Electrical Engineering 141, Homework 4

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Problem 1

a) From the Internal Model Principle, if $\frac{s}{s+3}C(s)$ includes all the poles of $R(s)$ as poles, then the closed loop system can track a unit step reference input with zero steady state error. Since $R(s) = \frac{1}{s}$ has a pole at $s = 0$, we can design a compensator $C(s)$ such that it tracks the input with zero steady state error.

b) From the Internal Model Principle, we can set $C(s) = \frac{1}{s^2}$ in order for the open loop transfer function to contain a pole at s.

c) We have the following transfer functions.

$$\begin{aligned}
Y(s) &= \frac{1}{s(s+3)}(R(s) - Y(s)) \\
\frac{Y(s)}{R(s)} &= \frac{1}{1 + s(s+3)} \\
H_{ry}(s) &= \frac{1}{s^2 + 3s + 1}
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{s}{s+3}\left(W(s) - \frac{1}{s^2}Y(s)\right) \\
\frac{Y(s)}{W(s)} &= \frac{s^2}{1 + s(s+3)} \\
H_{wy}(s) &= \frac{s^2}{s^2 + 3s + 1}
\end{aligned}$$

$$
\begin{aligned}
U(s) &= \frac{1}{s^2}\left(R(s) - \frac{s}{s+3}U(s)\right) \\
\frac{U(s)}{R(s)} &= \frac{s+3}{s(1+s(s+3))} \\
H_{ru}(s) &= \frac{s+3}{s(s^2+3s+1)}
\end{aligned}
$$

$$
\begin{aligned}
U(s) &= -\frac{1}{s(s+3)}(W(s)+U(s)) \\
\frac{U(s)}{W(s)} &= -\frac{1}{s(s+3)+1} \\
H_{wu}(s) &= -\frac{1}{s^2+3s+1}
\end{aligned}
$$

The transfer functions $H_{ry}(s)$ and $H_{wu}(s)$ are well behaved since they only have poles in the negative portion of the s plane. The transfer function H_{wy} will also cause unit step disturbances to go to zero, so it is stable. However, the transfer function $H_{ru}(s)$ has $s = 0$ as a pole. So this means that $u(t)$ will become unstable and grow without bound when $r(t)$ is the unit step function. So even though letting $C(s) = \frac{1}{s^2}$ allows us to track the input with zero steady state error, it causes our system to become unstable. Thus we cannot implement this compensator in real life.

Problem 2

a) Since $P(s)$ already contains $s = 0$ as a pole, the simplest compensator that ensures zero steady state error is $C(s) = 1$. This can be verified by solving the following equation.

$$
\begin{aligned}
Y(s) &= \frac{1}{s} + \frac{1}{s}\left(\frac{1}{s} - Y(s)\right) \\
Y(s) &= \frac{1}{s+1} + \frac{1}{s(s+1)} \\
Y(s) &= \frac{1}{s}
\end{aligned}
$$

So the output is indeed the unit step response.

b) If we use the compensator $C(s) = 1$, then our output is given by the following equation.

$$
\begin{aligned}
Y(s) &= \frac{1}{s} + \frac{1}{s}\left(\frac{2}{s} - Y(s)\right) \\
Y(s) &= \frac{1}{s+1} + \frac{2}{s(s+1)} \\
Y(s) &= -\frac{1}{s+1} + \frac{2}{s}
\end{aligned}
$$

This has an output with a steady state of 2, so our previous compensator would not work. We can instead choose a new compensator $C(s) = 2 + \frac{1}{s}$. This ensures that the system remains stable while also including the pole $s = 0$ of $W(s)$ in the poles of the compensator. The system remains stable because the sensitivity function becomes the following.

$$
\begin{aligned}
\frac{1}{1 + P(s)C(s)} &= \frac{1}{1 + \frac{1}{s}\left(2 + \frac{1}{s}\right)} \\
&= \frac{s^2}{s^2 + 2s + 1} \\
&= \frac{s^2}{(s+1)^2}
\end{aligned}
$$

This has two negative real poles, so it is stable. Furthermore, the gang of four becomes the following.

$$
\begin{aligned}
\frac{P(s)}{1 + P(s)C(s)} &= \frac{s}{(s+1)^2} \\
\frac{C(s)}{1 + P(s)C(s)} &= \frac{2s^2 + s}{(s+1)^2} \\
\frac{P(s)C(s)}{1 + P(s)C(s)} &= \frac{2s + 1}{(s+1)^2}
\end{aligned}
$$

These are all stable, so our system is stable.

c) We can solve the following equation to obtain the answer.

$$
\begin{aligned}
E(s) &= R(s) - (W_2(s) + P(s)(W_1(s) + C(s)E(s))) \\
E(s) &= \frac{1}{s^2} - \left(\frac{1}{s} + \frac{1}{s}\left(\frac{1}{s} + \left(2 + \frac{1}{s}\right)E(s)\right)\right) \\
E(s) &= \frac{1}{s^2} - \frac{1}{s} - \frac{1}{s^2} - \left(\frac{2s+1}{s^2}\right)E(s) \\
E(s) &= -\frac{s}{(s+1)^2}
\end{aligned}
$$

