Electrical Engineering 141, Homework 3

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Problem 1

a)

$$
\begin{aligned}
Y(s) &= \frac{K}{\tau s + 1}(R(s) - Y(s)) \\
\left(1 + \frac{K}{\tau s + 1}\right) Y(s) &= \frac{K}{\tau s + 1} R(s) \\
\frac{\tau s + 1 + K}{\tau s + 1} Y(s) &= \frac{K}{\tau s + 1} R(s) \\
\frac{Y(s)}{R(s)} &= \frac{K}{\tau s + 1 + K}
\end{aligned}
$$

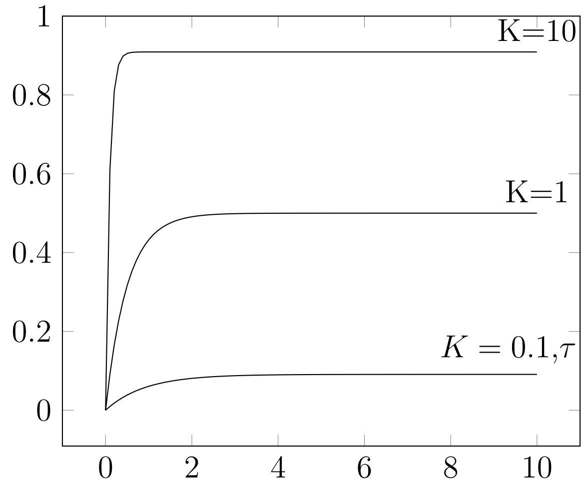
b) If the input is the unit step function, it will have a Laplace transform of $\frac{1}{s}$. Then we have

$$
\begin{aligned}
Y(s) &= \frac{K}{s(\tau s + 1 + K)} \\
&= \frac{K}{s(1+K)} - \frac{K\tau}{(1+K)(\tau s + 1 + K)}
\end{aligned}
$$

This has an inverse Laplace transform of

$$
y(t) = \frac{K}{1+K}\left(1 - e^{-\frac{(1+K)t}{\tau}}\right) u(t)
$$

This looks like the plot shown below.



c) The DC gain of this system is $\frac{K}{1+K}$. The time constant of this system is $\frac{\tau}{1+K}$.

d) The steady state error of this system with the unit step input is $\frac{1}{1+K}$.

e) For a maximum 10% steady state error we would need to have $\frac{1}{1+K} = 0.1$ which gives $K = 9$. Then can solve the following.

$$\begin{aligned} e^{-10\frac{2}{\tau}} &= 0.02 \\ -10\frac{2}{\tau} &= \ln(0.02) \\ \tau &= -\frac{20}{\ln(0.02)} \\ \tau &= \frac{20}{\ln(50)} \approx 5.112 \end{aligned}$$

Problem 2

a) It reached 2% error in 60 seconds. So we can solve the following.

$$
\begin{aligned}
e^{-\frac{60}{\tau}} &= 0.02 \\
-\frac{60}{\tau} &= \ln(0.02) \\
\tau &= -\frac{60}{\ln(0.02)} \\
\tau &= \frac{60}{\ln(50)} \approx 15.34
\end{aligned}
$$

So it has a time constant of approximately 15.34 seconds.

b) The output will have a Laplace transform of $\frac{2}{s^2(\tau s+1)}$. This is equivalent to the following.

$$Y(s) = \frac{2}{s^2} - \frac{2\tau}{s} + \frac{2\tau^2}{\tau s + 1}$$

This has the following inverse Laplace transform.

