Vectors

Length of Vector: $||u|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.
Dot Product of Two Vectors: $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$ where $a = \langle a_1, a_2, a_3\rangle$ and $b = \langle b_1, b_2, b_3\rangle$
Dot Product Angle Formula: $a \cdot b = ||a||\ ||b||cos\theta$. Two non-zero vectors a and b are **orthogonal** iff $a \cdot b = 0$. They have the same direction if $\theta = 0$, opposite direction if $\theta = \pi$, perpendicular if $\theta = \frac{\pi}{2}$.
Projection: $\text{comp}_a b = ||b||cos\theta = \frac{a\cdot b}{||a||}$. $\text{proj}_a b = \text{comp}_a b \times \frac{a}{||a||} = \frac{a\cdot b}{a\cdot a}a$
Cross Product: $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$. The vector $a \times b$ is orthogonal to both a and b. We can use cross product to find the **area of a parallelogram**, or to find the **distance from a point to a line** in $\mathbb{R}^3$.
Properties of cross product: If a, b and c are vectors and d is a scalar, then
(i) $a \times b = -b \times a$
(ii) $(da) \times b = d(a \times b) = a \times (db)$
(iii) $a \times (b + c) = a \times b + a \times c$
(iv) $(a + b) \times c = a \times c + b \times c$
Cross product angle formula: $||a \times b|| = ||a||\ ||b||sin\theta$.
Distance from Q to line through P and R:

$$||PQ||sin\theta = \frac{||PQ \times PR||}{||PR||}$$

Scalar Triple Product: $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. If θ is the angle between a and $b \times c$, then the height h of the parallelepiped is $h = ||a|| \cdot |cos\theta|$ and the volume of the parallelepiped is $V = |a \cdot (b \times c)|$. The area of the base parallelogram is $A = ||b \times c||$
Find if vectors are coplanar: Check if the volume of the parallelepiped determined by the vectors is equal to 0: $|a \cdot (b \times c)| = 0$
Parametric Equation of Line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
Vector Equation of Plane: $n \cdot (r - r_0) = n \cdot \langle x - a, y - b, z - c\rangle = 0$ or $n \cdot r = n \cdot r_0$
Linear Equation of Plane: $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.
Parallel Planes: Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line. The angle between the two planes is the angle θ between their normal vectors.
Derivative of Vector-valued Function: Let $r(t) = \langle f(t), g(t), h(t)\rangle$ and suppose that the components f, g and h are all differentiable at $t = a$. Then r is differentiable at $t = a$ and its derivative is given by $r'(a) = \langle f'(a), g'(a), h'(a)\rangle$.
Derivative Rules: Suppose $r(t)$ and $s(t)$ are differentiable vector-valued functions, $f(t)$ is a differentiable scalar function and c is a scalar constant. Then
$\frac{d}{dt}(r(t) + s(t)) = r'(t) + s'(t)$
$\frac{d}{dt}(cr(t)) = cr'(t)$
$\frac{d}{dt}(f(t)r(t)) = f'(t)r(t) + f(t)r'(t)$
$\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
$\frac{d}{dt}(r(t) \times s(t)) = r'(t) \times s(t) + r(t) \times s'(t)$
Arc Length Formula: Let C be the curve given by $r(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$ where f', g' and h' are continuous. If C is traversed exactly once as t increases from a to b, then its length is $s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2 dt} = \int_b^a ||r'(t)||dt$. This only applies for **smooth** curves.

Surfaces

Level Curve: $f(x, y) = k$
Contour Plots: Numerous $f(x, y) = k$
Cylinders: A surface is a cylinder if there is a plane P s.t. all planes parallel to P intersect the surface in the same curve.
Quadric Surface: $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
Elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ (symmetric about z-axis)
Hyperbolic paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Elliptic cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid of two sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$
Level Surface: $f(x, y, z) = k$

Limits and Continuity

Definition of Limit: $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for any $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x, y) - L| < \epsilon$ when $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$
Show limit does not exist: If $f(x, y)$ approaches L_1 along path P_1 and L_2 along path P_2 and $L_1 \neq L_2$ then $lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.
Limit Theorems: Suppose $f(x, y)$ and $g(x, y)$ both have limits as (x, y) approaches (a, b). Then $\lim(f(x, y) \pm g(x, y)) = \lim f(x, y) \pm \lim g(x, y)$, $\lim f(x, y)g(x, y) = (\lim f(x, y))(\lim g(x, y))$, and $\lim \frac{f(x,y)}{g(x,y)} = \frac{\lim f(x,y)}{\lim g(x,y)}$ provided $\lim g(x, y) \neq 0$
Squeeze: Suppose $|f(x, y) - L| \le g(x, y)$ for all (x, y) in the interior of some circle centered at (a, b), except possible at (a, b). If $lim_{(x,y)\to(a,b)} g(x, y) = 0$, then $\lim_{(x,y)\to(a,b)} f(x, y) = L$
Definition of Continuity: f is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. This is the **substitution** property. If $f(x, y)$ is not continuous at (a, b), then we call (a, b) a discontinuity (point) of f. f is said to be continuous on $D \subseteq \mathbb{R}^2$ if f is continuous at each point in D.
Continuity Theorems: If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then $f \pm g$, $f \cdot g$ are all continuous at (a, b). Further, $\frac{f}{g}$ is continuous at (a, b), provided $g(a, b) \neq 0$. Polynomial, Trigonometric, Exponential and Rational functions in x and y are continuous **in its domain**.
Continuity and Composition: Suppose $f(x, y)$ is continuous at (a, b) and $g(x)$ is continuous at $f(a, b)$. Then $h(x, y) = (g \circ f)(x, y) = g(f(x, y))$ is continuous at (a, b).

