MA1522 Notes (AY24/25 Sem1)

Michael Yang

September 3, 2024

Chapter 2: Matrix Algebra

2.1 Definition and Special types of Matrices

2.2 Matrix Algebra

2.3 Linear System and Matrix Equation

2.4 Inverse of Matrices

2.5 Elementary Matrices

2.6 Equivalent Statements for Invertibility

2.7 LU Factorization

2.8 Determinant by Cofactor Expansion

2.9 Determinant by Reduction

Theorem

Suppose $\mathbf{B}$ is obtained from $\mathbf{A}$ by a single elementary row operation, $\mathbf{A} \xrightarrow{r} \mathbf{B}$.
Then the determinant of $\mathbf{B}$ is obtained from the determinant of $\mathbf{A}$ as such.

- If $r = R_i + aR_j$, then $\det(\mathbf{B}) = \det(\mathbf{A})$;
- If $r = cR_i$, then $\det(\mathbf{B}) = c\ \det(\mathbf{A})$;
- If $r = R_i \leftrightarrow R_j$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.

Corollary

The determinant of an elementary matrix $\mathbf{E}$ is given as such.

- If $\mathbf{E}$ corresponds to $R_i + aR_j$, then $\det(\mathbf{E}) = 1$.
- If $\mathbf{E}$ corresponds to cR_i, then $\det(\mathbf{E}) = c$.
- If $\mathbf{E}$ corresponds to $R_i + aR_j$, then $\det(\mathbf{E}) = -1$.

Theorem

Let $\mathbf{A}$ and $\mathbf{R}$ be square matrices such that

$$\mathbf{R} = \mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

for some elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k$. Then

$$\det(\mathbf{R}) = \det(\mathbf{E}_k) \dots \det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{A}).$$

Corollary

Let $\mathbf{A}$ be a $n \times n$ square matrix.

Suppose $\mathbf{A} \xrightarrow{r_1} \xrightarrow{r_2} \dots \xrightarrow{r_k} \mathbf{R} = \begin{pmatrix} d_1 & * & \dots & * \\ 0 & d_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$, where $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$. Let $\mathbf{E}_1$ be the elementary matrix corresponding to the elementary row operation r_i, for $i = 1, \dots, k$. Then

$$\det(\mathbf{A}) = \frac{d_1 d_2 \dots d_n}{\det(\mathbf{E}_k) \dots \det(\mathbf{E}_2)\det(\mathbf{E}_1)}.$$

2.10 Properties of Determinant

Theorem

(Determinant of product is the product of determinant)

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. Then

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

By induction, for square matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k$ of the same size,

$$\det(\mathbf{A}_1\mathbf{A}_2 \dots \mathbf{A}_k) = \det(\mathbf{A}_1)\det(\mathbf{A}_2) \dots \det(\mathbf{A}_k).$$

Theorem

(Determinant of inverse is the inverse of determinant)

If $\mathbf{A}$ is invertible, then

$$\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$$

Theorem

(Determinant of scalar multiplication)

For any square matrix $\mathbf{A}$ of order n and scalar c,

$$\det(c\mathbf{A}) = c^n \det(\mathbf{A}).$$

Definition

Let $\mathbf{A}$ be a $n \times n$ square matrix. The adjoint of $\mathbf{A}$, denoted as $\mathbf{adj(A)}$, is the $n \times n$ square matrix whose (i, j) entry is the (j, i)-cofactor of $\mathbf{A}$,

$$\mathbf{adj}(\mathbf{A}) = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

Theorem

(Adjoint formula)

Let $\mathbf{A}$ be a square matrix and $\mathbf{adj(A)}$ be its adjoint. Then

$$\mathbf{A}(\mathbf{adj}(\mathbf{A})) = \det(\mathbf{A})\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

Corollary

(Adjoint formula for inverse)

Let $\mathbf{A}$ be an invertible matrix. Then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{adj}(\mathbf{A})$$

Chapter 3: Euclidean Vector Spaces

3.1 Euclidian Vector Spaces

Definition

A (real) n-**vector** is a collection of n ordered real numbers,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \textit{where } v_i \in \mathbb{R} \textit{ for } i = 1, \ldots, n.$$