$$Y(t) = 2tu(t) - 2\tau(1 - e^{-\frac{t}{\tau}})u(t)$$

So as time increases, the thermometer temperature will be 2τ or

$$2\frac{60}{\ln(50)}\frac{1}{60} = 0.5112$$

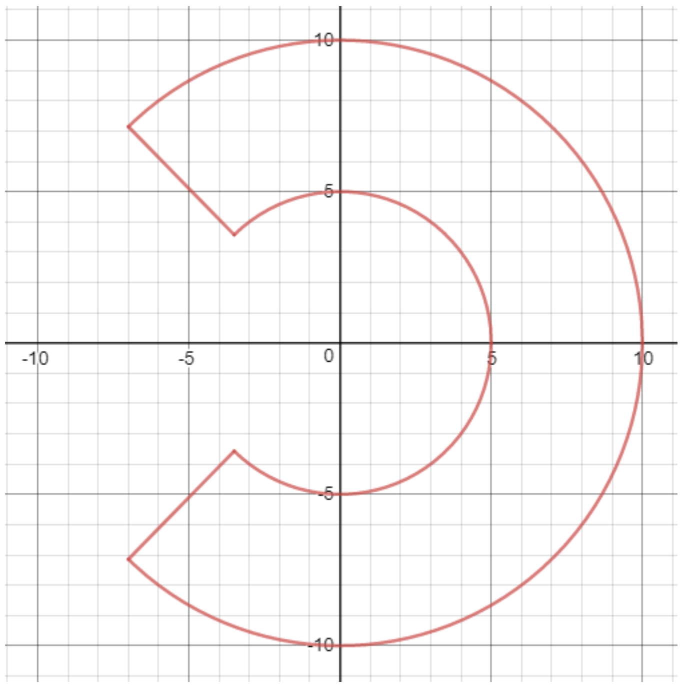
degrees below the tank temperature.

Problem 3

a) The poles are located at

$$\frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$

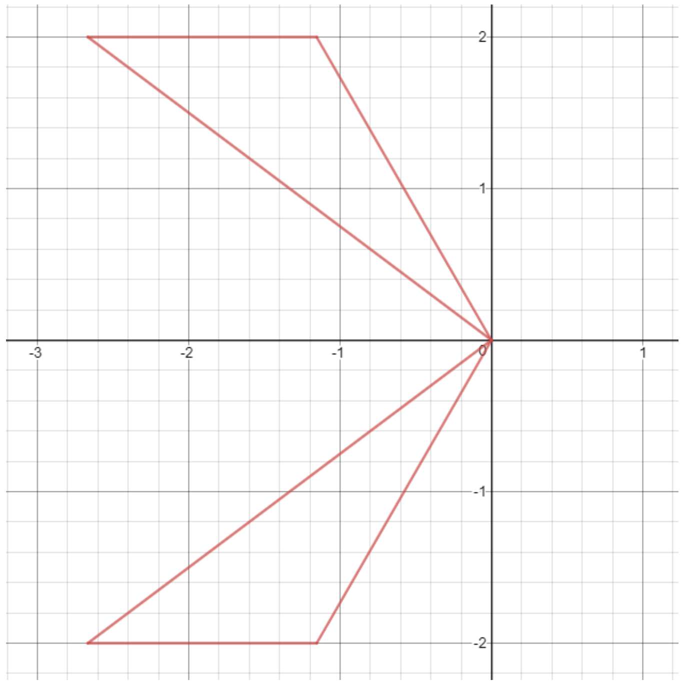
Since $\zeta^2 - 1$ must be negative in order to have imaginary poles, the minimum value that ζ can take is -1. Note that ω_n simply scales the roots by a constant factor. So the ζ traces out an arc in the imaginary plane, while the ω_n extends this out radially which leads to the following shape.



b) We have that $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$. Then our poles are located at

$$\omega_d \left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \pm j \right)$$

So this looks like the following.



