Nuclear Arms Races: India and Pakistan

Elisa Gonzalez, Ty Geri, Leah Margulies, Michael Chen

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1 Abstract

Nuclear arms races pose a great threat to humanity. Hostile nations, like India and Pakistan, have begun nuclear arms races and are on a potential trajectory for war. Being able to forecast future outcomes of an arms race is crucial to preserving geopolitical stability. To understand the dynamics of the nuclear arms race, we use a coupled ODE system (an adapted version of Richardson model with carrying capacity). Through our model, fitted to real data on nuclear stockpiles, we predict India and Pakistan will de-escalate and reach a stable equilibrium that is slightly less than current nuclear bv levels.

2 Introduction

Arms races are widely believed to have significant consequences for states' security, and in an increasingly interconnected world, hold far reaching implications. In a world where each nation only has three basic options to acquire the military capabilities it needs to achieve its international goals - (1) gain allies; (2) cooperate with its adversary to reduce threats; (3) build up arms - an arms race takes on quite an important role. Particularly pervasive in today's technological society is the escalation of nuclear weapons. Nuclear weapons are the most dangerous military arsenal a country can possess. As such, a rival country's escalation of nuclear weapons poses a severe and direct security risk. While traditional arms races may exhibit elements of increasing aggression towards a specific country, they often are confounded with upkeep of a wide range of armaments that are necessary for the general protection of the country against many nations. Nuclear races however are often more distinct in their target, as seen in the Cold War. For that reason, this paper will explore the drivers of nuclear escalation and examine the hostile tensions between India and Pakistan.

3 Description of the Model

3.1 Richardson and Caspary Models

Lewis Richardson was an English physicist who developed his arms race model after serving for France's medical corps in World War I [5]. A coupled pair of differential equations explains changes in levels of weapons in each of two nations as a function of the weapons of each side (x, y) including reaction (p_1, p_2), burden (c_1, c_2) and initial aggrivation (γ_1, γ_2) terms:

$$\begin{aligned} \frac{dx}{dt} &= p_1 y + c_1 x + \gamma_1 \\ \frac{dy}{dt} &= p_2 x + c_1 y + \gamma_2 \end{aligned} \tag{1}$$

Our arms race model is an adapted version of the Richardson model. The model follows similar assumptions and parameters, but includes an additional constraint C which is impacted by diminishing returns, inspired by Caspary's augmentation [1]. That is, the benefits of increasing nuclear weapons is subject to adverse effects via a combination of anti-proliferation legislature and decreasing returns to escalation. We define C to be the maximum the country can escalate to, in some senses this is the carrying capacity prior to pushback, via treaties and embargoes, from the international community. In addition to this carrying capacity, the benefits received from nuclear weapons, such as deterrence, decrease past a certain point of nuclear weapons. For example, at high levels, a unit increase in nuclear weapons provides less benefit to national security than at low levels. This concept is embodied by an exponential function, which at low levels is negligible but at high levels illustrates the decreasing returns phenomena. Lastly and importantly, we have kept this carrying capacity constant given we only intend to use this model for the next 5-10 years. This is one of the shortcomings of our model - the carrying capacity we chose for India and Pakistan is a local approximation given historical data that we have on the Cold War, current nukes in each country's respective arsenals, and spending data [2]. In a future paper we could change carrying capacity to be a function of time rather than fixed.

3.2 Parameters and Variables

$$p_1, p_2 \geq 0 \tag{2}$$

Represents the response factor or "reaction" from Country X to Country Y's nuclear arms levels. This indicates that the stronger the enemy, the more they need to spend to catch up.

$$C_1, C_2 \geq 0 \tag{3}$$

Represents the maximum amount of nuclear weapons a country can amass for each respective country. We define C_1 to be 465 nukes. Considering that Pakistan generally spends about 46 percent as much as India does on nukes every year, we set C_2 equal to $\frac{1}{2}C_1 = 213$ nukes. Our calculations for these values are shown in the appendix below.

