m-k}\delta^m.$$

Combining these two inequalities implies that

$$\left|\frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)} - (z-q)^{k-m}\right| \le K\hat{d}(q)$$
$$K = \frac{((|q|+2)^m - (|q|+1)^m)}{\eta^{m-k}\delta^m}.$$

□

Lemmas 4 and 5 prove useful identities related to partial fraction decompositions of approximating transfer functions.

Lemma 4. *Let k be a nonnegative integer, m a positive integer and $q, p_1, ..., p_m \in \mathbb{D}$. Let $c_{p_i} = \left(\prod_{\substack{j=1\\j\neq i}}^{m}(p_i - p_j)\right)^{-1}$. Then*

a. For $k < m$

$$\sum_{i=1}^{m}(p_i-q)^k c_{p_i}\frac{1}{z-p_i} = \frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)}.$$

b. For $k \ge m$

$$\sum_{i=1}^{m}(p_i-q)^k c_{p_i}\frac{1}{z-p_i} = \frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)} - \sum_{i=0}^{k-m} b_i(z-q)^i$$
$$b_i = \sum_{j=1}^{m}(p_j-q)^{k-1-i}c_{p_j}.$$

Proof of Lemma 4. First consider Case (a). Write the partial fraction decomposition

$$\frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)} = \sum\nolimits_{i=1}^{m}\kappa_i\frac{1}{z-p_i}.$$

Multiplying both sides by $\prod_{i=1}^{m}(z-p_i)$ and evaluating at $z = p_i$ implies that

$$\kappa_i = \frac{(p_i-q)^k}{\prod_{\substack{j=1\\j\neq i}}^{m}(p_i-p_j)} = (p_i-q)^k c_{p_i}$$

which completes the proof for Case (a).

Next consider Case (b). Write the partial fraction decomposition

$$\frac{(z-q)^k}{\prod_{i=1}^{m}(z-p_i)} = \sum\nolimits_{i=1}^{m}\kappa_i\frac{1}{z-p_i} + \sum\nolimits_{i=0}^{k-m} b_i(z-q)^i. \tag{14}$$

Multiplying by $\prod_{i=1}^{m}(z-p_i)$ and evaluating at $z = p_i$ implies

$$\kappa_i = \frac{(p_i-q)^k}{\prod_{\substack{j=1\\j\neq i}}^{m}(p_i-p_j)} = (p_i-q)^k c_{p_i}.$$

Differentiating (14) i times with respect to z and evaluating at $z = q$ implies that

$$0 = -i!\sum\nolimits_{j=1}^{m}(p_j-q)^{k-1-i}c_{p_j} + i!b_i$$

for $i \in \{0, ..., k-m\}$, so

$$b_i = \sum\nolimits_{j=1}^{m}(p_j-q)^{k-1-i}c_{p_j}.$$

□

For the remainder of this section, $\lambda \in \sigma$ will represent a stable eigenvalue of the A matrix of the plant. Let m and n be integers, and define the rising factorial $m^{(n)} = \prod_{k=0}^{n-1}(m+k)$ and the falling factorial $m_n = \prod_{k=0}^{n-1}(m-k)$. For m and n nonnegative, letting $m!$ denote the standard factorial, we have $m^{(n)} = \frac{(m+n-1)!}{(m-1)!}$ and $m_n = \frac{m!}{(m-n)!}$. Note:

Fact 3. $(-1)^n m^{(n)} = (-m)_n$

Fact 4. $\sum_{j=0}^{n}\binom{n}{j}m_j(m')_{n-j} = (m+m')_n$.

Lemma 5. *Let k and m be positive integers, $z \in \partial\mathbb{D}$, and $q, \lambda, p_1, ..., p_m \in \mathbb{D}$ with $d(\lambda, \{p_i\}_{i=1}^m) \ge \delta > 0$ and $d(\lambda, q) \ge \eta > 0$. Choose constants c_{p_i} as in Lemma 4. Then*

a. There exists $K > 0$ such that

$$\sum_{i=1}^{m}c_{p_i}\frac{(\lambda-p_i)^{-k}}{z-p_i} = \frac{(\lambda-z)^{-k}}{\prod_{i=1}^{m}(z-p_i)} - \frac{r(z)}{(\lambda-z)^k}$$
$$r(z) = \sum_{n=0}^{k-1}a_n(\lambda-z)^n$$

and

$$\lim_{z\to\lambda}\frac{d}{dz^l}\left(r(z)\prod_{i=1}^{m}(z-p_i)\right)$$
$$= \begin{cases} 1, & l = 0 \\ 0, & l \in \{1, ..., k-1\} \\ \left(\frac{(-1)^{l+1}m^{(l)}}{(\lambda-q)^{m+l}} + \epsilon\right)\prod_{i=1}^{m}(\lambda-p_i), & l = k \end{cases}$$
$$|\epsilon| \le K\hat{d}(q).$$

b. There exist $K'_0, ..., K'_{k-1} > 0$ such that

$$\sum_{i=1}^{m}c_{p_i}\frac{(p_i-q)^m}{(\lambda-p_i)^k}\frac{1}{z-p_i} = \frac{(z-q)^m}{(\lambda-z)^k\prod_{i=1}^{m}(z-p_i)}$$
$$- \sum_{n=0}^{k-1}\frac{a_n}{(\lambda-z)^{k-n}}$$

$$|a_0 - 1| \leq K_0'\hat{d}(q),\ |a_n| \leq K_n'\hat{d}(q),\ n \in \{1, ..., k-1\}.$$

Proof of Lemma 5. For $l \in \{0, m\}$, write the partial fraction decomposition

$$\frac{(z-q)^l}{(\lambda - z)^k \prod_{i=1}^m (z - p_i)} = \sum\nolimits_{i=1}^m \kappa_i \frac{1}{z - p_i} + \frac{r(z)}{(\lambda - z)^k}$$
$$r(z) = \sum\nolimits_{n=0}^{k-1} a_n (\lambda - z)^n. \tag{15}$$

Multiplying both sides by $(\lambda - z)^k \prod_{i=1}^m (z - p_i)$ yields

$$(z-q)^l = (\lambda - z)^k \sum_{i=1}^m \kappa_i \prod_{\substack{j=1 \\ j \neq i}}^m (z - p_j) + r(z) \prod_{i=1}^m (z - p_i). \tag{16}$$

Evaluating (16) at $z = p_i$ implies that

$$\kappa_i = c_{p_i} (p_i - q)^l (\lambda - p_i)^{-k}.$$

For n any nonnegative integer, define

$$b_n = \left(\frac{d}{dz^n} r(z)\right)(\lambda) = (-1)^n n! a_n \tag{17}$$

$$d_n = \left(\frac{d}{dz^n} \prod\nolimits_{i=1}^m (z - p_i)\right)(\lambda) = \sum_{\substack{v \in \mathbb{R}^n \\ v_i \in I_m \forall i \\ v_i \neq v_j \text{ for } i \neq j}} \prod_{\substack{k \in I_m \\ k \notin v}} (\lambda - p_k) \tag{18}$$

$$e_n = \left(\frac{d}{dz^n} (z - q)^l\right)(\lambda) = l_n (\lambda - q)^{l-n}. \tag{19}$$

Note that

$$\lim_{z \to \lambda} \frac{d}{dz^n} \left(r(z) \prod\nolimits_{i=1}^m (z - p_i)\right) = \sum\nolimits_{j=0}^n \binom{n}{j} d_j b_{n-j} \tag{20}$$

for any nonnegative integer n. Differentiating (16) n times with respect to z, and evaluating at $z = \lambda$ implies that

$$e_n = \lim_{z \to \lambda} \frac{d}{dz^n} \left(r(z) \prod_{i=1}^m (z - p_i)\right) = \sum_{j=0}^n \binom{n}{j} d_j b_{n-j} \tag{21}$$

for $n \in \{0, ..., k-1\}$. Dividing by d_0 and solving for b_n implies

$$b_n = \frac{e_n}{d_0} - \sum\nolimits_{j=1}^n \binom{n}{j} \frac{d_j}{d_0} b_{n-j}. \tag{22}$$

Note that

$$\frac{d_n}{d_0} = \sum_{\substack{v \in \mathbb{R}^n \\ v_i \in I_m \forall i \\ v_i \neq v_j \text{ for } i \neq j}} \prod_{k \in v} \frac{1}{\lambda - p_k}.$$

Define

$$\epsilon_n' = \frac{d_n}{d_0} - \frac{m_n}{(\lambda - q)^n} = \sum_{\substack{v \in \mathbb{R}^n \\ v_i \in I_m \forall i \\ v_i \neq v_j \text{ for } i \neq j}} \left(\prod_{k \in v} \frac{1}{\lambda - p_k} - \frac{1}{(\lambda - q)^n}\right)$$

since the number of terms in the sum is m_n. Thus, by Lemma 3 there exists $k_n' > 0$ such that

$$\frac{d_n}{d_0} = m_n \frac{1}{(\lambda - q)^n} + \epsilon_n', \quad |\epsilon_n'| \leq k_n' \hat{d}(q). \tag{23}$$

Consider first Case (a): $l = 0$. Then $e_0 = 1$ and $e_n = 0$ for $n \in \{1, ..., k-1\}$. By (21), this implies the desired result for $n \in \{0, ..., k-1\}$, so it suffices to prove the desired result for $n = k$. By (22), $b_0 = \frac{1}{d_0}$. We claim that there exists $k_n > 0$ such that

$$-\sum\nolimits_{j=1}^n \binom{n}{j} \frac{d_j}{d_0} b_{n-j} = \frac{(-1)^n m^{(n)}}{(\lambda - q)^{m+n}} + \epsilon_n, \quad |\epsilon_n| \leq k_n \hat{d}(q) \tag{24}$$

for all $n \in \{1, ..., k\}$. Note that by (22), this implies that

$$b_n = (-1)^n m^{(n)} \frac{1}{(\lambda - q)^{m+n}} + \epsilon_n, \quad |\epsilon_n| \leq k_n \hat{d}(q) \tag{25}$$

for $n \in \{1, ..., k-1\}$. We prove (24) by strong induction. For the base case, first note that

$$b_0 = \frac{1}{d_0} = \frac{1}{(\lambda - q)^m} + \left(\frac{1}{\prod_{i=1}^m (\lambda - p_i)} - \frac{1}{(\lambda - q)^m}\right)$$
$$= \frac{1}{(\lambda - q)^m} + \epsilon_0, \quad |\epsilon_0| \leq k_0 \hat{d}(q) \tag{26}$$

where such k_0 exists by Lemma 3. Then for $n = 1$ we have

$$-\frac{d_1}{d_0} b_0 = -\left(\frac{m}{\lambda - q} + \epsilon_1'\right)\left(\frac{1}{(\lambda - q)^m} + \epsilon_0\right)$$
$$= -\frac{m}{(\lambda - q)^m} - \frac{m}{\lambda - q}\epsilon_0 - \frac{1}{(\lambda - q)^m}\epsilon_1' - \epsilon_0 \epsilon_1'$$
$$= -\frac{m}{(\lambda - q)^m} + \epsilon_1, \quad |\epsilon_1| \leq k_1 \hat{d}(q)$$
$$k_1 = \frac{m k_0}{|\lambda - q|} + \frac{k_1'}{|\lambda - q|^m} + k_0 k_1'.$$

