Electrical Engineering 141, Homework 2

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January 25th, 2019

Problem 1

a) First we have the following.

$$\begin{aligned}
Y_1 &= G_2G_1(R_1 - H_1Y_1) \\
&= \frac{G_1G_2}{1 + G_1G_2H_1}R_1
\end{aligned}$$

Then we can write the transfer function for Y_2 as follows.

$$\begin{aligned}
Y_2 &= G_4(G_3(G_6Y_1 - H_2Y_2) + G_5G_1(R_1 - H_1Y_1)) \\
(1 + G_4G_3H_2)Y_2 &= G_4G_3G_6Y_1 + G_4G_5G_1R_1 - G_4G_5G_1H_1Y_1 \\
(1 + G_4G_3H_2)Y_2 &= (G_4G_3G_6 - G_4G_5G_1H_1)Y_1 + G_4G_5G_1R_1 \\
\frac{Y_2}{R_1} &= \left(\frac{G_4G_3G_6 - G_4G_5G_1H_1}{1 + G_4G_3H_2}\right)\frac{Y_1}{R_1} + \frac{G_4G_5G_1}{1 + G_4G_3H_2}
\end{aligned}$$

Then we can plug in our earlier expression to obtain the following.

$$\begin{aligned}
\frac{Y_2}{R_1} &= \left(\frac{G_4G_3G_6 - G_4G_5G_1H_1}{1 + G_4G_3H_2}\right)\frac{G_1G_2}{1 + G_1G_2H_1} + \frac{G_4G_5G_1}{1 + G_4G_3H_2} \\
&= \frac{G_1G_2G_4G_3G_6 - G_2G_4G_5G_1^2H_1 + G_4G_5G_1 + G_2G_4G_5G_1^2H_1}{(1 + G_4G_3H_2)(1 + G_1G_2H_1)} \\
&= \frac{G_1G_2G_3G_4G_6 + G_1G_4G_5}{(1 + G_3G_4H_2)(1 + G_1G_2H_1)}
\end{aligned}$$

b) If we let

$$G_5(s) = -G_2 G_3 G_6$$

then the previous transfer function will be zero.

Problem 2

a) Across the first resistor we have

$$V_i - V_a = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o - V_a = I_b Z_2$$

due to Ohm's law. Since the impedance of the OpAmp is infinity, we know that the current into the negative side is zero. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite using the previous equations to obtain the following.

$$\begin{aligned}
\frac{V_i - V_a}{Z_1} + \frac{V_o - V_a}{Z_2} &= 0 \\
(V_i - V_a) * Z_2 + (V_o - V_a) * Z_1 &= 0 \\
Z_2 V_i + Z_1 V_o &= (Z_1 + Z_2) V_a \\
V_a &= \frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o
\end{aligned}$$

b)

$$\begin{aligned}
V_o &= -a \left(\frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o \right) \\
\left(1 + \frac{a Z_1}{Z_1 + Z_2} \right) V_o &= \frac{-a Z_1}{Z_1 + Z_2} V_i \\
\frac{V_o}{V_i} &= \frac{\frac{-a Z_1}{Z_1 + Z_2}}{1 + \frac{a Z_1}{Z_1 + Z_2}}
\end{aligned}$$

c) This is equivalent to assuming that $V_a = 0$ in our previous calculations. Then we have across the first resistor

$$V_i = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o = I_b Z_2$$

due to Ohm's law. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite as follows.

$$\begin{aligned} \frac{V_i}{Z_1} + \frac{V_o}{Z_2} &= 0 \\ \frac{V_o}{Z_2} &= -\frac{V_i}{Z_1} \\ \frac{V_o}{V_i} &= -\frac{Z_2}{Z_1} \end{aligned}$$

d) Let I_1 be a current flowing down across Z_1. Let I_2 be a current flowing to the left across Z_2. Then we have that

$$V_o - V_a = I_2 Z_2$$

and that

$$V_a = I_1 Z_1$$

and from Kirchoff's law we have that

$$I_2 = I_1$$

This means that we know that

$$\begin{aligned} \frac{V_o - V_a}{Z_2} &= \frac{V_a}{Z_1} \\ \frac{V_o}{Z_2} &= \frac{V_a}{Z_1} + \frac{V_a}{Z_2} \\ V_a &= \frac{Z_1}{Z_1 + Z_2} V_o \end{aligned}$$