The real number v_i is called the i-th coordinate of the vector $\mathbf{v}$. The **Euclidean** n-**space**, denoted $\mathbb{R}^n$, is the collection of all n-vectors

$$\mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R} \textit{ for } i = 1, \ldots, n. \right\}$$

Properties of Vector Addition and Scalar Multiplication

Since vectors are matrices (column vectors are $n \times 1$ matrices and row vectors are $1 \times n$ matrices), the properties of matrix addition and scalar multiplication holds for vectors. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars $a, b \in \mathbb{R}$,

1. The sum $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$
2. (Commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. (Associative) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. (Zero vector) $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
5. The negative $-\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{v} - \mathbf{v} = \mathbf{0}$.
6. (Scalar multiple) $a\mathbf{v}$ is a vector in $\mathbb{R}^n$.
7. (Distribution) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. (Distribution) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
9. (Associativity of scalar multiplication) $(ab)\mathbf{u} = a(b\mathbf{u})$.
10. If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.

Definition

A **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ for some $c_1, c_2, \ldots, c_k \in \mathbb{R}^k$.

Definition

A set V equipped with **addition** and **scalar multiplication** is said to be a **vector space** over $\mathbb{R}$ if it satisfies the following axioms.

1. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

2. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$ for all vectors $\mathbf{v}$ in V.
5. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.
7. (Distribution) For any scalar a in $\mathbb{R}$ and vector $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.
8. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.
9. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u})=(ab)\mathbf{u}$.
10. For any vector $\mathbf{u}$ in V, $1\mathbf{u}=\mathbf{u}$.

3.2 Dot Product, Norm, Distance

Definition

The inner product (or dot product) of vectors $\mathbf{u}=(u_i)$ and $\mathbf{v}=(v_i)$ in $\mathbb{R}^n$ is defined to be

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2,+\cdots+u_nv_n.$$

Define the **norm** of a vector $\mathbf{u}\in\mathbb{R}^n$, $\mathbf{u}=(u_i)$, to be the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $||\mathbf{u}||$,

$$||u||=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{u_1^2+u_2^2+\cdots+u_n^2}.$$

This is also known as the **length** or **magnitude** of the vector.

Properties of inner product and norm

Let $\mathbf{u}, \mathbf{v}\in\mathbb{R}^n$ be vectors and $a, b, c\in\mathbb{R}$ be real numbers.

1. Inner product is **symmetric**,
$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}.$$
2. Inner product **commutes** with scalar multiple,
$$c\mathbf{u}\cdot\mathbf{v}=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v}).$$
3. Inner product is **distributive**,
$$\mathbf{u}\cdot(a\mathbf{v}+b\mathbf{w})=a\mathbf{u}\cdot\mathbf{v}+b\mathbf{u}\cdot\mathbf{w}$$
4. Inner product is **positive definite**, $\mathbf{u}\cdot\mathbf{u}\geq 0$ with equality if and only if $\mathbf{u}=\mathbf{0}$.
5. $||c\mathbf{u}||=|c|\ ||\mathbf{u}||$.

Definition

A vector $\mathbf{u} \in \mathbb{R}^n$ is a **unit vector** if its norm is 1,

$$||\mathbf{u}|| = 1$$

Normalizing a vector
Let $\mathbf{u}$ be a nonzero vector $\mathbf{u} \neq \mathbf{0}$. By multiplying by the reciprocal of the norm, we get a unit vector,

$$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{||\mathbf{u}||}$$

This is called **normalizing u**.

Definition

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $d(\mathbf{u}, \mathbf{v})$, is defined to be

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

Define the angle θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

3.3 Linear Combinations and Linear Spans

Definition

A linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k, \text{ for some } c_1, c_2, \ldots c_k \in \mathbb{R}.$$

The scalars $c_1, c_2, \ldots c_k$ are called **coefficients**.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The **span** (or **linear span**) of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Algorithm to Check for Linear Combination

1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_k)$ whose columns are the vectors in S.
2. Then a vector $\mathbf{v}$ in $\mathbb{R}^n$ is in $span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent.
3. If the system is consistent, then the solutions to the system are the possible coefficients of the linear combination. That is, if $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is a solution to $\mathbf{Ax} = \mathbf{v}$, then

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Algorithm to Check if $span(S) = \mathbb{R}^n$

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ whose columns are the vectors in S.
2. Then $span(S)$ $=\mathbb{R}^n$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent for all $\mathbf{v}$.
3. This is equivalent to the reduced row-echelon form of $\mathbf{A}$ having no zero rows.

