MATH 310L112
Introduction to Mathematical Reasoning
Assignment #10

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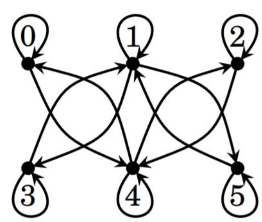
April 26th, 2020

Section 11.1

Exercise 4: Here is a diagram for a relation R on set A. Write the sets A and R.



Solution. The set $A = \{0, 1, 2, 3, 4, 5\}$. As a set, the relation $R = \{(0, 0), (0, 4), (1, 1),$ $(1, 3), (1, 5), (2, 2), (2, 4), (3, 3), (3, 1), (4, 4), (4, 0), (4, 2), (5, 5), (5, 1)\}$. □

Exercise 10: Consider the subset $R = (\mathbb{R} \times \mathbb{R}) - \{(x, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. What familiar relation on $\mathbb{R}$ is this? Explain.

Solution. In this example, we have removed all ordered pairs where the coordinates are the same. The relation can also be written as $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$. Thus, xRy if and only if $x \neq y$. □

Section 11.2

Exercise 4: Let $A = \{a, b, c, d\}$. Suppose R is the relation

$$R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(c,a),\\ (a,d),(d,a),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\}.$$

Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

Solution. Observe that R is **reflexive** because $(a,a),(b,b),(c,c),(d,d) \in R$. It also clearly follows that R is **symmetric** because xRy implies yRx for all $x, y \in A$. (i.e. $(a,b) \in R \Rightarrow (b,a) \in R$, and so on). Finally, observe that R is **transitive** because whenever xRy and yRz, then also xRz for every $x, y, z \in A$. (i.e. $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$). □

Exercise 16: Define a relation R on $\mathbb{Z}$ by declaring that xRy if and only if $x^2 \equiv y^2 \pmod 4$. Prove that R is reflexive, symmetric, and transitive.

Proof. Consider the set $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x^2 \equiv y^2 \pmod 4\}$.

First we will show that R is reflexive. Take any integer $x \in \mathbb{Z}$. Observe that $4 \mid 0$, and therefore $4 \mid (x^2 - x^2)$. By definition, we have that $x^2 \equiv x^2 \pmod 4$. Thus, xRx for every $x \in \mathbb{Z}$, which shows R is reflexive.

Next, we will show that R is symmetric. We must show that for all $x, y \in \mathbb{Z}$, the condition $x^2 \equiv y^2 \pmod 4$ implies $y^2 \equiv x^2 \pmod 4$. We use direct proof. Let $x, y \in \mathbb{Z}$ so that xRy. This means $x^2 \equiv y^2 \pmod 4$. By definition, $4 \mid (x^2 - y^2)$. Then $x^2 - y^2 = 4k$ for some $k \in \mathbb{Z}$. Multiplying both sides by -1 gives us $y^2 - x^2 = 4(-k)$. Therefore $4 \mid (y^2 - x^2)$ and $y^2 \equiv x^2 \pmod 4$. Since xRy implies yRx we have shown that R is symmetric.

Finally we show that R is transitive. We must show that if $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$, then $x^2 \equiv z^2 \pmod 4$. We use direct proof once more. Suppose that $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$. This means $4 \mid (x^2 - y^2)$ and $4 \mid (y^2 - z^2)$. Then $x^2 - y^2 = 4k$ and $y^2 - z^2 = 4l$ for some $k, l \in \mathbb{Z}$ by definition. Adding the two equations gets us $x^2 - z^2 = 4k + 4l$. Therefore, $x^2 - z^2 = 4(k + l)$, so $4 \mid (x^2 - z^2)$, thus $x^2 \equiv z^2 \pmod 4$. We have just shown R is transitive.

Since the relation R is reflexive, symmetric, and transitive, we are done. □

Section 11.3

Exercise 8: Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 + y^2$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

Proof. In order to show that R is an equivalence relation, we must prove that R is reflexive, symmetric, and transitive.

We begin by showing that R is reflexive. Let $x \in \mathbb{Z}$ such that xRx. Therefore we have $x^2 + x^2 = 2x^2$. Since $x^2 \in \mathbb{Z}$, the quantity $x^2 + x^2$ is even. Thus xRx which means that R is reflexive.

To show that R is symmetric, let $x, y \in \mathbb{Z}$ such that xRy. Obviously $x^2 + y^2 = y^2 + x^2$ is even by the commutative property. Because xRy implies yRx, it follows that R is symmetric.