Additionally across the OpAmp we know that

$$\begin{aligned}
V_o &= a(V_i - V_a) \\
V_o &= a\left(V_i - \frac{Z_1}{Z_1+Z_2}V_o\right) \\
\left(1 + \frac{aZ_1}{Z_1+Z_2}\right)V_o &= aV_i \\
\frac{V_o}{V_i} &= \frac{a}{1+\frac{aZ_1}{Z_1+Z_2}} \\
\frac{V_o}{V_i} &= \frac{a(Z_1+Z_2)}{(1+a)Z_1+Z_2}
\end{aligned}$$

As the gain a becomes very large the limit of this becomes

$$\frac{V_o}{V_i} = \frac{Z_1+Z_2}{Z_1}$$

e) This is the same as the inverting amplifier that we studied earlier, except with impedances

$$Z_1 = \frac{\frac{R_1}{j\omega C_1}}{R_1 + \frac{1}{j\omega C_1}}$$

and

$$Z_2 = R_2 + \frac{1}{j\omega C_2}$$

which can be rewritten in terms of s to be

$$\begin{aligned}
Z_1 &= \frac{\frac{R_1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{1+sR_1C_1} \\
Z_2 &= R_2 + \frac{1}{sC_2} = \frac{1+sR_2C_2}{sC_2}
\end{aligned}$$

Then our transfer function is given by the following

$$\begin{aligned}
\frac{V_0(s)}{V_i(s)} &= -\frac{(1+sR_2C_2)(1+sR_1C_1)}{sR_1C_2} \\
&= -\frac{1+s(R_1C_1+R_2C_2)+s^2R_1C_1R_2C_2}{sR_1C_2} \\
&= -\left(\frac{C_1}{C_2}+\frac{R_1}{R_2}\right) - R_2C_1s - \frac{1}{sR_1C_2}
\end{aligned}$$

Therefore this is equivalent to

$$\begin{aligned}\frac{V_0(s)}{V_i(s)} &= K_P + K_D s + \frac{K_I}{s}\\ K_P &= -\frac{C_1}{C_2} - \frac{R_1}{R_2}\\ K_D &= -R_2 C_1\\ K_I &= -\frac{1}{R_1 C_2}\end{aligned}$$

f) We have that

$$v_o = -\frac{R_b}{R_a} v_2$$

since there is an inverting amplifier in the middle of the circuit. Furthermore we have that $v_2 = \frac{Q_1}{C_1}$, and so

$$v_2' = \frac{I_1}{C_1}$$

where I_1 is the current going through the capacitor C_1. The current going through the resistor R_1 is equal to $I_1 + I_a$, where I_a is the current going through R_a. We can solve

$$\begin{aligned}v_2 - v_o &= I_a(R_a + R_b)\\ \left(1 + \frac{R_b}{R_a}\right) v_2 &= I_a(R_a + R_b)\\ I_a &= \frac{v_2}{R_a}\end{aligned}$$

to obtain the expression

$$v_1 - v_2 = \left(C_1 v_2' + \frac{v_2}{R_a}\right) R_1$$

from the left side of the circuit. Similarly the right side of the circuit gives us

$$-\frac{R_b}{R_a} v_2 - v_3 = C_2 v_3' R_2$$

Solving for v_2' and v_3' yields the system of equations shown below.

$$\begin{aligned} v_2' &= \left(-\frac{1}{R_1C_1} - \frac{1}{R_aC_1}\right)v_2 + \frac{1}{R_1C_1}v_1 \\ v_3' &= -\frac{R_b}{R_aR_2C_2}v_2 - \frac{1}{R_2C_2}v_3 \end{aligned}$$

which is equivalent to the State-Space form

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{R_1C_1} - \frac{1}{R_aC_1} & 0 \\ -\frac{R_b}{R_aR_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} v_1$$

Problem 3

a) Assume that movement can only occur along the horizontal axis so we can ignore gravity. Then the position of the payload is given by

$$x(t) + L\sin(\phi(t))$$

The force F on the payload must have an equal and opposite force $-F$ on the cart. Because this force acts along the horizontal plane, we multiply it by $\sin(\phi(t))$. Then adding together the forces on the cart yields the following equation.

