CSC165H1: Problem Set 3

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Question 1. Number representation.

For all $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, we define $C(n,k)$ to be:

$$\exists a_1, \ldots, a_k \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq k \Rightarrow a_i \leq i) \wedge (n = \sum_{i=1}^{k} a_i \cdot i!)$$

Prove using induction that $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}^+, n < (k+1)! \Rightarrow C(n,k)$.

Proof. We define the predicate $P(k)$: "$\forall n \in \mathbb{N}, n < (k+1)! \Rightarrow C(n,k)$", where $k \in \mathbb{Z}^+$.

Base case. Let $k = 1$. We want to prove $P(1)$. Let $n \in \mathbb{N}$. Assume $n < 2!$. Since $k = 1$, we know $n = \sum_{i=1}^{1} a_i \cdot i! = a_1$. And a_1 can be either 1 or 0, and both cases will result in $n < 2$. Thus the base case is True for $P(1)$.

Inductive step. Let $x \in \mathbb{Z}^+$. Assume $P(x)$ is True, such that $\forall n \in \mathbb{N}, n < (x+1)! \Rightarrow C(n,x)$. We want to prove that $P(x+1)$ is True, such that $\forall n \in \mathbb{N}, n < (x+2)! \Rightarrow C(n, x+1)$.

Let $n \in \mathbb{N}$. Assume $n < (x+2)!$. We want to prove that $C(n, x+1)$. Since we know that $n < (x+1)!$ or $n \geq (x+1)!$, we can divide the proof into two cases:

Case 1. Assume $n < (x+1)!$.

Then we know $C(n,x)$ is True by the induction hypothesis. So we know that $\exists a_1, \ldots, a_x \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq x \Rightarrow a_i \leq i)$, and $n = \sum_{i=1}^{x} a_i \cdot i!$. In other words, $a_1, \ldots, a_k$ satisfy $n = \sum_{i=1}^{x} a_i \cdot i!$.

Let $a'_1 = a_1, a'_2 = a_2, \ldots, a'_x = a_x, a'_{x+1} = 0$. Then we have

$$\begin{aligned} n &= a_1 \cdot 1! + a_2 \cdot 2! + \ldots + a_x \cdot x! + 0 \cdot (x+1)! \\ &= a'_1 \cdot 1! + a'_2 \cdot 2! + \ldots + a'_x \cdot x! + a'_{x+1} \cdot (x+1)! \\ &= \sum_{i=1}^{x+1} a_i \cdot i! \end{aligned}$$

We know that $a'_{x+1} = 0$ satisfies the first part of the predicate since $0 < i$ for all positive integers i. So $C(n, x+1)$ is True for $n < (x+1)!$.

Case 2. Assume $n \geq (x+1)!$.

By the assumptions of the statement we want to prove, we know that n can be represented using the Quotient Remainder Theorem: $\exists q, r \in \mathbb{Z}$ such that all n between $(x+1)! \leq n < (x+2)!$ can be written as $q(x+1)! + r$. We know that $0 \leq r < (x+1)!$ by the definition of theorem, so r can already be represented by $P(x)$ using the proof in Case 1 such that there exists $a_1, \ldots, a_x$ that satisfy $r = \sum_{i=1}^{x} a_i \cdot i!$.

We also know that in order for n to be less than $(x+2)!$, q has to be less than $(x+2)$, or in other words, $q \leq (x+1)$. Let $q = a_{x+1}$, then we get

$$\begin{aligned} n &= q \cdot (x+1)! + r \\ n &= a_{x+1} \cdot (x+1)! + \sum_{i=1}^{x} a_i \cdot i! \\ n &= \sum_{i=1}^{x+1} a_i \cdot i! \end{aligned}$$

We know that $a_{x+1} = q$ satisfies the first part of the statement since $q \leq (x+1)$.

So $C(n, x+1)$ is True by the assumption of $n < (x+2)!$.

Thus $P(x+1)$ is True by the assumption of $P(x)$, so we have proven the predicate $P(k)$: "$\forall n \in \mathbb{N}, n < (k+1)! \Rightarrow C(n,k)$" where $k \in \mathbb{Z}^+$ by induction.

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Question 2. Induction.

For all $m, n \in \mathbb{N}$, let $A_m = \{a \mid a \in \mathbb{N} \wedge a \leq m\}$ and $B_n = \{b \mid b \in \mathbb{N} \wedge b \leq n\}$, and define $F_{m,n}$ to be:

$$\{f : A_m \to B_n \mid [\forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)] \wedge f(m) = n\}$$

For all $m, n \in \mathbb{N}$, define $P(m, n)$ to be:

$$|F_{m,n}| = \frac{(m+n)!}{m! \cdot n!}$$

(a) Prove each following statement.

