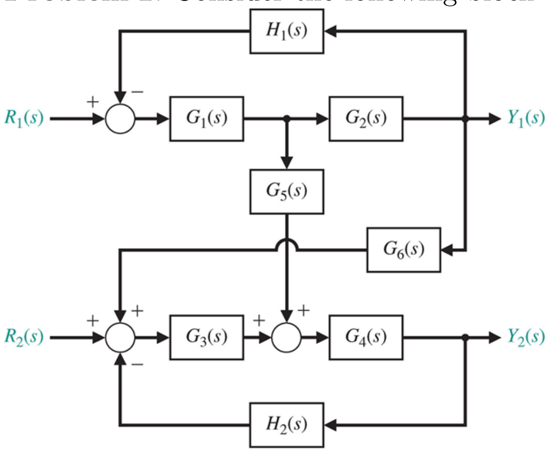
ECE141 - Principles of Feedback Control

Homework 2 , Due: 1/25/19, 9:00am

Problem 1. Consider the following block diagram of a coupled two-input two-output control system:

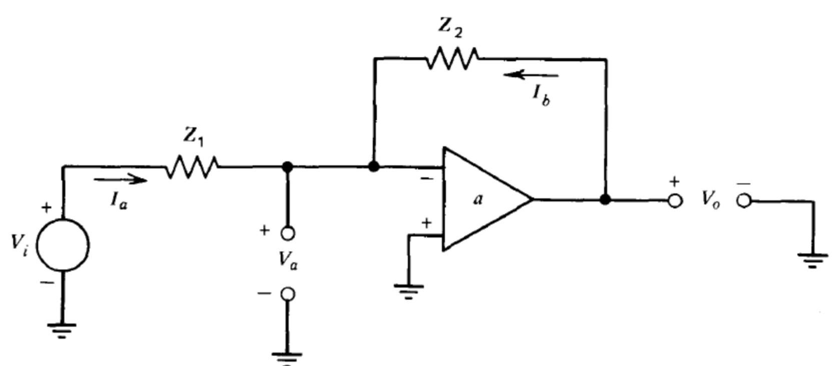


(a) Find the Transfer Function $T(s) = \frac{Y_2(s)}{R_1(s)}\Big|_{R_2(s)=0}$

(b) Find $G_5(s)$ in terms of the other transfer functions in order to make $T(s) = 0$ and thus decouple $Y_2(s)$ from $R_1(s)$.

Problem 2. Operational Amplifiers (OpAmps): OpAmps are key active building blocks in electrical circuits, and are commonly used in many applications such as amplifying sensor signals, or implementing various filters and compensators in electronic controllers. Due to very high open-loop gain, OpAmps are typically used with negative feedback, and with proper characteristics, the closed-loop gain of the amplifier can be controlled primarily by the relatively stable and accurate passive elements in the feedback path.

(a) Consider the following configuration. We typically assume an *ideal* OpAmp. This means that the input impedance is assumed to be infinity and the output can essentially act like a voltage source with zero impedance.



Using the superposition rule, and the voltage division, show that:

$$V_a = \frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o$$

These homework problems are compiled using the different textbooks listed on the course syllabus

(b) Given the polarity of the input connection in the above configuration, we have $V_o = -aV_a$ where a is the open-loop gain of the OpAmp. Substitude in the above equation and show that the transfer function $\frac{V_o}{V_i}$ can be obtained as:

$$\frac{V_o}{V_i} = \frac{-aZ_2/(Z_1+Z_2)}{1+aZ_1/(Z_1+Z_2)}$$

Remember that both the open loop gain a, as well as the impedances Z_i, can be frequency-dependent complex quantites. The quantity $L(j\omega)$ shown below is called the ***Loop Transfer Function*** and, as we shall see in future lectures, it is generally a very critical quanitity in any feedback system and can virtually impact all the performance aspects of the closed-loop system.

