MATH 310L112
Introduction to Mathematical Reasoning
Assignment #11

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Section 11.5

Exercise 8: Suppose $[a], [b] \in \mathbb{Z}_n$, and $[a] = [a']$ and $[b] = [b']$. Alice adds $[a]$ and $[b]$ as $[a] + [b] = [a + b]$. Bob adds them as $[a'] + [b'] = [a' + b']$. Show that their answers $[a + b]$ and $[a' + b']$ are the same.

Proof. Suppose $[a], [b] \in \mathbb{Z}_n$ and $n \in \mathbb{N}$. We must show that $[a + b] = [a' + b']$. Since $[a] = [a']$, we know that $a \equiv a' \pmod{n}$. Therefore, by definition, $n \mid (a - a')$, so $a - a' = nk$ for some $k \in \mathbb{Z}$. Likewise, because $[b] = [b']$, we know that $b \equiv b' \pmod{n}$. Thus, $n \mid (b - b')$, so $b - b' = nl$ for some $l \in \mathbb{Z}$. Therefore, $a = a' + nk$ and $b = b' + nl$. Adding these equations together gives us

$$\begin{aligned}(a + b) &= a' + nk + b' + nl \\ &= (a' + b') + n(k + l).\end{aligned}$$

We can subtract $(a' + b')$ from both sides to get $(a + b) - (a' + b') = n(k + l)$. Consequently, this means $n \mid \big[(a + b) - (a' + b')\big]$, and it follows that $(a + b) \equiv (a' + b') \pmod{n}$. From that we conclude that $[a + b] = [a' + b']$. $\square$

Section 12.1

Exercise 12: Is the set $\theta = \{\big((x,y),(3y,2x,x+y)\big) : x,y \in \mathbb{R}\}$ a function? If so, what is its domain and range? What can be said about the codomain?

Solution. Observe that the set $\theta = \{\big((x,y),(3y,2x,x+y)\big) : x,y \in \mathbb{R}\} \subseteq \mathbb{R}^2 \times \mathbb{R}^3$. Thus, θ is a relation from $\mathbb{R}^2$ to $\mathbb{R}^3$. For every $(x,y) \in \mathbb{R}^2$ there will be only one unique point $(3y,2x,x+y) \in \mathbb{R}^3$. Therefore, θ is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Since we have that $\theta : \mathbb{R}^2 \to \mathbb{R}^3$, we can identify the domain of θ as $\mathbb{R}^2$ and the codomain of θ as $\mathbb{R}^3$. Let $a,b,c \in \mathbb{R}$. Then the range of θ is $\{(a,b,c) \in \mathbb{R}^3 : a = 3y, b = 2x, c = x+y; (x,y) \in \mathbb{R}^2\}$. □

Section 12.2

Exercise 8: A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as $f(m,n) = (m+n, 2m+n)$. Verify whether this function is injective and whether it is surjective.

Proof. To show that f is injective, we will use contrapositive proof. We must prove that $f(m,n) = f(k,l)$ implies $(m,n) = (k,l)$. Assume that $(m,n),(k,l) \in \mathbb{Z} \times \mathbb{Z}$ and $f(m,n) = f(k,l)$. Therefore, $(m+n, 2m+n) = (k+l, 2k+l)$. We then get that $m+n = k+l$ and $2m+n = 2k+l$. By subtracting the first equation from the second we find that $m = k$. Then, subtracting $m = k$ from $m+n = k+l$ gets us $n = l$. Because $m = k$ and $n = l$, it means that $(m,n) = (k,l)$. Thus f is injective.