$$Mx''(t) = u(t) - bx'(t) - F\sin(\phi(t))$$

The payload's movement is defined by the following equation on the horizontal axis.

$$m(x(t) + L\sin(\phi(t)))'' = F\sin(\phi(t))$$

The payload's movement is defined by the following equation on the vertical axis.

$$mL\cos(\phi(t))'' = F\cos(\phi(t)) + mg$$

We can eliminate F in the horizontal equation as follows.

$$\begin{aligned} F\sin(\phi(t)) &= -Mx''(t) + u(t) - bx'(t) \\ F\sin(\phi(t)) &= mx''(t) + mL\cos(\phi(t))\phi''(t) - mL\sin(\phi(t))(\phi'(t))^2 \\ 0 &= (m+M)x''(t) - u(t) + bx'(t) \\ &\quad + mL\cos(\phi(t))\phi''(t) - mL\sin(\phi(t))(\phi'(t))^2 \end{aligned}$$

Then expanding the vertical equation gives the following.

$$
\begin{aligned}
F\cos(\phi(t)) &= -mL\sin(\phi(t))\phi''(t) - mL\cos(\phi(t))(\phi'(t))^2 - mg \\
F\sin(\phi(t)) &= -mL\sin(\phi(t))\tan(\phi(t))\phi''(t) - mL\sin(\phi(t))(\phi'(t))^2 \\
&\quad - mg\tan(\phi(t)) \\
0 &= -mx''(t) - mL\cos(\phi(t))\phi''(t) + mL\sin(\phi(t))(\phi'(t))^2 \\
&\quad - mL\sin(\phi(t))\tan(\phi(t))\phi''(t) - mL\sin(\phi(t))(\phi'(t))^2 \\
&\quad - mg\tan(\phi(t)) \\
0 &= -mx''(t) - mL\cos(\phi(t))\phi''(t) - mL\sin(\phi(t))\tan(\phi(t))\phi''(t) \\
&\quad - mg\tan(\phi(t)) \\
0 &= -mx''(t)\cos(\phi(t)) - mL\phi''(t)(\cos^2(\phi(t)) + \sin^2(\phi(t))) \\
&\quad - mg\sin(\phi(t)) \\
0 &= -mx''(t)\cos(\phi(t)) - mL\phi''(t) - mg\cos(\phi(t)) \\
0 &= x''(t)\cos(\phi(t)) + L\phi''(t) + g\sin(\phi(t))
\end{aligned}
$$

b) With ϕ being small, we can approximate $\sin(\phi)$ with ϕ and $\cos(\phi)$ with 1. Then our system of equations becomes

$$
\begin{aligned}
(m+M)x''(t) + mL\phi''(t) - mL\phi(t)(\phi'(t))^2 &= u(t) \\
x''(t) + L\phi''(t) + g\phi(t) &= 0
\end{aligned}
$$

This still contains a $(\phi'(t))^2$ term which is not linear, so we assume that ϕ changes slowly in order to make this term negligible. Then our linear system is

$$
\begin{aligned}
(m+M)x''(t) + mL\phi''(t) &= u(t) \\
x''(t) + L\phi''(t) + g\phi(t) &= 0
\end{aligned}
$$

Using the second equation and rewriting it as

$$v'(t) + L\phi''(t) + g\phi(t) = 0$$

we can obtain the transfer function

$$
\begin{aligned}
sV(s) + s^2L\Phi(s) + g\Phi(s) = 0& \\
sV(s) &= -(s^2L+g)\Phi(s) \\
\frac{\Phi(s)}{V(s)} &= -\frac{s}{s^2L+g}
\end{aligned}
$$

c) If $v(t)$ is the unit step function, then it has a Laplace transform of $\frac{1}{s}$. Then we have

$$\Phi(s) = -\frac{1}{s^2L + g} = -\frac{1}{\sqrt{gL}} \frac{\sqrt{\frac{g}{L}}}{s^2 + \frac{g}{L}}$$

which has an inverse Laplace transform

$$\phi(t) = -\frac{\sin\left(\sqrt{\frac{g}{L}}t\right)}{\sqrt{gL}}$$

so it does oscillate with frequency $\omega_0 = \sqrt{\frac{g}{L}}$.