Lastly, we show that R is transitive. Let $x, y, z \in \mathbb{Z}$ such that xRy and yRz. Then $x^2 + y^2$ is even and $y^2 + z^2$ is even. By definition, $x^2 + y^2 = 2k$ and $y^2 + z^2 = 2l$ for some $k, l \in \mathbb{Z}$. Therefore $x^2 = 2k - y^2$ and $z^2 = 2l - y^2$. Adding the equations together gives us

$$\begin{aligned} x^2 + z^2 &= 2k + 2l - 2y^2 \\ &= 2(k + l - y^2). \end{aligned}$$

Since $k + l - y^2$ is an integer, $x^2 + z^2$ is even by definition. Hence, xRz, which proves that R is transitive. Thus, the relation R is an equivalence relation. $\square$

Observe how that for the sum of two numbers to be even, they both must have the same parity. Consequently, for $x \in \mathbb{Z}$, we know x^2 is even if and only if x is even (which also holds true for odd integers). Thus we have two equivalence classes:

$$[0] = \{x \in \mathbb{Z} : x^2 \text{ is even}\}$$

$$[1] = \{x \in \mathbb{Z} : x^2 \text{ is odd}\}$$

where $[0] = [2] = [4] = \cdots$ and $[1] = [3] = [5] = \cdots$ similarly.

Exercise 12: Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

Disproof. This statement is **false** because of the following counterexample.

Let $A = \{1, 2, 3\}$. Now suppose R is a relation on A defined by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Let S be a relation on A defined by $S = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$. Both R and S are equivalence relations on A because they are reflexive, symmetric, and transitive. However, consider $R \cup S$:

$$R \cup S = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}.$$

Observe how $R \cup S$ contains $(2,1)$ and $(1,3)$, but not $(2,3)$. Because $R \cup S$ is not transitive, then it is not an equivalence relation on A. Thus, the statement is false. $\square$

Section 11.4

Exercise 2: List all the partitions of the set $A = \{a, b, c\}$. Compare your answer to the answer to Exercise 6 of Section 11.3.

Solution. Let the set $A = \{a, b, c\}$. The partitions of set A are:

$$\Big\{\{a\},\{b\},\{c\}\Big\},$$
$$\Big\{\{a,b\},\{c\}\Big\},$$
$$\Big\{\{b,c\},\{a\}\Big\},$$
$$\Big\{\{a,c\},\{b\}\Big\},$$
$$\Big\{\{a,b,c\}\Big\}.$$

The 5 partitions of A correspond to the 5 equivalence relations of A that are found in Exercise 6 of Section 11.3. $\square$

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% PREAMBLE
\documentclass[12pt]{article}
\usepackage{amssymb, amsmath, amsthm}
   % libraries of additional mathematics commands
\usepackage[paper=letterpaper, margin=1in]{geometry}
   % sets margins and space for headers
\usepackage{setspace, listings}
   % allow for adjusted line spacing and printing source code
\usepackage{graphicx}
\graphicspath{ {./images/} }
\title{MATH 310L112\\
       Introduction to Mathematical Reasoning\\
       Assignment \#10}
\author{Michael Wise}
\date{April 26th, 2020}
% END PREAMBLE

