

# Electrical Engineering 141, Homework 2

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January 25th, 2019

## Problem 1

a) First we have the following.

$$\begin{aligned} Y_1 &= G_2 G_1 (R_1 - H_1 Y_1) \\ &= \frac{G_1 G_2}{1 + G_1 G_2 H_1} R_1 \end{aligned}$$

Then we can write the transfer function for  $Y_2$  as follows.

$$\begin{aligned} Y_2 &= G_4 (G_3 (G_6 Y_1 - H_2 Y_2) + G_5 G_1 (R_1 - H_1 Y_1)) \\ (1 + G_4 G_3 H_2) Y_2 &= G_4 G_3 G_6 Y_1 + G_4 G_5 G_1 R_1 - G_4 G_5 G_1 H_1 Y_1 \\ (1 + G_4 G_3 H_2) Y_2 &= (G_4 G_3 G_6 - G_4 G_5 G_1 H_1) Y_1 + G_4 G_5 G_1 R_1 \\ \frac{Y_2}{R_1} &= \left( \frac{G_4 G_3 G_6 - G_4 G_5 G_1 H_1}{1 + G_4 G_3 H_2} \right) \frac{Y_1}{R_1} + \frac{G_4 G_5 G_1}{1 + G_4 G_3 H_2} \end{aligned}$$

Then we can plug in our earlier expression to obtain the following.

$$\begin{aligned} \frac{Y_2}{R_1} &= \left( \frac{G_4 G_3 G_6 - G_4 G_5 G_1 H_1}{1 + G_4 G_3 H_2} \right) \frac{G_1 G_2}{1 + G_1 G_2 H_1} + \frac{G_4 G_5 G_1}{1 + G_4 G_3 H_2} \\ &= \frac{G_1 G_2 G_4 G_3 G_6 - G_2 G_4 G_5 G_1^2 H_1 + G_4 G_5 G_1 + G_2 G_4 G_5 G_1^2 H_1}{(1 + G_4 G_3 H_2)(1 + G_1 G_2 H_1)} \\ &= \frac{G_1 G_2 G_3 G_4 G_6 + G_1 G_4 G_5}{(1 + G_3 G_4 H_2)(1 + G_1 G_2 H_1)} \end{aligned}$$

b) If we let

$$G_5(s) = -G_2G_3G_6$$

then the previous transfer function will be zero.

## Problem 2

a) Across the first resistor we have

$$V_i - V_a = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o - V_a = I_b Z_2$$

due to Ohm's law. Since the impedance of the OpAmp is infinity, we know that the current into the negative side is zero. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite using the previous equations to obtain the following.

$$\begin{aligned} \frac{V_i - V_a}{Z_1} + \frac{V_o - V_a}{Z_2} &= 0 \\ (V_i - V_a) * Z_2 + (V_o - V_a) * Z_1 &= 0 \\ Z_2 V_i + Z_1 V_o &= (Z_1 + Z_2) V_a \\ V_a &= \frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o \end{aligned}$$

b)

$$\begin{aligned} V_o &= -a \left( \frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o \right) \\ \left( 1 + \frac{a Z_1}{Z_1 + Z_2} \right) V_o &= \frac{-a Z_2}{Z_1 + Z_2} V_i \\ \frac{V_o}{V_i} &= \frac{\frac{-a Z_2}{Z_1 + Z_2}}{1 + \frac{a Z_1}{Z_1 + Z_2}} \end{aligned}$$

c) This is equivalent to assuming that  $V_a = 0$  in our previous calculations. Then we have across the first resistor

$$V_i = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o = I_b Z_2$$

due to Ohm's law. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite as follows.

$$\begin{aligned}\frac{V_i}{Z_1} + \frac{V_o}{Z_2} &= 0 \\ \frac{V_o}{Z_2} &= -\frac{V_i}{Z_1} \\ \frac{V_o}{V_i} &= -\frac{Z_2}{Z_1}\end{aligned}$$

d) Let  $I_1$  be a current flowing down across  $Z_1$ . Let  $I_2$  be a current flowing to the left across  $Z_2$ . Then we have that

$$V_o - V_a = I_2 Z_2$$

and that

$$V_a = I_1 Z_1$$

and from Kirchhoff's law we have that

$$I_2 = I_1$$

This means that we know that

$$\begin{aligned}\frac{V_o - V_a}{Z_2} &= \frac{V_a}{Z_1} \\ \frac{V_o}{Z_2} &= \frac{V_a}{Z_1} + \frac{V_a}{Z_2} \\ V_a &= \frac{Z_1}{Z_1 + Z_2} V_o\end{aligned}$$