d) Taking the Laplace transform of our system of equations yields the following

$$\begin{aligned}
(m+M)s^2X(s) + mLs^2\Phi(s) &= U(s) \\
s^2X(s) + Ls^2\Phi(s) + g\Phi(s) &= 0 \\
\Phi(s) &= -\frac{s^2}{Ls^2 + g}X(s) \\
(m+M)s^2X(s) - \frac{mLs^4}{Ls^2+g}X(s) &= U(s) \\
\frac{(m+M)s^2(Ls^2+g) - mLs^4}{Ls^2+g}X(s) &= U(s) \\
\frac{s^2(MLs^2 + (m+M)g)}{Ls^2+g}X(s) &= U(s) \\
\frac{X(s)}{U(s)} &= \frac{Ls^2+g}{s^2(MLs^2 + (m+M)g)}
\end{aligned}$$

e) If $u(t)$ is the unit step function, then it has a Laplace transform of $\frac{1}{s}$. Then we have

$$X(s) = \frac{Ls^2+g}{s^3(MLs^2 + (m+M)g)}$$

We can try using the final value theorem by taking the following limit.

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{Ls^2+g}{s^2(MLs^2 + (m+M)g)}$$

The denominator goes to 0 while the numerator goes to g, so this limit does not exist and the value goes to ∞. Therefore by the final value theorem no limit exists and the system's position increases without bound.

Problem 4

We can convert each equation into a linear approximation by taking the appropriate derivatives evaluated at our initial point. For the first equation we have

$$x_1' \approx u\frac{\partial f_1}{\partial u} + x_2\frac{\partial f_1}{\partial x_2} = 0 + x_2$$

For the second equation we have

$$x_2' \approx u\frac{\partial f_2}{\partial u} + x_4\frac{\partial f_2}{\partial x_4} = -\pi u - \pi x_4$$

For the third equation we have

$$x_3' \approx u\frac{\partial f_3}{\partial u} + x_2\frac{\partial f_3}{\partial x_2} = \pi u + \pi x_2$$

For the fourth equation we have

$$x_4' \approx x_1\frac{\partial f_4}{\partial x_1} + x_3\frac{\partial f_4}{\partial x_3} = 2x_1 + 4x_3$$

This yields the linear system

$$\delta x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & \pi & 0 & 0 \\ 2 & 0 & 4 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\pi \\ \pi \\ 0 \end{bmatrix} \delta u$$

Problem 5

The controller canonical form is shown below. The state is given by the following.

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{K_3} \end{bmatrix} \delta(t)$$

The output is given by the following.

$$\phi(t) = \begin{bmatrix} K_b & K_a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem 6

a)

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ (sI - A)X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= C(sI - A)^{-1}BU(s) \\ \frac{Y(s)}{U(s)} &= C(sI - A)^{-1}B \end{aligned}$$

b) The transfer function in the form

$$H(s) = K\frac{\Pi_i(s - z_i)}{\Pi_j(s - p_j)}$$

has the constants

$$\begin{aligned} K &= 0.0003 \\ z_1 &= -0.394 \\ z_2 &= -0.02 \\ p_1 &= -0.656 \\ p_2 &= -0.1889 \\ p_3 &= -0.0042 \end{aligned}$$

Problem 7

a)

$$\begin{aligned} (s-1)(s-2) - 1 &= 0 \\ s^2 - 3s + 1 &= 0 \\ s &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

One of these poles is positive and one is negative, so it is not stable. As time increases the positive pole will cause exponential growth.

b)

$$\begin{aligned}(s-1+k_1)(s-2)+(k_2-1) &= (s+2)^2+1\\ s^2-3s+k_1s+k_2-1 &= s^2+4s+5\end{aligned}$$

This gives us $k_1 = 7$ and $k_2 = 6$.

c) No you cannot stabilize the system in this case. The characteristic equation will always contain a term $s-2$ which means that a positive pole exists, leading to instability. The gain term k_2 will have no effect on the pole locations since the 0 in the bottom left makes it have no impact on the characteristic equation.