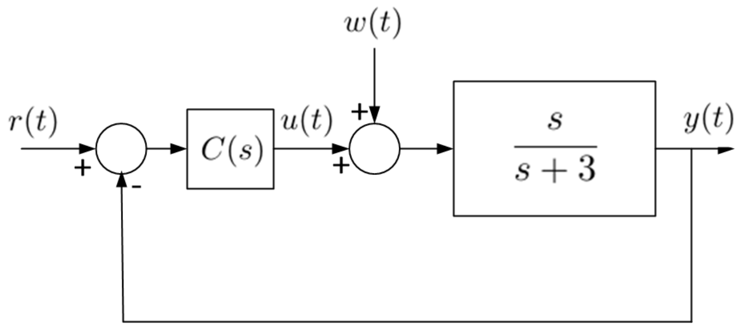
# ECE141 - Principles of Feedback Control

## Homework 4 , Due: 2/08/19, 9:00am

**Problem 1.** Consider the following closed-loop system:

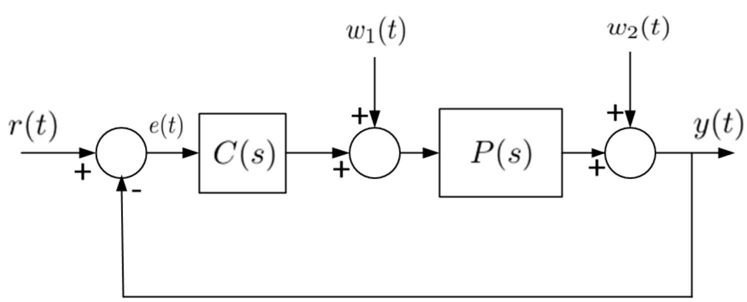



(a) Can you design a compensator $C(s)$ such that the closed-loop system track a unit step reference input (i.e., $r(t)$ is unit step) with zero steady-state error?

(b) If your answer to (a) is yes, suggest a form of a compensator $C(s)$ to achieve your objective, i.e., to drive the steady-state error to a step input to zero.

(c) For your suggested form of compensator in Part (b), find the four closed-loop transfer functions $H_{ry}(s) : r \to y$, $H_{wy}(s) : w \to y$, $H_{ru}(s) : r \to u$, $H_{wu}(s) : w \to u$. Compare these transfer functions and discuss.

**Problem 2.** Consider the following closed-loop system where $w_1(t)$ and $w_2(t)$ denote disturbance signals respectively at the input and the output of the plant.




(a) Assume our plant model is represented by a pure integrator (i.e., $P(s) = \frac{1}{s}$). Ignore the disturbance at the input (i.e., $w_1(t) = 0$). Our goal is to track a step reference input (i.e., $r(t) = u(t)$) with zero steady-state error in the presence of a step disturbance at the output of the plant (i.e., $w_2(t) = u(t)$). Can you suggest the simplest compensator to achieve that?

(b) With the same plant model, what if we want to also reject a step disturbance at the plant input as well?, i.e., we want to have $e_{ss} = e(\infty) = 0.0$ when $r(t) = w_1(t) = w_2(t) = u(t)$? Can your compensator in Part (a) achieve that? If not, why? What would be the simplest form of the compensator you can propose to achieve that? Compare with Part (a) and discuss. (*hint*: Don't forget that your closed-loop system should remain stable. Think of PI compensators).

These homework problems are compiled using the different textbooks listed on the course syllabus

(c) With the same plant model, and the same compensator you designed for Part (b), what would be the seady-state tracking error at the output when the reference input is a ramp, while both disturbance signals are still modeled as step signals?, i.e., find $e_{ss}$, given $r(t) = tu(t)$ and $w_1(t) = w_2(t) = u(t)$.

(d) With the same plant model, and the same compensator you designed for Part (b), what would be the *System Type* for this system? What would be the value of the corresponding *Error Constant*?

(e) Use the same assumptions as Part (c), except this time, the disturbance at the plant output also happens to behave like a ramp. What would be the steady-state tracking error in this case? i.e., find $e_{ss}$, given $r(t) = w_2(t) = tu(t)$ and $w_1(t) = u(t)$.