Partial Derivatives

Partial Derivative: If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by $f_x(x, y) = \lim_{h\to0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}$ and $f_y(x, y) = \lim_{h\to0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f}{\partial y}$
Clairaut's Theorem: Suppose f is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$. So long as the number of the same variable occurring in the subscript are the same, the corresponding partial derivatives are the same. E.g. $f_{xxyyzz} = f_{xyzxyz}$.
Equation of Tangent Plane: Suppose $f(x, y)$ has continuous first partial derivatives at (a, b). A normal vector to the tangent plane is $\langle f_x(a, b), f_y(a, b), 1\rangle$. Further, an equation of the tangent plane is given by

$$f_x(a, b)(x-a) + f_y(a, b)(y-b) - (z - f(a, b)) = 0$$

or $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.
Increment: Let $z = f(x, y)$. Suppose Δx and Δy are increments in the independent variable x and y respectively from a fixed point (a, b). Then the increment in z at (a, b) is defined by

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Differentiable - Two Variable: Let $z = f(x, y)$. We say that f is differentiable at (a, b) if the tangent plane at (a, b) is a **good** approximation to f at points close to **(a, b)**. Formally, f is differentiable if we can write

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy and $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. We say that f is differentiable on a region $R \subseteq \mathbb{R}^2$ if f is differentiable at every point in R.
Linear Approximation: Suppose $z = f(x, y)$ is differentiable at (a, b). Let Δx and Δy be small increments in x and y respectively from (a, b). Then $\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$. I.e. if Δx, Δy are small, then, provided $f(x, y)$ is differentiable, $f(a + \Delta x, b + \Delta y) \approx$

$$f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

As Δx, $\Delta y \to (0,0)$, tangent plane gets closer to the surface

Useful Facts:

f_x and f_y continuous $\Rightarrow f$ differentiable

f_x and f_y continuous $\not\Leftarrow f$ differentiable

f differentiable $\Rightarrow f$ continuous

f differentiable $\not\Leftarrow f$ continuous

f_x and f_y exist $\not\Rightarrow f$ differentiable

f_x and f_y exist $\Leftarrow f$ differentiable

Chain Rule - General Version: Suppose that u is a differentiable function of n variables $x_, \dots, x_n$, and each x_j is a differentiable function of m variables $t_1, \dots, t_m$. Then u is a function of $t_1, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, \dots, m$

Implicit Differentiation - Two Independent Variables: Suppose the equation $F(x,y,z) = 0$, where F is differentiable, defines z **implicitly** as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

provided $F_z(x,y,z) \neq 0$.

Directional Derivative: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of unit vector $u = \langle a, b\rangle$ is $D_u f(x_0,y_0) =$

$$\lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

provided this limit exists. This can be extended to 3 variables.

Computing Directional Derivative: If $f(x,y)$ is a differentiable function, then f has a directional derivative in the direction of any unit vector $u = \langle a,b\rangle$ and $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$. We can rewrite it in terms of vectors:

$$D_u f(x,y) = \langle f_x, f_y\rangle \cdot \langle a,b\rangle = \langle f_x, f_y\rangle \cdot u$$

Gradient: $\nabla f(x,y) = \langle f_x, f_y\rangle = fxi + fyj = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$ provided both partial derivatives exist. Thus $D_u f(x,y) = \nabla f(x,y)\cdot u$

Gradient

Level Curve/Surface vs ∇F: Suppose $f(x,y)$ is a differentiable function of x and y at (x_0,y_0) and $\nabla f(x_0,y_0) \neq 0$. Then $\nabla f(x_0,y_0) \neq 0$ is **normal to the level curve** $f(x,y) = k$ that contains the point (x_0,y_0). Similarly for $F(x,y,z)$, the gradient is normal to the level **surface** at (x_0,y_0,z_0).

Tangent Plane to Level Surface: $F(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$

Maximizing Rate of Increase/Decrease of F: Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P) \neq 0$. Let **u** be a unit vector making an angle θ with ∇f. Then $D_u f(P) = ||\nabla f(P)|| cos\theta$. $\nabla f(P)$ points in direction of **maximum** rate of change of f at P. $-\nabla f(P)$ points in direction of **minimum** rate of change of f at P.

