MA1522 Notes (AY24/25 Sem1)

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Chapter 2: Matrix Algebra

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Chapter 3: Euclidean Vector Spaces

3.1 Euclidian Vector Spaces

> **Definition**
>
> A (real) n-**vector** is a collection of n ordered real numbers,
>
> $$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, where\ v_i \in \mathbb{R}\ for\ i = 1, \ldots, n.$$
>
> The real number v_i is called the i-th coordinate of the vector $\mathbf{v}$. The **Euclidiean** n-**space**, denoted $\mathbb{R}^n$, is the collection of all n-vectors
>
> $$\mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R}\ for\ i = 1, \ldots, n. \right\}$$

> **Properties of Vector Addition and Scalar Multiplication**
>
> Since vectors are matrices (column vectors are $n \times 1$ matrices and row vectors are $1 \times n$ matrices), the properties of matrix addition and scalar multiplication holds for vectors. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars $a, b \in \mathbb{R}$,
>
> 1. The sum $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$
> 2. (Commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
> 3. (Associative) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
> 4. (Zero vector) $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
> 5. The negative $-\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{v} - \mathbf{v} = \mathbf{0}$.
> 6. (Scalar multiple) $a\mathbf{v}$ is a vector in $\mathbb{R}^n$.
> 7. (Distribution) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
> 8. (Distribution) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
> 9. (Associativity of scalar multiplication) $(ab)\mathbf{u} = a(b\mathbf{u})$.
> 10. If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.

> **Definition**
>
> A **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ for some $c_1, c_2, \ldots, c_k \in \mathbb{R}^k$.

> **Definition**
>
> A set V equipped with **addition** and **scalar multiplication** is said to be a **vector space** over $\mathbb{R}$ if it satisfies the following axioms.
>
> 1. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

2. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

4. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.

5. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.

7. (Distribution) For any scalar a in $\mathbb{R}$ and vector $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

8. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

9. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u}) = (ab)\mathbf{u}$.

10. For any vector $\mathbf{u}$ in V, $1\mathbf{u} = \mathbf{u}$.

3.2 Dot Product, Norm, Distance

> **Definition**
>
> The inner product (or dot product) of vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$ in $\mathbb{R}^n$ is defined to be
>
> $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2, + \cdots + u_n v_n.$$
>
> Define the **norm** of a vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_i)$, to be the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $||\mathbf{u}||$,
>
> $$||u|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$
>
> This is also known as the **length** or **magnitude** of the vector.

> **Properties of inner product and norm**
>
> Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors and $a, b, c \in \mathbb{R}$ be real numbers.
>
> 1. Inner product is **symmetric**,
> $$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
>
> 2. Inner product **commutes** with scalar multiple,
> $$c\mathbf{u} \cdot \mathbf{v} = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}).$$
>
> 3. Inner product is **distributive**,
> $$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$$
>
> 4. Inner product is **positive definite**, $\mathbf{u} \cdot \mathbf{u} \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.
>
> 5. $||c\mathbf{u}|| = |c| \, ||\mathbf{u}||$.

> **Definition**
>
> A vector $\mathbf{u} \in \mathbb{R}^n$ is a **unit vector** if its norm is 1,
>
> $$||\mathbf{u}|| = 1$$
>
> **Normalizing a vector**
>
> Let $\mathbf{u}$ be a nonzero vector $\mathbf{u} \neq \mathbf{0}$. By multiplying by the reciprocal of the norm, we get a unit vector,
>
> $$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{||\mathbf{u}||}$$
>
> This is called **normalizing u**.

> **Definition**
>
> The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $d(\mathbf{u}, \mathbf{v})$, is defined to be
>
> $$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$
>
> Define the angle θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that
>
> $$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||}$$

3.3 Linear Combinations and Linear Spans

> **Definition**
>
> A linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is
>
> $$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k, \text{ for some } c_1, c_2, \ldots c_k \in \mathbb{R}.$$
>
> The scalars $c_1, c_2, \ldots c_k$ are called **coefficients**.
>
> Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The **span** (or **linear span**) of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$,
>
> $$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

> **Algorithm to Check for Linear Combination**
>
> 1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_k)$ whose columns are the vectors in S.
>
> 2. Then a vector $\mathbf{v}$ in $\mathbb{R}^n$ is in $span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if the system $\mathbf{A}\mathbf{x} = \mathbf{v}$ is consistent.
>
> 3. If the system is consistent, then the solutions to the system are the possible coefficients of the linear combination. That is, if $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is a solution to $\mathbf{A}\mathbf{x} = \mathbf{v}$, then
>
> $$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Algorithm to Check if $span(S) = \mathbb{R}^n$

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ whose columns are the vectors in S.

