Dot Product: $a \cdot b = \| a \| \| b \| \cos\theta$. Orthogonal iff $a \cdot b = 0$.
Component of b along a: $\text{comp}_a b = \| b \| \cos\theta = \frac{a \cdot b}{b}$
Projection of b onto a: $proj_a b = \frac{a \cdot b}{a \cdot a} a$
Cross Product: $a \times b = |a||b| \sin\theta$. (1) $a \times b = -b \times a$, (2) $a \times (b + c) = a \times b + a \times c$, (3) $(a + b) \times c = a \times c + b \times c$
Area of Parallelogram: $||a \times b||$
Distance of Q to line through P and R: $|\vec{PQ}| \sin\theta = \frac{|\vec{PQ} \times \vec{PR}|}{|\vec{PR}|}$
Scalar Triple Product: $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
Volume of Parallelepiped: $V = |a \cdot (b \times c)|$ (b and c form the base)
Equation of Line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
Equations of Plane: $ax + by + cz = d$, $n \cdot (r - r_0) = 0$, $n \cdot r = n \cdot r_0$
Tangent Vector: $r'(a) = \langle f'(a), g'(a), h'(a) \rangle$. (1) $\frac{d}{dt}(r(t) + s(t) = r'(t) + s'(t)$. (2) $\frac{d}{dt}(cr(t)) = cr'(t)$. (3) $\frac{d}{dt} f(t)r(t) = f'(t)r(t) + f(t)r'(t)$. (4) $\frac{d}{dt} r(t) \cdot s(t) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$. (5) $\frac{d}{dt} r(t) \times s(t) = r'(t) \times s(t) + r(t) \times s'(t)$
Arc Length (only for smooth curves): $s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b ||r'(t)|| dt$

Cylinder: $x^2 + y^2 = k$
Elliptic Paraboloid (cup): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Hyperbolic Paraboloid (saddle): $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Ellipsoid (sphere-like): $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Elliptic Cone (two cups): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid of Two Sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$
Limit: does not exist if there exists two different approaches which give different values. To prove it exists, use squeeze theorem or use properties of limits.
Continuous: If $\lim f(x, y) = f(a, b)$ as $(x, y) \to (a, b)$ then f is continuous $f \pm g$. If f and g are continuous, $f \cdot g$, $\frac{f}{g}$, $h(x) = f(g(x))$ are continuous. Polynomial, trigonometric, exponential and rational functions are continuous in their domain.
Squeeze Theorem: If $|f(x, y) - L| \le g(x, y)$ for all (x, y) except possibly at (a, b) and $\lim g(x, y) = 0$ as $(x, y) \to (a, b)$, then $\lim f(x, y) = L$.

Partial Derivative: $f_x(x, y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$ (similarly for y)
Clairaut's Theorem: $f_{xy}(a, b) = f_{yx}(a, b)$. As long as the no. of occurrences of the variable stays the same
Tangent Plane: Normal Vector $\langle f_x(a, b), f_y(a, b), -1 \rangle$, Equation: $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
Differentiable: Tangent plane at (a, b) is a good approximation of f at points close to (a, b).
Linear Approximation: $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where ϵ_1, ϵ_2 are functions of Δx and Δy and $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$
Implicit Differentiation: $F(x, y, z) = 0$, $\frac{\partial z}{\partial x} = 0\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, $F_z \ne 0$
Directional Derivative: of $f(x, y)$ in direction of unit vector $\vec{u} = \langle a, b \rangle$ is $D_u f(x, y) = \langle f_x, f_y \rangle \cdot \vec{u}$ (can be extended to 3D)
Gradient: $\nabla f = \langle f_x, f_y \rangle$

Normal to Level Curve: $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = k$ that contains (x_0, y_0)
Tangnet Plane to Level Surface: $\nabla F(x_0, y_0, z_0)$ is normal to the level surface $F(x, y, z) = k$. Equation: $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Maximum Rate of Increase/Decrease: $D_u f = \nabla f \cdot u = |\nabla f| \cos\theta$. Max at $\theta = 0$, min at $\theta = \pi$
Critical Point: $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Local min/max implies critical point (but not vice versa)
Saddle Point: (a, b) is a critical point and there are points $f(x, y) > f(a, b)$ and $f(x, y) < f(a, b)$
Closed Set: $R \subseteq \mathbb{R}^2$ contains all its boundary points
Bounded Set: Finite in extent (contained in a disk)
Extreme Value Theorem: If $f(x, y)$ is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then there exists absolute max/min in D. To find abs max/min: (1) check critical points, (2) check extreme values on boundaries
Second Derivative Test: $D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$. (1) $D > 0$, $f_{xx} > 0$: local min. (2) $D > 0$, $f_{xx} < 0$: local max. (3) $D < 0$: saddle. (4) $D = 0$: min, max or saddle
Lagrange Multiplier: Min/max values of $f(x, y)$ subject to constraint curve $g(x, y) = k$. (1) Find all values of x, y such that $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = k$. (2) The largest is the max of f, the smallest is min