For the induction step, assume that (24) holds for all $j \in \{1, ..., n-1\}$, which, together with (26), implies that (25) holds for all $j \in \{0, ..., n-1\}$. By (23) and (25) we have

$$-\sum_{j=1}^n \binom{n}{j} \frac{d_j}{d_0} b_{n-j} = -\sum_{j=1}^n \binom{n}{j} \left(m_j \frac{1}{(\lambda - q)^j} + \epsilon_n'\right)$$
$$* \left((-1)^{n-j} m^{(n-j)} \frac{1}{(\lambda - q)^{m+n-j}} + \epsilon_{n-j}\right)$$
$$= -\sum\nolimits_{j=1}^n \binom{n}{j} m_j (-1)^{n-j} m^{(n-j)} \frac{1}{(\lambda - q)^{m+n}} + \epsilon_n$$
$$\epsilon_n = -\sum_{j=1}^n \binom{n}{j} \left(\frac{m_j \epsilon_{n-j}}{(\lambda - q)^j} + \frac{(-1)^{n-j} m^{(n-j)} \epsilon_n'}{(\lambda - q)^{m+n-j}} + \epsilon_n' \epsilon_{n-j}\right)$$
$$|\epsilon_n| \leq k_n \hat{d}(q)$$
$$k_n = \sum\nolimits_{j=1}^n \binom{n}{j} \left(\frac{m_j k_{n-j}}{|\lambda - q|^j} + \frac{m^{(n-j)} k_n'}{|\lambda - q|^{m+n-j}} + k_{n-j} k_n'\right).$$

So

$$-\sum\nolimits_{j=1}^n \binom{n}{j} \frac{d_j}{d_0} b_{n-j}$$
$$\overset{\substack{\text{above} \\ \text{identity}}}{=} -\frac{1}{(\lambda - q)^{m+n}} \sum\nolimits_{j=1}^n \binom{n}{j} m_j (-1)^{n-j} m^{(n-j)} + \epsilon_n$$
$$\overset{\text{Fact 3}}{=} -\frac{1}{(\lambda - q)^{m+n}} \sum\nolimits_{j=1}^n \binom{n}{j} m_j (-m)_{n-j} + \epsilon_n$$

$$\overset{\text{add }0}{=} \frac{1}{(\lambda-q)^{m+n}}\left((-m)_n - \sum_{j=0}^{n}\binom{n}{j} m_j(-m)_{n-j}\right) + \epsilon_n$$
$$\overset{\text{Fact }4}{=} \frac{1}{(\lambda-q)^{m+n}}\left((-m)_n - (m-m)_n\right) + \epsilon_n$$
$$\overset{0_n=0}{=} \frac{1}{(\lambda-q)^{m+n}}(-m)_n + \epsilon_n$$
$$\overset{\text{Fact }3}{=} (-1)^n m^{(n)} \frac{1}{(\lambda-q)^{m+n}} + \epsilon_n, \quad |\epsilon_n| \le k_n \hat{d}(q).$$

Thus, (24) holds. Note that $b_k = 0$ since $r(z)$ is a polynomial of order $k-1$. Therefore, by (20) and (24) we have that

$$\lim_{z\to\lambda} \frac{d}{dz^k}\left(r(z)\prod_{i=1}^{m}(z-p_i)\right) = \sum_{j=0}^{k}\binom{k}{j} d_j b_{k-j}$$
$$= d_0 b_k + \sum_{j=1}^{k}\binom{k}{j} d_j b_{k-j} = \sum_{j=1}^{k}\binom{k}{j} d_j b_{k-j}$$
$$= (-d_0)\left(-\sum_{j=1}^{k}\binom{k}{j}\frac{d_j}{d_0} b_{k-j}\right)$$
$$= (-d_0)\left((-1)^k m^{(k)}\frac{1}{(\lambda-q)^{m+k}} + \epsilon_k\right), \quad |\epsilon_k| \le k_k \hat{d}(q)$$

which yields the result for Case (a). Next consider Case (b): $l = m$. Then by (22) and (17)

$$a_0 = b_0 = \frac{e_0}{d_0} = \frac{(\lambda-q)^m}{\prod_{i=1}^{m}(\lambda-p_i)}$$

so

$$|a_0 - 1| = \left|\frac{(\lambda-q)^m}{\prod_{i=1}^{m}(\lambda-p_i)} - 1\right| \le k_0 \hat{d}(q)$$

where such $k_0 > 0$ exists by Lemma 3. We claim that there exist $k_n > 0$ such that

$$|b_n| \le k_n \hat{d}(q) \tag{27}$$

for $n \in \{1, ..., k-1\}$. We prove (27) by strong induction. For the base case, note that by (22) and (23)

$$b_1 = \frac{e_1}{d_0} - \frac{d_1}{d_0} b_0 = \frac{m(\lambda-q)^{m-1}}{d_0} - \left(m\frac{1}{\lambda-q} + \epsilon_1'\right)\frac{e_0}{d_0}$$
$$= \frac{m(\lambda-q)^{m-1}}{d_0} - m\frac{1}{\lambda-q}\frac{(\lambda-q)^m}{d_0} - \epsilon_1'\frac{(\lambda-q)^m}{\prod_{i=1}^{m}(\lambda-p_i)}$$
$$= \frac{m(\lambda-q)^{m-1}}{d_0} - \frac{m(\lambda-q)^{m-1}}{d_0} - \epsilon_1'\frac{(\lambda-q)^m}{\prod_{i=1}^{m}(\lambda-p_i)}$$
$$= -\epsilon_1'\frac{(\lambda-q)^m}{\prod\limits_{i=1}^{m}(\lambda-p_i)}, \quad |b_1| \le k_1\hat{d}(q), \quad k_1 = k_1'\frac{|\lambda-q|^m}{\prod\limits_{i=1}^{m}|\lambda-p_i|}.$$

For the induction step, assume (27) holds for all $j \in \{1, ..., n-1\}$. By (22), (23), (19), and the induction hypothesis we have

$$b_n \overset{(22)}{=} \frac{e_n}{d_0} - \sum_{j=1}^{n}\binom{n}{j}\frac{d_j}{d_0} b_{n-j}$$
$$\overset{\substack{\text{regrouping}\\\text{terms}}}{=} \frac{e_n}{d_0} - \frac{d_n}{d_0} b_0 - \sum_{j=1}^{n-1}\binom{n}{j}\frac{d_j}{d_0} b_{n-j}$$
$$\overset{\substack{(19)\\(23)}}{=} \frac{m_n(\lambda-q)^{m-n}}{d_0} - \left(m_n\frac{1}{(\lambda-q)^n} + \epsilon_n'\right)\frac{e_0}{d_0}$$
$$- \sum_{j=1}^{n-1}\binom{n}{j}\left(m_j\frac{1}{(\lambda-q)^j} + \epsilon_j'\right) b_{n-j}$$
$$\overset{\substack{(19)\\\text{regrouping}}}{=} \frac{m_n(\lambda-q)^{m-n}}{d_0} - \frac{m_n(\lambda-q)^{m-n}}{d_0}$$
$$- \frac{\epsilon_n'(\lambda-q)^m}{\prod\limits_{i=1}^{m}(\lambda-p_i)} - \sum_{j=1}^{n-1}\binom{n}{j}\left(m_j(\lambda-q)^{-j} + \epsilon_j'\right) b_{n-j}$$
$$\overset{\text{cancel}}{=} -\frac{\epsilon_n'(\lambda-q)^m}{\prod\limits_{i=1}^{m}(\lambda-p_i)} - \sum_{j=1}^{n-1}\binom{n}{j}\left(m_j(\lambda-q)^{-j} + \epsilon_j'\right) b_{n-j}$$

$|b_n| \le k_n \hat{d}(q)$

$$k_n = k_n'\frac{|\lambda-q|^m}{\prod\limits_{i=1}^{m}|\lambda-p_i|} + \sum_{j=1}^{n-1}\binom{n}{j}\left(m_j|\lambda-q|^{-j} + k_j'\right) k_{n-j}.$$

Thus, (27) holds. Combining (27) with (15) and (17) yields the result for Case (b). □

For $z \in \mathbb{C}$, let $J(z)$ denote an elementary Jordan block with eigenvalue z. In the proof of Lemma 2 this will refer to the elementary Jordan blocks of the A matrix of the plant. Corollary 2 will help bound the error between the optimal and approximating transfer functions in terms of $D(\mathcal{P})$.

Corollary 2. *Let k be a nonnegative integer, m a positive integer, $z \in \partial\mathbb{D}$, and $q, \lambda, p_1, ..., p_m \in \mathbb{D}$ with $d(\lambda, \{p_i\}_{i=1}^m) \ge \delta > 0$ and $d(\lambda, q) \ge \eta > 0$. Choose constants c_{p_i} as in Lemma 4. Then*

a. There exists $K > 0$ such that

$$\sum_{i=1}^{m}\frac{c_{p_i}}{(\lambda-p_i)^{(k+1)}} = \binom{m-1+k}{k}(\lambda-q)^{-(m+k)} + \epsilon$$
$$|\epsilon| \le K\hat{d}(q).$$

b. There exists $K > 0$ such that

$$\left\|\sum_{i=1}^{m} c_{p_i}(p_i-q)^m J(\lambda-p_i)^{-1}\frac{1}{z-p_i}\right\|_2 \le K\hat{d}(q).$$

Proof of Corollary 2. First we prove Case (a). By Lemma 5(a) we have

$$\sum_{i=1}^{m} c_{p_i}(\lambda-p_i)^{-(k+1)} = \lim_{z\to\lambda}\sum_{i=1}^{m} c_{p_i}(\lambda-p_i)^{-k}\frac{1}{z-p_i}$$
$$= \lim_{z\to\lambda}\frac{1 - r(z)\prod_{i=1}^{m}(z-p_i)}{(\lambda-z)^k\prod_{i=1}^{m}(z-p_i)}.$$

Furthermore, by Lemma 5(a), the numerator/deminator satisfy

$$\lim_{z\to\lambda} 1 - r(z)\prod_{i=1}^{m}(z-p_i) = 0, \ \lim_{z\to\lambda}(\lambda-z)^k\prod_{i=1}^{m}(z-p_i) = 0.$$