From the Final Value Theorem we can see that since $\lim_{s\to 0} sE(s) = 0$, the steady state tracking error will also be zero.

d) Our loop transfer function is the following.

$$
P(s)C(s) = \frac{2s+1}{s^2}
$$

Therefore our system is a type 2 system. The value of the corresponding error constant K_2 is the following.

$$
\lim_{s\to 0} 2s + 1 = 1
$$

e) We can solve the following equation to obtain the answer.

$$
\begin{aligned}
E(s) &= R(s) - (W_2(s) + P(s)(W_1(s) + C(s)E(s))) \\
E(s) &= \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\left(\frac{1}{s} + \left(2 + \frac{1}{s}\right)E(s)\right)\right) \\
E(s) &= \frac{1}{s^2} - \frac{1}{s^2} - \frac{1}{s^2} - \left(\frac{2s+1}{s^2}\right)E(s) \\
E(s) &= -\frac{1}{(s+1)^2}
\end{aligned}
$$

From the Final Value Theorem we can see that since $\lim_{s\to 0} sE(s) = 0$, the steady state tracking error will also be zero.

f) No we cannot achieve zero steady state tracking error with the same compensator as earlier. This is because the compensator $C(s)$ must include the unstable poles of $W(s)$ as poles. If $W(s)$ is the unit ramp function, it will have two poles at $s = 0$ while our compensator $C(s) = 2 + \frac{1}{s}$ only has one pole at $s = 0$. We could modify our compensator to become $C(s) = 3 + \frac{3}{s} + \frac{1}{s^2}$ in order to have two poles at $s = 0$. This leads to a stable system because the sensitivity function becomes the following.

$$\begin{aligned}
\frac{1}{1 + P(s)C(s)} &= \frac{1}{1 + \frac{1}{s}\left(3 + \frac{3}{s} + \frac{1}{s^2}\right)} \\
&= \frac{s^3}{s^3 + 3s^2 + 3s + 1} \\
&= \frac{s^3}{(s+1)^3}
\end{aligned}$$

The gang of four then becomes the following.

$$\begin{aligned}
\frac{P(s)}{1 + P(s)C(s)} &= \frac{s^2}{(s+1)^3} \\
\frac{C(s)}{1 + P(s)C(s)} &= \frac{3s^3 + 3s^2 + s}{(s+1)^3} \\
\frac{P(s)C(s)}{1 + P(s)C(s)} &= \frac{3s^2 + 3s + 1}{(s+1)^3}
\end{aligned}$$

Since there are no poles in the closed right half plane, our system is stable.

g) We have the following equations.

$$\begin{aligned}
Y(s) &= W_2(s) + P(s)C(s)(R(s) - Y(s)) \\
Y(s) &= \frac{W_2(s) + P(s)C(s)R(s)}{1 + P(s)C(s)} \\
R(s) &= \frac{1}{s} + \frac{1}{s^2 + 1} \\
W_2(s) &= \frac{s}{s^2 + \frac{1}{10000}}
\end{aligned}$$

Then plugging in yields the following expression.

$$Y(s) = \frac{10000s^2(s+1)}{(s^2+s+1)(10000s^2+1)} + \frac{1}{s(s^2+1)}$$

Notice that the right term has poles $s = \pm j$, therefore this output signal does not have a steady state value. It will oscillate around the value 1.

Problem 3

a) We have the following.

$$\begin{aligned} Y(s) &= H(s)(R(s) - F(s)Y(s)) \\ \frac{Y(s)}{R(s)} &= \frac{H(s)}{1 + H(s)F(s)} \end{aligned}$$

b)

$$\begin{aligned} S_H^{H_{cl}} &= \frac{\frac{dH_{cl}}{H_{cl}}}{\frac{dH}{H}} \\ &= \frac{dH_{cl}}{dH}\frac{H}{H_{cl}} \\ &= \frac{1}{(1+H(s)F(s))^2}(1+H(s)F(s)) \\ &= \frac{1}{1+L(s)} \end{aligned}$$

c)

$$
\begin{aligned}
S_F^{H_{cl}} &= \frac{\frac{dH_{cl}}{H_{cl}}}{\frac{dF}{F}} \\
&= \frac{dH_{cl}}{dF}\frac{F}{H_{cl}} \\
&= -\frac{H(s)^2}{(1+H(s)F(s))^2}\frac{F(s)(1+H(s)F(s))}{H(s)} \\
&= -\frac{H(s)F(s)}{1+H(s)F(s)} \\
&= -\frac{L(s)}{1+L(s)}
\end{aligned}
$$

d)

Problem 4

a)

b)

c)

Problem 5

a)

b)

c)

d)

e)

f)

g)

h)

i)

j)

k)

l)

m)