$$\gamma_1, \gamma_2 \geq 0 \tag{4}$$

Represents "grievance" or additional factors like ambitions or grudges impacting the escalation of armaments

$$x, y \geq 0 \tag{5}$$

Represents the amount of armaments possessed by country X and country Y at some time t

Below is our model - a system of nonlinear coupled ODEs.

$$\begin{aligned} \frac{dx}{dt} &= p_1 * y + C_1(1 - e^{\frac{x}{C_1}}) + \gamma_1 \\ \frac{dy}{dt} &= p_2 * x + C_2(1 - e^{\frac{y}{C_2}}) + \gamma_2 \end{aligned} \tag{6}$$

Note that thanks to this constraint, it is not possible for x or y to grow to infinity because for large enough values, the exponential will overtake the linear terms making $\frac{dx}{dt}$ and $\frac{dy}{dt}$ negative. Finally, we have made the assumption in our model that many of these effects (grievance, etc.) are additive. This assumes that the effect of each input variable on the output variable (total change in nuclear weapon capacity) is independent of the values of the other input variables. For example, there might be interactions between the terms that we are not aware of.

4 Results and Analysis

4.1 Results

In order to fit our model to the data, we solved the model for the optimal parameters that minimized the square error of the fit. These optimal parameters were $p_1 = 1.10, p_2 = 1.53, \gamma_1 = 0.0, \gamma_2 = 1.59$. These accurately reflect the trajectory of the nuclear arms race between India and Pakistan for a few reasons. The first reason is aggressiveness. Given more limited financial resources, the fact that Pakistan has kept levels of nuclear warheads comparable to that of India's indicates a higher aggressiveness. This denoted by the fact that $p_2 > p_1$. The second reason, perhaps related to the historical tensions between India and Pakistan is the fact that $\gamma_2 > \gamma_1$. In other words, Pakistan has greater grievance which may be motivated by the power asymmetries between the two nations. To better illustrate the influence of the model dynamics for these two countries, in Figure 1 we provide a phase portrait.

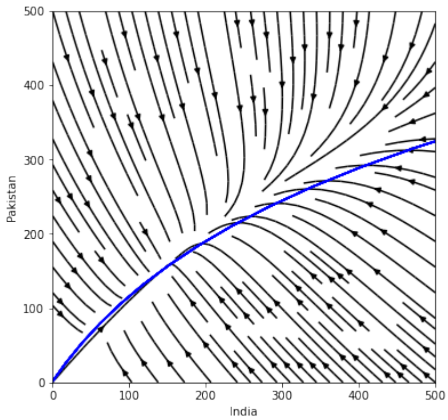

Figure 1: Phase Portrait

Currently, India and Pakistan have roughly equal amounts of armaments, in particular, most recently they had around 160 armaments each [2]. On the phase portrait, we see that, roughly over the past 20 years, nuclear arms have risen from levels close to zero to levels close to 160, following the nullcline (in blue) on the phase portrait. What future estimates we can derive from this is that, as they continue escalating, Pakistan will likely experience greater fiscal constraints (a consequence of their lower carrying capacity C_2) and have to begin de-escalating. Their deescalation is mirrored by steeper drops in armaments as levels rise. However, in response, we anticipate that India will also begin to decrease their armaments as indicated by less steep, but significant stagnation in India's level of nukes. Note that a shortcoming of this analysis is that due to the nature of the data we found on total armaments secured by each country vs. nuclear weapon data, we chose to focus our analysis on a nuclear weapons arm race. In an extension of this paper, we could modify our model to track total armaments, instead of just a specific subset of those armaments as seen above.

4.2 Jacobian and Fixed Point Analysis

Now that we have fitted our model to the real world data and have calculated the optimal parameters, we can plug in $p_1, p_2, C_1, C_2, \gamma_1, \gamma_2$ into our non-linear coupled ODE model from above. We now can calculate any equilibrium points and conduct a stability analysis.