As both the numerator and denominator approach zero as $z \to \lambda$, we can evaluate the limit using L'Hospital's rule. For any $l \in \{1, ..., k-1\}$, by Lemma 5(a), differentiating the numerator and denominator l times and taking the limit as $z \to \lambda$ implies

$$\lim_{z\to\lambda}\frac{d}{dz^l}\left(1 - r(z)\prod_{i=1}^{m}(z-p_i)\right)$$
$$= -\lim_{z\to\lambda}\frac{d}{dz^l}\left(r(z)\prod_{i=1}^{m}(z-p_i)\right) = 0$$
$$\lim_{z\to\lambda}\frac{d}{dz^l}\left((\lambda-z)^k\prod_{i=1}^{m}(z-p_i)\right) = 0.$$

Therefore, we apply L'Hospital's rule k times and use Lemma 5(a) to obtain

$$\sum_{i=1}^{m} c_{p_i}(\lambda-p_i)^{-(k+1)} = \frac{\lim_{z\to\lambda}\frac{d}{dz^k}\left(1-r(z)\prod_{i=1}^{m}(z-p_i)\right)}{\lim_{z\to\lambda}\frac{d}{dz^k}\left((\lambda-z)^k\prod_{i=1}^{m}(z-p_i)\right)}$$
$$= \frac{-\left((-1)^{k+1}m^{(k)}(\lambda-q)^{-(m+k)}+\epsilon'\right)\prod_{i=1}^{m}(\lambda-p_i)}{(-1)^k k!\prod_{i=1}^{m}(\lambda-p_i)}$$
$$= \frac{m^{(k)}}{k!}(\lambda-q)^{-(m+k)} - (-1)^{-k}\frac{1}{k!}\epsilon'$$
$$= \binom{m+k-1}{k}(\lambda-q)^{-(m+k)}+\epsilon, \quad |\epsilon|\le K\hat{d}(q).$$

This proves Case (a).

For Case (b), we first recall the following fact:
Fact 1. If there exist $k_{i,j}$ and d positive such that $|M_{i,j}|\le k_{i,j}d$ for all i, j then there exists $K>0$ such that $||M||_2\le Kd$.

By Fact 1 it suffices to show that for each $l\in\{0,...,m_q-1\}$ there exists $k_l>0$ such that the lth superdiagonal of the matrix in the desired result satisfies

$$\left|\sum_{i=1}^{m} c_{p_i}(p_i-q)^m(-1)^l(\lambda-p_i)^{-(l+1)}\frac{1}{z-p_i}\right|\le k_l\hat{d}(q).$$

By Lemma 5(b) and since $z\in\partial\mathbb{D}$,

$$\left|\sum_{i=1}^{m} c_{p_i}(p_i-q)^m(-1)^l(\lambda-p_i)^{-(l+1)}\frac{1}{z-p_i}\right|$$
$$\overset{\text{absolute value}}{=} \left|\sum_{i=1}^{m} c_{p_i}(p_i-q)^m(\lambda-p_i)^{-(l+1)}\frac{1}{z-p_i}\right|$$
$$\overset{\text{Lemma 5(b)}}{=} \left|\frac{(z-q)^m}{(\lambda-z)^{l+1}\prod_{i=1}^{m}(z-p_i)}-\sum_{n=0}^{l}a_n(\lambda-z)^{n-l-1}\right|$$
$$\overset{\text{triangle inequality}}{\le} \left|\frac{(z-q)^m-a_0\prod_{i=1}^{m}(z-p_i)}{(\lambda-z)^{l+1}\prod_{i=1}^{m}(z-p_i)}\right|+\sum_{n=1}^{l}\frac{|a_n|}{|\lambda-z|^{-n+l+1}}$$
$$\overset{a_0=1+(a_0-1)}{\le} \left|\frac{(z-q)^m-\prod_{i=1}^{m}(z-p_i)+(1-a_0)\prod_{i=1}^{m}(z-p_i)}{(\lambda-z)^{l+1}\prod_{i=1}^{m}(z-p_i)}\right|$$
$$+\sum_{n=1}^{l}|a_n|(1-|\lambda|)^{n-l-1}$$
$$\overset{\text{triangle inequality}}{\le} \left|\frac{(z-q)^m}{\prod_{i=1}^{m}(z-p_i)}-1\right||\lambda-z|^{-l-1}+\frac{|1-a_0|}{|\lambda-z|^{l+1}}$$
$$+\sum_{n=1}^{l}|a_n|(1-|\lambda|)^{n-l-1}$$
$$\overset{z\in\partial\mathbb{D}}{\le} \left|\frac{(z-q)^m}{\prod_{i=1}^{m}(z-p_i)}-1\right|(1-|\lambda|)^{-l-1}+\frac{|1-a_0|}{(1-|\lambda|)^{l+1}}$$
$$+\sum_{n=1}^{l}|a_n|(1-|\lambda|)^{n-l-1}\le k_l\hat{d}(q)$$

$$k_l = K'(1-|\lambda|)^{-l-1}+K_0'(1-|\lambda|)^{-l-1}+\sum_{n=1}^{l}K_n'(1-|\lambda|)^{n-l-1}$$

where such $K'>0$ exists by Lemma 3. This proves Case (b). $\square$

Lemma 6. *Let $\tilde{m}\in\{0,1\}$, let $m_q, m>0$ be integers, and let $q\in\mathbb{D}$. Suppose that for each $i\in\{1,...,m\}$ we have matrices G_i^* and H_i^*, and for each $j\in\{1,...,i\}$ we have matrices H_j^i and G_j^i, and poles p_j^i, such that the following hold:*

$$H_j^i = c_j^i H_i^*, \quad G_j^i = -J(q-p_j^i)^{-1}BH_j^i. \tag{28}$$

Then

$$\begin{aligned}
&\sum_{i=1+\tilde{m}}^{m}\sum_{j=1+\tilde{m}}^{i} G_j^i\frac{1}{z-p_j^i}\\
&+\tilde{m}\sum_{l=2}^{m_q+1}J(0)^{l-2}\sum_{i=1}^{m}BH_1^i\frac{1}{(z-q)^l}\\
&-\sum_{l=1}^{m_q}J(0)^{l-1}\sum_{i=1+\tilde{m}}^{m}\sum_{j=1+\tilde{m}}^{i}G_j^i\frac{1}{(z-q)^l}\\
&=\left(\sum_{l=0}^{m_q-1}J(0)^l\tilde{m}\frac{1}{(z-q)^{l+2}}\right)BH_1^*\\
&+\sum_{i=1+\tilde{m}}^{m}\left(\sum_{j=1+\tilde{m}}^{i}J(q-p_j^i)^{-1}\frac{-c_j^i}{z-p_j^i}\frac{(p_j^i-q)^{m_q}}{(z-q)^{m_q}}\right.\\
&+\sum_{l=0}^{m_q-i-1}\sum_{k=0}^{m_q-i-1-l}J(0)^l\sum_{j=1+\tilde{m}}^{i}\frac{c_j^i(p_j^i-q)^{m_q-2-k-l}}{(z-q)^{m_q-k}}\\
&\left.+\tilde{m}c_1^iJ(0)^{m_q-1}\frac{1}{(z-q)^{m_q+1}}\right)BH_i^*.
\end{aligned}\tag{29}$$

Furthermore, for each $i\in\{1,...,m_q\}$ and $l\in\{0,...,m_q-1\}$, there exists $K_{i,l}>0$ such that each element in the lth superdiagonal of the term multiplying BH_i^* in (29) has a difference from $\frac{1}{(z-q)^{i+l+1}}$ bounded in absolute value by $K_{i,l}D(\mathcal{P})$.

Proof of Lemma 6. We begin by proving (29). For any $i\in\{1,...m\}$, $j\in\{1,...,i\}$, and $k\in\{1,...,m_q-1\}$ we have that $J(0)G_j^i=-c_j^iJ(0)J(q-p_j^i)^{-1}BH_i^*$. Writing $J(0)=J(q-p_j^i)+(p_j^i-q)I$ implies that $J(0)G_j^i=-c_j^iBH_i^*-c_j^i(p_j^i-q)J(q-p_j^i)^{-1}BH_i^*$. Iterating this process yields

$$\begin{aligned}J(0)^kG_j^i=&\sum_{l=0}^{k-1}-c_j^iJ(0)^{k-1-l}(p_j^i-q)^lBH_i^*\\ &-c_j^i(p_j^i-q)^kJ(q-p_j^i)^{-1}BH_i^*.\end{aligned}\tag{30}$$

Then, applying Lemma 4(a) for $k\le i-1$, setting $z=q$, and noting that for $\tilde{m}=1$, c_j^i contains a factor of $\frac{1}{p_j^i-q}$ gives

$$\sum_{j=1+\tilde{m}}^{i}c_j^i(p_j^i-q)^l=0 \tag{31}$$

for any $i\in\{1+\tilde{m},...,m\}$ and $l\in\{1,...,i-2\}$. Furthermore, applying Lemma 4(a) for $k=1$ and setting $z=q$ implies

$$\sum_{j=1}^{i}c_j^i=0. \tag{32}$$

for any $i \in \{1+\tilde{m}, ..., m\}$ (and $l = 0$). Therefore, for any $i \in \{1+\tilde{m}, ..., m\}$ and $k \in \{1, ..., m_q - 1\}$

$$
\begin{aligned}
&\tilde{m} J(0)^{k-1} B H_1^i - J(0)^k \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \\
&\overset{(30)}{\overset{(28)}{=}} \sum\nolimits_{l=0}^{k-1} J(0)^{k-1-l} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^l \\
&+ \sum_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^k J(q - p_j^i)^{-1} B H_i^* + \tilde{m} J(0)^{k-1} c_1^i B H_i^* \\
&\overset{\substack{\text{regroup} \\ l=0 \text{ terms}}}{=} \sum\nolimits_{l=1}^{k-1} J(0)^{k-1-l} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^l \\
&+ \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^k J(q - p_j^i)^{-1} B H_i^* \\
&+ J(0)^{k-1} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i + \tilde{m} J(0)^{k-1} c_1^i B H_i^* \\
&\overset{\substack{\text{combine last} \\ \text{two terms}}}{=} \sum\nolimits_{l=1}^{k-1} J(0)^{k-1-l} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^l \\
&+ \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^k J(q - p_j^i)^{-1} B H_i^* \\
&+ J(0)^{k-1} B H_i^* \sum\nolimits_{j=1}^{i} c_j^i \\
&\overset{(31)}{\overset{(32)}{=}} \sum\nolimits_{l=i-1}^{k-1} J(0)^{k-1-l} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^l \\
&+ \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^k J(q - p_j^i)^{-1} B H_i^* \\
&\overset{(56)}{=} \sum\nolimits_{l=i-1}^{k-1} J(0)^{k-1-l} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} c_j^i (p_j^i - q)^l \\
&- \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^k G_j^i.
\end{aligned}
\tag{33}
$$

