Electrical Engineering 102, Homework 4

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Problem 1

a)

1. Since the left term has a period of $\frac{2}{3}$ and the right term has a period of $\frac{1}{2}$, we know that $f(t)$ has a period of 2. So we need to calculate the Fourier series coefficients c_k to obtain a solution in the form of

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t}$$

which is given by

$$c_k = \frac{1}{2} \int_0^2 f(t) e^{-jk\pi t} \, dt$$

Rewriting $f(t)$ in complex form yields

$$f(t) = \frac{1}{2} \left(e^{j3\pi t} + e^{-j3\pi t} \right) + \frac{1}{4j} \left(e^{j4\pi t} - e^{-j4\pi t} \right)$$

and so

$$c_k = \frac{1}{2} \int_0^2 \frac{1}{2} \left(e^{j(3-k)\pi t} + e^{-j(3+k)\pi t} \right) + \frac{1}{4j} \left(e^{j(4-k)\pi t} - e^{-j(4+k)\pi t} \right) \, dt$$

We see that $c_k = 0$ for all $k \neq -4, -3, 3, 4$. We obtain the following coefficients.

$$\begin{aligned} c_{-4} &= -\frac{1}{4j} \\ c_{-3} &= \frac{1}{2} \\ c_3 &= \frac{1}{2} \\ c_4 &= \frac{1}{4j} \end{aligned}$$

Therefore we have the Fourier series

$$f(t) = -\frac{1}{4j} e^{-j4\pi t} + \frac{1}{2} e^{-j3\pi t} + \frac{1}{2} e^{j3\pi t} + \frac{1}{4j} e^{j4\pi t}$$

which is the same as the complex form written above.

2. The period is 1 so we have a Fourier series of the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi t}$$

where the Fourier series coefficients are given by

$$\begin{aligned} c_k &= \int_0^1 e^{-2t} e^{-jk2\pi t} \, dt \\ &= \int_0^1 e^{-(2+jk2\pi)t} \, dt \\ &= -\frac{e^{-(2+jk2\pi)} - 1}{2 + jk2\pi} \\ &= -\frac{e^{-2} - 1}{2 + jk2\pi} \end{aligned}$$

3. The period is 3 so we have a Fourier series of the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\frac{2\pi}{3}t}$$

where the Fourier series coefficients are given by

$$\begin{aligned}
c_k &= \frac{1}{3}\left(\int_0^1 2e^{-jk\frac{2\pi}{3}t}\,dt + \int_1^2 e^{-jk\frac{2\pi}{3}t}\,dt\right) \\
&= \frac{1}{3}\left(-\frac{2(e^{-jk\frac{2\pi}{3}}-1)}{jk\frac{2\pi}{3}} - \frac{e^{-jk\frac{4\pi}{3}}-e^{-jk\frac{2\pi}{3}}}{jk\frac{2\pi}{3}}\right) \\
&= \frac{1-e^{-jk\frac{2\pi}{3}}}{jk\pi} + \frac{e^{-jk\frac{2\pi}{3}}-e^{-jk\frac{4\pi}{3}}}{jk2\pi}
\end{aligned}$$

b)

1. The function $z(t)$ will have a period of $T = T_1$ and have a Fourier series of the form
$$z(t) = \sum_{k=-\infty}^{\infty} Z_k e^{jk\frac{2\pi}{T}t}$$
Because the exponential terms are the exact same as in the Fourier series of $x(t)$ and $y(t)$, the coefficients for $z(t)$ are given by $Z_k = X_k + Y_k$.

2. The function $w(t)$ will have a period of $T = T_1 = 2T_2$ and have a Fourier series of the form
$$w(t) = \sum_{k=-\infty}^{\infty} W_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} W_k e^{jk\frac{\pi}{T_2}t}$$
The exponential at a given k corresponds to the same exponential in the Fourier series of $x(t)$, while it corresponds to the exponential with index $\frac{k}{2}$ in the Fourier series of $y(t)$ if k is even. Thus we have
$$W_k = \begin{cases} X_k & \text{if } k \text{ is odd} \\ X_k + Y_{\frac{k}{2}} & \text{if } k \text{ is even} \end{cases}$$

Problem 2

a) The period of this signal is T_0, and it has the Fourier series coefficients

$$g_k = \begin{cases} c_k + 1 & k = 0 \\ c_k & k \neq 0 \end{cases}$$

b) The period of this signal is T_0, and it has the Fourier series coefficients $g_k = c_{-k}$.

c) The period of this signal is T_0, and it has the Fourier series coefficients $g_k = c_k e^{-jk\omega_0 t_0}$.

d) The period of this signal is $\frac{T_0}{a}$, and it has the Fourier series coefficients $g_k = c_k$.

