Electrical Engineering 113, Bonus Project

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June 14th, 2019

1 Given Information

Let $\mathbf{x} = (2, y)$ be the position of the source, $\mathbf{x}_1 = (0, 4)$ be the position of the first receiver, and $\mathbf{x}_2 = (0, 4.5)$ be the position of the second receiver. Let $P_{r,l}$ be the power received by the lth receiver, P_t be the power at the source, and y_l be the y coordinate of the l receiver. We model the received power with the following formula.

$$P_{r,l} = \frac{P_t}{4 + (y - y_l)^2}$$

2 Normalizing Noise

Assume the noise in the received power signal is log-normal distributed, so we can take the logarithm and rewrite this as follows.

$$\begin{aligned} \ln(P_{r,l}) - \ln(P_t) &= -\ln\left(4 + (y - y_l)^2\right) + n_l \\ r_l &= -\ln(d_l^2) + n_l \\ r_l &= -2\ln(d_l) + n_l \end{aligned}$$

Now we can assume the noise n_l follows a Gaussian distribution $N(0, \lambda_l^2)$. Here we have $r_l = \ln(P_{r,l}) - \ln(P_t)$ and d_l is the distance formula from the source to the receiver.

3 Matrix Formulation

Letting $R = 4 + y^2$, we can rewrite the distance formula as follows.

$$\begin{aligned} d_l^2 &= 4 + (y - y_l)^2 \\ &= 4 + y^2 - 2yy_l + y_l^2 \\ R - 2yy_l &= d_l^2 - y_l^2 \end{aligned}$$

Equivalently in matrix form this is the following.

$$\begin{bmatrix} -2 \times 4 & 1 \\ -2 \times 4.5 & 1 \end{bmatrix} \begin{bmatrix} y \\ R \end{bmatrix} = \begin{bmatrix} d_1^2 - 4^2 \\ d_2^2 - 4.5^2 \end{bmatrix}$$

4 Unbiased Distance Estimator

Since we do not know d_l^2, can try to obtain an estimate of it using the following calculations.

$$\begin{aligned} r_l &= -2\ln(d_l) + n_l \\ \ln(d_l) &= \frac{n_l - r_l}{2} \\ d_l &= e^{\frac{n_l - r_l}{2}} \\ d_l^2 &= e^{n_l - r_l} \end{aligned}$$

Let $x = n_l - r_l$ be a random variable that is normally distributed with a mean at $-r_l$ and the variance λ_l^2 coming entirely from n_l. We calculate the expected value of d_l^2 as follows.

$$\begin{aligned} E\left[e^{n_l - r_l}\right] &= E\left[e^x\right] \\ &= \int e^x \frac{1}{\sqrt{2\pi\lambda_l^2}} e^{-\frac{1}{2}\frac{(x+r_l)^2}{\lambda_l^2}}\, dx \end{aligned}$$

We can make the change of variables $x = z\lambda_l - r_l$ in order to make the integral easier to evaluate.

$$
\begin{aligned}
E\left[e^x\right] &= \int e^{z\lambda_l - r_l} \frac{1}{\sqrt{2\pi\lambda_l^2}} e^{-\frac{1}{2}z^2} \lambda_l \, dz \\
&= e^{-r_l} \int e^{z\lambda_l} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz \\
&= e^{-r_l} \int \frac{1}{\sqrt{2\pi}} e^{z\lambda_l - \frac{1}{2}z^2} \, dz \\
&= e^{-r_l} \int \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\lambda_l^2 - \frac{1}{2}(z-\lambda_l)^2} \, dz \\
&= e^{-r_l} e^{\frac{1}{2}\lambda_l^2} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\lambda_l)^2} \, dz \\
&= e^{-r_l + \frac{1}{2}\lambda_l^2}
\end{aligned}
$$

The integral evaluates to 1 because it is the integral of a normal distribution. We can substitute this expression into the matrix formulation of the problem since it is an unbiased estimator of d_l^2.

$$
\begin{bmatrix} -2 \times 4 & 1 \\ -2 \times 4.5 & 1 \end{bmatrix} \begin{bmatrix} y \\ R \end{bmatrix} = \begin{bmatrix} e^{-r_1 + \frac{1}{2}\lambda_1^2} - 4^2 \\ e^{-r_2 + \frac{1}{2}\lambda_2^2} - 4.5^2 \end{bmatrix}
$$