We compute

$$
\begin{aligned}
&\Phi_x(z) \overset{(51)}{\overset{(54)}{=}} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - p_j^i} \\
&+ \tilde{m} \sum\nolimits_{l=2}^{m_q+1} J(0)^{l-2} \sum\nolimits_{i=1}^{m} B H_1^i \frac{1}{(z-q)^l} \\
&- \sum\nolimits_{l=1}^{m_q} J(0)^{l-1} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{(z-q)^l} \\
&\overset{\substack{\text{combining} \\ \text{terms in sum}}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - p_j^i} \\
&+ \sum_{l=2}^{m_q} \sum_{i=1+\tilde{m}}^{m} \left(\tilde{m} J(0)^{l-2} B H_1^i - J(0)^{l-1} \sum_{j=1+\tilde{m}}^{i} G_j^i \right) \frac{1}{(z-q)^l} \\
&- \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - q} \\
&+ \tilde{m} J(0)^{m_q - 1} \sum\nolimits_{i=1+\tilde{m}}^{m} B H_1^i \frac{1}{(z-q)^{m_q+1}} \\
&+ \sum\nolimits_{l=0}^{m_q - 1} \tilde{m} J(0)^l B H_1^1 \frac{1}{(z-q)^{l+2}} \\
&\overset{(33)}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - p_j^i} \\
&+ \sum_{l=2}^{m_q} \sum_{i=1+\tilde{m}}^{m} \sum_{k=i-1}^{l-2} J(0)^{l-2-k} B H_i^* \sum_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i - q)^k}{(z-q)^l} \\
&- \sum\nolimits_{l=2}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&- \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - q} \\
&+ \tilde{m} J(0)^{m_q - 1} \sum\nolimits_{i=1+\tilde{m}}^{m} B H_1^i \frac{1}{(z-q)^{m_q+1}} \\
&+ \sum\nolimits_{l=0}^{m_q - 1} \tilde{m} J(0)^l B H_1^1 \frac{1}{(z-q)^{l+2}}.
\end{aligned}
\tag{34}
$$

Note that

$$
\begin{aligned}
&\sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - p_j^i} \\
&- \sum\nolimits_{l=2}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&- \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z - q} \\
&\overset{\text{regrouping}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} G_j^i \left(\frac{1}{z - p_j^i} - \frac{1}{z - q} \right) \\
&- \sum\nolimits_{l=2}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&\overset{\text{simplifying}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q) G_j^i \frac{1}{z - q} \frac{1}{z - p_j^i} \\
&- \sum\nolimits_{l=2}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&\overset{\text{regrouping}}{=} \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{i} (p_j^i - q) G_j^i \frac{1}{z - q} \left(\frac{1}{z - p_j^i} - \frac{1}{z - q} \right) \\
&- \sum\nolimits_{l=3}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&\overset{\text{simplifying}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^2 G_j^i \frac{1}{(z - q)^2} \frac{1}{z - p_j^i} \\
&- \sum\nolimits_{l=3}^{m_q} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{l-1} G_j^i \frac{1}{(z-q)^l} \\
&\ldots \\
&\overset{\substack{\text{iterating} \\ \text{the above}}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{i} (p_j^i - q)^{m_q} G_j^i \frac{1}{z - p_j^i} \frac{1}{(z-q)^{m_q}} \\
&\overset{(56)}{=} \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{i} J(q - p_j^i)^{-1} B H_i^* \frac{-c_j^i}{z - p_j^i} \frac{(p_j^i - q)^{m_q}}{(z-q)^{m_q}}
\end{aligned}
\tag{35}
$$

Also, we compute

$$
\begin{aligned}
&\sum_{l=2}^{m_q} \sum_{i=1+\tilde{m}}^{m} \sum_{k=i-1}^{l-2} J(0)^{l-2-k} B H_i^* \sum_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i - q)^k}{(z-q)^l} \\
&\overset{\substack{\text{reverse} \\ \text{sums}}}{=} \sum_{i=1+\tilde{m}}^{m} \sum_{l=i+1}^{m_q} \sum_{k=i-1}^{l-2} J(0)^{l-2-k} B H_i^* \sum_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i - q)^k}{(z-q)^l} \\
&\overset{\substack{l'=l-2-k \\ k'=m_q-l}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{k'=0}^{m_q-i-1} \sum\nolimits_{l'=0}^{m_q-i-1-k'} \\
&\qquad J(0)^{l'} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i - q)^{m_q - 2 - k' - l'}}{(z-q)^{m_q - k'}} \\
&\overset{\substack{\text{reverse} \\ \text{sums}}}{=} \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{l'=0}^{m_q-i-1} \sum\nolimits_{k'=0}^{m_q-i-1-l'} \\
&\qquad J(0)^{l'} B H_i^* \sum\nolimits_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i - q)^{m_q - 2 - k' - l'}}{(z-q)^{m_q - k'}}.
\end{aligned}
\tag{36}
$$

Furthermore, we have

$$\begin{aligned}
&\tilde m J(0)^{m_q-1}\sum\nolimits_{i=1+\tilde m}^{m} BH_1^i\frac{1}{(z-q)^{m_q+1}}\\
&\qquad\overset{(28)}{=}\tilde m\sum\nolimits_{i=1+\tilde m}^{m} c_1^i J(0)^{m_q-1}BH_i^*\frac{1}{(z-q)^{m_q+1}}\\
&\sum\nolimits_{l=0}^{m_q-1}\tilde m J(0)^l BH_1^1\frac{1}{(z-q)^{l+2}}\\
&\qquad\overset{(28)}{=}\sum\nolimits_{l=0}^{m_q-1}\tilde m J(0)^l BH_1^*\frac{1}{(z-q)^{l+2}}.
\end{aligned}\tag{37}$$

Substituting (35), (36), and (37) into (34) yields

$$\begin{aligned}
\Phi_x(z)=&\left(\sum\nolimits_{l=0}^{m_q-1}J(0)^l\tilde m\frac{1}{(z-q)^{l+2}}\right)BH_1^*\\
&+\sum\nolimits_{i=1+\tilde m}^{m}\left(\sum\nolimits_{j=1+\tilde m}^{i}J(q-p_j^i)^{-1}\frac{-c_j^i}{z-p_j^i}\frac{(p_j^i-q)^{m_q}}{(z-q)^{m_q}}\right.\\
&+\sum_{l=0}^{m_q-i-1}\sum_{k=0}^{m_q-i-1-l}J(0)^l\sum_{j=1+\tilde m}^{i}\frac{c_j^i(p_j^i-q)^{m_q-2-k-l}}{(z-q)^{m_q-k}}\\
&\left.+\tilde m c_1^i J(0)^{m_q-1}\frac{1}{(z-q)^{m_q+1}}\right)BH_i^*.
\end{aligned}$$

This completes the proof of (29).

Next we derive the upper bound of the difference from $\frac{1}{(z-q)^{i+l+1}}$ for the elements of the superdiagonal of the term multiplying BH_i^*. Note that, by the form of (29), all elements on each such superdiagonal are identical. By (29), for $i\in\{1+\tilde m,...,m\}$ and $l\in\{m_q-i,...,m_q-2\}$, the lth superdiagonal of the term multiplying BH_i^* in $\Phi_x(z)$ is

$$\begin{aligned}
&\left(\sum\nolimits_{j=1+\tilde m}^{i}(-1)^l(q-p_j^i)^{-(l+1)}\frac{-c_j^i(p_j^i-q)^{m_q}}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&=\left(\sum_{j=1+\tilde m}^{i}\frac{(-1)^{l+1}}{(-1)^{l+1}}(p_j^i-q)^{-(l+1)}\frac{c_j^i(p_j^i-q)^{m_q}}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&=\left(\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-1-l}\frac{1}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&\overset{\text{Lemma 4(a)}}{=}\frac{(z-q)^{m_q-1-l-\tilde m}}{\prod_{j=1+\tilde m}^{i}(z-p_j^i)}\frac{1}{(z-q)^{m_q}}=\frac{(z-q)^{-(l+1+\tilde m)}}{\prod_{j=1+\tilde m}^{i}(z-p_j^i)}
\end{aligned}$$

where for Lemma 4(a) note that for $\tilde m=1$, c_j^i contains a factor of $\frac{1}{p_j^i-q}$. For $i\in\{1+\tilde m,...,m\}$, the (m_q-1)th superdiagonal (i.e. $l=m_q-1$) of the term multiplying BH_i^* in $\Phi_x(z)$ is

$$\begin{aligned}
&\left(\sum_{j=1+\tilde m}^{i}(-1)^{m_q-1}(q-p_j^i)^{-m_q}\frac{-c_j^i(p_j^i-q)^{m_q}}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&+\tilde m c_1^i\frac{1}{(z-q)^{m_q+1}}\\
&=\left(\sum_{j=1+\tilde m}^{i}\frac{(-1)^{m_q}}{(-1)^{m_q}}(p_j^i-q)^{-m_q}\frac{c_j^i(p_j^i-q)^{m_q}}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&+\tilde m c_1^i\frac{1}{(z-q)^{m_q+1}}\\
&=\left(\sum\nolimits_{j=1+\tilde m}^{i}c_j^i\frac{1}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}+\tilde m c_1^i\frac{1}{(z-q)^{m_q+1}}\\
&=\left(\sum\nolimits_{j=1+\tilde m}^{i}c_j^i\frac{1}{z-p_j^i}+\tilde m c_1^i\frac{1}{(z-q)}\right)\frac{1}{(z-q)^{m_q}}\\
&=\left(\sum_{j=1}^{i}c_j^i\frac{1}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\overset{\text{[1, Eq. 7]}}{=}\frac{1}{\prod_{j=1}^{i}(z-p_j^i)}\frac{1}{(z-q)^{m_q}}\\
&=\frac{1}{\prod_{j=1}^{i}(z-p_j^i)}\frac{1}{(z-q)^{l+1}}.
\end{aligned}$$

