Due Wednesday 22 March, *before* 10:00am

General instructions for Problem Sets

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Solutions must be typeset and submitted as a PDF with the correct filename. **Handwritten submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set3.pdf**.
- Each problem set may be completed in groups of up to three—**except for Problem Set 0**. If you are working in a group for this problem set, please consult https://github.com/MarkUsProject/Markus/wiki/Student-Guide for a brief explanation of how to create a group on MarkUs.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with one or more partner(s), you must form a group on MarkUs, and make one submission per group.
- Your submitted file(s) should not be larger than 19MB. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF; in that case, you should look into PDF compression tools to make your PDF smaller, but please make sure that your PDF is still legible before submitting!
- Submissions must be made *before* the due date on MarkUs. You may use *grace credits* to extend the deadline; please see the course syllabus for details on using grace credits.
- MarkUs may be slow when many students try to submit right before a deadline. **Aim to submit your work at least one hour before the deadline. It is your responsibility to meet the deadline.** You can submit your work more than once (and you are encouraged to do so); the most recent version submitted within the deadline (or within the late submission period) is the version that will be marked.
- The work you submit must be that of your group; you may not use or copy from the work of other groups, or external sources like websites or textbooks. Please see the section on Academic Integrity in the course syllabus for further details.

Additional instructions

- When writing a proof by induction, always label the step(s) that use the induction hypothesis.
- You may **not** use forms of induction that were not covered in lecture, except where indicated in the question. In particular, you may not use strong/complete induction.
- Please follow the same guidelines as in Problem Set 2 for all proofs.

1. [8 marks] Number representation.

For each $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, define $C(n,k)$ to be:

$$\exists a_1, \ldots, a_k \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq k \Rightarrow a_i \leq i) \wedge (n = \sum_{i=1}^{k} a_i \cdot i!)$$

Prove, using Induction, that: $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}^+, n < (k+1)! \Rightarrow C(n,k)$.

Here we use the factorial function, where for each $m \in \mathbb{N}$, $m! = \prod_{j=1}^{m} j$.

2. [12 marks] Induction.

For each $m, n \in \mathbb{N}$, let $A_m = \{a \mid a \in \mathbb{N} \wedge a \leq m\}$ and $B_n = \{b \mid b \in \mathbb{N} \wedge b \leq n\}$, and define $F_{m,n}$ to be:

$$\{f : A_m \to B_n \mid [\ \forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)\ ] \wedge f(m) = n\}$$

For each $m, n \in \mathbb{N}$, define $P(m,n)$ to be:

$$|F_{m,n}| = \frac{(m+n)!}{m! \cdot n!}$$

(a) [6 marks]

Prove each of the following statements:

i. $\forall m \in \mathbb{N}, P(m,0)$.

ii. $\forall n \in \mathbb{N}, P(0,n)$.

iii. $\forall m, n \in \mathbb{N}, P(m, n+1) \wedge P(m+1, n) \Rightarrow P(m+1, n+1)$.

(b) [2 marks]

Prove, using the results from part (a), that: $P(1,1) \wedge P(2,2)$.

(c) [3 marks]

For each $t \in \mathbb{N}$, define $Q(t)$ to be: $\forall m, n \in \mathbb{N}, m + n = t \Rightarrow P(m,n)$.

Prove, using Induction and the results from part (a), that: $\forall t \in \mathbb{N}, Q(t)$.

(d) [1 mark]

Prove, using the result from part (c), that: $\forall m, n \in \mathbb{N}, P(m,n)$.

3. [8 marks] Asymptotic notation.

For the following questions use the definitions of $\mathcal{O}$, Ω, and Θ, *not* our various results about them.

(a) [3 marks]

Prove or disprove that $n^n \in \mathcal{O}(n!)$.

(b) [5 marks]

Prove that if $a, b \in \mathbb{R}$ and $b > 0$, then $(n+a)^b \in \Theta(n^b)$.

4. [7 marks] More asymptotic notation.

For the following questions use the definitions of $\mathcal{O}$, Ω, and Θ, *not* our various results about them.

(a) [3 marks]
Prove or disprove that: if $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $k \in \mathbb{R}^+$, and $f(n) \in \mathcal{O}(n^k)$, then $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$.

(b) [4 marks]
Prove that: if $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_1 \in \mathcal{O}(g_1)$, and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$.
Here, $(f_1 + f_2)(n) = f_1(n) + f_2(n)$ and $\max(g_1, g_2)(n) = \max(g_1(n), g_2(n))$.