Additionally across the OpAmp we know that

$$\begin{aligned}
 V_o &= a(V_i - V_a) \\
 V_o &= a \left( V_i - \frac{Z_1}{Z_1 + Z_2} V_o \right) \\
 \left( 1 + \frac{aZ_1}{Z_1 + Z_2} \right) V_o &= aV_i \\
 \frac{V_o}{V_i} &= \frac{a}{1 + \frac{aZ_1}{Z_1 + Z_2}} \\
 \frac{V_o}{V_i} &= \frac{a(Z_1 + Z_2)}{(1 + a)Z_1 + Z_2}
 \end{aligned}$$

As the gain  $a$  becomes very large the limit of this becomes

$$\frac{V_o}{V_i} = \frac{Z_1 + Z_2}{Z_1}$$

e) This is the same as the inverting amplifier that we studied earlier, except with impedances

$$Z_1 = \frac{\frac{R_1}{j\omega C_1}}{R_1 + \frac{1}{j\omega C_1}}$$

and

$$Z_2 = R_2 + \frac{1}{j\omega C_2}$$

which can be rewritten in terms of  $s$  to be

$$\begin{aligned}
 Z_1 &= \frac{\frac{R_1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{1 + sR_1C_1} \\
 Z_2 &= R_2 + \frac{1}{sC_2} = \frac{1 + sR_2C_2}{sC_2}
 \end{aligned}$$

Then our transfer function is given by the following

$$\begin{aligned}
 \frac{V_o(s)}{V_i(s)} &= - \frac{(1 + sR_2C_2)(1 + sR_1C_1)}{sR_1C_2} \\
 &= - \frac{1 + s(R_1C_1 + R_2C_2) + s^2R_1C_1R_2C_2}{sR_1C_2} \\
 &= - \left( \frac{C_1}{C_2} + \frac{R_1}{R_2} \right) - R_2C_1s - \frac{1}{sR_1C_2}
 \end{aligned}$$

Therefore this is equivalent to

$$\begin{aligned}\frac{V_0(s)}{V_i(s)} &= K_P + K_D s + \frac{K_I}{s} \\ K_P &= -\frac{C_1}{C_2} - \frac{R_1}{R_2} \\ K_D &= -R_2 C_1 \\ K_I &= -\frac{1}{R_1 C_2}\end{aligned}$$

f) We have that

$$v_o = -\frac{R_b}{R_a} v_2$$

since there is an inverting amplifier in the middle of the circuit. Furthermore we have that  $v_2 = \frac{Q_1}{C_1}$ , and so

$$v'_2 = \frac{I_1}{C_1}$$

where  $I_1$  is the current going through the capacitor  $C_1$ . The current going through the resistor  $R_1$  is equal to  $I_1 + I_a$ , where  $I_a$  is the current going through  $R_a$ . We can solve

$$\begin{aligned}v_2 - v_o &= I_a(R_a + R_b) \\ \left(1 + \frac{R_b}{R_a}\right) v_2 &= I_a(R_a + R_b) \\ I_a &= \frac{v_2}{R_a}\end{aligned}$$

to obtain the expression

$$v_1 - v_2 = \left(C_1 v'_2 + \frac{v_2}{R_a}\right) R_1$$

from the left side of the circuit. Similarly the right side of the circuit gives us

$$-\frac{R_b}{R_a} v_2 - v_3 = C_2 v'_3 R_2$$

Solving for  $v'_2$  and  $v'_3$  yields the system of equations shown below.

$$\begin{aligned} v'_2 &= \left( -\frac{1}{R_1 C_1} - \frac{1}{R_a C_1} \right) v_2 + \frac{1}{R_1 C_1} v_1 \\ v'_3 &= -\frac{R_b}{R_a R_2 C_2} v_2 - \frac{1}{R_2 C_2} v_3 \end{aligned}$$

which is equivalent to the State-Space form

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{R_1 C_1} - \frac{1}{R_a C_1} & 0 \\ -\frac{R_b}{R_a R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} \\ 0 \end{bmatrix} v_1$$