Properties of linear span

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a finite set of vector. The span of S, $span(S)$ has the following properties.

1. The span of S **contains the origin**,
$$\mathbf{0} \in span(S).$$
2. The span of S is **closed under vector addition**, for any **u, v** $\in span(S)$, and real number $\alpha \in \mathbb{R}$,
$$\mathbf{u} + \mathbf{v} \in span(S)$$
3. The span S is **closed under scalar multiplication**, for any $\mathbf{u} \in span(S)$ and real number $\alpha \in \mathbb{R}$,
$$\alpha\mathbf{u} \in span(S).$$

   Properties (ii) and (iii) can be combined together into one property (ii'): The span is closed under linear combinations, that is, if **u, v** are vectors in $span(S)$ and α, β are any scalars, then the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is a vector in $span(S)$.

Theorem

(Linear span is closed under linear combinations)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $span(S)$, the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a subset of $span(S)$,

$$span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq span(S).$$

Algorithm to check for Set Relations between Spans

Suppose we are given 2 sets of vectors $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

1. By the corollary, if $\mathbf{v}_i \in span(S)$ for $i = 1, \ldots, m$, we can conclude that $span(T) \subseteq span(S)$.
2. Recall that to check if $\mathbf{v}_i \in span(S)$, we check that the system $(\ \mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k \mid \mathbf{v}_i\ )$ is consistent for all $i = 1, \ldots, m$.
3. There are in total m such linear systems to check. However, since they have the same coefficient matrix, we may combine and check them together, that is, check that
$$(\ \mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k \mid \mathbf{v}_1\ \mid \mathbf{v}_2 \mid\ \ldots\ \mid \mathbf{v}_m\ )$$
   is consistent.

Theorem

(Algorithm to check for set relations between spans)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be sets of vectors in $\mathbb{R}^n$. Then $span(T) \subseteq span(S)$ if and only if $(\ \mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \mid \mathbf{v}_1 \ \mid \mathbf{v}_2 \mid \ \ldots \ \mid \mathbf{v}_m \ )$ is consistent.

3.4 Subspaces

Definition

The set of solutions to a linear system $\mathbf{Ax} = \mathbf{b}$ can be expressed **implicitly** as

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{b}\}$$

or **explicitly** as

$$V = \{\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \mid s_1, s_2, \ldots, s_k \in \mathbb{R}\},$$

where $\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,\ s_1, s_2, \ldots, s_k \in \mathbb{R}$ is the general solution.

Definition

A subset V of $\mathbb{R}^n$ is a **subspace** if it satisfies the following properties.

1. V **contains the zero vector**, $\mathbf{0} \in V$.
2. V is **closed under scalar multiplication**. For any vector. v in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.
3. V is **closed under addition**. For any vectors **u, v** in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

Property (i) can be replaced with property (i'): V is **nonempty**.

Properties (ii) and (iii) is equivalent to property (ii'): V is **closed under linear combination**. For any **u, v** in V, and scalars α, β, the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is in V.

Theorem

(Solution set of a homogeneous system is a subspace)

The solution set $V = \{\mathbf{u} \mid \mathbf{Au} = \mathbf{b}\}$ to a linear system $\mathbf{Ax} = \mathbf{b}$ is a **subspace** if and only if $\mathbf{b} = \mathbf{0}$, that is, the system is **homogeneous**.

Definition

The **solution set** to a **homogeneous system** is call a **solution space.**

Theorem

(Subspaces are equivalent to linear spans)

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span, $V = span(S)$, for some finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

Check if a set is a subspace

To show that a set V is a subspace, we can either

- find a spanning set, that is, find a set S such that $V = span(s)$, or
- show that V satisfies the 3 conditions of being a subspace.