2. Then $span(S) = \mathbb{R}^n$ if and only if the system $\mathbf{A}\mathbf{x} = \mathbf{v}$ is consistent for all $\mathbf{v}$.

3. This is equivalent to the reduced row-echelon form of $\mathbf{A}$ having no zero rows.

Properties of linear span

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a finite set of vector. The span of S, $span(S)$ has the following properties.

1. The span of S **contains the origin**,
$$\mathbf{0} \in span(S).$$

2. The span of S is **closed under vector addition**, for any $\mathbf{u}, \mathbf{v} \in span(S)$, and real number $\alpha \in \mathbb{R}$,
$$\mathbf{u} + \mathbf{v} \in span(S)$$

3. The span S is **closed under scalar multiplication**, for any $\mathbf{u} \in span(S)$ and real number $\alpha \in \mathbb{R}$,
$$\alpha\mathbf{u} \in span(S).$$

Properties (ii) and (iii) can be combined together into one property (ii'): The span is closed under linear combinations, that is, if $\mathbf{u}, \mathbf{v}$ are vectors in $span(S)$ and α, β are any scalars, then the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is a vector in $span(S)$.

Theorem

(Linear span is closed under linear combinations)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $span(S)$, the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a subset of $span(S)$,

$$span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq span(S).$$

Algorithm to check for Set Relations between Spans

Suppose we are given 2 sets of vectors $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

1. By the corollary, if $\mathbf{v}_i \in span(S)$ for $i = 1, \ldots, m$, we can conclude that $span(T) \subseteq span(S)$.

2. Recall that to check if $\mathbf{v}_i \in span(S)$, we check that the system ($\mathbf{u}_1$ $\mathbf{u}_2$ $\ldots$ $\mathbf{u}_k$ | $\mathbf{v}_i$) is consistent for all $i = 1, \ldots, m$.

3. There are in total m such linear systems to check. However, since they have the same coefficient matrix, we may combine and check them together, that is, check that

$$(\mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \mid \mathbf{v}_1 \mid \mathbf{v}_2 \mid \ldots \mid \mathbf{v}_m)$$

is consistent.

> **Theorem**
>
> (Algorithm to check for set relations between spans)
>
> Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be sets of vectors in $\mathbb{R}^n$. Then $span(T) \subseteq span(S)$ if and only if $(\,\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k\ |\ \mathbf{v}_1\ |\ \mathbf{v}_2\ |\ \ldots\ |\ \mathbf{v}_m\,)$ is consistent.

3.4 Subspaces

> **Definition**
>
> The set of solutions to a linear system $\mathbf{Ax} = \mathbf{b}$ can be expressed **implicitly** as
>
> $$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{b}\}$$
>
> or **explicitly** as
>
> $$V = \{\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \mid s_1, s_2, \ldots, s_k \in \mathbb{R}\},$$
>
> where $\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,\ s_1, s_2, \ldots, s_k \in \mathbb{R}$ is the general solution.

> **Definition**
>
> A subset V of $\mathbb{R}^n$ is a **subspace** if it satisfies the following properties.
>
> 1. V **contains the zero vector**, $\mathbf{0} \in V$.
>
> 2. V is **closed under scalar multiplication**. For any vector. v in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.
>
> 3. V is **closed under addition**. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.
>
> Property (i) can be replaced with property (i'): V is **nonempty**.
>
> Properties (ii) and (iii) is equivalent to property (ii'): V is **closed under linear combination**. For any $\mathbf{u}, \mathbf{v}$ in V, and scalars α, β, the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is in V.

> **Theorem**
>
> (Solution set of a homogeneous system is a subspace)
>
> The solution set $V = \{\mathbf{u} \mid \mathbf{Au} = \mathbf{b}\}$ to a linear system $\mathbf{Ax} = \mathbf{b}$ is a **subspace** if and only if $\mathbf{b} = \mathbf{0}$, that is, the system is **homogeneous**.

> **Definition**
>
> The **solution set** to a **homogeneous system** is call a **solution space.**

> **Theorem**
>
> (Subspaces are equivalent to linear spans)
>
> A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span, $V = span(S)$, for some finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.
>
> **Check if a set is a subspace**

To show that a set V is a subspace, we can either

- find a spanning set, that is, find a set S such that $V = span(s)$, or

- show that V satisfies the 3 conditions of being a subspace.

To show that a subset V is not a subspace, we can either

- show that it does not contain the zero vector, $\mathbf{0} \notin V$,

- find a vector $\mathbf{v} \in V$ and a scalar $\alpha \in \mathbb{R}$ such that $\alpha \mathbf{v} \notin V$, or

- find vectors $\mathbf{u}, \mathbf{v} \in V$ such that the sum is not in V, $\mathbf{u} + \mathbf{v} \notin V$.