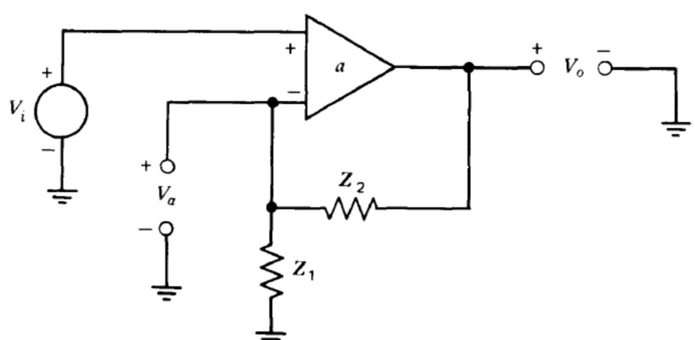
$$L(j\omega) \triangleq \frac{a(j\omega)Z_1(j\omega)}{Z_1(j\omega)+Z_2(j\omega)}$$

(c) With OpAmps, we typically assume ideal closed-loop gain by assuming very large open loop gain leading to $|L(j\omega)| \gg 1$, which will reduce our closed-loop transfer function for the above configuration to the following:

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Notice how our ideal closed-loop gain no longer depends on the open-loop gain of the OpAmp and can be controlled by the passive Z_i elements. This is a key advantage of using feedback with large loop gain. Show how you could have obtained this transfer function by simply writing a KCL node equation at the negative input terminal while assuming that negligible current flows into the OpAmp terminals and also that the differential voltage at the input to the OpAmp is negligible (i.e., in the above configuration, the negative terminal can be considered as a *virtual ground* with zero voltage).

(d) The configuration we studied above is called an *inverting amplifier* due to the opposite sign relationship between input and output voltages. Now, consider the following configuration:

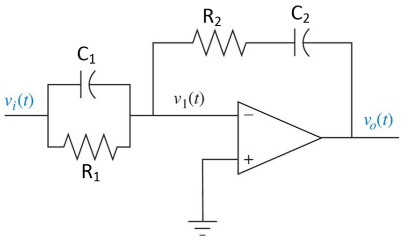


Using similar analysis and under the assumption of an ideal OpAmp along with an ideal closed-loop gain, show how the transfer function $\frac{V_o(s)}{V_i(s)}$ can be obtained as:

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)}$$

This configuration is called a *non-inverting amplifier*.

(e) Now that we have seen the basic OpAmp configurations, let's place a parallel RC for Z_1 and a series RC for Z_2, as shown below:

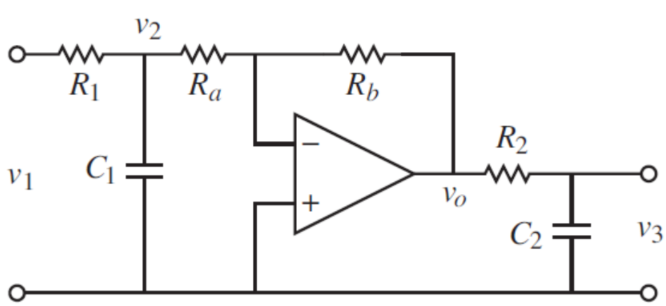


Again assume ideal OpAmp with very high open loop gain so that both the current flowing into the OpAmp terminals as well as the differential voltage at the input to the OpAmp may be neglected. Show that the Transfer Function $\frac{V_o(s)}{V_i(s)}$ may be written in the following form, and find the constants K_P, K_D, and K_I in terms of R's and C's:

$$\frac{V_o(s)}{V_i(s)} = K_P + K_D s + \frac{K_I}{s}$$

As we will see in future lectures, this is a simple basic ciruit realization of a so-called ***Proportional-Integral-Derivative (PID)*** controller.

(f) Finally, let's look at another configuration where a bit more complicated passive network forms the feedback around our OpAmp, and see how the dynamics of the network may be represented in State-Space form.

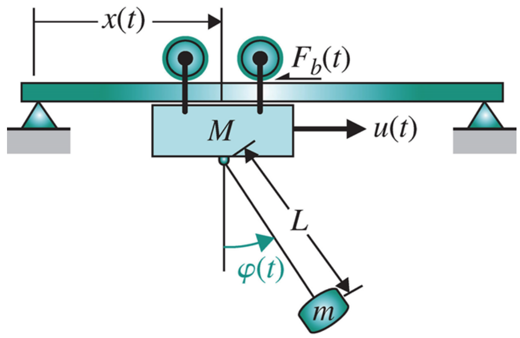


Under the same ideal assumptions as before, show that the dynamics of this circuit network may be written in the State-Space form as:

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{R_1C_1} - \frac{1}{R_aC_1} & 0 \\ -\frac{R_b}{R_a}\frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \tag{1}$$

where $u = v_1$ is the input voltage, and $y = v_3$ is the output voltage. (*Hint*: Use v_2, i.e., the voltage across the capacitor C_1, and the output voltage v_3 as your two state variables).

