Computer Science 180, Homework 2

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Chapter 1, Problem 5

a) The Gale-Shapely algorithm always produces a perfect matching with no strong instability, if it is modified so that a woman w rejects any man m' who asks her to be engaged if she is already engaged to a man m who she prefers equally to the man m'. In full the algorithm is as follows

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Initially all men m in M and women w in W are free
While there are free men m who haven't proposed to every woman
    Choose such a man m
    Let w be one of m's unasked and most preferred women
    If w is free then
        (m,w) become engaged
    Else (m',w) are already engaged
        If w prefers m' more than or equal to m then
            m remains free
        Else
            (m,w) become engaged
            m' becomes free
        Endif
    Endif
Endwhile
Return the set S of engaged pairs
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This algorithm must generate a matching because the algorithm terminates after no men exist who are free or have yet to propose to every woman. If no man is free, then everybody is matched. If a man has proposed to every woman, then every woman should have been proposed to at least once. This means that every woman is paired because they must remain paired after being proposed to once. We prove that this results in a perfect matching with no strong instability by contradiction. Assume that this algorithm produces a matching S with a strong instability. Thus there exists the pairs $(m, w) \in S$ and $(m', w') \in S$ where m prefers w' to w and where w' prefers m to m'. Then m has already asked w' to be engaged before asking w because he asks in order of decreasing preference. Afterwards, m' has asked w' to be engaged, causing her to cancel the engagement with m because she prefers m' to m. This is a contradiction because w' prefers m to m' in order for there to be a strong instability. Thus no strong instability can occur.

b) Here is a set of two men and two women with preference lists where any perfect matching always results in a weak instability. Man m_1 prefers w_1 to w_2. Man m_2 prefers w_1 to w_2. Woman w_1 prefers m_1 and m_2 equally. Woman w_2 prefers m_1 and m_2 equally. There are only two perfect matchings. The first is the perfect matching $(m_1, w_1), (m_2, w_2)$. A weak instability occurs because m_2 prefers w_1 to w_2 and w_1 is indifferent between m_2 and m_1. The second is the perfect matching $(m_1, w_2), (m_2, w_1)$. A weak instability occurs because m_1 prefers w_1 to w_2 and w_1 is indifferent between m_1 and m_2.

Chapter 1, Problem 8

The following preference list allows for a switch that would improve the partner of a woman w_1 who switched preferences.

person	1st pref.	2nd pref.	3rd pref.
m_1	w_1	w_3	w_2
m_2	w_1	w_2	w_3
m_3	w_3	w_1	w_2
w_1	m_3	m_1	m_2
w_2	m_2	m_3	m_1
w_3	m_1	m_3	m_2

This results in the stable matching $S = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. If w_1 switched m_1 and m_2, this would result in the stable matching $S' = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\}$. This is beneficial to w_1.

Chapter 2, Problem 4

1. $g_1(n) = 2^{\sqrt{\log n}}$
2. $g_3(n) = n(\log n)^3$
3. $g_4(n) = n^{\frac{4}{3}}$
4. $g_5(n) = n^{\log n}$
5. $g_2(n) = 2^n$
6. $g_7(n) = 2^{n^2}$
7. $g_6(n) = 2^{2^n}$

Chapter 2, Problem 5

a) True. $\log_2(n)$ is strictly increasing on $(0, \infty)$. As $f(n)$ grows large, $f(n) < cg(n)$ for some constant c. Thus

$$\log_2 f(n) < \log_2 cg(n) = \log_2 g(n) + \log_2(c)$$

We can ignore the constant value because it does not affect the growth rate at all, and so $\log_2 f(n)$ is $O(\log_2 g(n))$.

b) False. $f(n) = x$ and $g(n) = \frac{x}{2}$. Then $f(n)$ is $O(g(n))$ but 2^x is not $O(2^{\frac{x}{2}})$.

c) True. We have that $f(n) < cg(n)$ for some constant c as n grows large. Thus

$$f(n)^2 < c^2 g(n)^2$$

Because c^2 is just a constant, we have that $f(n)^2$ is $O(g(n)^2)$.