(i) $\forall m \in \mathbb{N}, P(m, 0)$.

Proof. Let $m \in \mathbb{N}$. We want to prove that $P(m, 0)$ is True.

Since the codomain is a set that contains only 0, the size of $F_{m,n}$ would only be 1, since the only function that would satisfy the conditions would be the one that mapped all m to 0. It would look like a horizontal line at the origin, since $f(0), \ldots, f(m)$ must all be natural numbers less than or equal to 0.

$$\begin{aligned} LHS &= |F_{m,n}| \\ &= 1 \\ RHS &= \frac{(m+0)!}{m! \cdot 0!} \\ &= \frac{m!}{m!} \\ &= 1 \\ LHS &= RHS = 1 \end{aligned}$$

Thus we have proven both sides of $P(m, 0)$ are equal, so the statement is True for all $m \in \mathbb{N}$.

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(ii) $\forall n \in \mathbb{N}, P(0, n)$.

Proof. Let $n \in \mathbb{N}$. We want to prove that $P(0, n)$ is True.

Since the domain is a set that only contains 0, we would have one function in $F_{m,n}$ since $A_m = \{0\}$. The function would only have $f(0) = n$.

$$\begin{aligned} LHS &= |F_{m,n}| \\ &= 1 \\ RHS &= \frac{(0+n)!}{0! \cdot n!} \\ &= \frac{n!}{n!} \\ &= 1 \\ LHS &= RHS = 1 \end{aligned}$$

Thus we have proven both sides of $P(0, n)$ are True, so the statement is True for all $n \in \mathbb{N}$.

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(iii) $\forall m, n \in \mathbb{N}, P(m, n+1) \wedge P(m+1, n) \Rightarrow P(m+1, n+1)$.

Proof. Let $m, n \in \mathbb{N}$. Assume $P(m, n+1) \wedge P(m+1, n)$. We want to show that $P(m+1, n+1)$, such that $|F_{m+1,n+1}| = \frac{(m+n+2)!}{(m+1)!(n+1)!}$.

We know every function in $F_{m,n+1}$ has $f(m) = n+1$. To convert these functions to members of $F_{m+1,n+1}$, we would have to add the assignment $f(m+1) = n+1$. These functions would include all cases where $f(m) = n+1$.

We need to also do this for $F_{m+1,n}$, and make $f(m+1) = n+1$ for all of these functions. This new set would have all cases where $f(m) \leq n$.

We are not introducing any new functions into these sets, but instead modifying the existing functions so $f(m+1) = n+1$. These functions would all satisfy the statement by the hypothesis.

There would not be any overlaps since the $F_{m,n+1}$ set includes of all cases where $f(m) = n+1$ and the $F_{m+1,n}$ group takes care of all cases where $f(m) \leq n$.

The cardinality of $F_{m+1,n+1}$ would be the total amount of functions that that can have their final term be $f(m+1) = n+1$, such that $f(m) \leq n+1$, which means the union of $F_{m,n+1}$ and $F_{m+1,n}$ would cover all cases where $f(m) \leq n+1$ and $f(m+1) = n+1$.

In other words, $|F_{m+1,n+1}| = |F_{m,n+1}| + |F_{m+1,n}|$.

Substituting into the equation, we have

$$\begin{aligned}
LHS &= |F_{m+1,n+1}| \\
&= \frac{(m+n+2)!}{(m+1)!(n+1)} && \text{(By definition of } P(m,n)) \\
RHS &= |F_{m,n+1}| + |F_{m+1,n}| \\
&= \frac{(m+n+1)!}{m!(n+1)!} + \frac{(m+n+1)!}{(m+1)!n!} && \text{(By assumption)} \\
&= \frac{(m+n+1)!(m+1)}{(m+1)!(n+1)!} + \frac{(m+n+1)!(n+1)}{(m+1)!(n+1)!} \\
&= \frac{(m+n+1)!((m+1)+(n+1))}{(m+1)!(n+1)!} \\
&= \frac{(m+n+1)!(m+n+2)}{(m+1)!(n+1)!} \\
&= \frac{(m+n+2)!}{(m+1)!(n+1)!} \\
LHS &= RHS
\end{aligned}$$

Both sides are equal assuming $P(m, n+1)$ and $P(m+1, n)$, thus $P(m+1, n+1)$ is True.

