MATH 310L112
Introduction to Mathematical Reasoning
Assignment #8

Michael Wise

April 10th, 2020

Chapter 7

Exercise 8: Suppose $a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{10}$ if and only if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.

Proof. First we show that $a \equiv b \pmod{10}$ implies that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. We use direct proof. Suppose that $a \equiv b \pmod{10}$. Then by definition, $10 \mid (a-b)$ which means $a = 10k + b$ for some $k \in \mathbb{Z}$. Factoring gives us $a = 2(5k) + b$, which shows that $a \equiv b \pmod{2}$ because $5k \in \mathbb{Z}$. In a similar manner, we can factor and get $a = 5(2k) + b$, which proves that $a \equiv b \pmod{5}$.

Conversely, we need to prove that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$ imply that $a \equiv b \pmod{10}$. We can prove this directly. Assume that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. Therefore, $2 \mid (a-b)$ and $5 \mid (a-b)$. By definition, we can say $a - b = 2k = 5l$ for some $k, l \in \mathbb{Z}$. We know that both 2 and 5 are prime. This means 2 has to divide one of the factors of $5l$. Since 2 doesn't divide 5, it must divide l. Thus, $l = 2x$ for some $x \in \mathbb{Z}$. Finally, we get that $a - b = 10x$ and therefore $a \equiv b \pmod{10}$. $\square$

Exercise 12: There exists a positive real number x for which $x^2 < \sqrt{x}$.

Proof. Let $x = \frac{1}{4}$. We know that $\frac{1}{4} \in \mathbb{R}$ and is positive. Then

$$\begin{aligned} x^2 &< \sqrt{x} \\ \left(\frac{1}{4}\right)^2 &< \sqrt{\frac{1}{4}} \\ \frac{1}{16} &< \frac{1}{2}. \end{aligned}$$

We have just shown an example of a positive real number for which $x^2 < \sqrt{x}$. $\square$

Chapter 8

Exercise 28: Prove that $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Proof. Let $A = \{12a + 25b : a, b \in \mathbb{Z}\}$. First we show $A \subseteq \mathbb{Z}$. Suppose $x \in A$. This means that $x = 12a + 25b$. Since $a, b \in \mathbb{Z}$, then $x = 12a + 25b$ will always be an integer by multiplication and addition rules. Therefore $x \in \mathbb{Z}$. Because $x \in A$ implies $x \in \mathbb{Z}$, it follows that $A \subseteq \mathbb{Z}$.

Next we show that $\mathbb{Z} \subseteq A$. Suppose $x \in \mathbb{Z}$. We need to show that $x \in A$. Notice how for $a = -2$ and $b = 1$ that $12(-2) + 25(1) = 1$. Multiplying by x gives us $12(-2x) + 25(x) = x$. This shows us that $x \in \{12a + 25b : a, b \in \mathbb{Z}\} = A$. Because $x \in \mathbb{Z}$ implies $x \in A$, it follows that $\mathbb{Z} \subseteq A$.

Therefore, since $A \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq A$, we have proven that $A = \mathbb{Z}$. $\square$

Chapter 9

Exercise 16: If A and B are finite sets, then $|A \cup B| = |A| + |B|$.

Disproof. This statement is **false** because of the following counterexample.

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Therefore, $A \cup B = \{1, 2, 3, 4, 5, 6\}$. Note that $|A| = 4, |B| = 4$, and $|A \cup B| = 6$. Consider the sum of the cardinalities

$$\begin{aligned} |A| + |B| &= 4 + 4 \\ &= 8 \\ &\neq |A \cup B|. \end{aligned}$$

In this example $|A \cup B| \neq |A| + |B|$, therefore making the statement false. $\square$

```
% PREAMBLE
\documentclass[12pt]{article}
\usepackage{amssymb, amsmath, amsthm}
   % libraries of additional mathematics commands
\usepackage[paper=letterpaper, margin=1in]{geometry}
   % sets margins and space for headers
\usepackage{setspace, listings}
   % allow for adjusted line spacing and printing source code
\title{MATH 310L112\\
       Introduction to Mathematical Reasoning\\
       Assignment \#8}
\author{Michael Wise}
\date{April 10th, 2020}
% END PREAMBLE

\begin{document}
\maketitle
%\thispagestyle{empty}
\begin{description}
\section*{Chapter 7}
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\end{description}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The commands in this section print the source code starting
% on a new page. Comment out or delete if you do not want to
% include the source code in your document.
%
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\lstset{
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   language=[LaTeX]{TeX}
   }
\lstinputlisting{Assignment8.tex} % Change to correct filename
%
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