\begin{document}
\maketitle
%\thispagestyle{empty}
\begin{description}
\section*{Section 11.1}
\item[Exercise 4:] Here is a diagram for a relation $R$ on set $A$. Write
   the sets $A$ and $R$.
\begin{center}
\includegraphics[scale = .4]{images/11.1ex4.JPG}
\end{center}
\begin{spacing}{2}
\begin{proof}[Solution]
The set $A = \{0,1,2,3,4,5\}$. As a set, the relation $R = \{(0,0), (0,4),
    (1,1),
\newline
(1,3), (1,5), (2,2), (2,4), (3,3), (3,1), (4,4), (4,0), (4,2), (5,5),
   (5,1)\}$.
\end{proof}
\end{spacing}
\item[Exercise 10:] Consider the subset $R = (\mathbb{R} \times \mathbb{R
   }) - \{(x,x):x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R
   }.$ What familiar relation on $\mathbb{R}$ is this? Explain.
\begin{spacing}{2}
\begin{proof}[Solution]
In this example, we have removed all ordered pairs where the coordinates
   are the same. The relation can also be written as $R = \{(x,y) \in \
   mathbb{R} \times \mathbb{R}: x \neq y\}$. Thus, $x R y$ if and only if
   $x \neq y$.
\end{proof}
\end{spacing}
\section*{Section 11.2}
\item[Exercise 4:] Let $A = \{a,b,c,d\}$. Suppose $R$ is the relation
\begin{align*}
    R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(c,a), \\
(a,d),(d,a),(b, c),(c,b),(b,d),(d,b),(c,d),(d, c)\}.
\end{align*}
Is $R$ reflexive? Symmetric? Transitive? If a property does not hold, say
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   why.
\begin{spacing}{2}
\begin{proof}[Solution]
Observe that $R$ is \textbf{reflexive} because $(a,a),(b,b),(c,c),(d,d) \
   in R$. It also clearly follows that $R$ is \textbf{symmetric} because
   $xRy$ implies $yRx$ for all $x,y \in A$. (i.e. $(a,b) \in R \Rightarrow
    (b,a) \in R$, and so on). Finally, observe that $R$ is \textbf{
   transitive} because whenever $xRy$ and $yRz$, then also $xRz$ for every
    $x,y,z \in A$. (i.e. $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$).
\end{proof}
\end{spacing}
\item[Exercise 16:] Define a relation $R$ on $\mathbb{Z}$ by declaring
   that $xRy$ if and only if $x^2 \equiv y^2 \pmod{4}$. Prove that $R$ is
   reflexive, symmetric, and transitive.
\begin{spacing}{2}
\begin{proof}
Consider the set $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z}: x^2 \equiv
    y^2 \pmod{4}\}$.
\newline
First we will show that $R$ is reflexive. Take any integer $x \in \mathbb{
   Z}$. Observe that $4 \mid 0$, and therefore $4 \mid (x^2 - x^2)$. By
   definition, we have that $x^2 \equiv x^2 \pmod{4}$. Thus, $xRx$ for
   every $x \in \mathbb{Z}$, which shows $R$ is reflexive.
\newline
Next, we will show that $R$ is symmetric. We must show that for all $x,y \
   in \mathbb{Z}$, the condition $x^2 \equiv y^2 \pmod{4}$ implies $y^2 \
   equiv x^2 \pmod{4}$. We use direct proof. Let $x,y \in \mathbb{Z}$ so
   that $xRy$. This means $x^2 \equiv y^2 \pmod{4}$. By definition, $4 \
   mid (x^2 - y^2)$. Then $x^2-y^2 = 4k$ for some $k \in \mathbb{Z}$.
   Multiplying both sides by $-1$ gives us $y^2 - x^2 = 4(-k)$. Therefore
   $4 \mid (y^2 - x^2)$ and $y^2 \equiv x^2 \pmod{4}$. Since $xRy$ implies
    $yRx$ we have shown that $R$ is symmetric.
\newline
Finally we show that $R$ is transitive.  We must show that if $x^2 \equiv
   y^2 \pmod{4}$ and $y^2 \equiv z^2 \pmod{4}$, then $x^2 \equiv z^2 \pmod
   {4}$. We use direct proof once more. Suppose that $x^2 \equiv y^2 \pmod
   {4}$ and $y^2 \equiv z^2 \pmod{4}$. This means $4 \mid (x^2-y^2)$ and
   $4 \mid (y^2-z^2)$. Then $x^2-y^2 = 4k$ and $y^2 - z^2 = 4l$ for some $
   k,l \in \mathbb{Z}$ by definition. Adding the two equations gets us $x
   ^2 - z^2 = 4k + 4l$. Therefore, $x^2 - z^2 = 4(k+l)$, so $4 \mid (x^2 -
    z^2)$, thus $x^2 \equiv z^2 \pmod{4}$. We have just shown $R$ is
   transitive.
\newline
Since the relation $R$ is reflexive, symmetric, and transitive, we are
   done.
\end{proof}
\end{spacing}
\section*{Section 11.3}
\item[Exercise 8:] Define a relation $R$ on $\mathbb{Z}$ as $xRy$ if and
   only if $x^2+y^2$ is even. Prove $R$ is an equivalence relation.