Next, we will show that f is surjective. Suppose $(b,c) \in \mathbb{Z} \times \mathbb{Z}$ is a random element. We must show that there exists some $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ for which $f(x,y) = (b,c)$. If we want $f(x, y) = (b, c)$, this means $(x+y, 2x+y) = (b, c)$, giving us a system of equations:

$$x + y = b$$

$$2x + y = c.$$

After solving, we find that $x = c-b$ and $y = 2b-c$. Consequently, $(x, y) = (c-b, 2b-c)$, so $f(c - b, 2b - c) = (b, c)$. Thus, f is surjective. $\square$

Exercise 12: Consider the function $\theta : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a, b) = a - 2ab + b$. Is θ injective? Is it surjective? Bijective? Explain.

Proof. We will use contrapositive proof to show that θ is injective. Therefore, we must show that $\theta(a, b) = \theta(k, l)$ implies $(a, b) = (k, l)$. Suppose that $\theta(a, b) = \theta(k, l)$. Then $a - 2ab + b = k - 2kl + l$. We know that the first coordinate of our ordered pair must be either 0 or 1. We can also check when $a \neq k$ to make sure that it holds true that $b \neq l$. There are three cases.

Case 1. Let $a = k = 0$. Then

$$a - 2ab + b = k - 2kl + l$$

$$0 - 0 + b = 0 - 0 + l$$

$$b = l.$$

Since $a = k$ and $b = l$, it follows that $(a, b) = (k, l)$.

Case 2. Let $a = k = 1$. Then

$$a - 2ab + b = k - 2kl + l$$
$$1 - 2b + b = 1 - 2l + l$$
$$-2b + b = -2l + l$$
$$b = l.$$

Since $a = k$ and $b = l$, it follows that $(a, b) = (k, l)$.

Case 3. Let $a = 1$ and $k = 0$. Then

$$a - 2ab + b = k - 2kl + l$$
$$1 - 2b + b = 0 - 0 + l$$
$$1 - b = l.$$

Regardless of whether a is equal to 0 or 1, we get that $(a, b) = (k, l)$. Thus θ is injective.

Next, we will show that θ is surjective. Suppose $x \in \mathbb{Z}$ is an arbitrary element. We want to show that there is an $(a, b) \in \{0, 1\} \times \mathbb{N}$ for which $\theta(a, b) = x$. This means that $a - 2ab + b = x$. We can again look at two cases where $a = 0$ and $a = 1$.

Case 1. Let $a = 0$. Then

$$\theta(0, b) = 0 - 0 + b = x$$
$$b = x.$$

Thus, if x is positive, we can imagine any $b \in \mathbb{N}$.

Case 2. Let $a = 1$. Then

$$\theta(1, b) = 1 - 2b + b = x$$

$$= 1 - b = x.$$

In a situation where x is negative, we can assume $b > 1$. Since $\theta(a, b) = x$ for $(a, b) \in \{0, 1\} \times \mathbb{N}$, it follows that θ is surjective.

Because θ is both injective and surjective, it is bijective as well. $\square$

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% PREAMBLE
\documentclass[12pt]{article}
\usepackage{amssymb, amsmath, amsthm}
   % libraries of additional mathematics commands
\usepackage[paper=letterpaper, margin=1in]{geometry}
   % sets margins and space for headers
\usepackage{setspace, listings}
   % allow for adjusted line spacing and printing source code
\usepackage{graphicx}
\graphicspath{ {./images/} }
\title{MATH 310L112\\
       Introduction to Mathematical Reasoning\\
       Assignment \#11}
\author{Michael Wise}
\date{May 3rd, 2020}
% END PREAMBLE