Theorem

(Affine spaces)

The solution set $W = \{ \ \mathbf{w} \mid \mathbf{A}\mathbf{w} = \mathbf{b} \ \}$ of a non-homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq 0$, is given by

$$\mathbf{u} + V := \{ \ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \ \}$$

where $V = \{ \ v \mid \mathbf{A}\mathbf{v} = \mathbf{0} \ \}$ is the solution space to the associated homogeneous system and $\mathbf{u}$ is a particular solution, $\mathbf{A}\mathbf{u} = \mathbf{b}$.

That is, vectors in $\mathbf{u} + V$ are of the form $\mathbf{u} + \mathbf{v}$ for some $\mathbf{v}$ in V.

3.5 Linear Independence

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent if the only coefficients $c_1, c_2, \ldots, c_k \in \mathbb{R}$ satisfying the equation

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_k \mathbf{u}_k = \mathbf{u}$$

are $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, we say that the set is linearly dependent.

Algorithm to Check for Linear Independence

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

- $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is **linearly independent** if and only if the homogeneous system $(\ \mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \ )\mathbf{x} = \mathbf{0}$ has only the **trivial solution**.

- The homogeneous system has only the **trivial solution** if and only if the reduce **row-echelon form** of $(\ \mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \ )$ has **no non-pivot column**.

Theorem

(Solution set of a homogeneous system is a subspace)

A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of $\mathbb{R}^n$ is **linearly independent** if and only if the **reduced row-echelon form** of $\mathbf{A} = (\ \mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \ )$ has **no non-pivot columns**.

3.6 Basis and Coordinates

> **Definition**
>
> Let V be a subspace of $\mathbb{R}^n$. A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a **basis** for V if
>
> - $span(S) = V$ and
>
> - S is linearly independent.

> **Theorem**
>
> Suppose S $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for . Then every vector $\mathbf{v}$ in the subspace V **can be written** as a linear combination of vectors in S **uniquely**.

> **Theorem**
>
> (Basis for Solution Set of Homogeneous System)
>
> Let $V = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{0}\}$ be the solution space to some homogeneous system. Suppose $s_1\mathbf{u}_1 + s_1\mathbf{u}_2 + \cdots + s_1\mathbf{u}_k$, $s_1, s_2, \ldots, s_k \in \mathbb{R}$ is a general solution to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for the subspace $V = \{\mathbf{u} | \mathbf{A}\mathbf{u} = \mathbf{0}\}$.

> **Theorem**
>
> Basis for the zero space $\{\mathbf{0}\}$ of $\mathbb{R}^n$ is the **empty set** $\{\}$ or $\emptyset$.

> **Theorem**
>
> A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns are linearly independent.

> **Theorem**
>
> A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns spans $\mathbb{R}^n$.

> **Corollary**
>
> Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of $\mathbb{R}^n$ containing n vectors. Then S is linearly independent if and only if S spans $\mathbb{R}^n$.

> **Corollary**
>
> Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of $\mathbb{R}^n$ and $\mathbf{A} = (\ \mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k\ )$ be the matrix whose columns are vectors in S. Then S is a basis for $\mathbb{R}^n$ if and only if $k = n$ and $\mathbf{A}$ is an invertible matrix.

> **Theorem**
>
> A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the columns of $\mathbf{A}$ form a basis for basis for $\mathbb{R}^n$.

> **Theorem**
>
> A $n \times n$ square matrix $\mathbf{A}$ is invertible if and only if the row of $\mathbf{A}$ form a basis for basis for $\mathbb{R}^n$.

> **Theorem**
>
> (Equivalent Statements for Invertibility)
>
> Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.
>
> 1. $\mathbf{A}$ is invertible.
>
> 2. $\mathbf{A}^T$ is invertible.
>
> 3. $\mathbf{A}$ has a left-inverse, that is, there is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$.
>
> 4. $\mathbf{A}$ has a right-inverse, that is, there is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$.
>
> 5. The reduced row-echelon form of $\mathbf{A}$ is the identity matrix.
>
> 6. $\mathbf{A}$ can be expressed as a product of elementary matrices.
>
> 7. The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
>
> 8. For any $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ is consistent.
>
> 9. The determinant of $\mathbf{A}$ is nonzero, $\det(\mathbf{A}) \neq 0$.
>
> 10. The columns/rows of $\mathbf{A}$ are linearly independent for $\mathbb{R}^n$.
>
> 11. The columns/rows of $\mathbf{A}$ spans $\mathbb{R}^n$.