For $i\in\{1+\tilde m,...,m\}$ and $l\in\{0,...,m_q-i-1\}$, the lth superdiagonal of the term multiplying BH_i^* in $\Phi_x(z)$ is

$$\begin{aligned}
&\left(\sum\nolimits_{j=1+\tilde m}^{i}(-1)^l(q-p_j^i)^{-(l+1)}\frac{-c_j^i(p_j^i-q)^{m_q}}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&+\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}\frac{1}{(z-q)^{m_q-k}}\\
&=\left(\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-l-1}\frac{1}{z-p_j^i}\right)\frac{1}{(z-q)^{m_q}}\\
&+\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}\frac{1}{(z-q)^{m_q-k}}\\
&\overset{\text{Lemma 4(b)}}{=}\frac{1}{(z-q)^{m_q}}\left(\frac{(z-q)^{m_q-l-1-\tilde m}}{\prod_{j=1+\tilde m}^{i}(z-p_j^i)}\right.\\
&\qquad\left.-\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}(z-q)^k\right)\\
&+\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}\frac{1}{(z-q)^{m_q-k}}\\
&\overset{\text{simplifying}}{=}\frac{1}{\prod_{j=1+\tilde m}^{i}(z-p_j^i)}\frac{1}{(z-q)^{l+1+\tilde m}}\\
&-\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}\frac{1}{(z-q)^{m_q-k}}\\
&+\sum\nolimits_{k=0}^{m_q-i-1-l}\sum\nolimits_{j=1+\tilde m}^{i}c_j^i(p_j^i-q)^{m_q-2-k-l}\frac{1}{(z-q)^{m_q-k}}\\
&\overset{\text{cancelling}}{=}\frac{1}{\prod_{j=1+\tilde m}^{i}(z-p_j^i)}\frac{1}{(z-q)^{l+1+\tilde m}}
\end{aligned}$$

where for Lemma 4(b) note that for $\tilde m=1$, c_j^i contains a factor of $\frac{1}{p_j^i-q}$. Finally, if $\tilde m=1$, then for $i=1$ and any $l\in\{0,...,m_q-1\}$, the lth superdiagonal of the term multiplying BH_1^* in $\Phi_x(z)$ is given by

$$\tilde m\frac{1}{(z-q)^{l+2}}=\tilde m\frac{1}{(z-q)^{i+l+1}}.$$

Thus, combining the cases above, by Lemma 3 for every $i\in\{1,...,m\}$ and $l\in\{0,...,m_q-1\}$, each term in the lth superdiagonal of the term multiplying BH_i^* in (29) has difference from $\frac{1}{(z-q)^{i+l+1}}$ bounded by $K_{i,l}D(\mathcal{P})$.

□

Proof of Lemma 2. The proof begins by selecting an optimal solution (Φ_x^*,Φ_u^*) to (2), and constructing $\Phi_u(z)=\sum_{p\in\mathcal{P}}H_p\frac{1}{z-p}$ by [1, Theorem 1] to approximate Φ_u^*. By [1, Theorem 1], this implies that the approximation error bounds for Φ_u of (12) are satisfied. Next, Φ_x is defined as the unique solution to the SLS constraint in (4). The remainder of the proof will show that Φ_x is a feasible solution to (7)-(9), and that it satisfies the approximation error bounds of (13).

Towards that end, first it is shown that it suffices to work in coordinates in which A is in Jordan normal form. Next it is shown that, in these coordinates, the approximation error bounds and the SLS constraints decouple according to each elementary Jordan block in A, so it suffices to prove the result for a single elementary Jordan block with eigenvalue λ. Afterwards, it is shown that the SLS constraint uniquely determines the poles and multiplicities of $\tilde{\Phi}_x$ from those of $\tilde{\Phi}_u$ for any transfer functions $(\tilde{\Phi}_x, \tilde{\Phi}_u)$ in $\frac{1}{z}\mathcal{RH}_\infty$ that satisfy it. From the choice of Φ_u, this immediately implies that Φ_x is a feasible solution to (7)-(9).

Subsequently, for each pole q in Φ_x^* that appears in Φ_u^*, by [1, Theorem 1] there exist poles in Φ_u for approximating the portion of Φ_u^* corresponding to pole q. By the relationship between Φ_u and Φ_x described above, we then consider the resulting poles that appear in Φ_x, and will show that the portion of Φ_x corresponding to these poles closely approximates the portion of Φ_x^* corresponding to the pole q. To do so, we fix a pole q in Φ_x^* and consider two cases: Case 1 where $q \neq \lambda$, and Case 2 where $q = \lambda$. For each of these cases we use the SLS constraints to determine the coefficients in the portions of Φ_x^* and Φ_x corresponding to pole q and the poles used to approximate it, respectively, and then bound the resulting approximation error. As q was arbitrary, this then yields the desired approximation error bounds for Φ_x of (13).

First we obtain an optimal solution to (2), and use [1, Theorem 1] to find Φ_u which closely approximates Φ_u^*. Let (Φ_x^*, Φ_u^*) be an optimal solution to (2), which exists by Assumption A6. By [1, Theorem 1], there exist coefficient matrices $\{H_p\}_{p\in\mathcal{P}}$ such that, if we define $\Phi_u(z) = \sum_{p\in\mathcal{P}} H_p \frac{1}{z-p}$ then $\Phi_u \in \frac{1}{z}\mathcal{RH}_\infty$, $||\Phi_u - \Phi_u^*||_{H_\infty} \leq K_\infty^u D(\mathcal{P})$, and $||\Phi_u - \Phi_u^*||_{H_2} \leq K_2^u D(\mathcal{P})$. Define $\Phi_x(z) = (zI - A)^{-1}(B\Phi_u(z) + I)$ and note that this implies (Φ_x, Φ_u) satisfy the SLS constraint in (4) by construction.

As $\mathcal{Q}$ and σ are finite, $\eta = \min_{q\in\mathcal{Q},\lambda\in\sigma,\lambda\neq q} d(\lambda, q) > 0$ and $d(\lambda, q) \geq \eta$ for all such $\lambda \neq q$. By Assumption A5, for every $q \in \mathcal{Q}$ and $\lambda \in \sigma$ with $\lambda \neq q$, $d(\lambda, \mathcal{P}(q)) > 0$, where $\mathcal{P}(q)$ are the m_q closest poles in $\mathcal{P}$ to q. This implies that $\delta = \min_{q\in\mathcal{Q},\lambda\in\sigma,\lambda\neq q} d(\lambda, \mathcal{P}(q)) > 0$ and that $d(\lambda, \mathcal{P}(q)) \geq \delta$ for all such $\lambda \neq q$.

Next we show that it suffices to work in coordinates in which A is in Jordan normal form, and that in these coordinates the SLS constraints decouple according to each elementary Jordan block. There exist matrices J in Jordan normal form and V invertible such that $J = VAV^{-1}$. Fix $z \in \partial\mathbb{D}$ for the remainder of the proof. We will show that there exists $K > 0$ such that

$$||V\Phi_x(z) - V\Phi_x^*(z)||_2 \leq KD(\mathcal{P}). \tag{38}$$

This will imply that

$$\begin{aligned} ||\Phi_x - \Phi_x^*||_{H_\infty} &= \sup_{z\in\partial\mathbb{D}} ||V^{-1}(V\Phi_x(z) - V\Phi_x^*(z))||_2 \\ &\leq ||V^{-1}||_2 \sup_{z\in\partial\mathbb{D}} ||V\Phi_x(z) - V\Phi_x^*(z)||_2 \leq K_\infty^x D(\mathcal{P}) \\ ||\Phi_x - \Phi_x^*||_{H_2} &\leq \sqrt{n}||\Phi_x - \Phi_x^*||_{H_\infty} \leq K_2^x D(\mathcal{P}) \end{aligned}$$

$$K_\infty^x = ||V^{-1}||_2 K, \quad K_2^x = \sqrt{n}K_\infty^x.$$

So, to prove the lemma it suffices to show that (38) holds and that (Φ_x, Φ_u) is a feasible solution to (7)-(9). Let $J(\lambda)$ denote an elementary Jordan block with eigenvalue λ in J, $M|_{J(\lambda)}$ the restriction of the matrix M to the rows corresponding to the rows of $J(\lambda)$ in J, and $M|_{J(\lambda)}^0$ the concatenation of $M|_{J(\lambda)}$ with rows of zeros. Decomposing $V\Phi_x, V\Phi_x^*$ by rows gives

$$\begin{aligned} &||V\Phi_x(z) - V\Phi_x^*(z)||_2 \\ &= \left|\left|\sum_{\lambda\in\sigma}\sum_{J(\lambda) \text{ in } J} V\Phi_x(z)|_{J(\lambda)}^0 - V\Phi_x^*(z)|_{J(\lambda)}^0\right|\right|_2 \\ &\leq \sum_{\lambda\in\sigma}\sum_{J(\lambda) \text{ in } J} \left|\left|(V\Phi_x(z))|_{J(\lambda)} - (V\Phi_x^*(z))|_{J(\lambda)}\right|\right|_2. \end{aligned}$$

Thus, to prove (38) it suffices to show that for each elementary Jordan block $J(\lambda)$ in J there exists $K_{J(\lambda)} > 0$ such that

$$\left|\left|(V\Phi_x(z))|_{J(\lambda)} - (V\Phi_x^*(z))|_{J(\lambda)}\right|\right|_2 \leq K_{J(\lambda)} D(\mathcal{P}). \tag{39}$$

Premultiplying the SLS constraint in (4) by V implies that $(zI - J)V\Phi_x - VB\Phi_u = V$, and note that this is satisfied by both $(V\Phi_x, \Phi_u)$ and $(V\Phi_x^*, \Phi_u^*)$. As $(zI - J)$ is block diagonal, this equation decouples into independent equations for each elementary Jordan block $J(\lambda)$ in J given by $(zI - J(\lambda))(V\Phi_x(z))|_{J(\lambda)} - (VB)|_{J(\lambda)}\Phi_u(z) = V|_{J(\lambda)}$. Therefore, both our objective (39) and the SLS constraints become decoupled for each $J(\lambda)$, so it suffices to prove (39) for a single elementary Jordan block since the same argument applies to all the elementary Jordan blocks in J. Thus, for the remainder of the proof we fix a particular $\lambda \in \sigma$ and $J(\lambda)$ in J. For notational convenience, for the remainder of the proof we abuse notation and let Φ_x, Φ_x^*, B, and V denote $(V\Phi_x)|_{J(\lambda)}$, $(V\Phi_x^*)|_{J(\lambda)}$, $(VB)|_{J(\lambda)}$, and $V|_{J(\lambda)}$, respectively. Then the objective (39) and the SLS constraint become

$$||\Phi_x(z) - \Phi_x^*(z)||_2 \leq KD(\mathcal{P}) \tag{40}$$

$$(zI - J(\lambda))\Phi_x(z) - B\Phi_u(z) = V. \tag{41}$$

To complete the proof it suffices to show that there exists $K > 0$ such that (40) holds, and that (Φ_x, Φ_u) is a feasible solution to (7)-(9).

