Electrical Engineering 113, Homework 4

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Problem 1

We can write that $z[n] = x[n] \circledast x[n]$, which can be easily seen by using a flip and drag method. Since convolution in the time domain corresponds to multiplication in the frequency domain with a scale of N, we obtain the following result. Let r_k be the DTFS coefficients for $z[n]$ and s_k be the DTFS coefficients for $x[n]$.

$$r_k = 3s_k^2$$

Problem 2

a) We know that the DTFT of $a^n u[n]$ is $\frac{1}{1-ae^{-j\Omega}}$. Then we will rewrite the function $x[n]$ as follows.

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^{n-3} u[n] + \left(\frac{1}{3}\right)^{n} u[n-1] \\ &= 8\left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u[n-1] \end{aligned}$$

By substituting in the expression for the DTFT of $a^n u[n]$ and applying the time shifting property we can obtain the following result.

$$X(\Omega) = \frac{8}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{e^{-j\Omega}}{3\left(1 - \frac{1}{3}e^{-j\Omega}\right)}$$

b) For the cosine term, we begin with the DTFT for $x[n] = 1$ which is shown below.

$$2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$$

We can then apply the modulation property to this to obtain the DTFT for the cosine term.

$$\pi \sum_{m=-\infty}^{\infty} \left(\delta \left(\Omega - \frac{\pi}{3} - 2\pi m \right) + \delta \left(\Omega + \frac{\pi}{3} - 2\pi m \right) \right)$$

We will use the DTFT of $a^n u[n]$ again which gives us the following result.

$$X(\Omega) = \frac{4}{1 - \frac{1}{4}e^{-j\Omega}} + \pi \sum_{m=-\infty}^{\infty} \left(\delta \left(\Omega - \frac{\pi}{3} - 2\pi m \right) + \delta \left(\Omega + \frac{\pi}{3} - 2\pi m \right) \right)$$

Problem 3

a) Begin by finding the IDTFT of $\text{rect}\left(\frac{\Omega}{\frac{\pi}{6}}\right)$.

$$\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{rect}\left(\frac{\Omega}{\frac{\pi}{6}}\right) e^{j\Omega n}\, d\Omega \\
&= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{j\Omega n}\, d\Omega \\
&= \frac{1}{2\pi j n}\, e^{j\Omega n}\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{1}{2\pi j n}(e^{j\frac{\pi}{6}n} - e^{j\frac{\pi}{6}n}) \\
&= \frac{\sin(\frac{\pi}{6}n)}{\pi n}
\end{aligned}$$

Then we can apply time shifting and scale by a constant to obtain the following result.

$$x[n] = \frac{\sin(\frac{\pi}{6}(n-2))}{2(n-2)}$$

b) From Parseval's Theorem we have the following relationship.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{\pi}{2}\text{rect}\left(\frac{\Omega}{\frac{\pi}{6}}\right) e^{-j2\Omega} \right|^2 \, d\Omega$$

We then obtain the following result.

$$\frac{\pi}{8}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\Omega = \frac{\pi^2}{24}$$

Problem 4

a) Take the case where $n \neq 0$.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi j n}\; e^{j\Omega n}\Big|_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2\pi j n}(e^{j\omega_c n} - e^{j\omega_c n}) \\ &= \frac{\sin(\omega_c n)}{\pi n} \end{aligned}$$

Take the case where $n = 0$.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\Omega \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

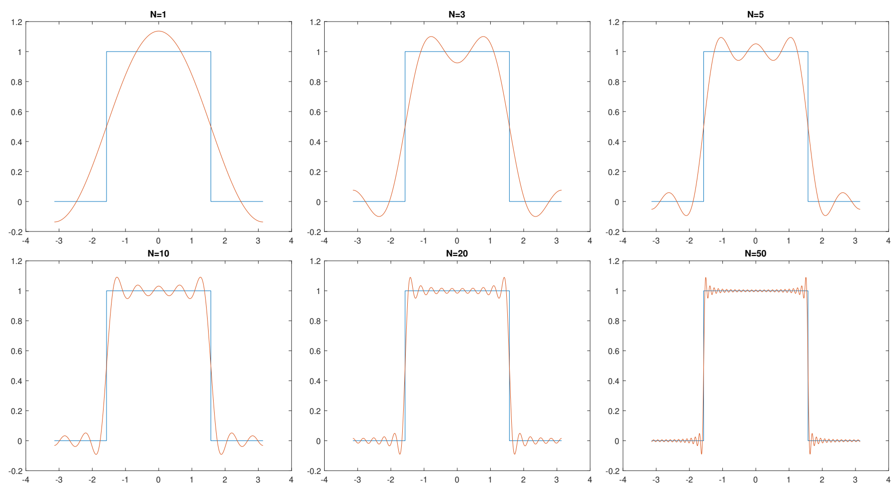
Thus we have the following result.

$$x[n] = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

b)

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$
$$= \frac{\omega_c}{\pi} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sin(\omega_c n)}{\pi n} e^{-j\Omega n}$$

c) The plots are shown below.



d) The plots of $\tilde{X}(\Omega)$ begin to more closely resemble $X(\Omega)$ as N increases, but there is some overshoot at the discontinuities in $X(\Omega)$. This is the Gibbs phenomenon.

Problem 5

a) I ran the following script.

```
A=double(imread('Lena.bmp'));
[U, S, V]=svd(A);
singvals=diag(S);
indices=find(singvals >= 0.01 * max(singvals));
U_red=U(:, indices);
```

```
S_red=S(indices, indices);
V_red=V(:, indices);
A_red=U_red * S_red * V_red';
imshow(uint8(A_red));
export_fig problem5a.pdf
```

This produced the following output.

b) We need $53 \times 512 + 53^2 + 53 \times 512 = 57081$ real values to store this image. This is 21.775% the size of the size of the original image A.