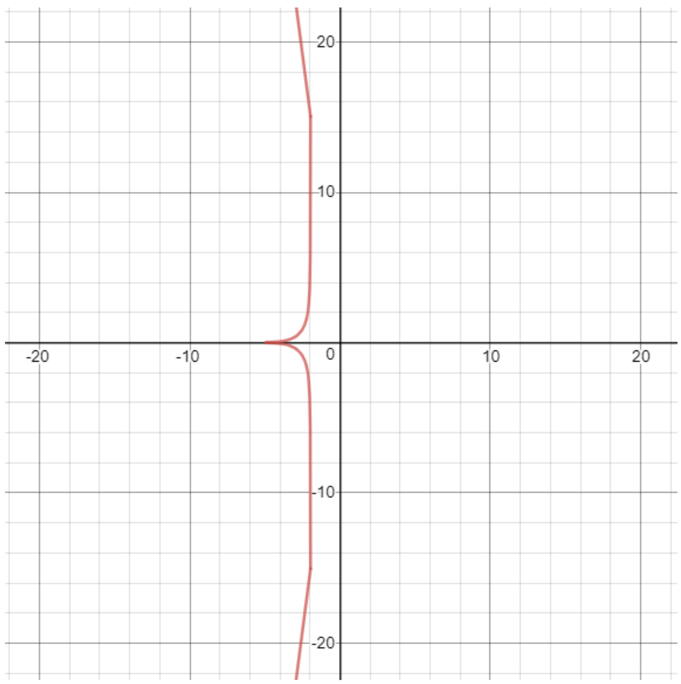
c) From the maximum overshoot we have the following.

$$
\begin{aligned}
e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} &< 0.2 \\
-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &< \ln(0.2) \\
\frac{\zeta}{\sqrt{1-\zeta^2}} &> -\frac{\ln(0.2)}{\pi} \\
\zeta^2 &> (1-\zeta^2)\frac{\ln^2(5)}{\pi^2} \\
\left(1+\frac{\ln^2(5)}{\pi^2}\right)\zeta^2 &> \frac{\ln^2(5)}{\pi^2} \\
\zeta &> \frac{\ln(5)}{\sqrt{\pi^2+\ln^2(5)}}
\end{aligned}
$$

From the settling time we have the following.

$$
\begin{aligned}
-\frac{1}{\zeta\omega_n}\ln\left(0.02\sqrt{1-\zeta^2}\right) &< 2 \\
-\frac{1}{2\zeta}\ln\left(0.02\sqrt{1-\zeta^2}\right) &< \omega_n
\end{aligned}
$$

This region is left of the following boundary.



Problem 4

a) The step response has a Laplace transform of

$$\frac{2(1-s)}{s(s+1)(s+2)}$$

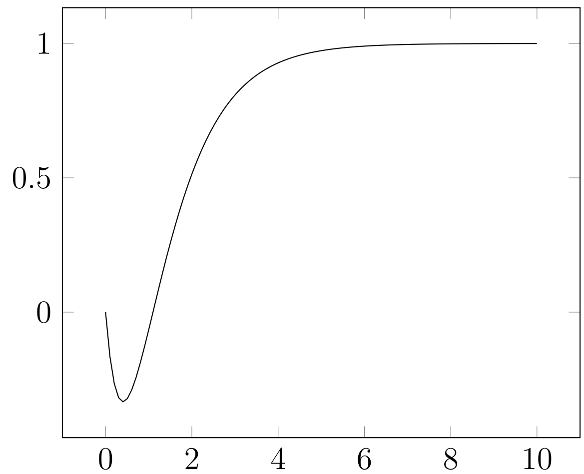
which can be rewritten using partial fraction decomposition as follows.

$$\frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2}$$

This has an inverse Laplace transform of

$$(1 - 4e^{-t} + 3e^{-2t})u(t)$$

which looks like the following plot.



b) The derivative of the step response at $t = 0$ is

$$4e^{-t} - 6e^{-2t}\Big|_{t=0} = -2$$

c) The step response initially goes downwards, which is unexpected because we would want the step response to move towards 1, its steady state response. This is most likely due to the correction for a large move pushing the response in the negative direction. After a while as the input stays the same, the response moves towards the steady state value of 1.

Problem 5

a) Let us find the transfer function for the inner feedback loop.

$$\begin{aligned}
\Omega(s) &= \frac{100}{0.2s+1}K(X(s) - K_g\Omega(s)) \\
\left(1 + KK_g\frac{100}{0.2s+1}\right)\Omega(s) &= \frac{100}{0.2s+1}KX(s) \\
\frac{\Omega(s)}{X(s)} &= \frac{100K}{0.2s+1+100KK_g}
\end{aligned}$$

Then we can solve for the transfer function of the outer feedback loop.