---

Double Integral over Type I Domain: $\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy\, dx$
Type II: $\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx\, dy$
E.g. Find the volume/Evaluate $\iint_D (\dots) dA$ where D is the region bounded by …
Additivity of Regions: $\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \dots + \iint_{D_n} f(x, y) dA$
Area of Plane Region: $A(D) = \iint_D 1 dA$

Double Integral in Polar Coordinates $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$ Replace x, y in $f(x, y)$, integrate over $r\, dr\, d\theta$. General region: $h_1(\theta) \le r \le h_2(\theta)$ or $g_1(r) \le \theta \le g_2(r)$, integrate these out first
Triple Integral Type 1/2/3: Convert to double integral
Volume of Solid: $\iiint_V 1 dV$
Rewrite Order of Integral: Sketch the graph first, then express boundary curves in terms of the variable to integrate over
Cylindrical Coordinates: Polar coordinate + z-coordinate. $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$, $z = z$. Replacements: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(\dots) r\, dz\, dr\, d\theta$$

Spherical Coordinates: θ (angle on xy-plane), ϕ (angle from $+$ve z axis), p (distance). $p = x^2 + y^2 + z^2$. Use for triple integrals for spheres or cones. Replacements: $x = p\sin\phi\cos\theta$, $y = p\sin\phi\sin\theta$, $z = p\cos\phi$.

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\dots) p^2 \sin\phi\, dp\, d\theta\, d\phi = \int_c^d \int_\alpha^\beta \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\dots) p^2 \sin\phi\, dp\, d\theta\, d\phi$$

Plane Transformation: from uv-plane to xy-plane, $x = x(u, v)$, $y = y(u, v)$, $T : (u, v) \to (x, y)$. To find the image, check along the boundaries. E.g. $x = u^2 - v^2$, $y = 2uv$, $\{(u, v) | 0 \le u \le 1, 0 \le v \le 1\}$
2D Jacobian: $x = x(u, v)$, $y = y(u, v)$, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$. $dA = |\frac{\partial(x,y)}{\partial(u,v)}| du\, dv$. $\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) |\frac{\partial(x,y)}{\partial(u,v)}| du\, dv$
E.g. Use change of variables $x = \dots, y = \dots$ to compute $\iint_R (x^2 + y^2) dA$

3D Jacobian: $dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}| du\, dv\, dw$

$$\Big|\frac{\partial(x, y, z)}{\partial(u, v, w)}\Big| = \frac{\partial x}{\partial u}(\frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial v}) - \frac{\partial x}{\partial v}(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial u}) + \frac{\partial x}{\partial w}(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial u})$$

E.g. Find volume of solid bound by $x + y + z = 1$, $x + y + z = 2$, $x + 2y = 0$, $x + 2y = 1$. (Let $x = u - w$, $y = \dots$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, compute integral with new bounds)

Line Integral of Scalar Field: $\int_C f(x, y) dS = \int_a^b f(x(t), y(t)) ||r'(t)|| dt$. Line integral of scalar field is independent of orientation of $r(t)$. Can be extended into 3D.
E.g. Evaluate $\int_C f(\dots) dS$ where C consists of the arc C_1, C_2 from …
Work Done by Force Field (Line Integral of Vector Field): $W = \int_C F(x, y, z) \cdot T(x, y, z) dS = \int_a^b F(x(t), y(t), z(t)) \cdot r'(t) dt = \int_C F \cdot dr$. $T = \frac{r'(t)}{||r'(t)||}$ is the unit tangent vector, $\frac{dS}{dt} = ||r'(t)||$
Value of work done depends on orientation. $\int_C F \cdot dr = -\int_{-C} F \cdot dr$
Component Form: $\int_C F \cdot dr = \int_C \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt = \int_C P dx + \int_C Q dy + \int_C R dz$
Union of Curves: $\int_C F \cdot dr = \int_{C_1} F \cdot dr + \dots + \int_{C_n} F \cdot dr$
Fundamental Theorem for Line Integrals: If $F = \nabla f$ for a scalar function f, F is a conservative vector field, f is the potential function, and the line integral of a conservative vector field can be evaluated knowing only f at the endpoints.
$\int_C \nabla f \cdot dr = f(x(b), y(b), z(b)) - f(x(a), y(a), b(a))$
Test for Conservative Vector Field in the Plane: If $F(x, y) = P(x, y)i + Q(x, y)j$ in an open and simply-connected region D, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ at each point in D iff F is conservative on D. Also can just integrating ∇f to try and obtain F to show they are equal.
To prove a field is not conservative, show there exists two paths with the same start and end points but different line integral values.
Test for Conservative Field in Space: F is conservative on D iff $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all points in D