Problem 3. Mechanical System Modeling: The hanging crane structure supporting the Space Shuttle Atlantis, along with its simple schematic representation are shown below, where M is the mass of the cart, m is the mass of the payload, L is the length of the massless rigid connector, $x(t)$ is the cart displacement, $F_b(t) = -b\dot{x}(t)$ is the friction force, $\phi(t)$ is the connector angle with respect to the vertical, and $u(t)$ is the force applied to the cart.



(a) Write the equations of motion describing the motion of the cart and the payload. (*Hint*: Consider the reaction force between the cart and the payload along the connector. Write separate equations of motion for the cart and the payload, and eliminate the reaction force from your equations in order to obtain your final equations of motion).

(b) Assume $\phi \approx 0$, to linearize your equations. Also assume no friction (i.e., $b = 0$). Find the transfer function from cart velocity $v(t) = \dot{x}(t)$ to the connector angle $\phi(t)$, i.e., $\frac{\Phi(s)}{V(s)}$

(c) Assume the cart starts moving at a constant speed, i.e., $v(t)$ is a unit step function. Using the above transfer function, find the resulting connector angle $\phi(t)$. Show that in this case, the payload will oscillate with a frequency $\omega_0 = \sqrt{\frac{g}{L}}$ where g is gravity.

(d) Find the transfer function from the applied force to the cart position, i.e., $\frac{X(s)}{U(s)}$.

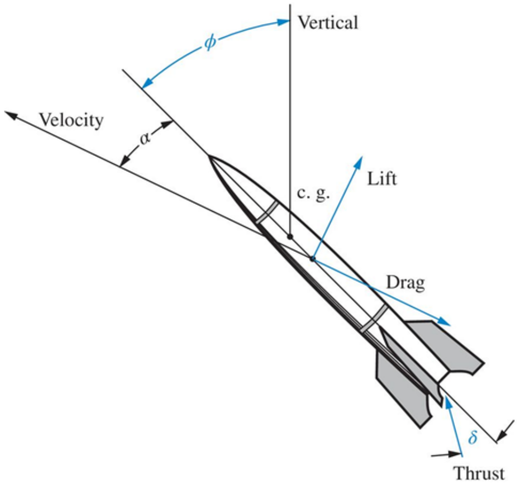
(e) Show that if a constant force is applied to the cart (i.e., if $u(t)$ a unit step function), its velocity will increase without bound as $t \to \infty$.

Problem 4. Linearization: Consider the following nonlinear system:

$$\underline{\dot{\mathbf{x}}} = \underline{\mathbf{f}}(\underline{\mathbf{x}}, \underline{\mathbf{u}}) \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} u \ln x_2 \\ \cos(\pi(u + x_4)) \\ \sin(\pi(u + x_2)) \\ x_3^2 + x_1^2 \end{bmatrix} \tag{2}$$

Linearize the system around $x_1 = x_2 = 1$, $x_3 = 2$, $x_4 = \frac{-1}{2}$, and $u = 1$, and express the linearized system in the State Space form: $\delta\underline{\dot{\mathbf{x}}} = \mathbf{A}\delta\underline{\mathbf{x}} + \mathbf{B}\delta\underline{\mathbf{u}}$

Problem 5. Transfer Function to State-Space: A missile in flight, as shown below, is subject to four forces: thrust, lift, drag, and gravity.



The missile flies at an angle of attack, α, from its longitudinal axis, creating lift. For steering, the body angle from vertical, ϕ, is controlled by rotating the engine at the tail. The transfer function relating the body angle, ϕ, to the angular displacement of the engine, δ, is in the following form:

$$H(s) = \frac{\Phi(s)}{\delta(s)} = \frac{K_a s + K_b}{K_3 s^3 + K_2 s^2 + K_1 s + K_0}$$

Represent the missile steering control system in the State-Space form. (*Hint*: Use the controller-canonical, *a.k.a.* phase-variable, form)

Problem 6. State Space to Transfer Function: In the past, Type-1 diabetes patients had to inject themselves with insulin three to four times a day. New delayed-action and long-lasting insulin analogues such as insulin glargine require a single daily dose. Dynamic models can be developed for the drug flow rate and absorbtion within the body. One such model for the time evolution of plasma concentration of glargine for a specific patient is represented by the following State-Space form:

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{A}\underline{\mathbf{x}} + \mathbf{B}u = \begin{bmatrix} -0.435 & 0.209 & 0.02 \\ 0.268 & -0.394 & 0 \\ 0.227 & 0 & -0.02 \end{bmatrix} \underline{\mathbf{x}}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \mathbf{C}\underline{\mathbf{x}} = \begin{bmatrix} 0.0003 & 0 & 0 \end{bmatrix} \underline{\mathbf{x}}$$

where the state variables $\underline{\mathbf{x}} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and the input u and output y are defined as:

$$\begin{aligned} x_1 &= \text{insulin amount in plasma compartment} \\ x_2 &= \text{insulin amount in liver compartment} \\ x_3 &= \text{insulin amount in body tissue compartment} \\ u &= \text{external insulin flow} \\ y &= \text{plasma insulin concentration} \end{aligned}$$

(a) Write the transfer function $H(s) = \frac{Y(s)}{U(s)}$ parametrically in terms of the matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$.

