ECE102, Fall 2018 **Homework #6**

Signals & Systems Prof. J.C. Kao

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Due Thursday, 29 Nov 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 13.
100 points total.

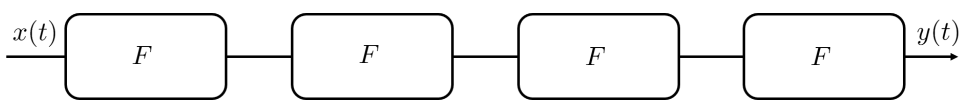
1. (15 points) **More properties of Fourier transform**

   (a) (10 points) Use Parseval's theorem to evaluate the following integral:

   $$\int_{-\infty}^{\infty} \text{sinc}^2(t)\cos(2\pi t)dt$$

   Hint: Use the fact that $\cos(2u) = 2\cos^2(u) - 1$

   (b) (5 points) Suppose we apply Fourier transform four times to signal $x(t)$ as shown below:



   How is $y(t)$ related to $x(t)$?

2. (31 points) **Frequency Response**

   Parts (a), (b) and (c) are independent.

   (a) (18 points) Consider the LTI system depicted in figure 1 whose response to an unknown input, $x(t)$, is

   $$y(t) = \left(4e^{-t} - 4e^{-4t}\right)u(t)$$

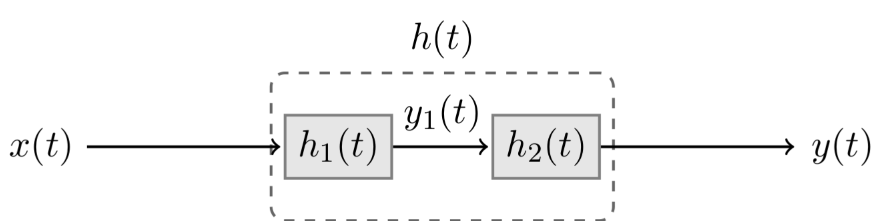


Figure 1: System for Problem 2.

   We know that for the same unknown input $x(t)$, the intermediate signal, $y_1(t)$, is given by:

   $$y_1(t) = 2e^{-t}u(t)$$

The overall LTI system is described by the following differential equation:

$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 3x(t)$$

i. (3 points) Find the frequency response, $H(j\omega)$, of the overall system $h(t)$.

ii. (6 points) Find the frequency responses $H_1(j\omega)$ of the first LTI system and $H_2(j\omega)$ of the second LTI system.

iii. (9 points) Find the impulse responses $h(t)$, $h_1(t)$ and $h_2(t)$.

(b) (6 points) Assume $x(t)$ a real signal that is baseband, i.e., its Fourier transform $X(j\omega)$ is non-zero for $|\omega| \leq \omega_0$ and zero for $|\omega| > \omega_0$. We process this signal through an LTI system. Let $y(t)$ denote the corresponding output and let $Y(j\omega)$ denote the Fourier transform of $y(t)$. Does $y(t)$ have frequency components different than those of $x(t)$? i.e., is $Y(j\omega) \neq 0$ for some $|\omega| > \omega_0$? What if we process $x(t)$ through a non-LTI system?

(c) (7 points) Consider the following two LTI systems with impulse responses:

$$h_1(t) = \text{sinc}\left(\frac{t}{2}\right)\cos(\pi t)$$

and

$$h_2(t) = 2\,\text{sinc}\,(2t)$$

Find the output of each system to the following input $x(t) = \cos(3\pi t)\cos(4\pi t)$. If we are given an input-output pair of an unknown LTI system, can we always identify this system?

3. (17 points) **Filters**

(a) (5 points) Consider an ideal low-pass filter $h_{LP,1}(t)$ with frequency response $H_{LP,1}(j\omega)$ depicted below in figure 2.

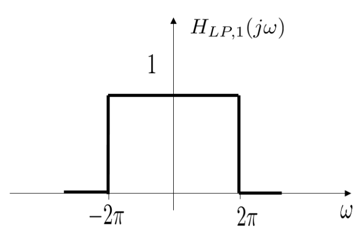


Figure 2: An ideal low pass filter

Using this filter, we construct the following new system:

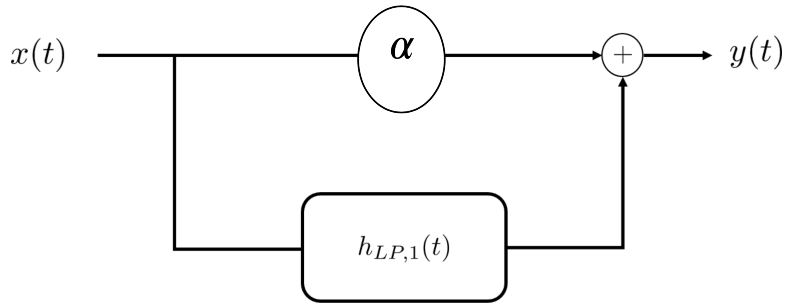


Figure 3: New system

We are given two choices for α: 1 or -1. Which value should we choose so that the new system is a high-pass filter? Does the new filter have any phase in its frequency response?

(b) (3 points) Why are the ideal filters non-realizable systems?

(c) (5 points) We want to design a causal non-ideal low-pass filter $h_{LP,2}(t)$, using the following frequency response:

$$H_{LP,2}(j\omega) = \frac{k}{\beta + j\omega}$$

Find k and β so that $H_{LP,2}(j\omega)$ is unity for $\omega = 0$ and its cutoff frequency is $\omega_0 = 2\pi$ rad/s, (i.e., the magnitude of $H_{LP,2}(j\omega)$ is $1/\sqrt{2}$ for $\omega = 2\pi$ rad/s).