\begin{document}
\maketitle
%\thispagestyle{empty}
\begin{description}
\section*{Section 11.5}
\item[Exercise 8:] Suppose $[a], [b] \in \mathbb{Z}_n$, and $[a] = [a']$
   and $[b] = [b']$. Alice adds $[a]$ and $[b]$ as $[a] + [b] = [a+b]$.
   Bob adds them as $[a'] + [b'] = [a'+b']$. Show that their answers $[a+b
   ]$ and $[a'+ b']$ are the same.
\begin{spacing}{2}
\begin{proof}
Suppose $[a], [b] \in \mathbb{Z}_n$ and $n \in \mathbb{N}$. We must show
   that $[a+b] = [a' + b']$. Since $[a] = [a']$, we know that $a \equiv a'
    \pmod{n}$. Therefore, by definition, $n \mid (a - a')$, so $a-a' = nk$
    for some $k \in \mathbb{Z}$. Likewise, because $[b] = [b']$, we know
   that $b \equiv b' \pmod{n}$. Thus, $n \mid (b - b')$, so $b-b' = nl$
   for some $l \in \mathbb{Z}$. Therefore, $a = a' + nk$ and $b = b' + nl
   $. Adding these equations together gives us
\begin{align*}
    (a + b) &= a' + nk + b' + nl \\
    &= (a' + b') + n(k+l).
\end{align*}
We can subtract $(a' + b' )$ from both sides to get $(a + b) - (a' + b') =
    n(k+l)$. Consequently, this means $n \mid \big[(a + b) - (a' + b')\big
   ]$, and it follows that $(a+b) \equiv (a' + b') \pmod{n}$. From that we
    conclude that $[a+b] = [a' + b']$.
\end{proof}
\end{spacing}
\section*{Section 12.1}
\item[Exercise 12:] Is the set $\theta = \{\big((x,y), (3y,2x,x+y)\big) :
   x,y \in \mathbb{R}\}$ a function? If so, what is its domain and range?
   What can be said about the codomain?
\begin{spacing}{2}
\begin{proof}[Solution]
Observe that the set $\theta = \{\big((x,y), (3y,2x,x+y)\big) : x,y \in \
   mathbb{R}\} \subseteq \mathbb{R}^2 \times \mathbb{R}^3$. Thus, $\theta$
    is a relation from $\mathbb{R}^2$ to $\mathbb{R}^3$. For every $(x,y)
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   \in \mathbb{R}^2$ there will be only one unique point $(3y,2x,x+y) \in
   \mathbb{R}^3$. Therefore, $\theta$ is a function from $\mathbb{R}^2$ to
    $\mathbb{R}^3$.
\newline
Since we have that $\theta:\mathbb{R}^2 \to \mathbb{R}^3$, we can identify
    the domain of $\theta$ as $\mathbb{R}^2$ and the codomain of $\theta$
   as $\mathbb{R}^3$. Let $a,b,c \in \mathbb{R}$. Then the range of $\
   theta$ is $\{(a,b,c) \in \mathbb{R}^3: a=3y, b=2x, c=x+y; (x,y) \in \
   mathbb{R}^2\}$.
\end{proof}
\end{spacing}
\section*{Section 12.2}
\item[Exercise 8:] A function $f:\mathbb{Z} \times \mathbb{Z} \to \mathbb{
   Z} \times \mathbb{Z}$ is defined as $f(m,n) = (m+n,2m+n)$. Verify
   whether this function is injective and whether it is surjective.
\begin{spacing}{2}
\begin{proof}
To show that $f$ is injective, we will use contrapositive proof. We must
   prove that $f(m,n)=f(k,l)$ implies $(m,n)=(k,l)$. Assume that $(m,n),(k
   ,l) \in \mathbb{Z} \times \mathbb{Z}$ and $f(m,n)=f(k,l)$. Therefore,
   $(m+n,2m+n) = (k+l,2k+l)$. We then get that $m+n = k+l$ and $2m+n = 2k+
   l$. By subtracting the first equation from the second we find that $m=k
   $. Then, subtracting $m=k$ from $m+n = k+l$ gets us $n=l$. Because $m=k
   $ and $n=l$, it means that $(m,n)=(k,l)$. Thus $f$ is injective.
\newline
Next, we will show that $f$ is surjective. Suppose $(b,c) \in \mathbb{Z} \
   times \mathbb{Z}$ is a random element. We must show that there exists
   some $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ for which $f(x,y) = (b,c)
   $. If we want $f(x,y) = (b,c)$, this means $(x+y,2x+y) = (b,c)$, giving
    us a system of equations:
\begin{align*}
    x + y &= b \\
    2x + y &= c.
\end{align*}
After solving, we find that $x= c-b$ and $y = 2b- c$. Consequently, $(x,y)
    = (c-b,2b-c)$, so $f(c-b,2b-c)=(b,c)$. Thus, $f$ is surjective.
\end{proof}
\end{spacing}