> **Definition**
>
> (Coordinates Relative to a Basis)
>
> Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for a subspace V of $\mathbb{R}^n$.
> Then given any vector $\mathbf{v} \in V$, we can write $\mathbf{v}$ unique as
>
> $$c_1\mathbf{u}_1, c_2\mathbf{u}_2, \ldots, c_k\mathbf{u}_k.$$
>
> The coordinates of $\mathbf{v}$ relative to the basis S is defined to be the vector
>
> $$[\mathbf{v}]_s = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

> **Algorithm for Computing Relative Coordinate**
>
> Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}$ be a basis for a subspace V of $\mathbb{R}^n$.
>
> For $\mathbf{v} \in V$, find real numbers $c_1, c_2, \ldots, c_k \in \mathbb{R} \in \mathbb{R}$ such that
>
> $$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v}.$$
>
> That is, we are solving for
> $$(\mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_k \mid \mathbf{v}).$$

> **Theorem**
>
> Let V be a subspace of $\mathbb{R}^n$ and B a basis for V.
>
> 1. For any vectors $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$.
>
> 2. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$,
>
> $$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m]_B = c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + \cdots + c_m[\mathbf{v}_m]_B.$$

3.7 Dimensions

> **Theorem**
>
> Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$. Let $v_1, v_2, \ldots, v_k$ be vectors in V. Then
>
> 1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ is linearly independent (respectively, dependent) in $\mathbb{R}^k$; and
>
> 2. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ spans $\mathbb{R}^k$.

> **Corollary**
>
> Let V be a subspace of $\mathbb{R}^n$ and V a basis for B. Suppose B contains k vectors, $|B| = k$.
>
> 1. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m > k$, then S is linearly dependent.
>
> 2. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m < k$, then S cannot span V.

> **Corollary**
>
> Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Then $k = m$.

> **Definition**
>
> Let V be a subspace of $\mathbb{R}^n$. The dimension of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

> **Theorem**
>
> (Dimension of solution space)
>
> Let $\mathbf{A}$ be a $m \times n$ matrix. The number of non-pivot columns in the reduced row-echelon form of A is the dimension of the solution space
> $$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = \mathbf{0}\}.$$

> **Theorem**
>
> (Spanning Set Theorem)
>
> Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = span(S)$. Suppose V is not the zero space, $V \neq \{\mathbf{0}\}$. Then there must be a subset of S that is a basis for V.

> **Theorem**
>
> (Linear Independence Theorem)
>
> Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

> **Theorem**
>
> Let U and V be subspaces of $\mathbb{R}^n$.
>
> 1. If $U \subseteq V$, then $dim(U) \leq dim(V)$.
>
> 2. If $U \subseteq V$, and $U \neq V$, then $dim(U) < dim(V)$ That is, $U \subseteq V$, then $dim(U) \leq dim(V)$ with equality if and only if $U = V$.

> **Theorem**
>
> (B1)
>
> Let V be a k-dimensional subspace of $\mathbb{R}^n$, $dim(V) = k$. Suppose $U \subseteq V$ is a linearly independent subset containing k vectors, $|S| = k$. Then S is a basis for V.

> **Theorem**
>
> (B2)
>
> Let V be a k dimensional subspace of $\mathbb{R}^n$, $dim(V) = k$. Suppose S is a set containing k vectors, $|S| = k$, such that $V \subseteq span(S)$. Then S is a basis for V.

3.8 Transition Matrices

> **Definition**
>
> Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are **basis** for the subspace V. Define the transition matrix from T to S to be
>
> $$\mathbf{P} = (\ [\mathbf{v}_1]_S \quad [\mathbf{v}_2]_S \quad \ldots \quad [\mathbf{v}_k]_S\ ),$$
>
> the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

> **Theorem**
>
> (Transition Matrix)
>
> Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ are **bases** for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector w in V,
>
> $$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Algorithm to find Transition Matrix

Let $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ be a basis for a subspace V in $\mathbb{R}^n$. To find $\mathbf{P}$, the transition matrix from T to S,

$$("S"|"T") = \begin{pmatrix} u_1 & u_2 & \ldots u_k & | & v_1 & v_2 & \ldots v_k \end{pmatrix} \xrightarrow{\text{rref}} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k) \times k} & \mathbf{0}_{(n-k) \times k} \end{array} \right)$$

Theorem

(Inverse of Transition Matrix)

Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are bases for a subspace V of $\mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then P^{-1} is the transition matrix from S to T.

Chapter 4: Subspaces Associated to a Matrix

4.1 Column Space, Row Space, and Nullspace

4.2 Rank