MA1522 Notes (AY24/25 Sem1)

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Chapter 2: Matrix Algebra

2.1 Definition and Special types of Matrices

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Chapter 3: Euclidean Vector Spaces

3.1 Euclidian Vector Spaces

Definition

A (real) n-**vector** is a collection of n ordered real numbers,

$$\mathbf{v} \;=\; \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, where\ v_i \in \mathbb{R}\ for\ i = 1, \dots, n.$$

The real number v_i is called the i-th coordinate of the vector $\mathbf{v}$. The **Euclidiean** n**-space**, denoted $\mathbb{R}^n$, is the collection of all n-vectors

$$\mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R}\ for\ i = 1, \dots, n. \right\}$$

Properties of Vector Addition and Scalar Multiplication

Since vectors are matrices (column vectors are $n \times 1$ matrices and row vectors are $1 \times n$ matrices), the properties of matrix addition and scalar multiplication holds for vectors. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars $a, b \in \mathbb{R}$,

1. The sum $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$
2. (Commutative) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. (Associative) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. (Zero vector) $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
5. The negative $-\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{v} - \mathbf{v} = \mathbf{0}$.
6. (Scalar multiple) $a\mathbf{v}$ is a vector in $\mathbb{R}^n$.
7. (Distribution) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. (Distribution) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
9. (Associativity of scalar multiplication) $(ab)\mathbf{u} = a(b\mathbf{u})$.
10. If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.

Definition

A **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in \mathbb{R}^n$ is $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ for some $c_1, c_2, \dots, c_k \in \mathbb{R}^k$.

Definition

A set V equipped with **addition** and **scalar multiplication** is said to be a **vector space** over $\mathbb{R}$ if it satisfies the following axioms.

1. For any vectors $\mathbf{u}, \mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

2. (Commutative) For any vectors $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. (Associative) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. (Zero vector) There is a vector $\mathbf{0}$ in V such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$ for all vectors $\mathbf{v}$ in V.
5. (Negative) For any vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For any scalar a in $\mathbb{R}$ and vector $\mathbf{v}$ in V, $a\mathbf{v}$ is a vector in V.
7. (Distribution) For any scalar a in $\mathbb{R}$ and vector $\mathbf{u}, \mathbf{v}$ in V, $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.
8. (Distribution) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.
9. (Associativity of scalar multiplication) For any scalars a, b in $\mathbb{R}$ and vector $\mathbf{u}$ in V, $a(b\mathbf{u})=(ab)\mathbf{u}$.
10. For any vector $\mathbf{u}$ in V, $1\mathbf{u}=\mathbf{u}$.

3.2 Dot Product, Norm, Distance

Definition

The inner product (or dot product) of vectors $\mathbf{u}=(u_i)$ and $\mathbf{v}=(v_i)$ in $\mathbb{R}^n$ is defined to be

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2,+\cdots+u_nv_n.$$

Define the **norm** of a vector $\mathbf{u}\in\mathbb{R}^n$, $\mathbf{u}=(u_i)$, to be the square root of the inner product of $\mathbf{u}$ with itself, and is denoted as $||\mathbf{u}||$,

$$||u||=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{u_1^2+u_2^2+\cdots+u_n^2}.$$

This is also known as the **length** or **magnitude** of the vector.

Properties of inner product and norm

Let $\mathbf{u}, \mathbf{v}\in\mathbb{R}^n$ be vectors and $a, b, c\in\mathbb{R}$ be real numbers.

1. Inner product is **symmetric**,
$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}.$$
2. Inner product **commutes** with scalar multiple,
$$c\mathbf{u}\cdot\mathbf{v}=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v}).$$
3. Inner product is **distributive**,
$$\mathbf{u}\cdot(a\mathbf{v}+b\mathbf{w})=a\mathbf{u}\cdot\mathbf{v}+b\mathbf{u}\cdot\mathbf{w}$$
4. Inner product is **positive definite**, $\mathbf{u}\cdot\mathbf{u}\geq 0$ with equality if and only if $\mathbf{u}=\mathbf{0}$.
5. $||c\mathbf{u}||=|c|\ ||\mathbf{u}||$.

Definition

A vector $\mathbf{u} \in \mathbb{R}^n$ is a **unit vector** if its norm is 1,

$$||\mathbf{u}|| = 1$$

Normalizing a vector
Let $\mathbf{u}$ be a nonzero vector $\mathbf{u} \neq \mathbf{0}$. By multiplying by the reciprocal of the norm, we get a unit vector,

$$\mathbf{u} \longrightarrow \frac{\mathbf{u}}{||\mathbf{u}||}$$

This is called **normalizing u**.