Let m_λ denote the multiplicity of λ in $J(\lambda)$. As $(zI - A)^{-1}$ is strictly proper real rational and $(B\Phi_u(z) + I)$ is proper real rational, their product Φ_x is strictly proper real rational. Therefore, Φ_x has a partial fraction decomposition which does not include any constant or polynomial terms, and in which all poles have finite multiplicity.

Now we derive the relationship between the poles and multiplicities of any pair of transfer functions which satisfy the SLS constraint. Let $(\tilde{\Phi}_x, \tilde{\Phi}_u)$ be any transfer functions which satisfy (41) and such that $\tilde{\Phi}_u \in \frac{1}{z}\mathcal{RH}_\infty$ and $\tilde{\Phi}_x$ is strictly proper rational. Let q be any pole of $\tilde{\Phi}_x$, $m_q = m_\lambda$ if $q = \lambda$ or $m_q = 0$ otherwise, and m the multiplicity of q in $\tilde{\Phi}_u$. Note that $m = 0$ if q is not a pole of $\tilde{\Phi}_u$. Let $\hat{m}$ be the multiplicity of q in $\tilde{\Phi}_x$. Then the terms in the partial fraction decompositions of $\tilde{\Phi}_u$ and $\tilde{\Phi}_x$ corresponding to pole q are given by $\sum_{i=1}^m H^i \frac{1}{(z-q)^i}$ and $\sum_{i=1}^{\hat{m}} G^i \frac{1}{(z-q)^i}$, respectively. By uniqueness of the partial fraction decomposition, (41) therefore implies that

$$G^{i+1} = J(\lambda - q)G^i + BH^i, \quad i \in \{1, ..., m\} \tag{42}$$

$$G^{i+1} = J(\lambda - q)G^i, \quad i \in \{m+1, ..., \hat{m}-1\} \tag{43}$$
$$0 = J(\lambda - q)G^{\hat{m}}. \tag{44}$$

First consider the case where $\lambda \neq q$. It is straightforward to verify the following fact:
Fact 2. If $J(\lambda)G = 0$ for $\lambda \neq 0$ then $G = 0$.

Then by Fact 2 and (44), $G^{\hat{m}} = 0$. Proceeding downwards in i, repeated application of Fact 2 and (43) imply that $G^i = 0$ for $i \in \{m+1, ..., \hat{m}\}$. So, in this case the order of q in $\tilde{\Phi}_x$ is $m = m + m_q$. Next consider the case where $\lambda = q$. Then, by (43), $G^i = J(0)^{i-(m+1)}G^{m+1}$ for $i \in \{m+1, ..., \hat{m}-1\}$. As $J(0)^{m_\lambda} = 0 = J(0)^{m_q}$, this implies that $G^i = 0$ for $i \geq m_q + m + 1$. So, in this case the order of q in $\tilde{\Phi}_x$ is $m + m_q$. Combining the above cases implies the following fact: that $\tilde{\Phi}_x$ only contains poles in σ and $\tilde{\Phi}_u$, and that their multiplicities are given by $m + m_q$. Applying this fact to (Φ_x, Φ_u) implies that $\Phi_x \in \frac{1}{z}\mathcal{RH}_\infty$ and is a feasible solution to (7)-(9). Hence, to complete the proof it suffices to prove (40).

In what follows, we show that to prove (13) it suffices to fix a particular pole in Φ_x^*, and to show that a certain portion of Φ_x closely approximates the portion of Φ_x^* corresponding to this pole. This is done by using the construction of [1, Theorem 1] to approximate Φ_u^* by Φ_u. Let $\mathcal{Q}$ denote the poles of Φ_x^*. For each $q \in \mathcal{Q}$, its contribution to the partial fraction decompositions of Φ_x^* and Φ_u^* is given, respectively, by

$$\sum\nolimits_{i=1}^{m_q+m} G_i^* \frac{1}{(z-q)^i}, \quad \sum\nolimits_{i=1}^{m} H_i^* \frac{1}{(z-q)^i} \tag{45}$$

by the above fact. Since Φ_u was constructed as in [1, Theorem 1], the portion of Φ_u that was chosen to approximate the pole at q in Φ_u^* is given by

$$\sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{i} H_j^i \frac{1}{z - p_j^i}, \quad H_j^i = c_j^i H_i^* \tag{46}$$

for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., i\}$, where $\{c_j^i\}_{j=1}^i$ are the constants chosen as in [1, Corollary 2] for approximating the pole q with multiplicity i by the poles $\{p_j^i\}_{j=1}^i$. Let $\tilde{m} = 1$ if $q = \lambda$ and Φ_u contains a pole at q, and $\tilde{m} = 0$ otherwise. If $\tilde{m} = 1$, reorder the poles in $\{p_j^i\}_{j=1}^i$ for each $i \in \{1, ..., m\}$ such that $p_1^i = q$. Then the above fact implies that the portion of Φ_x corresponding to the above portion of Φ_u is given by

$$\sum\nolimits_{i=1}^{m_q+\tilde{m}} G_i \frac{1}{(z-q)^i} + \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)}. \tag{47}$$

Hence, from (47) and (45) we compute

$$\begin{aligned} &||\Phi_x(z) - \Phi_x^*(z)||_2 \\ &= \left|\left| \sum_{q\in\mathcal{Q}} \sum_{i=1}^{m_q+\tilde{m}} G_i \frac{1}{(z-q)^i} + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \right.\right. \\ &\quad \left.\left. - \sum\nolimits_{q\in\mathcal{Q}} \sum\nolimits_{i=1}^{m_q+m} G_i^* \frac{1}{(z-q)^i} \right|\right|_2 \\ &\leq \sum_{q\in\mathcal{Q}} \left|\left| \sum_{i=1}^{m_q+\tilde{m}} G_i \frac{1}{(z-q)^i} + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \right.\right. \\ &\quad \left.\left. - \sum\nolimits_{i=1}^{m_q+m} G_i^* \frac{1}{(z-q)^i} \right|\right|_2 \end{aligned}$$

so, since $\mathcal{Q}$ is finite, to prove (40) it suffices to show that there exists $K_q > 0$ such that

$$\begin{aligned} &\left|\left| \sum\nolimits_{i=1}^{m_q+\tilde{m}} G_i \frac{1}{(z-q)^i} + \sum\nolimits_{i=1+\tilde{m}}^{m} \sum\nolimits_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \right.\right. \\ &\left.\left. - \sum\nolimits_{i=1}^{m_q+m} G_i^* \frac{1}{(z-q)^i} \right|\right| \leq K_q D(\mathcal{P}) \end{aligned} \tag{48}$$

for each $q \in \mathcal{Q}$. Towards that end, fix $q \in \mathcal{Q}$ and for the remainder of the proof let $\Phi_x(z)$ and $\Phi_x^*(z)$ denote the contributions to $\Phi_x(z)$ and $\Phi_x^*(z)$ given by (47) and (45), respectively, as in (48). We consider two cases.

Case 1: $q \neq \lambda$. Substituting (47), (46), and (45) into (42)-(44) implies that

$$\begin{aligned} &-J(\lambda - q)G_m^* = BH_m^* \\ &\qquad G_{i-1}^* = J(\lambda - q)^{-1}(G_i^* - BH_{i-1}^*), \quad i \in \{2, ..., m\} \\ &-J(\lambda - p_j^i)G_j^i = BH_j^i, \quad i \in \{1, ..., m\}, j \in \{1, .., i\} \end{aligned}$$

and all other coefficients in Φ_x and Φ_x^* are zero. The above implies that $G_l^* = -\sum_{i=l}^m J(\lambda - q)^{-(i+1-l)}BH_i^*$ for all $l \in \{1, ..., m\}$. Define $G_{(l,i)}^* = -J(\lambda - q)^{-(i+1-l)}BH_i^*$ for all $l \in \{1, ..., m\}$ and $i \in \{l, ..., m\}$, and note that $G_l^* = \sum_{i=l}^m G_{(l,i)}^*$. Write $G_j^i = c_j^i(G_{(i,i)}^* + \Delta G_j^i)$. Then, by (46)

$$\begin{aligned} J(\lambda - q)G_{(i,i)}^* &= -BH_i^* = -\frac{1}{c_j^i}BH_j^i = \frac{1}{c_j^i}J(\lambda - p_j^i)G_j^i \\ &= J(\lambda - p_j^i)(G_{(i,i)}^* + \Delta G_j^i) \\ &= (J(\lambda - q) - (p_j^i - q)I)(G_{(i,i)}^* + \Delta G_j^i) \end{aligned}$$

so $0 = -(p_j^i - q)G_{(i,i)}^* + J(\lambda - p_j^i)\Delta G_j^i$ and
$\Delta G_j^i = (p_j^i - q)J(\lambda - p_j^i)^{-1}G_{(i,i)}^*$. In summary,

$$\begin{aligned} \Phi_x^* &= \sum\nolimits_{l=1}^{m} G_l^* \frac{1}{(z-q)^l} = \sum\nolimits_{i=1}^{m} \sum\nolimits_{l=1}^{i} G_{(l,i)}^* \frac{1}{(z-q)^l} \\ G_{(l,i)}^* &= -J(\lambda - q)^{-(i+1-l)}BH_i^*, \quad l \in \{1, ..., m\} \\ \Phi_x &= \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{i} G_j^i \frac{1}{z - p_j^i} \\ G_j^i &= c_j^i(I + (p_j^i - q)J(\lambda - p_j^i)^{-1})G_{(i,i)}^*. \end{aligned}$$

For any $i \in \{1, ..., m\}$ and $l \in \{2, ..., i\}$ write
$J(\lambda - p_j^i)^{-1}G_{(l,i)}^* = G_{(l-1,i)}^* + \Delta G$. Note that for $i \in \{1, ..., m\}$ and $l \in \{2, ..., i\}$, $J(\lambda - q)G_{(l-1,i)}^* = G_{(l,i)}^*$. Then

$$\begin{aligned} &J(\lambda - q)G_{(l-1,i)}^* = G_{(l,i)}^* = J(\lambda - p_j^i)(G_{(l-1,i)}^* + \Delta G) \\ &= (J(\lambda - q) - (p_j^i - q)I)(G_{(l-1,i)}^* + \Delta G) \end{aligned}$$

so $0 = -(p_j^i - q)G_{(l-1,i)}^* + J(\lambda - p_j^i)\Delta G$ and
$\Delta G = (p_j^i - q)J(\lambda - p_j^i)^{-1}G_{(l-1,i)}^*$ which implies that
$J(\lambda - p_j^i)^{-1}G_{(l,i)}^* = (I + (p_j^i - q)J(\lambda - p_j^i)^{-1})G_{(l-1,i)}^*$. Applying this equation recursively implies that for $i \in \{1, ..., m\}$ and $j \in \{1, ..., i\}$

$$\begin{aligned} G_j^i &= c_j^i G_{(i,i)}^* + c_j^i(p_j^i - q)J(\lambda - p_j^i)^{-1}G_{(i,i)}^* \\ &= c_j^i G_{(i,i)}^* + c_j^i(p_j^i - q)(I + (p_j^i - q)J(\lambda - p_j^i)^{-1})G_{(i-1,i)}^* \\ &= c_j^i G_{(i,i)}^* + c_j^i(p_j^i - q)G_{(i-1,i)}^* \\ &+ c_j^i(p_j^i - q)^2 J(\lambda - p_j^i)^{-1}G_{(i-1,i)}^* = ... \end{aligned}$$

$$= \sum_{l=1}^{i} c_j^i (p_j^i - q)^{i-l} G^*_{(l,i)} + c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} G^*_{(1,i)}.$$