(b) Use `ss2zp` function in MATLAB and obtain the transfer function in the form:

$$H(s) = \frac{Y(s)}{U(s)} = K \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

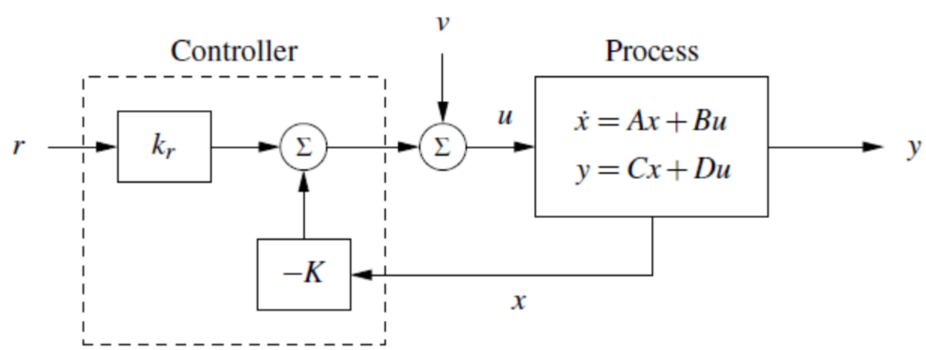
Problem 7. State Feedback: Consider a linear SISO system represented in the following State-Space form:

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{A}\underline{\mathbf{x}}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\underline{\mathbf{x}}(t)$$

If all the states are accessible and can be measured by sensors, and if the system is *controllable* (i.e., the states in $\underline{\mathbf{x}}$ can be moved from any initial state to any desired final state by the input u in a finite time), then a common feedback control strategy is the so-called *state feedback* control where the control input is formed by a linear combination of the states and possibly the external reference input $r(t)$:

$$u(t) = -\mathbf{K}\underline{\mathbf{x}}(t) + K_r r(t)$$

where $\underline{\mathbf{K}}$ is called the feedback gain matrix. The concept is shown in the following block diagram where v is added to model any noise at the input.



Substituting for $u(t)$, ignoring the noise input v, and assuming $k_r = 1$, we can write the State-Space equations for the closed-loop feedback system as:

$$\dot{\underline{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{BK})\underline{\mathbf{x}}(t) + \mathbf{B}r(t)$$
$$y(t) = \mathbf{C}\underline{\mathbf{x}}(t)$$

Remember that the open-loop pole locations for the system are given by the open-loop characteristic equation $\det(s\mathbf{I} - \mathbf{A}) = 0$, i.e., the eigenvalues of matrix $\mathbf{A}$. So, by using state feedback, we have moved the poles of the closed-loop system to the eignevalues of the matrix $\mathbf{A} - \mathbf{BK}$. Therefore, by proper selection of the gain matrix $\mathbf{K}$, we can place the closed-loop poles in our desired locations on the complex left-half plane (LHP) in order to ensure stability and hopefully meet our target performance specifications. As a simple example, let's consider the following system:

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{A}\underline{\mathbf{x}} + \mathbf{B}u = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \underline{\mathbf{x}}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \mathbf{C}\underline{\mathbf{x}} = \begin{bmatrix} 2 & 1 \end{bmatrix} \underline{\mathbf{x}}$$

(a) Find the open-loop poles. Is the open-loop system stable?

(b) Now, design a state feedback controller $u(t) = -\mathbf{K}\underline{\mathbf{x}}(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{\mathbf{x}}(t)$. Find the gain values k_1 and k_2 in order to stabilize the closed-loop system and move its poles to $s_{1,2} = -2.0 \pm j1.0$

(c) Now, assume the system has a slightly different model:

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{A}\underline{\mathbf{x}} + \mathbf{B}u = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \underline{\mathbf{x}}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \mathbf{C}\underline{\mathbf{x}} = \begin{bmatrix} 2 & 1 \end{bmatrix} \underline{\mathbf{x}}$$

Can you still stabilize the closed-loop system with full-state feedback in this case? Please discuss.