Problem 3

a) We say that $f(t)$ is an eigenfunction of an LTI system S if and only if $y(t) = S(f(t)) = af(t)$ where a is some constant. Given an impulse response $h(t)$ we have that

$$S(f(t)) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\, d\tau$$

Thus we want to show that

$$af(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\, d\tau$$

Given that $f(t) = \cos(\omega_0 t)$, we can rewrite $f(t)$ in its Fourier series form which is $f(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$, which allows us to plug into this formula and evaluate the following.

$$\begin{aligned}
a\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right) &= \int_{-\infty}^{\infty} h(\tau)\frac{1}{2}e^{j\omega_0 (t-\tau)} + \frac{1}{2}e^{-j\omega_0 (t-\tau)}\, d\tau \\
&= \left(\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau}\, d\tau\right)\frac{1}{2}e^{j\omega_0 t} \\
&\quad + \left(\int_{-\infty}^{\infty} h(\tau) e^{j\omega_0 \tau}\, d\tau\right)\frac{1}{2}e^{-j\omega_0 t} \\
&= b\frac{1}{2}e^{j\omega_0 t} + c\frac{1}{2}e^{-j\omega_0 t}
\end{aligned}$$

In this case we have that b and c are constants because the integrals do not depend on t. But assuming $\omega_0 \neq 0$, the two integrals can differ and $b \neq c$. Therefore there may be no constant $a = b = c$ that makes the above equation true. So $\cos(\omega_0 t)$ is not an eigenfunction of all LTI systems.

b) Consider the LTI system S characterized by the impulse response

$$h(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If the function $f(t) = t$ is an eigenfunction of this system S, we would expect that $at = S(f(t)) = h(t) * t$. Evaluating the convolution integral we get the following.

$$\begin{aligned} h(t) * t &= \int_0^1 t - \tau \, d\tau \\ &= t - \frac{1}{2} \end{aligned}$$

Because of the $\frac{1}{2}$ term, this cannot take the form of at which means that t is not an eigenfunction of S. Therefore t is not an eigenfunction of all LTI systems.

Problem 4

a) The output of the first multiplication will be $f(t) = e^t x(t)$. Then the output of the LTI system is $g(t) = S_1(f(t)) = f(t) * h(t)$, given the impulse function $h(t)$. Then the output of the second multiplication will be $y(t) = e^{-t} g(t)$. Using substitution this yields

$$y(t) = e^{-t}(f(t) * h(t)) = [(e^t x(t)) * h(t)]e^{-t}$$

.

b) Let $h'(t) = e^{-t} h(t)$. Then we can rewrite $y(t)$ as follows.

$$\begin{aligned} y(t) &= [(e^t x(t)) * h(t)]e^{-t} \\ &= e^{-t} \int_{-\infty}^{\infty} e^{\tau} x(\tau) h(t - \tau) \, d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-(t-\tau)} h(t - \tau) \, d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) h'(t - \tau) \, d\tau \\ &= \int_{-\infty}^{\infty} h'(\tau) x(t - \tau) \, d\tau \end{aligned}$$

c) The previous result indicates that the system that maps $x(t)$ to $y(t)$ is defined by a convolution integral, namely $h'(t) * x(t)$. Convolution is by definition linear and time invariant, so this is an LTI system. The impulse response of this system $h_{\text{eq}}(t)$ can be found by evaluating $h'(t) * \delta(t)$, giving us

$$h_{eq}(t) = h'(t) = e^{-t}h(t)$$

d) The impulse response is the derivative of the step response. This means that the impulse response is given by

$$h(t) = u(t-1)$$

So the system S_1 is causal because the impulse response is zero when $t < 0$. The system S_1 is not stable. An example that shows this is the step response, which is is a bounded input that produces the ramp function, an unbounded output.