   Describe its equivalence classes.
\begin{spacing}{2}
\begin{proof}
In order to show that $R$ is an equivalence relation, we must prove that $
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   R$ is reflexive, symmetric, and transitive.
\newline
We begin by showing that $R$ is reflexive. Let $x \in \mathbb{Z}$ such
   that $xRx$. Therefore we have $x^2 + x^2 = 2x^2$. Since $x^2 \in \
   mathbb{Z}$, the quantity $x^2 + x^2$ is even. Thus $xRx$ which means
   that $R$ is reflexive.
\newline
To show that $R$ is symmetric, let $x,y \in \mathbb{Z}$ such that $xRy$.
   Obviously $x^2 + y^2 = y^2 + x^2$ is even by the commutative property.
   Because $xRy$ implies $yRx$, it follows that $R$ is symmetric.
\newline
Lastly, we show that $R$ is transitive. Let $x,y,z \in \mathbb{Z}$ such
   that $xRy$ and $yRz$. Then $x^2 + y^2$ is even and $y^2 + z^2$ is even.
    By definition, $x^2 + y^2 = 2k$ and $y^2 + z^2 = 2l$ for some $k,l \in
    \mathbb{Z}$. Therefore $x^2 = 2k - y^2$ and $z^2 = 2l - y^2$. Adding
   the equations together gives us
\begin{align*}
    x^2 + z^2 &= 2k + 2l - 2y^2 \\
    &= 2(k + l - y^2).
\end{align*}
Since $k + l - y^2$ is an integer, $x^2 + z^2$ is even by definition.
   Hence, $xRz$, which proves that $R$ is transitive. Thus, the relation $
   R$ is an equivalence relation.
\end{proof}
Observe how that for the sum of two numbers to be even, they both must
   have the same parity. Consequently, for $x \in \mathbb{Z}$, we know $x
   ^2$ is even if and only if $x$ is even (which also holds true for odd
   integers). Thus we have two equivalence classes:
\begin{align*}
    [0] &= \{x \in \mathbb{Z} : x^2 \text{ is even}\} \\
    [1] &= \{x \in \mathbb{Z} : x^2 \text{ is odd}\}
\end{align*}
where $[0] = [2] = [4] = \cdots$ and $[1] = [3] = [5] = \cdots$ similarly.
\end{spacing}
\item[Exercise 12:] Prove or disprove: If $R$ and $S$ are two equivalence
   relations on a set $A$, then $R \cup S$ is also an equivalence relation
    on $A$.
\begin{spacing}{2}
\begin{proof}[Disproof]
This statement is \textbf{false} because of the following counterexample.
\newline
Let $A = \{1,2,3\}$. Now suppose $R$ is a relation on $A$ defined by $R =
   \{(1,1),(2,2),
\newline
(3,3),(1,2),(2,1)\}$. Let $S$ be a relation on $A$ defined by $S = \{(1,1)
   ,(2,2), (3,3),
\newline
(1,3),(3,1)\}$. Both $R$ and $S$ are equivalence relations on $A$ because
   they are reflexive, symmetric, and transitive. However, consider $R \
   cup S$:
\begin{align*}
    R \cup S = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}.
\end{align*}
Observe how $R \cup S$ contains $(2,1)$ and $(1,3)$, but not $(2,3)$.
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   Because $R \cup S$ is not transitive, then it is not an equivalence
   relation on $A$. Thus, the statement is false.
\end{proof}
\end{spacing}
\section*{Section 11.4}
\item[Exercise 2:] List all the partitions of the set $A = \{a,b,c\}$.
   Compare your answer to the answer to Exercise 6 of Section 11.3.
\begin{spacing}{2}
\begin{proof}[Solution]
Let the set $A = \{a,b,c\}$. The partitions of set $A$ are:
\begin{align*}
    &\Big\{\{a\},\{b\},\{c\}\Big\}, \\
    &\Big\{\{a,b\},\{c\}\Big\}, \\
    &\Big\{\{b,c\},\{a\}\Big\}, \\
    &\Big\{\{a,c\},\{b\}\Big\}, \\
    &\Big\{\{a,b,c\}\Big\}.
\end{align*}
The 5 partitions of $A$ correspond to the 5 equivalence relations of $A$
   that are found in Exercise 6 of Section 11.3.
\end{proof}
\end{spacing}
\end{description}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The commands in this section print the source code starting
% on a new page. Comment out or delete if you do not want to
% include the source code in your document.
%
\newpage
\lstset{
   basicstyle=\footnotesize\ttfamily,
   breaklines=true,
   language=[LaTeX]{TeX}
   }
\lstinputlisting{Assignment10.tex} % Change to correct filename
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\end{document}
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