\item[Exercise 12:] Consider the function $\theta:\{0,1\} \times \mathbb{N
   } \to \mathbb{Z}$ defined as $\theta(a,b)=a-2ab+b$. Is $\theta$
   injective? Is it surjective? Bijective? Explain.
\begin{spacing}{2}
\begin{proof}
We will use contrapositive proof to show that $\theta$ is injective.
   Therefore, we must show that $\theta(a,b) = \theta(k,l)$ implies  $(a,b
   ) = (k,l)$. Suppose that $\theta(a,b) = \theta(k,l)$. Then $a-2ab+b = k
   -2kl+l$. We know that the first coordinate of our ordered pair must be
   either $0$ or $1$. We can also check when $a \neq k$ to make sure that
   it holds true that $b \neq l$. There are three cases.
\newline
\textbf{Case 1.} Let $a = k = 0$. Then
\begin{align*}
    a-2ab+b &= k-2kl+l \\
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    0-0+b &= 0-0+l \\
    b &= l.
\end{align*}
Since $a=k$ and $b=l$, it follows that $(a,b) = (k,l)$.
\newline
\textbf{Case 2.} Let $a = k = 1$. Then
\begin{align*}
    a-2ab+b &= k-2kl+l \\
    1-2b+b &= 1-2l+l \\
    -2b+b &= -2l+l \\
    b &= l.
\end{align*}
Since $a=k$ and $b=l$, it follows that $(a,b) = (k,l)$.
\newline
\textbf{Case 3.} Let $a = 1$ and $k = 0$. Then
\begin{align*}
    a-2ab+b &= k-2kl+l \\
    1-2b+b &= 0-0+l \\
    1-b &= l.
\end{align*}
Regardless of whether $a$ is equal to $0$ or $1$, we get that $(a,b) = (k,
   l)$. Thus $\theta$ is injective.
\newline
Next, we will show that $\theta$ is surjective. Suppose $x \in \mathbb{Z}
   $ is an arbitrary element. We want to show that there is an $(a,b) \in
   \{0,1\} \times \mathbb{N}$ for which $\theta(a,b) = x$. This means that
    $a -2ab + b = x$. We can again look at two cases where $a = 0$ and $a
   =1$.
\newline
\textbf{Case 1.} Let $a = 0$. Then
\begin{align*}
    \theta(0,b) = 0 - 0 + b = x \\
    b = x.
\end{align*}
Thus, if $x$ is positive, we can imagine any $b \in \mathbb{N}$.
\newline
\textbf{Case 2.} Let $a = 1$. Then
\begin{align*}
    \theta(1,b) &= 1 - 2b + b = x \\
    &= 1 - b = x. \\
\end{align*}
In a situation where $x$ is negative, we can assume $b > 1$.
Since $\theta(a,b) = x$ for $(a,b) \in \{0,1\} \times \mathbb{N}$, it
   follows that $\theta$ is surjective.
\newline
Because $\theta$ is both injective and surjective, it is bijective as well
   .
\end{proof}
\end{spacing}
\end{description}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The commands in this section print the source code starting
% on a new page. Comment out or delete if you do not want to
% include the source code in your document.
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%
\newpage
\lstset{
   basicstyle=\footnotesize\ttfamily,
   breaklines=true,
   language=[LaTeX]{TeX}
   }
\lstinputlisting{Assignment11.tex} % Change to correct filename
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\end{document}
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