Therefore,

$$\begin{aligned}
\Phi_x(z) &= \sum_{i=1}^{m}\sum_{j=1}^{i} G_j^i \frac{1}{z-p_j^i} \\
&\overset{\substack{\text{above}\\ \text{identity}}}{=} \sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{l=1}^{i} c_j^i (p_j^i - q)^{i-l} G^*_{(l,i)} \frac{1}{z-p_j^i} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{i} c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} G^*_{(1,i)} \frac{1}{z-p_j^i} \\
&\overset{\substack{\text{reverse}\\ \text{sum order}}}{=} \sum_{i=1}^{m}\sum_{l=1}^{i} G^*_{(l,i)} \left(\sum_{j=1}^{i} c_j^i (p_j^i - q)^{i-l} \frac{1}{z-p_j^i} \right) \\
&\quad + \sum_{i=1}^{m} \left(\sum_{j=1}^{i} c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} \frac{1}{z-p_j^i} \right) G^*_{(1,i)} \\
&\overset{\text{Lemma 4(a)}}{=} \sum_{i=1}^{m}\sum_{l=1}^{i} G^*_{(l,i)} \frac{(z-q)^{i-l}}{\prod_{j=1}^{i}(z-p_j^i)} \\
&\quad + \sum_{i=1}^{m} \left(\sum_{j=1}^{i} c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} \frac{1}{z-p_j^i} \right) G^*_{(1,i)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi_x(z) - \Phi_x^*(z) &= \sum_{i=1}^{m}\sum_{l=1}^{i} G^*_{(l,i)} \left(\frac{(z-q)^{i-l}}{\prod_{j=1}^{i}(z-p_j^i)} - \frac{1}{(z-q)^l} \right) \\
&\quad + \sum_{i=1}^{m} \left(\sum_{j=1}^{i} c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} \frac{1}{z-p_j^i} \right) G^*_{(1,i)}.
\end{aligned}$$

This implies

$$\begin{aligned}
&||\Phi_x(z) - \Phi_x^*(z)||_2 \\
&\overset{\substack{\text{triangle}\\ \text{inequality}}}{\leq} \sum_{i=1}^{m}\sum_{l=1}^{i} ||G^*_{(l,i)}||_2 \left| \frac{(z-q)^{i-l}}{\prod_{j=1}^{i}(z-p_j^i)} - \frac{1}{(z-q)^l} \right| \\
&\quad + \sum_{i=1}^{m} ||G^*_{(1,i)}||_2 \left|\left| \sum_{j=1}^{i} c_j^i (p_j^i - q)^i J(\lambda - p_j^i)^{-1} \frac{1}{z-p_j^i} \right|\right|_2 \\
&\overset{\substack{\text{Lemma 3}\\ \text{Corollary 2(b)}}}{\leq} K D(\mathcal{P})
\end{aligned}$$

$$K = \sum_{i=1}^{m}\sum_{l=1}^{i} ||G^*_{(l,i)}||_2 k_{(l,i)} + \sum_{i=1}^{m} ||G^*_{(1,i)}||_2 k_i$$

which proves (48) for Case 1.

It will be useful to derive an additional bound for use in the proof of Case 2. In particular, we want to show that there exist constants $K_i > 0$ for $i \in \{1, ..., m\}$ such that

$$\left|\left| \sum_{j=1}^{i} G_j^i - G^*_{(1,i)} \right|\right|_2 \leq K_i D(\mathcal{P}). \tag{49}$$

Write $G_j^i = c_j^i (G^*_{(1,i)} + \Delta G)$. Then

$$\begin{aligned}
J(\lambda - q)^i G^*_{(1,i)} = -BH_i^* &= \frac{1}{c_j^i} J(\lambda - p_j^i) G_j^i \\
&= J(\lambda - p_j^i)(G^*_{(1,i)} + \Delta G)
\end{aligned}$$

so $\Delta G = J(\lambda - p_j^i)^{-1} J(\lambda - q)^i G^*_{(1,i)} - G^*_{(1,i)}$, which implies that $G_j^i = c_j^i J(\lambda - p_j^i)^{-1} J(\lambda - q)^i G^*_{(1,i)}$. Thus,

$$\begin{aligned}
\left|\left| \sum_{j=1}^{i} G_j^i - G^*_{(1,i)} \right|\right|_2 &= \left|\left| \sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} J(\lambda - q)^i G^*_{(1,i)} \right.\right. \\
&\qquad \left.\left. - J(\lambda - q)^{-i} J(\lambda - q)^i G^*_{(1,i)} \right|\right|_2 \\
&= \left|\left| \left(\sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} - J(\lambda - q)^{-i} \right) J(\lambda - q)^i G^*_{(1,i)} \right|\right|_2 \\
&\leq \left|\left| \sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} - J(\lambda - q)^{-i} \right|\right|_2 \left|\left| J(\lambda - q)^i G^*_{(1,i)} \right|\right|_2
\end{aligned}$$

Therefore, in order to prove (49) it suffices to show that

$$\left|\left| \sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} - J(\lambda - q)^{-i} \right|\right|_2 \leq K_i' D(\mathcal{P}) \tag{50}$$

for some constants $K_i' > 0$. For $l \in \{0, ..., m_q - 1\}$, the lth superdiagonal of $\sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1}$ is given by

$$\begin{aligned}
&\sum_{j=1}^{i} c_j^i (-1)^l (\lambda - p_j^i)^{-(l+1)} = (-1)^l \sum_{j=1}^{i} c_j^i (\lambda - p_j^i)^{-(l+1)} \\
&= \binom{i-1+l}{l} \frac{(-1)^l}{(\lambda - q)^{(i+l)}} + \epsilon_{(i,l)}, \quad |\epsilon_{(i,l)}| \leq K_{(i,l)} D(\mathcal{P})
\end{aligned}$$

where we evaluate the sum by Corollary 2(a). Consider the function $f(x) = x^{-i}$ and note that $f(J(\lambda - q)) = J(\lambda - q)^{-i}$. By [22, Theorem 11.1.1], for $l \in \{0, ..., m_q - 1\}$, the lth superdiagonal of $J(\lambda - q)^{-i} = f(J(\lambda - q))$ is given by

$$\frac{1}{l!} f^{(l)}(\lambda - q) = \frac{i^{(l)}}{l!} \frac{(-1)^l}{(\lambda - q)^{(i+l)}} = \binom{i-1+l}{l} \frac{(-1)^l}{(\lambda - q)^{(i+l)}}.$$

Thus, for each $i \in \{1, ..., m\}$ and $l \in \{0, ..., m_q - 1\}$, the difference between terms in superdiagonal l of the matrix in (50) is $\epsilon_{(i,l)}$, which satisfies $|\epsilon_{(i,l)}| \leq K_{(i,l)} D(\mathcal{P})$. Therefore, by Fact 1 in the proof of Corollary 2, this implies that there exist $K_i' > 0$ such that (50) holds.

Case 2: $q = \lambda$. Let $\hat{\mathcal{Q}}$ denote the poles in Φ_x. Substituting (47), (46), and (45) into (42)-(44) and (41) implies that

$$\begin{aligned}
J(0) G^*_{m_q+m} = 0, &\quad G_1^* = V - \textstyle\sum_{\substack{\hat{q} \in \mathcal{Q} \\ \hat{q} \neq q}} G^*_{(\hat{q},1)} \\
G^*_{i+1} = J(0) G_i^*, &\quad i \in \{m+1, ..., m_q + m - 1\} \\
G^*_{i+1} = J(0) G_i^* + BH_i^*, &\quad i \in \{1, ..., m\} \\
J(0) G_{m_q+\tilde{m}} = 0, &\quad G_1 = V - \textstyle\sum_{\substack{\hat{q} \in \hat{\mathcal{Q}} \\ \hat{q} \neq q}} G_{(\hat{q},1)} \\
G_{i+1} = J(0) G_i, &\quad i \in \{2, ..., m_q + \tilde{m} - 1\} \\
G_2 = J(0) G_1 &+ \tilde{m} \textstyle\sum_{i=1}^{m} BH_1^i
\end{aligned}$$

$$
\begin{aligned}
-J(q-p_j^i)G_j^i = BH_j^i, \quad & i \in \{1+\tilde{m},...,m\}, \\
& j \in \{1+\tilde{m},...,i\}
\end{aligned}
$$

where $G^*_{(\hat{q},1)}$ and $G_{(\hat{q},1)}$ denote the coefficients of $\frac{1}{z-\hat{q}}$ in Φ_x^* and Φ_x, respectively, for the pole $\hat{q}$. For $l \in \{1,...,m_q\}$, define Define $\hat{G}_1 = G_1 + \sum_{i=1+\tilde{m}}^m \sum_{j=1+\tilde{m}}^i G_j^i$, $\hat{G}_l = J(0)^{l-1}\hat{G}_1$, and $\hat{G}_l^* = J(0)^{l-1}G_1^*$. By (49) from Case 1,

$$
\begin{aligned}
||\hat{G}_1 - G_1^*||_2 &\leq \sum\nolimits_{\substack{\hat{q}\in\mathcal{Q} \\ \hat{q}\neq q}} \sum\nolimits_{i=1}^{m_{\hat{q}}} \left|\left| G^*_{(\hat{q},i,1)} - \sum\nolimits_{j=1}^{i} G^j_{(\hat{q},i,1)} \right|\right|_2 \\
&\leq K_1 D(\mathcal{P}), \quad K_1 = \sum\nolimits_{\substack{\hat{q}\in\mathcal{Q} \\ \hat{q}\neq q}} \sum\nolimits_{i=1}^{m_{\hat{q}}} K_{(\hat{q},i)}.
\end{aligned}
$$