Our system of ODEs has now become:

$$\begin{aligned} \frac{dx}{dt} &= 1.1 * y + 465(1 - e^{\frac{x}{465}}) + 0 \\ \frac{dy}{dt} &= 1.53 * x + 213(1 - e^{\frac{y}{213}}) + 1.59 \end{aligned} \tag{7}$$

We begin by calculating the nullclines, which can be done by setting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ The two nullclines become:

$$\begin{aligned} y &= 422.73 * e^{\frac{x}{465}} - 422.73 \\ y &= 213 * \ln(0.0072 * x + 1.0075) \end{aligned} \tag{8}$$

We now solve this nonlinear system of equations to find the equilibrium point (intersection of the two nullclines) and solve for $x*$ and $y*$. Note that this systems of equations is virtually impossible to solve mathematically, so we end up plotting the two nullclines and determining which x value lends itself to the same y value for both nullcline equations.

$$\boxed{X = 141.64, Y = 150.528}$$

Finally, we determine the stability of this point by linearization. We calculate the Jacobian matrix below and evaluate at our equilibrium point:

$$J(141.64, 150.528) = \begin{bmatrix} -e^{\frac{x}{465}} & 1.1 \\ 1.53 & -e^{\frac{y}{213}} \end{bmatrix}$$

$$\implies J = \begin{bmatrix} -1.356 & 1.1 \\ 1.53 & -2.027 \end{bmatrix}$$

Computing the eigenvalues for this matrix from Matlab yields:

$$\boxed{\lambda = -0.3515, -3.0315}$$

We can conclude that this equilibrium point is a stable equilibrium point because the real parts of both eigenvalues are less than zero. Moreover, given these levels are lower than India and Pakistan's current levels of nuclear weapons, we anticipate that they will deescalate weapons to reach this stable equilibrium as predicted in the phase portrait. While initially surprising, the inclusion of our C parameter forces there to be an upper limit to escalation, thus it makes sense that our model would reach a point of stable equilibrium.

4.3 Model Fit

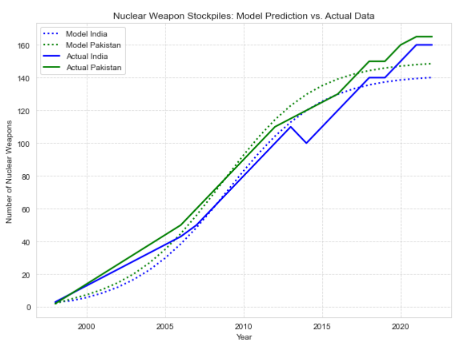

Figure 2: Actual Data versus Model

Our model differs from the original data at a few key points. In particular, the early period of escalation is significantly more steep, likely due to a fixed amount of budgetary allocation for nuclear expenditure. Additionally, there is volatility in the data around 2015 which is not captured in the model. However these jumps and their somewhat erratic behavior could be excess noise. Lastly, as to be expected the model somewhat under approximates the leveling off of the arms race in recent years. This is likely due to our fixed carrying capacity, a potentially limiting assumption we have made, and something that we will explore later in our sensitivity analysis.

5 Sensitivity Analysis

5.1 P_1, P_2 Sensitivities

In our sensitivity analysis, we were able to determine how both India and Pakistan's predicted current arms level would change with fluctuations in the parameters given. The parameters we explored below were the response coefficient (p_1 and p_2) which determined the proportional retaliation level of each nation in response to an increase in nukes from the other. In the first graph, we find each nation's estimated nuclear arsenal in the last year with respect to their reaction parameter, keeping the other country's reaction parameter constant. In the heat map, we fluctuate both nations' reaction parameters and find the total number of nukes present in 2021 across both countries.

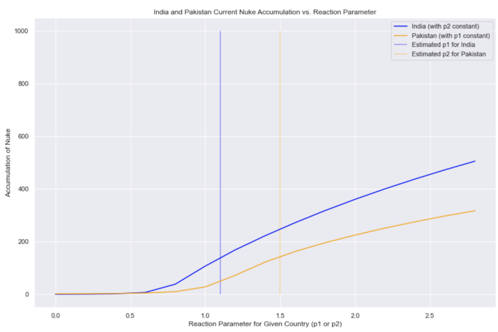

Figure 3: India and Pakistan 2021 nuclear forecast vs each nation's reaction parameter (holding other nation's parameter constant)