Definition

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $d(\mathbf{u}, \mathbf{v})$, is defined to be

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

Define the angle θ between two nonzero vectors, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ to be such that

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

3.3 Linear Combinations and Linear Spans

Definition

A linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^n$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k, \text{ for some } c_1, c_2, \ldots c_k \in \mathbb{R}.$$

The scalars $c_1, c_2, \ldots c_k$ are called **coefficients**.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. The **span** (or **linear span**) of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is the subset of $\mathbb{R}^n$ containing all the linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Algorithm to Check for Linear Combination

1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_k)$ whose columns are the vectors in S.
2. Then a vector $\mathbf{v}$ in $\mathbb{R}^n$ is in $span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent.
3. If the system is consistent, then the solutions to the system are the possible coefficients of the linear combination. That is, if $\mathbf{u} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is a solution to $\mathbf{Ax} = \mathbf{v}$, then
$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Algorithm to Check if $span(S) = \mathbb{R}^n$

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

1. Form the $n \times k$ matrix $\mathbf{A} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ whose columns are the vectors in S.
2. Then $span(S) = \mathbb{R}^n$ if and only if the system $\mathbf{Ax} = \mathbf{v}$ is consistent for all $\mathbf{v}$.
3. This is equivalent to the reduced row-echelon form of $\mathbf{A}$ having no zero rows.

Properties of linear span

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a finite set of vector. The span of S, $span(S)$ has the following properties.

1. The span of S **contains the origin**,
$$\mathbf{0} \in span(S).$$
2. The span of S is **closed under vector addition**, for any $\mathbf{u}, \mathbf{v} \in span(S)$, and real number $\alpha \in \mathbb{R}$,
$$\mathbf{u} + \mathbf{v} \in span(S)$$
3. The span S is **closed under scalar multiplication**, for any $\mathbf{u} \in span(S)$ and real number $\alpha \in \mathbb{R}$,
$$\alpha\mathbf{u} \in span(S).$$

   Properties (ii) and (iii) can be combined together into one property (ii'): The span is closed under linear combinations, that is, if $\mathbf{u}, \mathbf{v}$ are vectors in $span(S)$ and α, β are any scalars, then the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is a vector in $span(S)$.

Theorem

(Linear span is closed under linear combinations)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $span(S)$, the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a subset of $span(S)$,

$$span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq span(S).$$

Algorithm to check for Set Relations between Spans

Suppose we are given 2 sets of vectors $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

1. By the corollary, if $\mathbf{v}_i \in span(S)$ for $i = 1, \ldots, m$, we can conclude that $span(T) \subseteq span(S)$.
2. Recall that to check if $\mathbf{v}_i \in span(S)$, we check that the system $\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k & \mathbf{v}_i \end{array}\right)$ is consistent for all $i = 1, \ldots, m$.
3. There are in total m such linear systems to check. However, since they have the same coefficient matrix, we may combine and check them together, that is, check that
$$\left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_m \end{array}\right)$$
   is consistent.

Theorem

(Algorithm to check for set relations between spans)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be sets of vectors in $\mathbb{R}^n$. Then $span(T) \subseteq span(S)$ if and only if $(\ \mathbf{u}_1\ \mathbf{u}_2\ \dots\ \mathbf{u}_k \mid \mathbf{v}_1\ \mid \mathbf{v}_2 \mid\ \dots\ \mid \mathbf{v}_m\ )$ is consistent.

3.4 Subspaces

Definition

The set of solutions to a linear system $\mathbf{Ax} = \mathbf{b}$ can be expressed **implicitly** as

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{Au} = \mathbf{b}\}$$

or **explicitly** as

$$V = \{\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \mid s_1, s_2, \dots, s_k \in \mathbb{R}\},$$

where $\mathbf{u} + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,\ s_1, s_2, \dots, s_k \in \mathbb{R}$ is the general solution.

Definition

A subset V of $\mathbb{R}^n$ is a **subspace** if it satisfies the following properties.

1. V **contains the zero vector**, $\mathbf{0} \in V$.
2. V is **closed under scalar multiplication**. For any vector. v in V and scalar α, the vector $\alpha\mathbf{v}$ is in V.
3. V is **closed under addition**. For any vectors **u, v** in V, the sum $\mathbf{u} + \mathbf{v}$ is in V.

Property (i) can be replaced with property (i'): V is **nonempty**.

Properties (ii) and (iii) is equivalent to property (ii'): V is **closed under linear combination**. For any **u, v** in V, and scalars α, β, the linear combination $\alpha\mathbf{u} + \beta\mathbf{v}$ is in V.

Theorem

(Solution set of a homogeneous system is a subspace)

The solution set $V = \{\mathbf{u} \mid \mathbf{Au} = \mathbf{b}\}$ to a linear system $\mathbf{Ax} = \mathbf{b}$ is a **subspace** if and only if $\mathbf{b} = \mathbf{0}$, that is, the system is **homogeneous**.