### Problem 3

a) Assume that movement can only occur along the horizontal axis so we can ignore gravity. Then the position of the payload is given by

$$x(t) + L \sin(\phi(t))$$

The force  $F$  on the payload must have an equal and opposite force  $-F$  on the cart. Because this force acts along the horizontal plane, we multiply it by  $\sin(\phi(t))$ . Then adding together the forces on the cart yields the following equation.

$$Mx''(t) = u(t) - bx'(t) - F \sin(\phi(t))$$

The payload's movement is defined by the following equation on the horizontal axis.

$$m(x(t) + L \sin(\phi(t)))'' = F \sin(\phi(t))$$

The payload's movement is defined by the following equation on the vertical axis.

$$mL \cos(\phi(t))'' = F \cos(\phi(t)) + mg$$

We can eliminate  $F$  in the horizontal equation as follows.

$$\begin{aligned} F \sin(\phi(t)) &= -Mx''(t) + u(t) - bx'(t) \\ F \sin(\phi(t)) &= mx''(t) + mL \cos(\phi(t))\phi''(t) - mL \sin(\phi(t))(\phi'(t))^2 \\ 0 &= (m + M)x''(t) - u(t) + bx'(t) \\ &\quad + mL \cos(\phi(t))\phi''(t) - mL \sin(\phi(t))(\phi'(t))^2 \end{aligned}$$

Then expanding the vertical equation gives the following.

$$\begin{aligned}
F \cos(\phi(t)) &= -mL \sin(\phi(t))\phi''(t) - mL \cos(\phi(t))(\phi'(t))^2 - mg \\
F \sin(\phi(t)) &= -mL \sin(\phi(t)) \tan(\phi(t))\phi''(t) - mL \sin(\phi(t))(\phi'(t))^2 \\
&\quad - mg \tan(\phi(t)) \\
0 &= -mx''(t) - mL \cos(\phi(t))\phi''(t) + mL \sin(\phi(t))(\phi'(t))^2 \\
&\quad - mL \sin(\phi(t)) \tan(\phi(t))\phi''(t) - mL \sin(\phi(t))(\phi'(t))^2 \\
&\quad - mg \tan(\phi(t)) \\
0 &= -mx''(t) - mL \cos(\phi(t))\phi''(t) - mL \sin(\phi(t)) \tan(\phi(t))\phi''(t) \\
&\quad - mg \tan(\phi(t)) \\
0 &= -mx''(t) \cos(\phi(t)) - mL\phi''(t)(\cos^2(\phi(t)) + \sin^2(\phi(t))) \\
&\quad - mg \sin(\phi(t)) \\
0 &= -mx''(t) \cos(\phi(t)) - mL\phi''(t) - mg \cos(\phi(t)) \\
0 &= x''(t) \cos(\phi(t)) + L\phi''(t) + g \sin(\phi(t))
\end{aligned}$$

**b)** With  $\phi$  being small, we can approximate  $\sin(\phi)$  with  $\phi$  and  $\cos(\phi)$  with 1. Then our system of equations becomes

$$\begin{aligned}
(m + M)x''(t) + mL\phi''(t) - mL\phi(t)(\phi'(t))^2 &= u(t) \\
x''(t) + L\phi''(t) + g\phi(t) &= 0
\end{aligned}$$

This still contains a  $(\phi'(t))^2$  term which is not linear, so we assume that  $\phi$  changes slowly in order to make this term negligible. Then our linear system is

$$\begin{aligned}
(m + M)x''(t) + mL\phi''(t) &= u(t) \\
x''(t) + L\phi''(t) + g\phi(t) &= 0
\end{aligned}$$

Using the second equation and rewriting it as

$$v'(t) + L\phi''(t) + g\phi(t) = 0$$

we can obtain the transfer function

$$\begin{aligned}
sV(s) + s^2L\Phi(s) + g\Phi(s) &= 0 \\
sV(s) &= -(s^2L + g)\Phi(s) \\
\frac{\Phi(s)}{V(s)} &= -\frac{s}{s^2L + g}
\end{aligned}$$

c) If  $v(t)$  is the unit step function, then it has a Laplace transform of  $\frac{1}{s}$ . Then we have

$$\Phi(s) = -\frac{1}{s^2L + g} = -\frac{1}{\sqrt{gL}} \frac{\sqrt{\frac{g}{L}}}{s^2 + \frac{g}{L}}$$

which has an inverse Laplace transform

$$\phi(t) = -\frac{\sin\left(\sqrt{\frac{g}{L}}t\right)}{\sqrt{gL}}$$

so it does oscillate with frequency  $\omega_0 = \sqrt{\frac{g}{L}}$ .