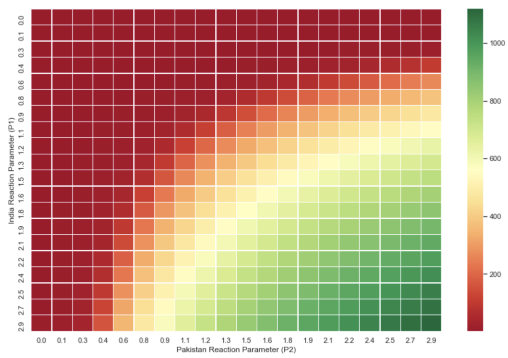

Figure 4: India and Pakistan 2021 nuclear forecast vs reaction parameters

From the first graph, increasing the retaliation parameter will increase India's stock at a faster rate than Pakistan's, which is unsurprising given India's higher carrying capacity. The second graph indicates the magnitude of collected arms grows quite dangerous at high p values (> 1000 arms total between the two countries when India and Pakistan have retaliation parameters over around 2.6). This is not concerning currently, given that retaliation parameters are closer to 1.1 and 1.5 respectively, which of course does not account for depreciation.

5.2 C_1, C_2 Sensitivities

The first plot is interesting as it depicts Pakistan acquiring a larger number of nukes than India despite lower economic capacity. This makes sense, however, given that Pakistan has a higher reaction parameter (by about 50 percent) than India does. Realistically, Pakistan's economic capacity will never reach the limits of the orange line in the graph. Equally as telling however,

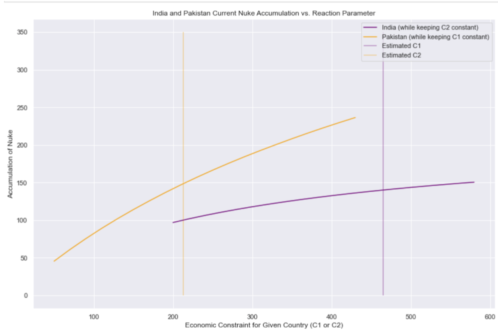

Figure 5: India and Pakistan 2021 nuclear forecast vs each nation's economic capacity (holding other capacity constant)

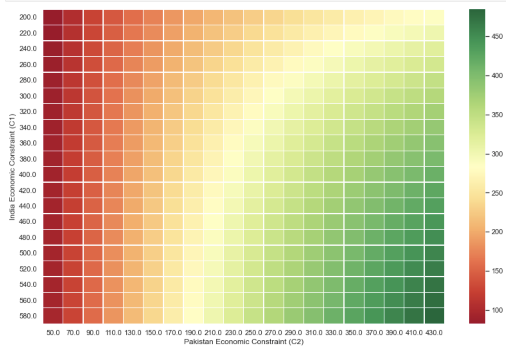

Figure 6: India and Pakistan 2021 nuclear forecast vs economic capacities

is how quickly Pakistan's estimated nukes decreases with a lower economic capacity. Overall this graph implies that Pakistan is far more sensitive to the economic constraint variable than India in both the up and down cases, suggesting that Pakistan's nuke estimation could be highly volatile in times of economic turbulence. The heat map supports this notion. As rows have more variation than columns, it would appear that Pakistan's economic constraint has more predictive power than India's which aligns with the previous plot where Pakistan's estimated nuke inventory was far more volatile with respect to its economic constraint than India's.
One limitation with the graphs above is that we do not have the knowledge to account for the rate of depreciation amongst nukes over the 23 year period since 1998. Therefore, our heat maps that take an estimated amount of nuke purchases are likely overeager. With that said, we believe that the amount of nukes *purchased* across both countries, even if they are not in use today, is still highly correlated to the probability of conflict or an outbreak of war. Further steps would be to explore the relationship between weapons accumulated and probability of war over the 25 year period.