Local extremum: If f has a local maximum or minimum at (a,b) and the first-order derivatives of f exist there, then $f_x(a,b) = f_y(a,b) = 0$.

Critical/Stationary Point: Let $f(x,y) : D \to \mathbb{R}$. Then a point (a,b) is called a **critical point** of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$. But being a critical point does not mean it is a local min/max.

Saddle Point: Let $f(x,y) : D \to \mathbb{R}$. A point (a,b) is called a **saddle point** of f if 1. it is a critical point, and 2. every open disk centered at (a,b) contains points $(x,y) \in D$ for which $f(x,y) < f(a,b)$ and points $(x,y) \in D$ for which $f(x,y) > f(a,b)$.

Second Derivative Test: Let $D = D(a,b) = f_{xx}(a,b) - f_{yy}(a,b) - [f_{xy}(a,b)]^2$. If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum. If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum. If $D < 0$ then (a,b) is a saddle point. If $D = 0$ then the point may be a min, max or saddle point.

Closed Set: A set $R \subseteq \mathbb{R}^2$ is **closed** if it contains all its boundary points. (A **boundary point** of R is a point (a,b) such that every disk with center (a,b) contains points in R and also points in $\mathbb{R}^2 \setminus R$).

Bounded Set: A set $R \subseteq \mathbb{R}^2$ is **bounded** if it is contained within some disk. In other words, it is finite in extent.

Extreme Value Theorem: If $f(x,y)$ is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$, AND an absolute minimum value $f(x_2,y_2)$ at some points $(x1,y1)$ and $(x2,y2)$ in D.

Lagrange Multiplier – Two Variables: Suppose $f(x,y)$ and $g(x,y)$ are differentiable functions such that $\nabla g(x,y) \neq 0$ on the constraint curve $g(x,y) = k$. Suppose that the minimum/maximum value of $f(x,y)$ subject to the constraint $g(x,y) = k$ occurs at (x_0,y_0). Then $\nabla f(x_0,y_0) = \lambda\nabla g(x_0,y_0)$ for some constant λ (called a **Lagrange Multiplier**).

Common Integrals

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq 1$

$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + c$

$\int \cos(u)du = \sin(u) + c$

$\int \sin(u)du = -\cos(u) + c$

$\int \sec^2(u)du = \tan(u) + c$

$\int \sec(u)\tan(u)du = \sec(u) + c$

$\int \tan(u)du = -\ln|cos(u)| + c = \ln|\sec(u)| + c$

$\int \csc(u)\cot(u)du = -\csc(u) + c$

$\int \csc^2(u)du = -\cot(u) + c$

$\int \cot(u)du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$

$\int \csc(u)du = \ln|\csc(u) - \cot(u)| + c$

$\int \sec(u)du = \ln|\sec(u) + \tan(u)| + c$

$\int \cot(u)du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$

$\int e^u du = e^u + c$

$\int a^u du = \frac{a^u}{\ln(a)} + c$

$\int \ln(u)du = u\ln(u) - u + c$

$\int ue^u du = (u-1)e^u + c$

$\int \frac{1}{u\ln(u)} du = \ln|\ln(u)| + c$

$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}(\frac{u}{a}) + c$

$\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + c$

$\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a}\sec^{-1}(\frac{u}{a}) + c$

$\int \sin^{-1}(u)du = u\sin^{-1}(u) + \sqrt{1-u^2} + c$

$\int \tan^{-1}(u)du = u\tan^{-1}(u) - \frac{1}{2}\ln(1+u^2) + c$

$\int \cos^{-1}(u)du = u\cos^{-1}(u) - \sqrt{1-u^2} + c$

$\int \frac{1}{a^2-u^2} du = \frac{1}{2a}\ln|\frac{u+a}{u-a}| + c$

$\int \frac{1}{u^2-a^2} du = \frac{1}{2a}\ln|\frac{u-a}{u+a}| + c$

$\int \sqrt{a^2+u^2}du =$
$\frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln|u + \sqrt{a^2+u^2}| + c$

$\int \sqrt{u^2-a^2}du =$
$\frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u + \sqrt{u^2-a^2}| + c$

$\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}(\frac{u}{a}) + c$

Integration by Parts: $\int udv = uv - \int vdu$.

LIATE (Order to differentiate):
Log/Inverse Trig/Algebraic/Trig/Exp

Partial Fractions: $\int \frac{P(x)}{Q(x)}dx$ where degree of $P(x)$ < degree of $Q(x)$

$$\frac{px+q}{(x-a)(x-b)}, a \neq b \to \frac{A}{x-a} + \frac{B}{x-b}$$

$$\frac{px+q}{(x-a)^2} \to \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\frac{px^2+qx+r}{(x-a)^2(x-b)} \to \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

$$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} \to \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

Common Derivatives

$\frac{d}{dx}a^x = a^x\ln(a)$

$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}, x > 0$

$\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

$\frac{d}{dx}\tan(x) = \sec^2(x)$

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\cot(x) = -\csc^2(x)$

$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$