Electrical Engineering 141, Homework 1

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January 25th, 2019

Problem 1

a) First we have the following.

$$\begin{aligned} Y_1 &= G_2G_1(R_1 - H_1Y_1) \\ &= \frac{G_1G_2}{1+G_1G_2H_1}R_1 \end{aligned}$$

Then we can write the transfer function for Y_2 as follows.

$$\begin{aligned} Y_2 &= G_4(G_3(G_6Y_1 - H_2Y_2) + G_5G_1(R_1 - H_1Y_1)) \\ (1+G_4G_3H_2)Y_2 &= G_4G_3G_6Y_1 + G_4G_5G_1R_1 - G_4G_5G_1H_1Y_1 \\ (1+G_4G_3H_2)Y_2 &= (G_4G_3G_6 - G_4G_5G_1H_1)Y_1 + G_4G_5G_1R_1 \\ \frac{Y_2}{R_1} &= \left(\frac{G_4G_3G_6 - G_4G_5G_1H_1}{1+G_4G_3H_2}\right)\frac{Y_1}{R_1} + \frac{G_4G_5G_1}{1+G_4G_3H_2} \end{aligned}$$

Then we can plug in our earlier expression to obtain the following.

$$\begin{aligned} \frac{Y_2}{R_1} &= \left(\frac{G_4G_3G_6 - G_4G_5G_1H_1}{1+G_4G_3H_2}\right)\frac{G_1G_2}{1+G_1G_2H_1} + \frac{G_4G_5G_1}{1+G_4G_3H_2} \\ &= \frac{G_1G_2G_4G_3G_6 - G_2G_4G_5G_1^2H_1 + G_4G_5G_1 + G_2G_4G_5G_1^2H_1}{(1+G_4G_3H_2)(1+G_1G_2H_1)} \\ &= \frac{G_1G_2G_3G_4G_6 + G_1G_4G_5}{(1+G_3G_4H_2)(1+G_1G_2H_1)} \end{aligned}$$

b) If we let

$$G_5(s) = -G_2 G_3 G_6$$

then the previous transfer function will be zero.

Problem 2

a) Across the first resistor we have

$$V_i - V_a = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o - V_a = I_b Z_2$$

due to Ohm's law. Since the impedance of the OpAmp is infinity, we know that the current into the negative side is zero. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite using the previous equations to obtain the following.

$$\begin{aligned}
\frac{V_i - V_a}{Z_1} + \frac{V_o - V_a}{Z_2} &= 0 \\
(V_i - V_a) * Z_2 + (V_o - V_a) * Z_1 &= 0 \\
Z_2 V_i + Z_1 V_o &= (Z_1 + Z_2) V_a \\
V_a &= \frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o
\end{aligned}$$

b)

$$\begin{aligned}
V_o &= -a \left(\frac{Z_2}{Z_1 + Z_2} V_i + \frac{Z_1}{Z_1 + Z_2} V_o \right) \\
\left(1 + \frac{a Z_1}{Z_1 + Z_2}\right) V_o &= \frac{-a Z_1}{Z_1 + Z_2} V_i \\
\frac{V_o}{V_i} &= \frac{\frac{-a Z_1}{Z_1 + Z_2}}{1 + \frac{a Z_1}{Z_1 + Z_2}}
\end{aligned}$$

c) This is equivalent to assuming that $V_a = 0$ in our previous calculations. Then we have across the first resistor

$$V_i = I_a Z_1$$

due to Ohm's law. Across the second resistor we also have

$$V_o = I_b Z_2$$

due to Ohm's law. Then Kirchhoff's law gives us

$$I_a + I_b = 0$$

which we can rewrite as follows.

$$\begin{aligned} \frac{V_i}{Z_1} + \frac{V_o}{Z_2} &= 0 \\ \frac{V_o}{Z_2} &= -\frac{V_i}{Z_1} \\ \frac{V_o}{V_i} &= -\frac{Z_2}{Z_1} \end{aligned}$$

d) Let I_1 be a current flowing down across Z_1. Let I_2 be a current flowing to the left across Z_2. Then we have that

$$V_o - V_a = I_2 Z_2$$

and that

$$V_a = I_1 Z_1$$

and from Kirchoff's law we have that

$$I_2 = I_1$$

This means that we know that

$$\begin{aligned} \frac{V_o - V_a}{Z_2} &= \frac{V_a}{Z_1} \\ \frac{V_o}{Z_2} &= \frac{V_a}{Z_1} + \frac{V_a}{Z_2} \\ V_a &= \frac{Z_1}{Z_1 + Z_2} V_o \end{aligned}$$