Positive Orientation: Single counterclockwise traversal of C.
Green's Theorem: Only applies where F is a two-dimensional vector field and C is a piecewise smooth, simple closed curve with positive orientation. $\int_C F \cdot dr = \int_C P dx + Q dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$.
Notation: $\oint_C F \cdot dr$ indicates the integral is calculated using the positive orientation.
Green's Theorem relates a line integral around a simple closed curve C with a double integral over the plane region D.
E.g. Evaluate $\int_C \langle \dots \rangle dr$ where C is the curve consisting of line segments...
Area of Plane Region: $A = \int_C x\, dy = -\int_C y\, dx = \frac{1}{2} \int_C x\, dy - y\, dx$
Parametric Surface: The set of all points (x, y, z) in $\mathbb{R}^3$ such that $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ as (u, v) varies through D. $r(u, v) = \langle x(u, v), y(u, v) z(u, v) \rangle$ is a parameterization of S.
Surface Integral of Scalar Field:

$$\iint_S f(x, y, z) dS = \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

$$\Delta S_{ij} \approx \| r_u \times r_v \| \Delta u \Delta v, \quad dS = \| r_u \times r_v \| du\, dv$$

Smooth Surface: A surface is smooth if it has parameterization $r(u, v)$ such that r_u and r_v are continuous and $r_u \times r_v \neq 0$ for all points in D.

Tangent Plane of Smooth Surface

For $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $r_u(a,b) \times r_v(a,b)$ is normal to the tangent plane at $\langle x(a,b), y(a,b), z(a,b)\rangle$. The tangent plane can be approximated by $\| r_u \times r_v \| \, dvdu$.

Surface Integral of Scalar Field Formula: $\iint_S f(x,y,z)dS = \iint_S f(x(u,v), y(u,v), z(u,v))||r_u \times r_v||dA$

E.g. Evaluate $\iint_S z dS$, where S is the surface whose sides are ..., bottom lies above ...

Union of Smooth Surfaces: $\iint_S f(x,y,z)dS = \iint_{S_1} f(x,y,z)dS + \cdots + \iint_{S_n} f(x,y,z)dS$

Surface Integral of Scalar Field with parameterization using function of two variables: Suppose S is given by $z = g(x,y)$, then a parameterization is $r(u,v) = r(x,y) = \langle x, y, g(x,y)\rangle$. Then $\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\frac{\partial g}{\partial x}^2 + \frac{\partial g}{\partial y}^2 + 1}\, dA$

Surface Area: $Area(S) = \iint_S 1 dS = ||r_u \times r_v||dA$ where $r_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ and $r_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$

E.g. find the surface area of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

E.g. find the surface area of the intersection of $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$. Parameterize $y^2 + z^2 = 1$ first: $x = x$, $y = \cos\theta$, $z = \sin\theta$. Then solve $x^2 + z^2 \leq 1 \rightarrow |x| < |\cos\theta|$. Then let $r(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$, find r_x and r_θ. Then $A = \int_0^{2\pi}\int_{-|\cos\theta|}^{|\cos\theta|} 1\, dx\, d\theta$

Orientable Surface: if it is possible to define a unit normal vector n at each point (x,y,z) not on the boundary of the surface such that n is a continuous function of (x,y,z) (has a top/bottom, inside/outside). $n = \frac{r_u \times r_v}{||r_u \times r_v||}$.

Special case when $z = g(x,y)$, $n = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+g_x^2+g_y^2}}$, the k component is positive, giving the upward orientation. The downward orientation is $-n$.

Positive Orientation for Closed Surface: For a closed surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E. Inward pointing normals give the negative orientation.

Surface Integral of Vector Field: Flux of F across $S = \iint_S F \cdot d\mathbf{S} = \iint_S F \cdot n\, dS$

Formula for Surface Integral of Vector Field: $\iint_S F \cdot d\mathbf{S} = \iint_D F \cdot (r_u \times r_v)dA$. Check that S is traced out by $r(u,v)$ and the orientation n is correct.

Special case when $r(x,y) = \langle x, y, g(x,y)\rangle$, then $\iint_S F \cdot d\mathbf{S} = \iint_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)\, dA$. This assumes upward orientation of S. For downward orientation, multiply by -1.

E.g. Evaluate $\iint_S \langle y, x, z\rangle d\mathbf{S}$ where S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$, and S has positive orientation. (Split into S_1 and S_2, compute the flux across each and sum)

Divergence: div $\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$

Divergence is positive $\rightarrow$ Net outflow

Divergence is negative $\rightarrow$ Net inflow

Divergence = Flux / Volume, i.e. flux density. It is a number.

Divergence / Gauss Theorem

Flux over $S = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E$ div $\mathbf{F}\, dV$.

E is piecewise smooth with positive (outward) orientation.

Flux is the sum of divergence over the volume V.

E.g. Find the flux of $F(x,y,z) = zi + yj + xk$ across the unit sphere $x^2 + y^2 + z^2 = 1$ with positive orientation.

div $F(P) > 0$: net outflow at P. div $F(P) < 0$: net inflow at P. div $F(P) = 0$: no net flow at P.

Curl: curl $F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})i + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})j + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})k = \nabla \times F$. Unlike Divergence, curl is a vector field.

Stoke's Theorem:

$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)d\mathbf{A}$

Stoke's Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve)

E.g. Evaluate $\int_C F \cdot dr$ where C is the intersection of the plane $y + z = 2$ and the cylinder ...