5 Polynomial Formulation

We can expand the matrix formulation of the problem to obtain the following equations.

$$
\begin{aligned}
y^2 - 8y + 20 - e^{-r_1 + \frac{1}{2}\lambda_1^2} &= 0 \\
y^2 - 9y + 24.25 - e^{-r_2 + \frac{1}{2}\lambda_2^2} &= 0 \\
y - 4.25 - e^{-r_1 + \frac{1}{2}\lambda_1^2} + e^{-r_2 + \frac{1}{2}\lambda_2^2} &= 0 \\
y = 4.25 + e^{-r_1 + \frac{1}{2}\lambda_1^2} - e^{-r_2 + \frac{1}{2}\lambda_2^2}
\end{aligned}
$$

This is a closed form solution for y, so all we need to do is obtain the variances and r_1 and r_2.

6 Code Implementation

Obtaining r_1 and r_2 from our data is straightforward. We will take the average power over intervals of a whole period which is one second or 200 samples at a time. By doing this we are able to reduce the effect of noise. The following code generates the power in log form.

```
window=200;
N=length(transmitted_signal)-window;
power1=zeros(1,N);
power2=zeros(1,N);
powerT=zeros(1,N);
tPrime=zeros(1,N);
for i=1:N
    indices=i:i+window;
    power1(i)=log(mean(received_signal1(indices).^2));
    power2(i)=log(mean(received_signal2(indices).^2));
    powerT(i)=log(mean(transmitted_signal(indices).^2));
    tPrime(i)=mean(t(indices));
end
```

Since our given data contains only noise from $-5 \leq t \leq 0$, we can use those data points to obtain the variances λ_1 and λ_2 with the following code.

```
var1=var(power1(1:1000-window));
var2=var(power2(1:1000-window));
```

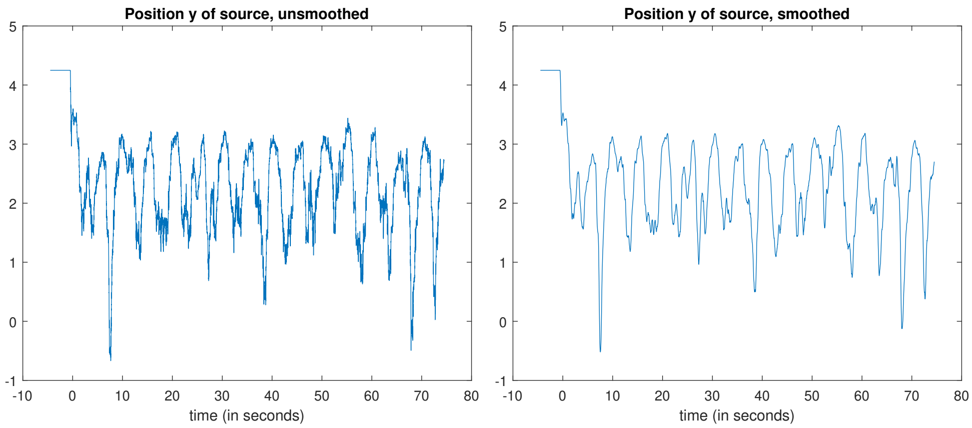
Finally we can calculate the expression for y with the following code.

```
r1=power1-powerT;
r2=power2-powerT;
y=4.25+exp(-r1+var1/2)-exp(-r2+var2/2);
```

We can also smooth the data so it is easier to work with. I found the following moving average filter with window size 50 worked well.

```
y=smoothdata(y,'movmean',50);
```

This results in the following unsmoothed and smoothed signals.