Additionally across the OpAmp we know that

$$
\begin{aligned}
V_o &= a(V_i - V_a) \\
V_o &= a\left(V_i - \frac{Z_1}{Z_1+Z_2}V_o\right) \\
\left(1+\frac{aZ_1}{Z_1+Z_2}\right)V_o &= aV_i \\
\frac{V_o}{V_i} &= \frac{a}{1+\frac{aZ_1}{Z_1+Z_2}} \\
\frac{V_o}{V_i} &= \frac{a(Z_1+Z_2)}{(1+a)Z_1+Z_2}
\end{aligned}
$$

As the gain a becomes very large the limit of this becomes

$$\frac{V_o}{V_i} = \frac{Z_1+Z_2}{Z_1}$$

e) This is the same as the inverting amplifier that we studied earlier, except with impedances

$$Z_1 = \frac{\frac{R_1}{j\omega C_1}}{R_1+\frac{1}{j\omega C_1}}$$

and

$$Z_2 = R_2 + \frac{1}{j\omega C_2}$$

which can be rewritten in terms of s to be

$$
\begin{aligned}
Z_1 &= \frac{\frac{R_1}{sC_1}}{R_1+\frac{1}{sC_1}} = \frac{R_1}{1+sR_1C_1} \\
Z_2 &= R_2 + \frac{1}{sC_2} = \frac{1+sR_2C_2}{sC_2}
\end{aligned}
$$

Then our transfer function is given by the following

$$
\begin{aligned}
\frac{V_0(s)}{V_i(s)} &= -\frac{(1+sR_2C_2)(1+sR_1C_1)}{sR_1C_2} \\
&= -\frac{1+s(R_1C_1+R_2C_2)+s^2R_1C_1R_2C_2}{sR_1C_2} \\
&= -\left(\frac{C_1}{C_2}+\frac{R_1}{R_2}\right) - R_2C_1s - \frac{1}{sR_1C_2}
\end{aligned}
$$

Therefore this is equivalent to

$$\frac{V_0(s)}{V_i(s)} = K_P + K_D s + \frac{K_I}{s}$$
$$K_P = -\frac{C_1}{C_2} - \frac{R_1}{R_2}$$
$$K_D = -R_2 C_1$$
$$K_I = -\frac{1}{R_1 C_2}$$

f) We have that

$$v_o = -\frac{R_b}{R_a} v_2$$

since there is an inverting amplifier in the middle of the circuit. Furthermore we have that$v_2 = \frac{Q_1}{C_1}$, and so

$$v_2' = \frac{I_1}{C_1}$$

where I_1 is the current going through the capacitor C_1. The current going through the resistor R_1 is equal to $I_1 + I_a$, where I_a is the current going through R_a. We can solve

$$v_2 - v_o = I_a(R_a + R_b)$$
$$\left(1 + \frac{R_b}{R_a}\right) v_2 = I_a(R_a + R_b)$$
$$I_a = \frac{v_2}{R_a}$$

to obtain the expression

$$v_1 - v_2 = \left(C_1 v_2' + \frac{v_2}{R_a}\right) R_1$$

from the left side of the circuit. Similarly the right side of the circuit gives us

$$-\frac{R_b}{R_a} v_2 - v_3 = C_2 v_3' R_2$$

Solving for v_2' and v_3' yields the system of equations shown below.

$$v_2' = \left(-\frac{1}{R_1C_1} - \frac{1}{R_aC_1}\right)v_2 + \frac{1}{R_1C_1}v_1$$
$$v_3' = -\frac{R_b}{R_aR_2C_2}v_2 - \frac{1}{R_2C_2}v_3$$

which is equivalent to the State-Space form

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{R_1C_1} - \frac{1}{R_aC_1} & 0 \\ -\frac{R_b}{R_aR_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} v_1$$

Problem 3

a)

b)

c)

d)

e)

Problem 4

We can convert each equation into a linear approximation by taking the appropriate derivatives evaluated at our initial point. For the first equation we have

$$x_1' \approx u\frac{\partial f_1}{\partial u} + x_2\frac{\partial f_1}{\partial x_2} = 0 + x_2$$

For the second equation we have

$$x_2' \approx u\frac{\partial f_2}{\partial u} + x_4\frac{\partial f_2}{\partial x_4} = -\pi u - \pi x_4$$

For the third equation we have

$$x_3' \approx u\frac{\partial f_3}{\partial u} + x_2\frac{\partial f_3}{\partial x_2} = \pi u + \pi x_2$$

For the fourth equation we have

$$x_4' \approx x_1 \frac{\partial f_4}{\partial x_1} + x_3 \frac{\partial f_4}{\partial x_3} = 2x_1 + 4x_3$$

This yields the linear system

$$\delta x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & \pi & 0 & 0 \\ 2 & 0 & 4 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\pi \\ \pi \\ 0 \end{bmatrix} \delta u$$

Problem 5

Problem 6

a)

b)

Problem 7

a)

b)

c)