Thus, for $l \in \{1,...,m_q\}$ we have

$$
||\hat{G}_l - \hat{G}_l^*||_2 \leq ||J(0)^{l-1}||_2 ||\hat{G}_l - \hat{G}_l^*||_1 \leq K_l D(\mathcal{P})
$$

where $K_l = ||J(0)^{l-1}||_2 K_1$. For $l \in \{2,...,m_q+\tilde{m}\}$ define

$$
\begin{aligned}
\tilde{G}_1 &= G_1 - \hat{G}_1 = -\sum\nolimits_{i=1+\tilde{m}}^m \sum\nolimits_{j=1+\tilde{m}}^i G_j^i \\
\tilde{G}_l &= G_l - \hat{G}_l = -J(0)^{l-1} \sum\nolimits_{i=1+\tilde{m}}^m \sum\nolimits_{j=1+\tilde{m}}^i G_j^i \qquad (51) \\
&\quad + J(0)^{l-2}\tilde{m} \sum\nolimits_{i=1}^m BH_1^i
\end{aligned}
$$

$$
\begin{aligned}
\tilde{G}_1^* &= G_1^* - \hat{G}_1^* = 0, \ \tilde{G}_l^* = G_l^* - \hat{G}_l^* = \sum\nolimits_{i=1}^{\min\{l-1,m\}} G^*_{(l,i)} \\
G^*_{(l,i)} &= J(0)^{l-(i+1)} BH_i^*, \quad i \in \{1,...,\min\{l-1,m\}\}.
\end{aligned} \tag{52}
$$

Then we have

$$
\begin{aligned}
&||\Phi_x(z) - \Phi_x^*(z)||_2 \\
\leq& \left|\left| \sum_{i=1}^{m_q+\tilde{m}} \tilde{G}_i \frac{1}{(z-q)^i} + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \right.\right. \\
&\left.\left. - \sum_{i=1}^{m_q+m} \tilde{G}_i^* \frac{1}{(z-q)^i} \right|\right|_2 + \left|\left| \sum_{i=1}^{m_q} (\hat{G}_i - \hat{G}_i^*) \frac{1}{(z-q)^i} \right|\right|_2 \\
\leq& \left|\left| \sum_{i=1}^{m_q+\tilde{m}} \tilde{G}_i \frac{1}{(z-q)^i} + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \right.\right. \\
&\left.\left. - \sum_{i=1}^{m_q+m} \tilde{G}_i^* \frac{1}{(z-q)^i} \right|\right|_2 + \hat{K} D(\mathcal{P}), \quad \hat{K} = \sum_{l=1}^{m_q} K_l \frac{1}{(1-|q|)^l}.
\end{aligned} \tag{53}
$$

So, for the remainder of the proof let Φ_x and Φ_x^* denote

$$
\Phi_x(z) = \sum_{i=1}^{m_q+\tilde{m}} \tilde{G}_i \frac{1}{(z-q)^i} + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G_j^i \frac{1}{(z-p_j^i)} \tag{54}
$$

$$
\Phi_x^*(z) = \sum\nolimits_{i=1}^{m_q+m} \tilde{G}_i^* \frac{1}{(z-q)^i}, \tag{55}
$$

and let G_i and G_i^* denote $\tilde{G}_i$ and $\tilde{G}_i^*$, respectively. Thus, to prove (48), by (53) it suffices to show that there exists $K > 0$ such that $||\Phi_x(z) - \Phi_x^*(z)||_2 \leq KD(\mathcal{P})$. By (46) we have

$$
G_j^i = -J(q-p_j^i)^{-1} BH_j^i = -c_j^i J(q-p_j^i)^{-1} BH_i^* \tag{56}
$$

for all $i \in \{\tilde{m}+1,...,m\}$ and $j \in \{\tilde{m}+1,...,i\}$. We compute

$$
\begin{aligned}
\Phi_x(z) \overset{\substack{(51)\\(54)}}{=}& \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{z-p_j^i} \\
&+ \tilde{m} \sum_{l=2}^{m_q+1} J(0)^{l-2} \sum_{i=1}^{m} BH_1^i \frac{1}{(z-q)^l} \\
&- \sum_{l=1}^{m_q} J(0)^{l-1} \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{i} G_j^i \frac{1}{(z-q)^l} \\
\overset{\text{Lemma 6}}{=}& \left(\sum_{l=0}^{m_q-1} J(0)^l \tilde{m} \frac{1}{(z-q)^{l+2}} \right) BH_1^* \\
&+ \sum_{i=1+\tilde{m}}^{m} \left(\sum_{j=1+\tilde{m}}^{i} J(q-p_j^i)^{-1} \frac{-c_j^i}{z-p_j^i} \frac{(p_j^i-q)^{m_q}}{(z-q)^{m_q}} \right. \\
&+ \sum_{l=0}^{m_q-i-1} \sum_{k=0}^{m_q-i-1-l} J(0)^l \sum_{j=1+\tilde{m}}^{i} \frac{c_j^i (p_j^i-q)^{m_q-2-k-l}}{(z-q)^{m_q-k}} \\
&\left. + \tilde{m} c_1^i J(0)^{m_q-1} \frac{1}{(z-q)^{m_q+1}} \right) BH_i^*.
\end{aligned}
$$

By (55) and (52) we have

$$
\Phi_x^* = \sum\nolimits_{i=1}^{m} \sum\nolimits_{l=0}^{m_q-1} J(0)^l BH_i^* \frac{1}{(z-q)^{i+l+1}}.
$$

Therefore, for $i \in \{1,...,m\}$ and $l \in \{0,...,m_q-1\}$, the lth superdiagonal of the term multiplying BH_i^* in Φ_x^* is given by $\frac{1}{(z-q)^{i+l+1}}$. Thus, by Lemma 6, for every $j, j' \in \{1,...,m_q\}$,

$$
\left|(\Phi_x(z) - \Phi_x^*(z))_{(j,j')}\right| \leq \sum_{i=1}^{m} \sum_{l=0}^{m_q-j} K_{i,l} D(\mathcal{P}) |(BH_i^*)_{(j+l,j')}|
$$

$$
\leq K^{(j,j')} D(\mathcal{P}), \quad K^{(j,j')} = \sum_{i=1}^{m} \sum_{l=0}^{m_q-j} K_{i,l} ||BH_i^*||_F.
$$

By Fact 1 in the proof of Corollary 2, this implies that $||\Phi_x(z) - \Phi_x^*(z)||_2 \leq KD(\mathcal{P})$ for some $K > 0$, which proves (48) for Case 2. $\square$

Theorem 1 applies the approximation error bounds of Lemma 2 to the optimal solution of (2) to obtain the desired suboptimality bounds.

Proof of Theorem 1. Let (Φ_x^*, Φ_u^*) be an optimal solution to (2). By Lemma 2, there exist $\Phi_x, \Phi_u \in \frac{1}{z}\mathcal{RH}_\infty$ which are a feasible solution to (7)-(9) and satisfy the approximation error bounds (12)-(13). Letting $J(\Phi_x, \Phi_u)$ denote the value of the objective of (2) for (Φ_x, Φ_u), we compute

$$
\begin{aligned}
J(\mathcal{P}) &\overset{\substack{\text{definition}\\\text{of optimum}}}{\leq} J(\Phi_x, \Phi_u) \\
&\overset{\substack{\text{adding}\\\text{zero}}}{=} \left|\left| C(\Phi_x(z) - \Phi_x^*(z) + \Phi_x^*(z))\hat{B} \right.\right. \\
&\quad \left.\left. + D(\Phi_u(z) - \Phi_u^*(z) + \Phi_u^*(z))\hat{B} - T_{\text{desired}}(z) \right|\right|_{\mathcal{H}_2} \\
&\quad + \lambda \left|\left| C(\Phi_x(z) - \Phi_x^*(z) + \Phi_x^*(z))\hat{B} \right.\right. \\
&\quad \left.\left. + D(\Phi_u(z) - \Phi_u^*(z) + \Phi_u^*(z))\hat{B} - T_{\text{desired}}(z) \right|\right|_{\mathcal{H}_\infty}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\substack{\text{triangle}\\\text{inequality}}}{\leq} \left|\left|C\Phi_x^*(z)\hat{B} + D\Phi_u^*(z)\hat{B} - T_{\text{desired}}(z)\right|\right|_{\mathcal{H}_2} \\
&\quad + \lambda \left|\left|C\Phi_x^*(z)\hat{B} + D\Phi_u^*(z)\hat{B} - T_{\text{desired}}(z)\right|\right|_{\mathcal{H}_\infty} \\
&+ \left|\left|C(\Phi_x(z) - \Phi_x^*(z))\hat{B}\right|\right|_{\mathcal{H}_2} + \left|\left|D(\Phi_u(z) - \Phi_u^*(z))\hat{B}\right|\right|_{\mathcal{H}_2} \\
&+ \lambda \left|\left|C(\Phi_x(z) - \Phi_x^*(z))\hat{B}\right|\right|_{\mathcal{H}_\infty} + \lambda \left|\left|D(\Phi_u(z) - \Phi_u^*(z))\hat{B}\right|\right|_{\mathcal{H}_\infty} \\
&\overset{\substack{(12)\\(13)}}{\leq} J^* + KD(\mathcal{P}) \\
&K = ||C||_F K_2^x ||\hat{B}||_F + ||D||_F K_2^u ||\hat{B}||_F \\
&\quad + \lambda ||C||_2 K_\infty^x ||\hat{B}||_2 + \lambda ||D||_2 K_\infty^u ||\hat{B}||_2.
\end{aligned}
$$

This yields the desired bound

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \frac{K}{J^*} D(\mathcal{P})$$

where $K = K(\mathcal{Q}, G^*_{(q,j)}, H^*_{(q,j)}, r, \delta)$ by the proofs of [1, Theorem 1] and Lemma 2. □

Proof of Corollary 1. Combining Theorem 1 with the result and proof of [1, Theorem 5] yields the desired result. □

VI. Conclusion

This work combined SLS with SPA to develop a new control design method. Unlike DBC, SPA does not result in deadbeat control, feasibility is automatic so it does not require slack variables which lead to additional suboptimality, and it can be solved by a single SDP, as opposed to the iterative algorithm that DBC requires. A suboptimality certificate was provided for SPA which, unlike the DBC bound, does not require a sufficiently long time horizon that the optimal impulse response has already decayed, and does not depend on this decay rate. The bound is specialized for the Archimedes spiral pole selection [1]. An example shows that SPA achieves much better matching with the optimal solution than DBC with orders of magnitude fewer poles. Future work should address extensions to state and input constraints, application of SPA to output feedback, extensions to continuous-time, static controllers, and extensions to time-varying systems and uncertainty.

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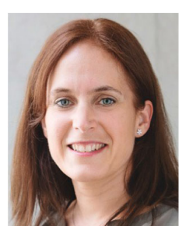

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