7 Parameter Estimation

We will now try to fit the generated signal to the following form.

$$y = A\sin(\omega t + \phi) + b$$

To estimate the mean and amplitude of the source position, begin by observing that there are approximately two peaks every 10 seconds. Thus I will split up the signal into windows that are 5 seconds or 1000 samples long, which is approximately one period. I want the window to start when the sine component of y is close to zero. I ignore the first few secondns, since these look inaccurate due to the start of the transmission signal. Thus I chose $t = 8.75$s as the start of the window. Then I can find the average of the signal to compute b and half of the difference between the maximum and the minimum to compute A during this window. I implement this with the following code.

```
startIndex=2651;
window=1000;
N=idivide(length(y)-startIndex+1,int32(window));
A=zeros(1,N);
B=zeros(1,N);
for i=1:N
    indices=(i-1)*window+startIndex:i*window+startIndex-1;
    A(i)=(max(y(indices))-min(y(indices)))/2;
```

```
    B(i)=mean(y(indices));
end
```

Next I will compute ω by finding the period T between successive peaks and troughs of each window and using $\omega = \frac{2\pi}{T}$. I implement this with the following code.

```
w=zeros(1,2*(N-1));
for i=1:N-1
    indices1=(i-1)*window+startIndex:i*window+startIndex-1;
    indices2=i*window+startIndex:(i+1)*window+startIndex-1;
    tCur=tPrime(indices1);
    tNext=tPrime(indices2);
    [curMax,peakCurIndex]=max(y(indices1));
    [nextMax,peakNextIndex]=max(y(indices2));
    [curMin,troughCurIndex]=min(y(indices1));
    [nextMin,troughNextIndex]=min(y(indices2));
    period1=tNext(peakNextIndex)-tCur(peakCurIndex);
    period2=tNext(troughNextIndex)-tCur(troughNextIndex);
    w(2*i-1)=2*pi/period1;
    w(2*i)=2*pi/period2;
end
```

Finally we can compute ϕ by finding the difference between the peaks of our signal y and the peaks of the non phase shifted version $y' = A\sin(\omega t) + b$. Assume that ϕ is positive, so we would want to find the next peak of y' at t' such that $t' > t$ where t is the time of the peak in the current window. At our first window we have the following expression for t'.

$$t' = 2.25T$$

Thus we can obtain ϕ as follows.

$$\phi = \frac{2.25\frac{2\pi}{\omega} - t}{\omega}$$

Each window we will add one more period to t', so the following code implements this method.

```
phi=zeros(1,N);
```

```
for i=1:N
    indices=(i-1)*window+startIndex:i*window+startIndex-1;
    tCur=tPrime(indices);
    [curMax,peakIndex]=max(y(indices));
    phi(i)=(2*pi*(double(i)+5/4)/mean(w)-tCur(peakIndex))/mean(w);
end
```

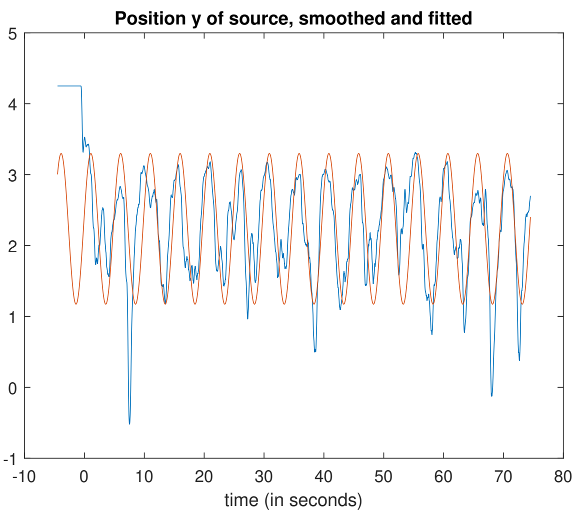
I tried to do the same method with the troughs of y except with the first $t' = 2.75T$, but I found that this led to higher variance and a worse fit. Thus I stuck with only using the peaks.

8 Results

I obtained the following values for the constants.

	μ	σ
A	1.0625	0.23684
b	2.2354	0.14709
ω	1.2643	0.14677
ϕ	0.21287	0.44424

Plotting $y = A\sin(\omega t + \phi) + b$ using the mean of these values against the smoothed data results in the following plot.



The first few seconds before the signal starts transmitting fully can be safely ignored. This is a fairly good fit, and confirms that my methods worked well. The standard deviations of my results are reasonably small. The only value with relatively high standard deviation is ϕ. Two standard deviations around the mean for ϕ has a range of about $\frac{\pi}{2}$, which is somewhat high. I attempted many methods of calculating ϕ but this seems to be around the best fit that I could achieve.