(f) Use the same assumptions as Part (c), except this time, we want to reject ramp-like disturbances at both the input and the output of the plant, i.e., our design objective is to have $e_{ss} = 0.0$ when $r(t) = w_1(t) = w_2(t) = tu(t)$. Can we achieve zero steady-state tracking error with the same compensator as the one in Part (b)? If not, why? How would you propose to modify your compensator in Part (b) to achieve that?

(g) Now, assume the plant model is $P(s) = \frac{1}{s+1}$ and we use a pure integrator as our compensator, i.e., $C(s) = \frac{1}{s}$. Assume our reference input is: $r(t) = (1 + \sin(t))u(t)$, the disturbance at the output is: $w_2(t) = \cos(0.01t)u(t)$, and the disturbance at the input to the plant is ignored ($w_1(t) = 0.0$). Find the steady-state output response $y_{ss}$.

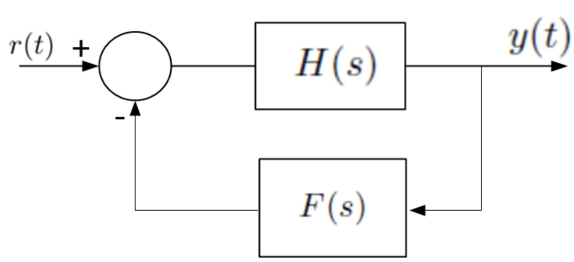
**Problem 3. Sensitivity to variations in Feedback vs Feedforward Elements**: We have discussed how the *sensitivity* of any parameter $M$ to variations in some parameter $K$ may be defined as:

$$S_K^M \triangleq \frac{\text{Relative/fractional change in } M}{\text{Relative/fractional change in } K} = \frac{\frac{dM}{M}}{\frac{dK}{K}} = \frac{dM}{dK} \cdot \frac{K}{M} = \frac{d(\ln M)}{d(\ln K)}$$

We also considered a unity-feedback system and showed how the sensitivity function of the closed-loop transfer function $T(s)$ to the changes in the the loop transfer function $L(s)$ can be obtained as:

$$S_L^T = \frac{\frac{dT}{T}}{\frac{dL}{L}} = \frac{1}{1+L}$$

And we therefore concluded that for the purpose of reducing the closed-loop sensitivity to loop gain variations, we would desire $|L(j\omega)| \gg 1$ at those frequencies where we would expect the loop gain variations to occur. Now, what if we have elements in the feedback path as well? How sensitivie would our closed-loop transfer function be to the changes in the parameters of the feedback elements vs the feedforward elements? Specifically, let's consider the following closed-loop feedback system:




Let: $H_{cl}(s) = H_{ry}(s) = \frac{Y(s)}{R(s)}$ be the closed-loop transfer function, and $L(s) \triangleq H(s)F(s)$ be our loop transfer function.

(a) Obtain $H_{cl}(s)$ in terms of $H(s)$ and $F(s)$.

(b) Obtain the sensitivity of $H_{cl}(s)$ to changes in the feedforward path transfer function $H(s)$, i.e., $S_H^{H_{cl}}$, and write it in terms of the loop transfer function $L(s)$.

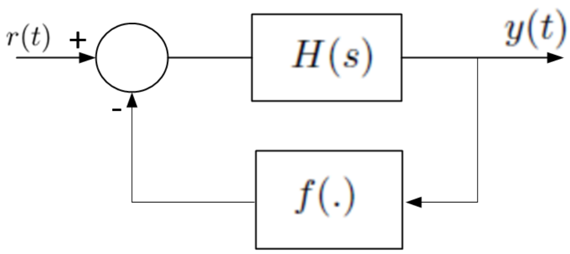
(c) Obtain the sensitivity of $H_{cl}(s)$ to changes in the feedback path transfer function $F(s)$, i.e., $S_F^{H_{cl}}$, and write it in terms of the loop transfer function $L(s)$.