$$\begin{aligned}
\Theta(s) &= \frac{100K(\Theta_r(s) - \Theta(s))}{20s(0.2s + 1 + 100KK_g)} \\
\left(1 + \frac{100K}{20s(0.2s + 1 + 100KK_g)}\right)\Theta(s) &= \frac{100K}{20s(0.2s + 1 + 100KK_g)}\Theta_r(s) \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{25K}{s^2 + (5 + 500KK_g)s + 25K}
\end{aligned}$$

This is a second order system, so we will try to solve for the following.

$$\begin{aligned}
5 + 500KK_g &= 2\zeta\omega_n \\
25K &= \omega_n^2
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
K_g &= \frac{2\zeta\omega_n - 5}{20\omega_n^2} \\
K &= \frac{\omega_n^2}{25}
\end{aligned}$$

Then the percentage overshoot yields the following relationship.

$$\begin{aligned}
e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} &= 0.1 \\
-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &= \ln(0.1) \\
\frac{\zeta}{\sqrt{1-\zeta^2}} &= -\frac{\ln(0.1)}{\pi} \\
\zeta^2 &= (1-\zeta^2)\frac{\ln^2(10)}{\pi^2} \\
\left(1 + \frac{\ln^2(10)}{\pi^2}\right)\zeta^2 &= \frac{\ln^2(10)}{\pi^2} \\
\zeta &= \frac{\ln(10)}{\sqrt{\pi^2 + \ln^2(10)}}
\end{aligned}$$

The settling time yields the following relationship.

$$
\begin{aligned}
0.05 &= -\frac{1}{\zeta\omega_n}\ln\left(0.02\sqrt{1-\zeta^2}\right)\\
\omega_n &= \frac{20}{\zeta}\ln\left(\frac{50}{\sqrt{1-\zeta^2}}\right)\\
\omega_n &= \frac{20}{\frac{\ln(10)}{\sqrt{\pi^2+\ln^2(10)}}}\ln\left(\frac{50}{\sqrt{1-\left(\frac{\ln(10)}{\sqrt{\pi^2+\ln^2(10)}}\right)^2}}\right)\\
\omega_n &= \frac{20\sqrt{\pi^2+\ln^2(10)}}{\ln(10)}\ln\left(\frac{50\sqrt{\pi^2+\ln^2(10)}}{\pi}\right)
\end{aligned}
$$

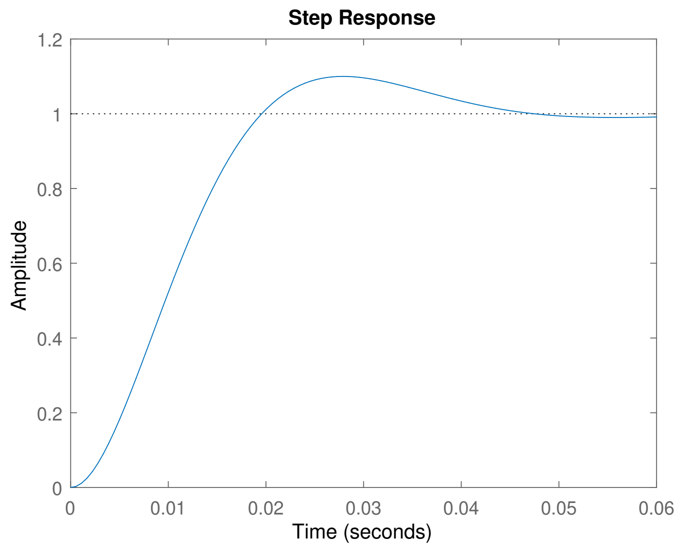
Plugging into the earlier expressions yields the following.