6 Summary

Our goal in this paper was to analyze tensions in the nuclear arms race between Pakistan and India. Through our adapted Richardson model with carrying capacity based on economic and geopolitical factors, fitted to real world data on nuclear stockpiles for the two countries, we determine that Pakistan and India will eventually reach a stable equilibrium point of de-escalation. India will have 142 nukes and Pakistan will have 151 nukes. We also examine how changes in retaliation between the two countries, and changes in carrying capacity between the two countries will affect this arms race. We find that Pakistan is far more sensitive to the economic constraint parameter than India, and that India is more sensitive to the retaliation parameter than Pakistan. This model has a wide variety of real world applications given various other nuclear arms conflicts and potential future ones, including US/Russia, Russia/Ukraine, and the wild card that is North Korea.

7 Attribution of Work

7.1 Elisa Gonzalez

Paper Writing and Editing, Model Creation and Assumptions, Compiled Nuclear Data, Initial Code for Nuclear Data Fitting, Results and Model Fit Sections and Connection to Richardson/Residual Analysis (in appendix)

7.2 Michael Chen

Paper Writing and Editing, Model Creation and Assumptions, Jacobian and Fixed Point Analysis of Equilibrium Points, Shortcomings and Potential Extensions

7.3 Leah Margulies

Paper Writing and Editing, Model Creation and Assumptions, C1 and C2 calculation, Sensitivity Analysis of p1, p2 and c1, c2 (including heatmaps in Appendix)

7.4 Ty Geri

Paper Writing and Editing, Model Creation and Assumptions, Background Research on Richardson's and Caspary, Data Pipeline, Adapting initial code for Nuclear Data Fitting

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8 Appendix

8.1 Richardson Model Continued

Richardson made three important assumptions about conditions under which nations will increase or decrease their armaments: (1) out of fear of military insecurity, country A will make increases in its “armaments” proportional to the level of country B’s armaments, and B will then respond in turn; (2) The burden of armaments upon the economy of the country imposes a restraint on further expenses, and is proportional to the size of the existing armament stock; (3) There are hostilities that drive nations to arm themselves at a constant rate in the absence of a direct military threat from another country [1]. We see this in the present with Russia/Ukraine, and persistent conflicts like India/Pakistan and Turkey/Greece. There are generally two views on the consequences of arms races - the first view states that arms races increase the probability of war by undermining military stability and straining political relations. The other view holds that engaging in an arms race is often a state’s best option for avoiding war when faced with an aggressive adversary.

8.2 Assumptions and Justifications

In our model we made the following critical assumptions. First, X and Y are measured as the total estimate of nuclear warheads possessed by India and Pakistan per year [1998-2021] respectively. While these are estimates, they accurately reflect the trajectory of nuclear escalation en masse and we predict that, although there are likely monthly variations in the data, yearly fluctuations are enough to represent general trends in armament escalation. Second, C_1 and C_2, the estimated carrying capacity in number of nukes for India and Pakistan, represent the economic constraints of each country and maximum point of escalation. To calculate India’s carrying capacity for nukes, we find that the country currently has around 160 nukes in its arsenal. To estimate their nuke accumulation during war time or times of geopolitical stress, we find the highest percentage change in the US’s total nuclear stockpile over 1991, the year that the Cold War officially ended and the nuclear stockpile decreased to a stable amount. The US, in the peak of the cold war conflict, was able to raise its nuke stockpile around 290 percent over the mean nuclear stockpile of 11 thousand nukes which has remained constant to this day. Applying this percentage to India’s current nuke arsenal, we get a carrying capacity of $160 * 2.9 = 465$ nukes. Considering that Pakistan generally spends about 46 percent as much as India does on nukes every year, we set C_2 equal to $\frac{1}{2}C_1 = 213$ nukes.

8.3 Connection to Richardson Model

While the non-linear nature of equation 5 departs from the traditional Richardson Model, for very small x and y, a Taylor expansion around 0 reveals the closeness of the economic model with the Richardson Model.

