1.4 Raisins

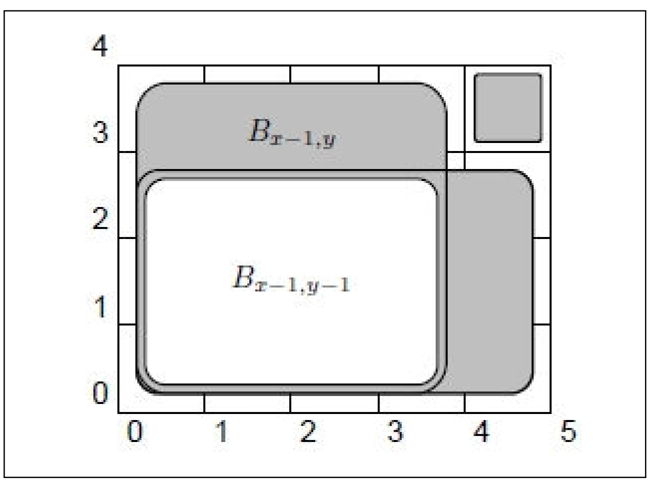
At any moment during the cutting, we have a set of independent sub-problems — blocks of chocolate. If we find the optimal solution for each of the blocks, together we get the optimal solution for the whole chocolate. This clearly hints at a dynamic programming solution.

Each sub-problem we may encounter corresponds to a rectangular part of the chocolate, and it can be described by four coordinates: specifically, two x and two y coordinates — the coordinates of its upper left and lower right corner. Hence we have $O(N^4)$ sub-problems to solve.

Now to solve a given sub-problem, we have to try all possible cuts. There are $O(N)$ possible cuts to try — at most $N-1$ horizontal and $N-1$ vertical ones. Each possible cut gives us two new, smaller sub-problems we solve recursively. Obviously, the recursion stops as soon as we reach a 1×1 block.

Assume that someone has given us a function $S(x_1, y_1, x_2, y_2)$ that returns the number of raisins in the rectangle given by coordinates (x_1, y_1) and (x_2, y_2) in constant time.

Using this function we can solve the entire problem in $O(N^5)$. We will use recursion with memoization. Given any of the $O(N^4)$ sub-problems, first check the memoization table to see whether we have computed it already. If yes, simply return the previously computed value. Otherwise, proceed as follows: The cost of the first cut is $S(x_1, y_1, x_2, y_2)$, which we have supposed can be computed in $O(1)$ time. For each possible placement of the first cut, recursively determine the cost of the remaining cuts in each sub-problem, and pick the optimal choice, storing the answer in the memoization table.



We are only missing one piece of the puzzle: the function S. All possible values can easily be precomputed in $O(N^4)$ and stored in an array.

Alternatively, we can use two-dimensional prefix sums: let A be the input array, and let $B_{x,y} = \sum_{i<x} \sum_{j<y} A_{i,j}$. The values B are called two-dimensional prefix sums. They can be computed using the formula

$$\forall x, y > 0 : \; B_{x,y} = B_{x-1,y} + B_{x,y-1} - B_{x-1,y-1} + A_{x-1,y-1}.$$

Having the two-dimensional prefix sums, we can compute the sum in any rectangle, using a similar formula. The sum in the rectangle with corners (x_1, y_1) and (x_2, y_2) is

$$S(x_1, y_1, x_2, y_2) = B_{x_2,y_2} - B_{x_1,y_2} - B_{x_2,y_1} + B_{x_1,y_1}.$$