$$
\begin{aligned}
K_g &= \frac{\left(8\ln\left(\frac{50\sqrt{\pi^2+\ln^2(10)}}{\pi}\right)-1\right)\ln^2(10)}{1600(\pi^2+\ln^2(10))\ln^2\left(\frac{50\sqrt{\pi^2+\ln^2(10)}}{\pi}\right)} \approx 0.00041056\\
K &= \frac{16(\pi^2+\ln^2(10))\ln^2\left(\frac{50\sqrt{\pi^2+\ln^2(10)}}{\pi}\right)}{\ln^2(10)} \approx 779.8
\end{aligned}
$$

b) I translated my previous results into the following code.

```
K = (16*(pi^2+log(10)^2)*log((50*sqrt(pi^2 +log(10)^2))/pi)^2)/(log(10)^2);
Kg = (8*log(50*sqrt(pi^2 +log(10)^2)/pi)-1)*log(10)^2/(1600*(pi^2+log(10)^2)...
*log(50*sqrt(pi^2+log(10)^2)/pi)^2);
s = tf('s');
H = 25*K/(s^2 + (5+500*K*Kg)*s + 25*K);
set(gcf,'color','w');
step(H);
export_fig problem5b.pdf;
stepinfo(H)
```

This produced a settling time of 0.0424 seconds and a 10% overshoot, as expected. The slight error in the settling time is due to the settling time formula using the envelope only. The real signal settles earlier due to the sinusoidal term bringing the response within the settling range before the envelope is entirely within the settling range. The step response is shown below.



Problem 6

a) The inner loop has the following transfer function.

$$\begin{aligned} C(s) &= \frac{1}{s(s+2)(s+4)}(X(s) - K_2 s C(s)) \\ \left(1 + \frac{K_2 s}{s(s+2)(s+4)}\right) C(s) &= \frac{1}{s(s+2)(s+4)} X(s) \\ \frac{C(s)}{X(s)} &= \frac{1}{s(s^2+6s+8+K_2)} \end{aligned}$$

Setting $K_2 = 1$ would cause the inner loop to have two equal negative real poles at $s = -3$. Then the entire system has the following transfer function.

$$\begin{aligned}
C(s) &= \frac{1}{s^3 + 6s^2 + 9s} K_1 (R(s) - C(s)) \\
\left(1 + \frac{K_1}{s^3 + 6s^2 + 9s}\right) C(s) &= \frac{1}{s^3 + 6s^2 + 9s} K_1 R(s) \\
\frac{C(s)}{R(s)} &= \frac{K_1}{s^3 + 6s^2 + 9s + K_1}
\end{aligned}$$

This will have the following Routh-Hurwitz table.

$$\begin{array}{c|cc}
s^3 & 1 & 9 \\
s^2 & 6 & K_1 \\
s & \frac{54-K_1}{6} & 0 \\
1 & K_1 & 0
\end{array}$$

This will have no roots with a positive real value if $0 < K_1 < 54$.

b) At $K_1 = 54$ the transfer function will have $(s+6)(s^2+9)$ in its denominator so it will have roots at $\pm 3j$. So the system will oscillate with an angular frequency of 3.

c) At $K_1 = 20$ the transfer function will have $(s+5)(s^2+s+4)$ in its denominator which corresponds to a real root at $s = -5$. The complex roots are given by

$$s = -\frac{1 \pm \sqrt{15}j}{2}$$

and have a absolute value of 2, much less than the real root at $s = -5$. So this can be approximated by a second-order, underdamped response using dominant pole approximation. The following transfer function describes the approximated system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + s + 4}$$

d) We have that $\omega_n = 2$ and therefore $\zeta = \frac{1}{4}$. From this we can find the percentage overshoot as follows.

$$M_p = e^{-\frac{\pi}{4\sqrt{1-\frac{1}{16}}}} = e^{-\frac{\pi}{\sqrt{15}}} \approx 44.43\%$$

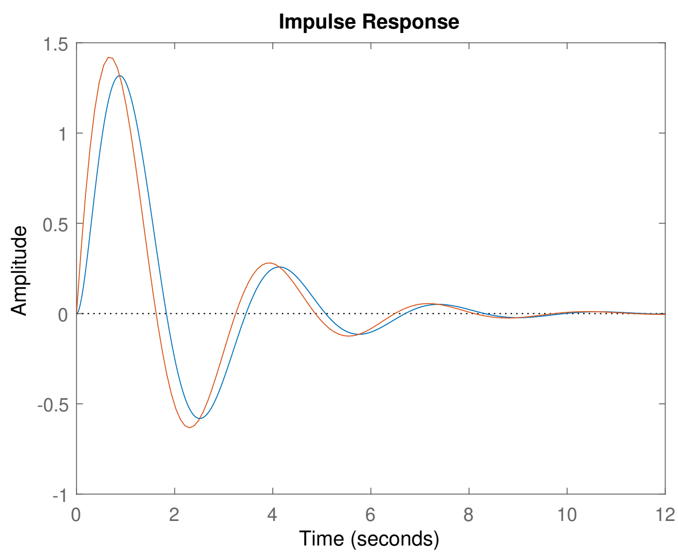
We can find the 2% settling time as follows.