(d) (4 points) We again consider the system of part (a) where instead of the ideal low-pass filter, we are going to use the non-ideal low-pass filter of part (c).

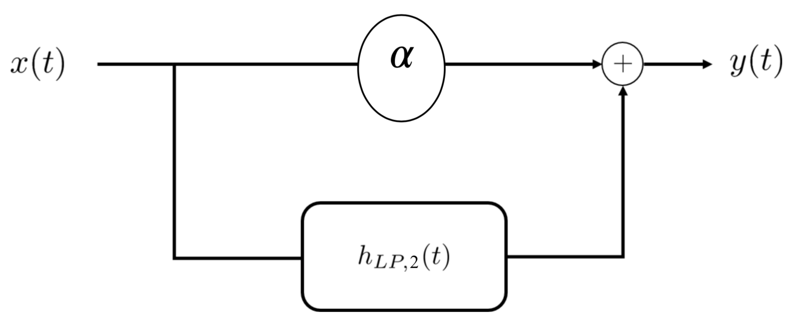
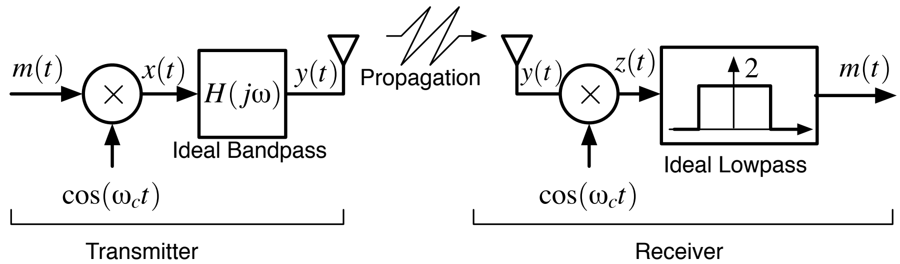


Figure 4: The system of part (a) with the non-ideal low pass filter

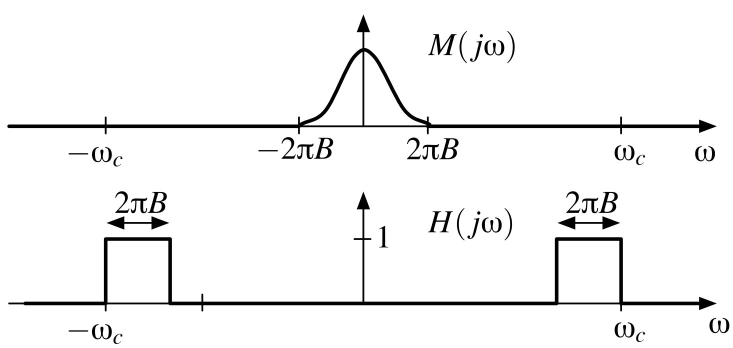
For the same value of α you found in part (a), find the frequency response of the equivalent system. Explain if the new system behaves as a high-pass filter.

4. (25 points) **Modulation and Demodulation**

(a) (15 points) Consider the communication system shown below:



The signal $m(t)$ is first modulated by $\cos(\omega_c t)$, and then passed through an ideal bandpass filter. The spectrum of the input $M(j\omega)$ and the frequency response of the ideal bandpass filter $H(j\omega)$ are:



The modulated signal is $x(t)$, and the output of the ideal bandpass is $y(t)$. This signal is transmitted through a channel. We assume that this channel does not introduce distortion into $y(t)$. The received signal $y(t)$ is then processed by a receiver. Sketch the signal spectrum at

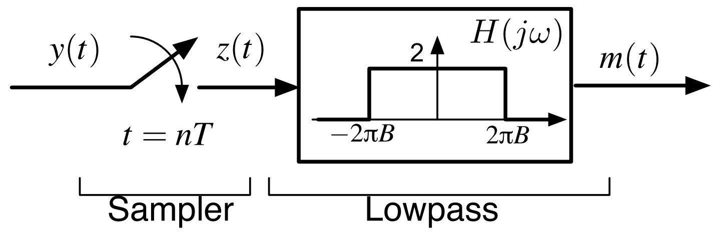
i. the output of the modulator, i.e., $X(j\omega)$,

ii. the output of the ideal bandpass, $Y(j\omega)$, and

iii. the output of the demodulator, $Z(j\omega)$

Does this system recover $m(t)$? (Assume $\omega_c \gg 2\pi B$)

(b) (10 points) In the first part of this problem, you have seen that to demodulate the received signal, we multiply $y(t)$ by $\cos(\omega_c t)$, and then to recover $m(t)$, we low-pass filter the result. In this part, you will show that you can achieve the same effect with an ideal sampler. In other words, we assume instead the following block diagram of the receiver: where the ideal sampler is drawn as a switch that closes instantaneously every T seconds to acquire a new sample.



Show that we can recover $m(t)$ if the ideal sampler operates at a frequency ω_c (i.e. samples at a rate of $\omega_c/2\pi$ samples/s). Draw the spectrum of the signals before and after the lowpass filter $Z(j\omega)$.

(Also assume $\omega_c \gg 2\pi B$)

5. (12 points) **Sampling at the Nyquist rate**

Assume $x(t)$ a real bandlimited signal where $X(j\omega)$ is non-zero for $|\omega| \leq 2\pi B$ rad/s. If F_s Hz is the Nyquist rate of $x(t)$, determine the Nyquist rate in Hz of the following signals in terms of B:

(a) $x(t-1)$

(b) $\cos(2\pi Bt)x(t)$

(c) $x(t) + x(t/2)$