(d) Compare the two results, and discuss how the magnitude of our loop transfer function $L(j\omega)$ would impact our closed-loop sensitivity to variations in the feedforward elements versus the feedback elements. Based on this result, say, if you are designing a feedback amplifier, where would you put your more accurate components?

**Problem 4. Building an inverse system**: Consider the closed loop system in the previous problem again.

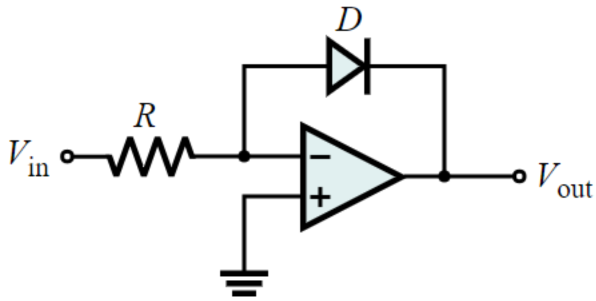
(a) Let $H(S) = H$ be a constant gain. For what range of values of $H$, can our closed-loop transfer function be approximated as $\frac{1}{F(s)}$? This, in fact, can be considered as a basic technique to implement an inverse of a system.

(b) Interestingly, the result in (a) would still hold even if we have a nonlinear element in the feedback path. Specifically, consider the following closed loop system:




where $H(s) = H$ is a constant gain in the feedforward path, and $f(.)$ is a nonlinear function in the feedback path. Let's assume that $f^{-1}(.)$ exists, i.e., if $x = f(y)$, one could obtain $y = f^{-1}(x)$. Also assume that our closed-loop system remains stable in the presence of nonlinearity. Write the input-output relationship for this closed-loop system. The loop gain in this case may be defined as $\frac{Hf(y)}{y}$. For what values of the this loop gain, will our closed-loop system act like the inverse of the feedback element, i.e., $y(t) = f^{-1}(r(t))$?

(c) A nice exampe application of the result in Part (b) is in designing logarithmic amplifiers where the output voltage is equal to $K$ times the natural logarithm of the input voltage. From the result in Part (b), we know that one can build a feedback system with proper feedforward gain values in order to implement the inverse of the characteristics in the feedback path. So to implement a logarithmic amplifier, we need to get an exponential characteristics in our feedback path. And that is exactly what the following basic logarithmic amplifier circuit does:



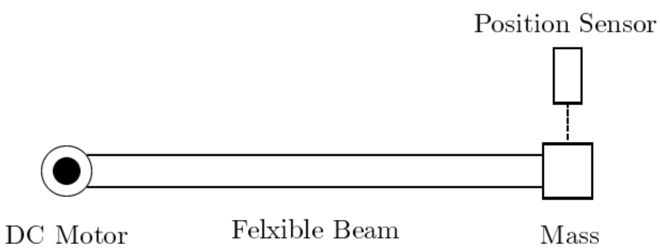

The voltage-current relationship of the diode may be approximated as:

$$I_d \approx I_s e^{\frac{V_d}{V_T}}$$

where $I_d$ is the diode current in the forward direction, $V_d$ is the voltage across the diode, $I_s$ is the so-called *reverse saturation current* of the diode, and $V_T$ is a constant called *thermal voltage*. This is a decent approximation when the diode is operating in the so-called forward bias region and the voltage across it is much bigger than the thermal voltage $V_T$. Assume an ideal OpAmp with very large open-loop gain $A$, so the current flowing into the OpAmp terminals as well as the differential voltage at the input to the OpAmp may be assumed negligible. Show that the input-output voltage relationship for the above amplifier can be obtained as follows:

$$V_{out} = -V_T \ln\left(\frac{V_{in}}{I_s R}\right)$$

**Problem 5. Robust Stability (The solution is Extra Credit to submit but mandatory to learn!)** Consider a flexible beam, connected to a mass and controlled by a motor as shown below:

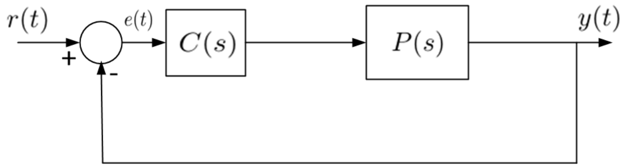



The nominal rigid body model from the motor control input to the beam tip deflection is given by:

$$P(s) = \frac{6.28}{s^2}$$

In order to stabilize the beam and keep the mass at the beam tip at a desired position, the following compensator in a unity feedback loop (shown below) had been designed:

$$C(s) = \frac{500(s+10)}{s+100}$$




(a) What is the Loop Transfer Function $L(s)$?

(b) What is the Sensitivity Function $S(s)$?

(c) Plot the magnitude of $|S(j\omega)|$ over frequency. What is the crossover frequency for $S(j\omega)$ in rad/sec? Remember that the crossover frequency is the frequency $\omega_{sc}$ at which $|S(j\omega)| = 1 = 0\text{dB}$. (Use `bodemag` function in MATLAB to plot $|S(j\omega)|$. Then use `margin` function to find the crossover frequency. Call it in this format `[gm,pm,wcg,wsc] = margin(S)` and the last value in the returned vector would be the magnitude crossover frequency.)

(d) Based on your answer to Part (c), if we had any disturbance modeled at the input to the plant, which disturbance frequencies would be attenuated by feedback?

(e) What is the closed-loop transfer function $T(s)$ from the reference input to the system output? Remember $T(s)$ is also the Complementary Sensitivity Function. Where are the closed-loop poles located? Is the feedback design based on the nominal rigid body model of the beam stable? (You can use the `feedback` function, followed by `pole` and `zero` functions in MATLAB).

(f) Plot the step response of the closed-loop system. (Use `step` function in MATLAB).

(g) Notice the time scale and how fast the step response is. It turns out that the bandwidth specification was not part of the design specification for the above compensator. As a result, this compensator would target too fast of a response such that it would excite the additional flexible modes of the beam which were not accounted for in the rigid body model of the beam. Assume the "true" model for the beam was:

$$P_1(s) = \frac{6.28}{s^2} + \frac{12.56}{s^2 + 0.707s + 28},$$

where the second term shows the unmodeled dynamics of the beam when designing the above compensator $C(s)$. Find out the multiplicative plant perturbation $\delta_p(s)$.

(h) Find the closed-loop transfer function with $P_1(s)$ and the same compensator $C(s)$ above. Where are the closed-loop poles located? Is the closed-loop system still stable? (Similar to Part (e), you can use MATLAB here).

(i) Plot the magnitudes of $1/|T(j\omega)|$ as well as $|\delta_p(j\omega)|$ over frequency on the same figure. Was the Robust Stability condition satisfied for all frequencies? Please discuss. (You can again use `bodemag` function in MATLAB here, and don't forget to use `hold on` to keep both plots on the same figure).

(j) Now, let's try a new compensator:

$$C_1(s) = \frac{(5 \times 10^{-4})(s + 0.01)}{s + 0.1}$$

Find the new closed-loop transfer function $T_1(s)$ assuming compensator $C_1(s)$ and the original nominal rigid body model for the beam $P(s) = \frac{6.28}{s^2}$. Where are the closed-loop poles located? Confirm that the nominal closed-loop system is still stable with this new compensator.

(k) On the same figure as Part (i), add the magnitude plot for $1/|T_1(j\omega)|$. (Again, use `bodemag` function in MATLAB). Does the perturbation due to the unmodeled dynamics, as given in Part (g), satisfy the robust stability condition now with this new compensator?

(l) Plot the step response of $T_1(s)$. Compare the time scale with the step response in Part (f), and discuss. (Use `step` command in MATLAB).

(m) Finally, obtain the closed-loop transfer function $T_2(s)$ with the new compensator $C_1(s)$ and the "true" beam model $P_1(s)$. Where are the closed-loop poles located? Notice how this new compensator has led to a closed-loop system that is robust enough to ensure stability despite the unmodeled flexible modes of the beam.