$$t_s = -\frac{4}{2}\ln\left(0.02\sqrt{1-\frac{1}{16}}\right) = 2\ln\left(\frac{200}{\sqrt{15}}\right) \approx 7.89$$

e) I plotted the impulse response with the following code.

```
s = tf('s');
H1 = 20/(s^3 + 6*s^2 + 9*s + 20);
H2 = 4/(s^2 + s + 4);
set(gcf,'color','w');
hold on;
impulse(H1);
impulse(H2);
hold off;
export_fig problem6e.pdf;
```

This generated the following plot.



The second order approximation is the response with the slightly higher peak. It closely matches the third order response, so it is a good approximation.

Problem 7

We can shift the characteristic equation to the right by 1 by substituting $s-1$ in for s. Then the Routh-Hurwitz method will indicate how many poles are on the right of the $\sigma = -1$ line. Our new characteristic equation is the following.

$$\begin{aligned} (s-1)^3 + (6+K)(s-1)^2 + (5+6K)(s-1) + 5K + 1 = 0 \\ (s^3 - 3s^2 + 3s - 1) + (6+K)(s^2 - 2s + 1) + (5+6K)(s-1) + 5K + 1 = 0 \\ s^3 + (K+3)s^2 + (4K-4)s + 1 = 0 \end{aligned}$$

Then the Routh-Hurwitz table can be written as shown below.

$$\begin{array}{c|cc} s^3 & 1 & 4K-4 \\ s^2 & K+3 & 1 \\ s & \frac{4K^2+8K-13}{K+3} & 0 \\ 1 & 1 & 0 \end{array}$$

In order for there to be no poles with a real value greater than -1, we must have $-3 < K$ due to the second line of this table. So the denominator of the third line must be positive which yields the following.

$$\begin{aligned} 4K^2 + 8K - 13 > 0 \\ K > -1 + \frac{\sqrt{17}}{2} \end{aligned}$$

The other root is slightly below -3, so this does not affect our range for K. Thus we must set $K > -1 + \frac{\sqrt{17}}{2}$ in order to satisfy the desired criteria.

Problem 8

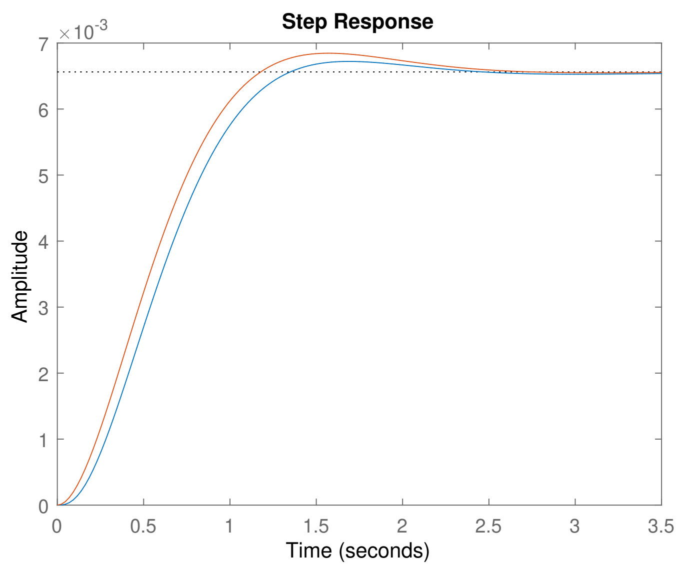
a) The pole at $s = -20$ can be ignored because it dies out very fast. The one at $s = -1$ is almost canceled by the zero at $s = -1.05$ so we will not include it in our approximation. This leaves $s^2 + 4s + 8$ in the denominator. We want the approximation $G_a(s)$ to have the same value as $G(s)$ at $s = 0$ so we choose $K = \frac{21}{400}$. So we have the following.