To show that a subset V is not a subspace, we can either

- show that it does not contain the zero vector, $\mathbf{0} \notin V$,
- find a vector $\mathbf{v} \in V$ and a scalar $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin V$, or
- find vectors $\mathbf{u}, \mathbf{v} \in V$ such that the sum is not in V, $\mathbf{u} + \mathbf{v} \notin V$.

Theorem

(Affine spaces)

The solution set $W = \{ \mathbf{w} \mid \mathbf{Aw} = \mathbf{b} \}$ of a non-homogeneous linear system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq 0$, is given by

$$\mathbf{u} + V := \{ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \}$$

where $V = \{ v \mid \mathbf{Av} = \mathbf{0} \}$ is the solution space to the associated homogeneous system and $\mathbf{u}$ is a particular solution, $\mathbf{Au} = \mathbf{b}$.

That is, vectors in $\mathbf{u} + V$ are of the form $\mathbf{u} + \mathbf{v}$ for some $\mathbf{v}$ in V.

3.5 Linear Independence

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is linearly independent if the only coefficients $c_1, c_2, \dots, c_k \in \mathbb{R}$ satisfying the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, we say that the set is linearly dependent.

Algorithm to Check for Linear Independence

Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is linearly independent if and only if the homogeneous system $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{pmatrix}\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The homogeneous system has only the trivial solution if and only if the reduced row-echelon form of $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{pmatrix}$ has no non-pivot column.

Theorem

(Solution set of a homogeneous system is a subspace)

A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ of $\mathbb{R}^n$ is linearly independent if and only if the reduced row-echelon form of $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{pmatrix}$ has no non-pivot columns.

3.6 Basis and Coordinates

Definition

Let V be a subspace of $\mathbb{R}^n$. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a **basis** for V if

- $span(S) = V$ and
- S is linearly independent.

Theorem

Suppose $S\ \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for V. Then every vector $\mathbf{v}$ in the subspace V can be written as a linear combination of vectors in S uniquely.

Theorem

(Basis for Solution Set of Homogeneous System)

Let $V = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{0}\}$ be the solution space to some homogeneous system. Suppose

$$s_1\mathbf{u}_1 + s_1\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k, s_1, s_2, \ldots, s_k \in \mathbb{R}$$

is a general solution to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.
Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for the subspace $V = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{0}\}$.

Theorem

Basis for the zero space $\{\mathbf{0}\}$ of $\mathbb{R}^n$ is the empty set $\{\}$ or $\emptyset$.

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns are linearly independent.

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns spans $\mathbb{R}^n$.

Corollary

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of $\mathbb{R}^n$ containing n vectors. Then S is linearly independent if and only if S spans $\mathbb{R}^n$.

Corollary

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of $\mathbb{R}^n$ and $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k \end{pmatrix}$ be the matrix whose columns are vectors in S. Then S is a basis for $\mathbb{R}^n$ if and only if $k = n$ and $\mathbf{A}$ is an invertible matrix.

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns of $\mathbf{A}$ form a basis for basis for $\mathbb{R}^n$.

Theorem

A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the row of $\mathbf{A}$ form a basis for basis for $\mathbb{R}^n$.

Theorem

(Equivalent Statements for Invertibility)

Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.

1. $\mathbf{A}$ is invertible.
2. $\mathbf{A}^T$ is invertible.
3. $\mathbf{A}$ has a left-inverse, that is, there is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.
4. $\mathbf{A}$ has a right-inverse, that is, there is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.
5. The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.
6. $\mathbf{A}$ can be expressed as a product of elementary matrices.
7. The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
8. For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ is consistent.
9. The determinant of $\mathbf{A}$ is nonzero, $\det(\mathbf{A}) \neq 0$.
10. The columns/rows of $\mathbf{A}$ are linearly independent for $\mathbb{R}^n$.
11. The columns/rows of $\mathbf{A}$ spans $\mathbb{R}^n$.