d) Taking the Laplace transform of our system of equations yields the following

$$(m + M)s^2X(s) + mLs^2\Phi(s) = U(s)$$

$$s^2X(s) + Ls^2\Phi(s) + g\Phi(s) = 0$$

$$\Phi(s) = -\frac{s^2}{Ls^2 + g}X(s)$$

$$(m + M)s^2X(s) - \frac{mLs^4}{Ls^2 + g}X(s) = U(s)$$

$$\frac{(m + M)s^2(Ls^2 + g) - mLs^4}{Ls^2 + g}X(s) = U(s)$$

$$\frac{s^2(MLs^2 + (m + M)g)}{Ls^2 + g}X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{Ls^2 + g}{s^2(MLs^2 + (m + M)g)}$$

e) If  $u(t)$  is the unit step function, then it has a Laplace transform of  $\frac{1}{s}$ . Then we have

$$X(s) = \frac{Ls^2 + g}{s^3(MLs^2 + (m + M)g)}$$

We can try using the final value theorem by taking the following limit.

$$\lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} \frac{Ls^2 + g}{s^2(MLs^2 + (m + M)g)}$$

The denominator goes to 0 while the numerator goes to  $g$ , so this limit does not exist and the value goes to  $\infty$ . Therefore by the final value theorem no limit exists and the system's position increases without bound.



## Problem 4

We can convert each equation into a linear approximation by taking the appropriate derivatives evaluated at our initial point. For the first equation we have

$$x'_1 \approx u \frac{\partial f_1}{\partial u} + x_2 \frac{\partial f_1}{\partial x_2} = 0 + x_2$$

For the second equation we have

$$x'_2 \approx u \frac{\partial f_2}{\partial u} + x_4 \frac{\partial f_2}{\partial x_4} = -\pi u - \pi x_4$$

For the third equation we have

$$x'_3 \approx u \frac{\partial f_3}{\partial u} + x_2 \frac{\partial f_3}{\partial x_2} = \pi u + \pi x_2$$

For the fourth equation we have

$$x'_4 \approx x_1 \frac{\partial f_4}{\partial x_1} + x_3 \frac{\partial f_4}{\partial x_3} = 2x_1 + 4x_3$$

This yields the linear system

$$\delta x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & \pi & 0 & 0 \\ 2 & 0 & 4 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\pi \\ \pi \\ 0 \end{bmatrix} \delta u$$

## Problem 5

The controller canonical form is shown below. The state is given by the following.

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{K_3} \end{bmatrix} \delta(t)$$

The output is given by the following.

$$\phi(t) = \begin{bmatrix} K_b & K_a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

## Problem 6

a)

$$\begin{aligned}sX(s) &= AX(s) + BU(s) \\(sI - A)X(s) &= BU(s) \\X(s) &= (sI - A)^{-1}BU(s) \\Y(s) &= C(sI - A)^{-1}BU(s) \\\frac{Y(s)}{U(s)} &= C(sI - A)^{-1}B\end{aligned}$$

b) The transfer function in the form

$$H(s) = K \frac{\Pi_i(s - z_i)}{\Pi_j(s - p_j)}$$

has the constants

$$\begin{aligned}K &= 0.0003 \\z_1 &= -0.394 \\z_2 &= -0.02 \\p_1 &= -0.656 \\p_2 &= -0.1889 \\p_3 &= -0.0042\end{aligned}$$

## Problem 7

a)

$$\begin{aligned}(s - 1)(s - 2) - 1 &= 0 \\s^2 - 3s + 1 &= 0 \\s &= \frac{3 \pm \sqrt{5}}{2}\end{aligned}$$

One of these poles is positive and one is negative, so it is not stable. As time increases the positive pole will cause exponential growth.

**b)**

$$\begin{aligned}(s - 1 + k_1)(s - 2) + (k_2 - 1) &= (s + 2)^2 + 1 \\ s^2 - 3s + k_1s + k_2 - 1 &= s^2 + 4s + 5\end{aligned}$$

This gives us  $k_1 = 7$  and  $k_2 = 6$ .

**c)** No you cannot stabilize the system in this case. The characteristic equation will always contain a term  $s - 2$  which means that a positive pole exists, leading to instability. The gain term  $k_2$  will have no effect on the pole locations since the 0 in the bottom left makes it have no impact on the characteristic equation.