Definition

The **solution set** to a **homogeneous system** is call a **solution space.**

Theorem

(Subspaces are equivalent to linear spans)

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span, $V = span(S)$, for some finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$.

Check if a set is a subspace

To show that a set V is a subspace, we can either

- find a spanning set, that is, find a set S such that $V = \mathit{span}(s)$, or
- show that V satisfies the 3 conditions of being a subspace.

To show that a subset V is not a subspace, we can either

- show that it does not contain the zero vector, $\mathbf{0} \notin V$,
- find a vector $\mathbf{v} \in V$ and a scalar $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin V$, or
- find vectors $\mathbf{u}, \mathbf{v} \in V$ such that the sum is not in V, $\mathbf{u} + \mathbf{v} \notin V$.

Theorem

(Affine spaces)

The solution set $W = \{ \ \mathbf{w} \mid \mathbf{A}\mathbf{w} = \mathbf{b} \ \}$ of a non-homogeneous linear system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq 0$, is given by

$$\mathbf{u} + V := \{ \ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \ \}$$

where $V = \{ \ v \mid \mathbf{A}\mathbf{v} = \mathbf{0} \ \}$ is the solution space to the associated homogeneous system and $\mathbf{u}$ is a particular solution, $\mathbf{Au} = \mathbf{b}$.

That is, vectors in $\mathbf{u} + V$ are of the form $\mathbf{u} + \mathbf{v}$ for some $\mathbf{v}$ in V.

3.5 Linear Independence

3.6 Basis and Coordinates

3.7 Dimensions

Theorem

Let V be a subspace of $\mathbb{R}^n$ and B a basis for V. Suppose B contains k vectors, $|B| = k$. Let $v_1, v_2, \ldots, v_k$ be vectors in V. Then

1. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ is linearly independent (respectively, dependent) in $\mathbb{R}^k$; and
2. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V if and only if $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ spans $\mathbb{R}^k$.

Corollary

Let V be a subspace of $\mathbb{R}^n$ and V a basis for B. Suppose B contains k vectors, $|B| = k$.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m > k$, then S is linearly dependent.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is a subset of V with $m < k$, then S cannot span V.

Corollary

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Then $k = m$.

Definition

Let V be a subspace of $\mathbb{R}^n$. The **dimension** of V, denoted by $\dim(V)$, is defined to be the number of vectors in any basis of V.

Theorem

(Dimension of solution space)

Let $\mathbf{A}$ be a $m \times n$ matrix. The **number of non-pivot columns** in the reduced row-echelon form of A is the **dimension** of the solution space

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = \mathbf{0}\}.$$

Theorem

(Spanning Set Theorem)

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of vectors in $\mathbb{R}^n$, and let $V = \mathit{span}(S)$. Suppose V is not the zero space, $V \neq \{\mathbf{0}\}$. Then there must be a subset of S that is a basis for V.

Theorem

(Linear Independence Theorem)

Let V be a subspace of $\mathbb{R}^n$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a linearly independent subset of V, $S \subseteq V$. Then there must be a set T containing S, $S \subseteq T$ such that T is a basis for V.

3.8 Transition Matrices

Definition

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are **basis** for the subspace V. Define the **transition matrix** from T to S to be

$$\mathbf{P} = \begin{pmatrix} [\mathbf{v}_1]_S & [\mathbf{v}_2]_S & \ldots & [\mathbf{v}_k]_S \end{pmatrix},$$

the matrix whose columns are the coordinates of the vectors in T relative to the basis S.

Theorem

(Transition Matrix)

Let V be a subspace of $\mathbb{R}^n$. Suppose $S = \{u_1, \ldots, u_k\}$ are **bases** for the subspace V. Let $\mathbf{P}$ be the transition matrix from T to S. Then for any vector w in V,

$$[\mathbf{w}]_S = \mathbf{P}[\mathbf{w}]_T.$$

Algorithm to find Transition Matrix

Let $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ be a basis for a subspace V in $\mathbb{R}^n$. To find $\mathbf{P}$, the transition matrix from T to S,

$$(\text{“}S''|\text{“}T'') = (u_1 \quad u_2 \quad \ldots u_k \quad | \quad v_1 \quad v_2 \quad \ldots v_k) \xrightarrow{\text{rref}} \left(\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{(n-k)\times k} \end{array} \right)$$

Theorem

(Inverse of Transition Matrix)

Suppose $S = \{u_1, \ldots, u_k\}$ and $T = \{v_1, \ldots, v_k\}$ are **bases** for a subspace V of $\mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from T to S. Then P^{-1} is the transition matrix from S to T.

Chapter 4: Subspaces Associated to a Matrix

4.1 Column Space, Row Space, and Nullspace

4.2 Rank