Week 4: Representation of Natural Numbers

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1 Decimal representation of natural numbers

A number in decimal is expressed in digits of $d_{k-1} \ldots d_1 d_0$, where each digit d_i is in $\{1, \ldots, 9\}$. The number that we get using this sequence of digits is $\sum_{i=0}^{k-1} d_i \times 10^i$. For example, 569 is $5 \times 10^2 + 6 \times 10^1 + 9 \times 10^0$.

Some useful properties of decimal representation:

1. To multiply by 10, we can just add a 0 to the right end
2. With k decimal digit positions, there are exactly 10^k numbers that can be represented. For example 10^3 is from 0 to 999.

2 Binary representation of natural numbers

Binary or base 2 representation uses binary digits $\{0, 1\}$ instead of the 10 decimal digits. The corresponding value of the numbers in a binary sequence is $\sum_{i=0}^{k-1} d_i \times 2^i$. For example, the decimal number 139 would be 10001011 in binary: $1 \times 2^7 + 1 \times 2^3 + 1 \times 2^1 + 1 \times 2^0$.

2.1 Converting from binary to decimal

We can use the sum given above to convert binary to decimal:

$$100101 = 1 \times 2^5 + 1 \times 2^2 + 1 \times 2^0 = 32 + 4 + 1 = 37$$

2.2 Properties of binary representation

Theorem 4.1. For every natural number n, there exists $p \in \mathbb{N}$ and bits $b_p, \ldots, b_0 \in \{0, 1\}$ such that $n = \sum_{i=0}^{p} b_i 2^i$.

Proof. We will prove an equivalent statement that is easily to use strong induction on:

$$\forall m \in \mathbb{N}, \left(\forall n \in \mathbb{N}, n \leq m \Rightarrow (\exists p \in \mathbb{N}, \exists b_0, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i 2^i) \right)$$

We can define $P(m)$ as the part after $\forall m \in \mathbb{N}$, and then use induction to prove it.

Base case. Let $m = 0$. There would be only one number to consider, which is zero. Let $p = 0$ and $b_0 = 0$, then $\sum_{i=0}^{p} b_i 2^i = 0 \times 2^0 = 0$. Thus the base case is True.

Inductive step. Let $m \in \mathbb{N}$, and assume $P(m)$ is True: that every natural number less than equal to m has a binary representaiton. WTS $P(m+1)$ is True.

Let $n \in \mathbb{N}$ and assume that $n \leq m+1$. If $n \leq m$ then by induction hypothesis n has a binary representation. All we need to do is prove $n = m+1$. We will use proof by cases for even and odd terms:

Case 1. Assume n is even: $\exists k \in \mathbb{N}, n = 2k$.

By the properties of divisibility, we know that since $k \mid n$, $k < n$. Then by the hypothesis, there exists $p \in \mathbb{N}$ and $b_p, \ldots, b_0 \in \{0, 1\}$ such that $k = \sum_{i=0}^{p} b_i 2^i$. Then n would be

$$\begin{aligned} n &= 2\sum_{i=0}^{p} b_i 2^i \\ &= \sum_{i=0}^{p} b_i 2^i \cdot 2 \\ &= \sum_{i=0}^{p} b_i 2^{i+1} \end{aligned}$$

We want to shift the sum so that we can go from 0 to $p+1$.

Let $p' = p+1$, and let $b'_0 = 0$, and for all $i \in \{1, 2, \ldots, p+1\}$, let $b'_i = b_{i-1}$ Then $n = \sum_{i=0}^{p'} b'_i 2^i$, which is equal to the former since $b'_0 = 0$.

Case 2. Assume that n is odd: $\exists k \in \mathbb{N}, n = 2k+1$.

Similar to the last case, we get $n = \left(\sum_{i=0}^{p} b_i 2^{i+1}\right) + 1$.

We can do the same as above, but make $b'_0 = 1$ to incorporate the $+1$ term, then $n = \sum_{i=0}^{p'} b'_i 2^i$.

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The problem with this representation is that they are not unique.[1] For example, we could represent the number 2 as 10, or 010, or 0010. So computer scientists can use the following theorem to induce unique representations:

[1] Since we proved an existential, it only states *at least one*, not exactly one.

Theorem 4.2. For every number $n \in \mathbb{Z}^+$, there exist **unique** values $p \in \mathbb{N}$ and $b_p, \ldots, b_0 \in \{0, 1\}$ such that both the following hold:

1. $n = \sum_{i=0}^{p} b_i 2^i$ (this is a binary representation of n)
2. $b_p = 1$ (this representation has no leading zeroes)

2.3 Dividing by two

Lemma 4.3. let $n \in \mathbb{N}$, and assume $n \geq 2$. Let the binary representation of n be $b_p, \ldots, b_0$, where $b_p = 1$. Then the binary represnetaiton of $\lfloor n/2 \rfloor = b_p, \ldots, b_1$ (rightmost digit removed).

Proof. Let $n \in \mathbb{N}$, and assume $n \geq 2$. Let $p \in \mathbb{N}$ and $b_0, \ldots, b_p \in \{0, 1\}$ such that $n = \sum_{i=0}^{p} b_i 2^i$ and $b_p = 1$. We divide the proof into even or odd cases:

Case 1. Assume n is even. Then $b_0 = 0$, and thus

$$
\begin{aligned}
\left\lfloor \frac{n}{2} \right\rfloor &= \frac{n}{2} \\
&= \frac{\sum_{i=0}^{p} b_i 2^i}{2} && \text{since } b_0 = 0 \\
&= \frac{\sum_{i=1}^{p} b_i 2^i}{2} \\
&= \sum_{i=1}^{p} b_i 2^{i-1} \\
&= \sum_{i=1}^{p-1} b_{i+1} 2^i
\end{aligned}
$$

Case 2. Assume n is odd. Then $b_0 = 1$, and thus

$$
\begin{aligned}
\left\lfloor \frac{n}{2} \right\rfloor &= \frac{n-1}{2} \\
&= \frac{\left(\sum_{i=0}^{p} b_i 2^i\right) - 1}{2} \\
&= \frac{\left(\sum_{i=1}^{p} b_i 2^i\right) + 1 \cdot 2^0 - 1}{2} && \text{since } b_0 = 1 \\
&= \frac{\sum_{i=1}^{p} b_i 2^i}{2} \\
&= \sum_{i=1}^{p} b_i 2^{i-1} \\
&= \sum_{i=0}^{p-1} b_{i+1} 2^i
\end{aligned}
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