ECE102, Fall 2018 **Homework #7**

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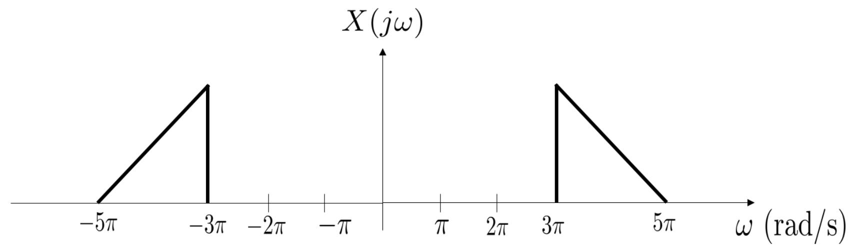
Due Thursday, 06 Dec 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 16.
100 points total.

1. (12 points) **Bandpass sampling**
   The figure below shows the Fourier transform of a real bandpass signal, i.e., a signal whose frequencies are not centered around the origin.



   We want to sample this signal. Let F_s in Hz represent the sampling frequency.

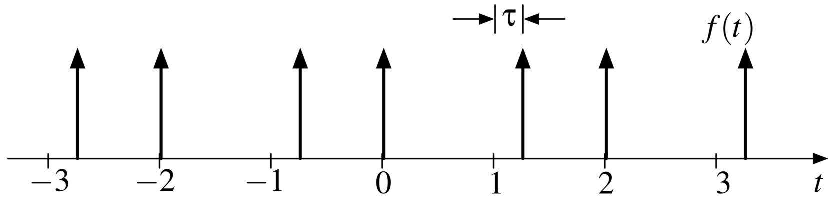
   (a) (2 points) One option is to sample this signal at the Nyquist rate. Then to recover the signal, we pass its sampled version through a low pass filter. What is the Nyquist rate of this signal? What is the cutoff frequency of the low pass filter?

   (b) (10 points) Since the signal might have high frequency components, Nyquist rate for this signal can be high. In other words, we need to have a lot of samples of the signal, which means that the sampling scheme is costly. It turns out that for this type of signal, we can sample it at a sampling frequency lower that the Nyquist rate and we can still recover the signal, however in this case, we will use a **bandpass** filter to recover the signal. To see this, we have the following two options for the sampling frequency:

   - $F_s = 2$ Hz;
   - $F_s = 2.5$ Hz;

   For each case, draw the spectrum of the signal after sampling it. To recover the signal, which F_s can we use? How we should choose the frequencies of the bandpass filter? What is the minimum F_s we can use and still recover the signal?

2. (18 points) **Sampling with imperfect sampler**
   Imperfections in a sampler cause characteristic artifacts in the sampled signal. In this problem we will look at the case where the sample timing is non-uniform, as shown below:



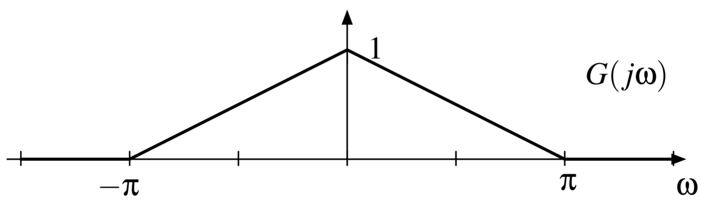
The sampling function $f(t)$ has its odd samples delayed by a small time τ.

(a) (6 points) Write an expression for $f(t)$ in terms of two uniformly spaced sampling functions.

(b) (6 points) Find $F(j\omega)$, the Fourier transform of $f(t)$. Express the impulse trains as sums, and simplify.

(c) (6 points) Find $F(j\omega)$, for the case where $\tau = 0$, and show that this is what you expect.

(d) (*Optional*) Assume the signal we are sampling has a Fourier transform



Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$

(e) (*Optional*) If we know $g(t)$ is real and even, can we recover $g(t)$ from the non-uniform samples $g(t)f(t)$?

3. (21 points) **Laplace Transform**

(a) (15 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. $f(t) = te^{-at}(\sin\omega_0 t)^2 u(t)$

ii. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ e^{-2(t-3)} & 2 \le t \end{cases}$

iii. $f(t) = \begin{cases} \sin(2\pi t), & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$

(b) (6 points) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: } \mathrm{Re}\{s\} > -1$$

Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

i. $\mathcal{F}\{x(t)e^{-t}\}$

ii. $\mathcal{F}\{x(t)e^{3t}\}$

4. (15 points) **Inverse Laplace Transform**

   Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

   (a) $F(s) = \dfrac{e^{-s}(s+1)}{(s-2)^2(s-3)}$

   (b) $F(s) = \dfrac{s+4}{s^3+4s}$

   (c) $F(s) = \dfrac{1}{(s+1)(s^2+2s+2)}$

5. (19 points) **LTI system**

   Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

   where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$. (Note: this is not a causal exponential, $e^t u(t)$. Rather, you should consider using the eigenfunction property of LTI systems.)

   (a) (7 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

   (b) (5 points) Find the output $y(t)$ when the input is $x(t) = u(t)$.

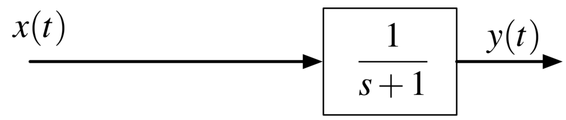
   (c) (7 points) The system $\mathcal{S}_1$ is linearly cascaded with another causal LTI system $\mathcal{S}_2$. The system $\mathcal{S}_2$ is given by the following input-output pair:

$$(\mathcal{S}_2) \quad \text{input} : u(t) - u(t-1) \rightarrow \text{output} : r(t) - 2r(t-1) + r(t-2)$$

   Find the overall impulse response.

6. (15 points) **Feedback systems**

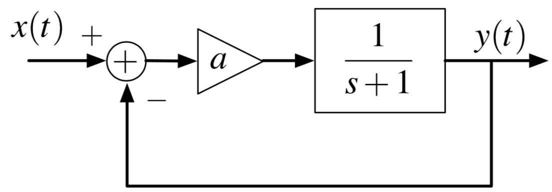
   The response to a remote manipulator can be modeled by this system:



$x(t)$ is the position we request, and $y(t)$ is the position of the manipulator. Its impulse response is a decaying exponential function with a time constant of 1 s (the time constant of $e^{-\lambda t}u(t)$ is $1/\lambda$), which is too slow to be practically usable. In order for a manipulator to feel immediate and interactive, we would like the response time to be no more than 100 ms.

(a) (5 points) Find the step response of the system, and plot it.

(b) (5 points) To speed up the response, we add a feedback loop around the system, along with a gain stage:



Find the transfer function of this system.

(c) (5 points) Choose a such that the time constant is 100 ms. Solve for the step response, and plot it on the same graph as part (a).