Looking first at $\frac{dx}{dt}$ a Taylor expansion yields the following. By symmetry, a similar result also holds for $\frac{dy}{dt}$.

$$\begin{aligned} e^{\frac{x}{C_1}} &= 1 + \frac{x}{C_1} + O(x^2) \\ \frac{dx}{dt} &= p_1 * y + C_1(1 - (1 + \frac{x}{C_1} + O(x^2)) + \gamma_1 \\ &= p_1 * y - x + \gamma_1 \end{aligned} \tag{9}$$

8.4 Residual Plot

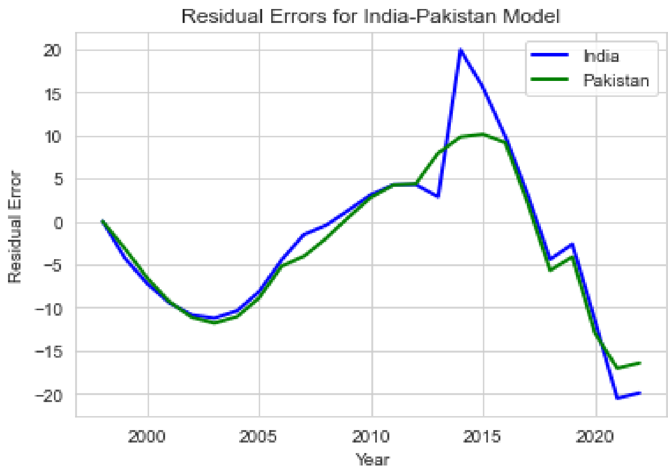

Figure 7: Residual Errors Plot

9 Additional Sensitivity Tables

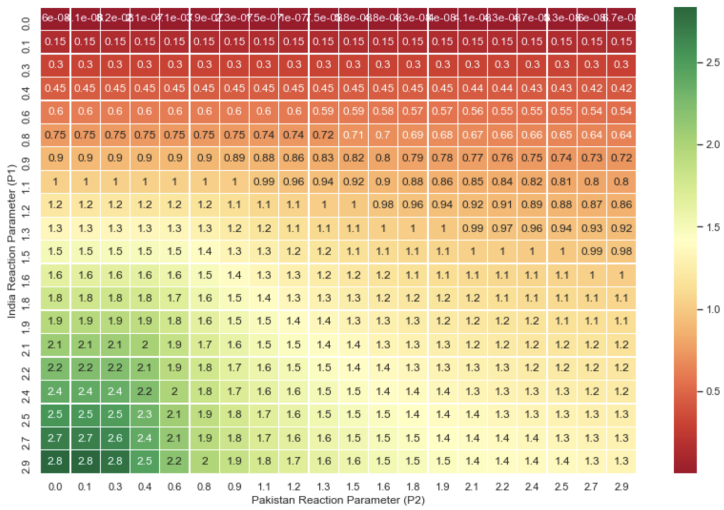

Figure 8: Mean Ratio of India's Annual Nuke Additions over Pakistan's Over Time vs Reaction Parameters

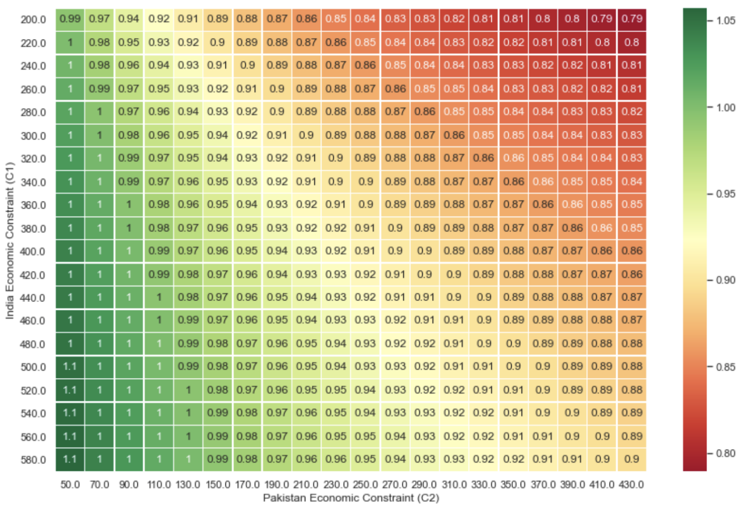

Figure 9: Mean Ratio of India's Annual Nuke Additions over Pakistan's Over Time vs Economic Constraints