MA1522 Notes (AY24/25 Sem1)

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Chapter 2: Matrix Algebra

2.1 Definition and Special types of Matrices

2.2 Matrix Algebra

2.3 Linear System and Matrix Equation

2.4 Inverse of Matrices

2.5 Elementary Matrices

2.6 Equivalent Statements for Invertibility

2.7 LU Factorization

2.8 Determinant by Cofactor Expansion

2.9 Determinant by Reduction

2.10 Properties of Determinant

Chapter 3: Euclidean Vector Spaces

3.1 Euclidian Vector Spaces

Definition

A (real) n-**vector** is a collection of n ordered real numbers,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, where\ v_i \in \mathbb{R}\ for\ i = 1, \dots, n.$$

The real number v_i is called the i-th coordinate of the vector $\mathbf{v}$. The **Euclidiean n-space**, denoted $\mathbb{R}^n$, is the collection of all n-vectors

$$\mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R}\ for\ i = 1, \dots, n. \right\}$$

Properties of Vector Addition and Scalar Multiplication

Since vectors are matrices (column vectors are $n \times 1$ matrices and row vectors are $1 \times n$ matrices), the properties of matrix addition and scalar multiplication holds for vectors. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars $a, b \in \mathbb{R}$,

1. The sum $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$
2. (Commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. (Associative) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. (Zero vector) $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
5. The negative $-\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{v} - \mathbf{v} = \mathbf{0}$.
6. (Scalar multiple) $a\mathbf{v}$ is a vector in $\mathbb{R}^n$.
7. (Distribution) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. (Distribution) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
9. (Associativity of scalar multiplication) $(ab)\mathbf{u} = a(b\mathbf{u})$.
10. If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.

Definition

A **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in \mathbb{R}^n$ is $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ for some $c_1, c_2, \dots, c_k \in \mathbb{R}^k$.

Definition

A set V equipped with **addition** and **scalar multiplication** is said to be a **vector space** over $\mathbb{R}$ if it satisfies the following axioms.

1. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.
2. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.
5. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.
7. (Distribution) For any scalar a in $\mathbb{R}$ and vector $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
9. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.
10. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$.

3.2 Dot Product, Norm, Distance

Definition

The inner product (or dot product) of vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$ in $\mathbb{R}^n$ is defined to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2, + \cdots + u_n v_n.$$

Define the **norm** of a vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_i)$, to be the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $||\mathbf{u}||$,

$$||u|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

This is also known as the **length** or **magnitude** of the vector.

Properties of inner product and norm

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors and $a, b, c \in \mathbb{R}$ be real numbers.

1. Inner product is **symmetric**,
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
2. Inner product **commutes** with scalar multiple,
$$c\mathbf{u} \cdot \mathbf{v} = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}).$$
3. Inner product is **distributive**,
$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$$
4. Inner product is **positive definite**, $\mathbf{u} \cdot \mathbf{u} \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.
5. $||c\mathbf{u}|| = |c| \; ||\mathbf{u}||$.

Definition

A vector $\mathbf{u} \in \mathbb{R}^n$ is a **unit vector** if its norm is 1,
$$||\mathbf{u}|| = 1$$

Normalizing a vector
Let $\mathbf{u}$ be a nonzero vector $\mathbf{u} \neq \mathbf{0}$. By multiplying by the reciprocal of the norm, we get a unit vector,
$$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{||\mathbf{u}||}$$
This is called **normalizing u**.

Definition

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $d(\mathbf{u}, \mathbf{v})$, is defined to be
$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$
Define the angle θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that
$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||}$$

3.3 Linear Combinations and Linear Spans

Definition

A linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k, \text{ for some } c_1, c_2, \ldots c_k \in \mathbb{R}.$$

The scalars $c_1, c_2, \ldots c_k$ are called **coefficients**.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The **span** (or **linear span**) of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Algorithm to Check for Linear Combination

3.4 Subspaces

3.5 Linear Independence

3.6 Basis and Coordinates

3.7 Dimensions

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$. Let $v_1, v_2, \ldots, v_k$ be vectors in V. Then

1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ is linearly independent (respectively, dependent) in $\mathbb{R}^k$; and
2. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ spans $\mathbb{R}^k$.

Corolalry

Let V be a subspace of $\mathbb{R}^n$ and V a basis for B. Suppose B contains k vectors, $|B| = k$.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m > k$, then S is linearly dependent.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m < k$, then S cannot span V.

Corolalry

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Then $k = m$.

Definition

Let V be a subspace of $\mathbb{R}^n$. The **dimension** of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

Theorem

(Dimension of solution space) Let $\mathbf{A}$ be a $m \times n$ matrix. The **number of non-pivot columns** in the reduced row-echelon form of A is the **dimension** of the solution space

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{0}\}.$$

Theorem

(Spanning Set Theorem) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = span(S)$. Suppose V is not the zero space, $V \neq \{\mathbf{0}\}$. Then there must be a subset of S that is a basis for V.

Theorem

(Linear Independence Theorem) Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

3.8 Transition Matrices

Definition

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are **basis** for the subspace V. Define the **transition matrix** from T to S to be

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \ldots & [\mathbf{v}_k]_S \end{pmatrix},$$

the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

Theorem

(Transition Matrix) Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ are **bases** for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector w in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Algorithm to find Transition Matrix

Let $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ be a basis for a subspace V in $\mathbb{R}^n$. To find $\mathbf{P}$, the transition matrix from T to S,

$$(\text{``}S\text{''} | \text{``}T\text{''}) = \left(\begin{array}{cccc|cccc} u_1 & u_2 & \ldots u_k & & v_1 & v_2 & \ldots v_k \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array}\right)$$

Theorem

(Inverse of Transition Matrix) Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are **bases** for a subspace V of $\mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then P^{-1} is the transition matrix from S to T.

Chapter 4: Subspaces Associated to a Matrix

4.1 Column Space, Row Space, and Nullspace

4.2 Rank