An alternative solution to the seat problem at Thanksgiving party*

Michael Wang
Department of Mathematics, Crank University.
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A host invited $N-1$ guest to attend a Thanksgiving party. The host assigned seat for each of the guest and himself, and placed the name card on the table in front the seat. However, among the guests there were m dotards who would disregard the arrangement but grab whatever seat available at random, while other guests sit where they were assigned if the seat was not taken, or choose an available seat at random. The host would sit last. What is the probability that the would sit at his own seat?

I. THE CLASSICAL SOLUTION

The classical solution is based on the reasoning that:

1. The seat available for the host are the seat assigned to himself or to the dotards because if the seat assigned to a guest is available, then it should have been taken by the guest;

2. The seats assigned to host and to dotards are equivalent, so the host would have equal opportunity to take any of the seat.

From these, we get the answer to the question is $\frac{1}{m+1}$.

II. THE QUANTUM MECHANICS SOLUTION

According to Schrödinger's cat, a dotard does not occupy one seat, but $\frac{1}{N}$ of all the seat.

Guest A, seeing his seat taken by the those m dotards, will have no choice but take portion of seat of all other $N-1$ people. Guest B, with portion of seat taken not only by m dotards but also by guest A, will take a bigger portion of seat from guest A, guest C, etc. A nuclear reaction started, when equilibrium were reached at the end, the portion of the seat taken by each other guest should be equal, and we shall denote this by x. The equilibrium equation is:

$$x = \frac{\frac{1}{N} * m + x * (N-m-2)}{N-1}$$

Solving for x, we have:

$$x = \frac{m}{(m+1)N}$$

The portion of dotard's and host's seat taken is:

$$\frac{m}{N} + \frac{m}{(m+1)N} * (N-m-1) = \frac{m}{m+1}$$

The portion not taken, *i.e.*, the possibility available to the host is $\frac{1}{m+1}$.

III. MATHEMATICAL INDUCTION SOLUTION

It is possible to find the solution for this problem by using the exact steps. Suppose we have m dotards, g *normal* guest, we want to find the possibility $P(m,g)$.

There are two variables, m and g. Let us first fix g by setting it to zero. Obviously we have:

Since $P(m,0) = \frac{m}{m+1} * P(m-1,0)$, so if $P(m-1,0) = \frac{1}{m}$, then $P(m+1,0) = \frac{1}{m+1}$. Obviously the initial condition $P(0,0) = 1$ is true, there we established that

$$P(m,0) = \frac{1}{m+1}$$

for all m's. Actually we do not need to use mathematical induction to prove this equation for this case, but this serves as a warm-up.

When number of *normal* guest is g, we have:

$$\begin{aligned} P(m,g) = \frac{m}{m+g} * (&\frac{m}{m+g+1} * P(m-1,g) \\ &+ \frac{g}{m+g+1} * P(m,g-1)) \\ + \frac{g}{m+g} * P(m,g-1)& \end{aligned}$$

where $\frac{g}{m+g+1} * P(m,g-1)$ means when a seat of *normal* guest is taken by a dotard, the *normal* guest became a dotard as he can take any seat.

Set g to 1, we know that $P(m,0) = \frac{1}{m+1}$ which has been proved in previous step and not a part of assumption for induction. If we assume $P(m-1,1) = \frac{1}{m}$, then $P(m,1) = \frac{1}{m+1}$. Obviously the initial condition $P(0,1) = 1$ is true, there we established that $P(m,1) = \frac{1}{m+1}$. The above logic applies for any value of g, so we have

* A problem posed by Professor Wan and extended by Professor Yuan

$$P(m,g)=\frac{1}{m+1}$$

for all m's.

In above derivation, we loop through the variable g (*i.e.*, for $g=0,1,...$), and use inductive step over m. We could also loop through the variable m, and use inductive step g.

IV. CONCLUSION

If the result is independent of the actual process, it is easier and more clear to concentrate on the end result instead of step through the process. In this case, it is possible but generally it is not. This happens in physics problems as well, and in real life.