Definition

(Coordinates Relative to a Basis)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for a subspace V of $\mathbb{R}^n$.
Then given any vector $\mathbf{v} \in V$, we can write $\mathbf{v}$ unique as

$$c_1\mathbf{u}_1, c_2\mathbf{u}_2, \ldots, c_k\mathbf{u}_k.$$

The coordinates of $\mathbf{v}$ relative to the basis S is defined to be the vector

$$[\mathbf{v}]_s = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Algorithm for Computing Relative Coordinate

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u_k}\}$ be a basis for a subspace V of $\mathbb{R}^n$.

For $\mathbf{v} \in V$, find real numbers $c_1, c_2, \ldots, c_k \in \mathbb{R} \in \mathbb{R}$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v}.$$

That is, we are solving for

$$(\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_k \mid \mathbf{v}).$$

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V.

1. For any vectors $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$.
2. For any $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in V$,

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B.$$

3.7 Dimensions

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$. Let $v_1, v_2, \dots, v_k$ be vectors in V. Then

1. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is linearly independent if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \dots, [\mathbf{v}_k]_B$ is linearly independent (respectively, dependent) in $\mathbb{R}^k$; and
2. $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ spans V if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \dots, [\mathbf{v}_k]_B$ spans $\mathbb{R}^k$.

Corollary

Let V be a subspace of $\mathbb{R}^n$ and V a basis for B. Suppose B contains k vectors, $|B| = k$.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_m\}$ is a subset of V with $m > k$, then S is linearly dependent.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_m\}$ is a subset of V with $m < k$, then S cannot span V.

Corollary

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Then $k = m$.

Definition

Let V be a subspace of $\mathbb{R}^n$. The dimension of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

Theorem

(Dimension of solution space)

Let $\mathbf{A}$ be a $m \times n$ matrix. The number of non-pivot columns in the reduced row-echelon form of A is the dimension of the solution space

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = \mathbf{0}\}.$$

Theorem

(Spanning Set Theorem)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = span(S)$. Suppose V is not the zero space, $V \neq \{\mathbf{0}\}$. Then there must be a subset of S that is a basis for V.

Theorem

(Linear Independence Theorem)

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

Theorem

Let U and V be subspaces of $\mathbb{R}^n$.

1. If $U \subseteq V$, then $dim(U) \leq dim(V)$.
2. If $U \subseteq V$, and $U \neq V$, then $dim(U) < dim(V)$ That is, $U \subseteq V$, then $dim(U) \leq dim(V)$ with equality if and only if $U = V$.

Theorem

(B1)
Let V be a k-dimensional subspace of $\mathbb{R}^n$, $dim(V) = k$. Suppose $U \subseteq V$ is a linearly independent subset containing k vectors, $|S| = k$. Then S is a basis for V.

1. $|S| = dim(V)$
2. $S \subseteq V$
3. S is linearly independent

Theorem

(B2)
Let V be a k dimensional subspace of $\mathbb{R}^n$, $dim(V) = k$. Suppose S is a set containing k vectors, $|S| = k$, such that $V \subseteq span(S)$. Then S is a basis for V.

1. $|S| = dim(V)$
2. $V \subseteq span(S)$

3.8 Transition Matrices

Definition

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \dots, u_k\}$ and $T = \{v_1, \dots, v_k\}$ are **basis** for the subspace V. Define the transition matrix from T to S to be

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \dots & [\mathbf{v}_k]_S \end{pmatrix},$$

the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

Theorem

(Transition Matrix)

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ are **bases** for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector w in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Algorithm to find Transition Matrix

Let $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ be a basis for a subspace V in $\mathbb{R}^n$. To find $\mathbf{P}$, the transition matrix from T to S,

$$(``S"|``T") = (u_1 \quad u_2 \quad \ldots u_k \quad | \quad v_1 \quad v_2 \quad \ldots v_k) \xrightarrow{\text{rref}} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array}\right)$$

Theorem

(Inverse of Transition Matrix)

Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are bases for a subspace V of $\mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then P^{-1} is the transition matrix from S to T.

Chapter 4: Subspaces Associated to a Matrix

4.1 Column Space, Row Space, and Nullspace

4.2 Rank