The overall equivalent system that maps $x(t)$ to $y(t)$ has the impulse response given by

$$h_{\text{eq}}(t) = e^{-t}u(t-1)$$

It is also causal since the impulse response is zero when $t < 0$. We can check if it is stable by finding the bounds of $y(t)$ given a bounded $x(t)$. Let $|x(t)| < B$ for all t. Then we have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} h_{\text{eq}}(\tau)x(t-\tau)\, d\tau \right| &\leq B \left| \int_{-\infty}^{\infty} h_{eq}(\tau)\, d\tau \right| \\ &= B \left| \int_{1}^{\infty} e^{-\tau}\, d\tau \right| \\ &= \frac{B}{e} \end{aligned}$$

Therefore we have that given an input bounded by B, we get an output bounded by $\frac{B}{e}$. So the overall equivalent system is stable.

Problem 5

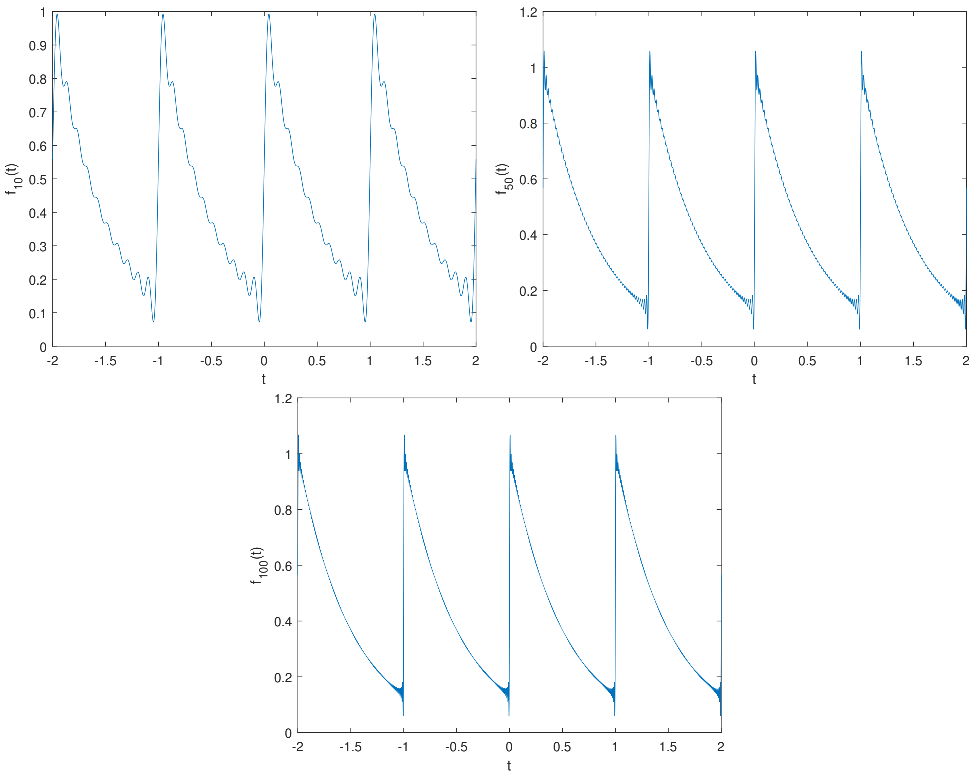
a) I implemented the function using the following code.

```
function fn=problem5a(Dn,omega0,t)
    function ft=evalAtPoint(Dn, omega0, t)
    ft=0;
    N=(length(Dn)-1)/2;
    for n=-N:N
        ft=ft+Dn(n+N+1)*exp(1i*omega0*n*t);
    end
    end
fn=arrayfun(@(t)(evalAtPoint(Dn, omega0, t)), t);
end
```

b) I graphed the plots using the following code.

```
index=-10:10;
c10=-(exp(-2)-1)./(2+2i.*index*pi);
index=-50:50;
c50=-(exp(-2)-1)./(2+2i.*index*pi);
index=-100:100;
c100=-(exp(-2)-1)./(2+2i.*index*pi);
t=-2:0.001:2;
f10=problem5a(c10,2*pi,t);
f50=problem5a(c50,2*pi,t);
f100=problem5a(c100,2*pi,t);
set(gcf,'color','w');
plot(t,f10);
xlabel('t');
ylabel('f_{10}(t)');
export_fig problem5b-10.pdf;
plot(t,f50);
xlabel('t');
ylabel('f_{50}(t)');
export_fig problem5b-50.pdf;
plot(t,f100);
xlabel('t');
ylabel('f_{100}(t)');
export_fig problem5b-100.pdf;
```

The plots are shown below.


f10(t)
t
f50(t)
t
f100(t)
t