$$G_a(s) = \frac{21}{400(s^2 + 4s + 8)}$$

b) I plotted the step response with the following code.

```
s = tf('s');
G = (s + 1.05)/((s + 1)*(s + 20)*(s^2 + 4*s + 8));
Ga = 21/(400*(s^2 + 4*s + 8));
set(gcf,'color','w');
hold on;
step(G);
step(Ga);
hold off;
export_fig problem8b.pdf;
```

This generated the following plot.



The second order approximation is the response with the slightly higher peak. It closely matches the fourth order response, so it is a good approximation.

Problem 9

a)

$$
\begin{aligned}
F_{\text{out}}(s) &= \frac{K_v K_f K_p}{s\left(1+\frac{s}{\omega_p}\right)}(F_{\text{in}}(s) - F_{\text{out}}(s)) \\
\frac{F_{\text{out}}}{F_{\text{in}}} &= \frac{K_v K_f K_p}{\frac{s^2}{\omega_p} + s + K_v K_f K_p} \\
\frac{F_{\text{out}}}{F_{\text{in}}} &= \frac{K_v K_f K_p \omega_p}{s^2 + \omega_p s + K_v K_f K_p \omega_p}
\end{aligned}
$$

b) If we solve the following system we can put the transfer function into the standard second-order form.

$$
\begin{aligned}
\omega_p &= 2\zeta\omega_n \\
K_v K_f K_p \omega_p &= \omega_n^2
\end{aligned}
$$

This has the following solution.

$$
\begin{aligned}
\omega_n &= \sqrt{K_v K_f K_p \omega_p} \\
\zeta &= \frac{1}{2}\sqrt{\frac{\omega_p}{K_v K_f K_p}}
\end{aligned}
$$

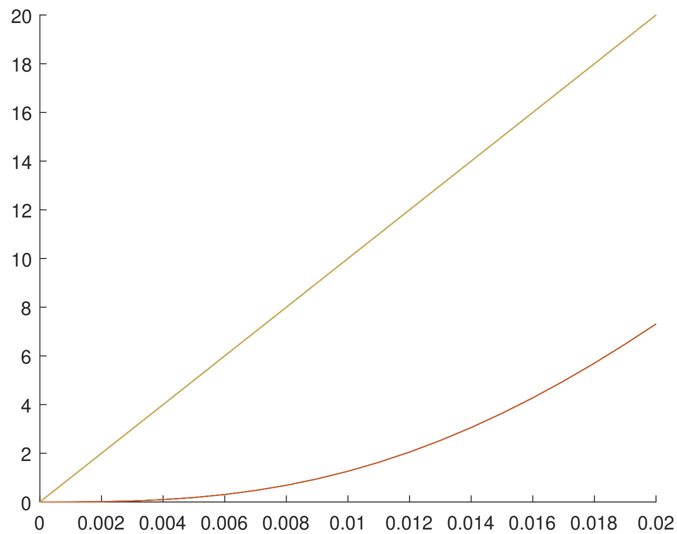
Since ω_p is positively correlated with both ω_n and ζ while the other constants are positively correlated with ω_n and negatively correlated with ζ, we have two degrees of freedom. Thus we can make any arbitrary combination of ω_n and ζ and this transfer function can be rewritten in the standard second order form.

c) I plotted the ramp response with the following code.

```
t = 0:0.001:0.02;
ramp = 1000*t;
s = tf('s');
H = 100^2/(s^2 + 100*s + 100^2);
[y,t] = lsim(H,ramp,t);
set(gcf,'color','w');
```

```
hold on;
plot(t,y);
plot(t,ramp);
hold off;
export_fig problem9c.pdf;
result = ramp(21) - y(21);
```

This generated the following plot.



The PLL frequency tracking error after 20 ms was 12.6871 Hz.