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(b) Prove, using results from part (a), that $P(1,1) \wedge P(2,2)$.

Proof. We want to prove $P(1,1) \wedge P(2,2)$. Let $m, n \in \mathbb{N}$. By part (a), we know that $P(0,n)$ and $P(m,0)$ are both True. If we let $m = 1$ and $n = 1$, then $P(0,1)$ and $P(1,0)$ are both True. Similarly for $P(0,2)$ and $P(2,0)$.

By part (a), we know that if $P(m, n+1) \wedge P(m+1, n)$, then $P(m+1.n+1)$ would be also be True. If we let $m = 0$ and $n = 0$, we get $P(0,1) \wedge P(1,0)$, which we know are both True, then we know that $P(1,1)$ is also True.

We can also do the same for $P(2,0)$ and $P(1,1)$ by letting $m = 1$ and $n = 0$ and proving $P(2,1)$. And the same goes for $P(0,2)$ and $P(1,1)$ by letting $m = 0$, $n = 1$, proving $P(1,2)$.

Finally, using $m = 1$ and $n = 1$, we have $P(2,1) \wedge (1,2) \Rightarrow P(2,2)$.

Thus both $P(1,1)$ and $P(2,2)$ are True by the results from part (a).

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(c) For each $t \in \mathbb{N}$, define $Q(t)$ to be $\forall m, n \in \mathbb{N}, m + n = t \Rightarrow P(m,n)$. Prove using induction and the results from part (a), that: $\forall t \in \mathbb{N}, Q(t)$

Proof. Let $t \in \mathbb{N}$. We want to prove $Q(t)$ such that for all $m, n \in \mathbb{N}$ if $m + n = t$, then $P(m,n)$.

Base case. Let $t = 0$. Assume that $t = m + n$. We want to show that $P(m,n)$ is True. Since m, n are both natural numbers, they must both be 0 since $m + n = 0$. Then $P(0,0)$ is True by the results in part (a).

Inductive step. Let $k \in \mathbb{N}$. Assume that $Q(k)$ is True such that $m_1 + n_1 = k \Rightarrow P(m_1, n_1)$. We want to prove that $Q(k+1)$ is True, such that $m_2 + n_2 = k + 1 \Rightarrow P(m_2, n_2)$. We will use a proof by cases.

We know that if $m_2 + n_2 = k + 1$, either $m_2 = m_1 + 1 \wedge n_2 = n_1$, or $m_2 = m_1 \wedge n_2 = n_1 + 1$.

Case 1. Assume $m_2 = m_1 + 1 \wedge n_2 = n_1$.

Using the induction hypothesis, we have $P(m_1, n_1) = P(m_2 - 1, n_2)$. We also have $P(m_2, n_2 - 1)$ by rearranging $k = m_1 + n_1 = m_2 - 1 + n_2 = m_2 + n_2 - 1$. By part (a), we can conclude that since $P(m_2 - 1, n_2) \wedge P(m_2, n_2 - 1)$, that $P(m_2, n_2)$ is True in this case.

Case 2. Assume $m_2 = m_1 \wedge n_2 = n_1 + 1$.

Following the same procedure as Case 1, we have $P(m_1, n_1) = P(m_2, n_2 - 1)$, and rearranging $k = m_2 + n_2 - 1$ we get $P(m_2 - 1, n_2)$. By part (a), we can conclude that that since $P(m_2 - 1, n_2) \wedge P(m_2, n_2 - 1)$, that $P(m_2, n_2)$ is also True in this case.

Thus we have proven that $P(m_2, n_2)$ is True in both cases by the assumption that $m_2 + n_2 = k + 1$, such that $Q(k+1)$ is True by the assumption that $Q(k)$ is True. We have proven $Q(t)$ by induction.

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(d) Prove, using results form part (c), that: $\forall m, n \in \mathbb{N}, P(m,n)$.

Proof. Let $m, n \in \mathbb{N}$. We want to prove that $P(m,n)$.

Let $t \in \mathbb{N}$. From part (c), we know that if $m + n = t$ then $P(m,n)$ is True.

If you add two natural numbers, the result will always be a natural number, so there will always be a natural number t such that $m + n = t$.

Thus, by part (c), $P(m,n)$ is True for any natural numbers m, n.

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Question 3. Asymptotic notation.

(a) Prove or disprove that $n^n \in \mathcal{O}(n!)$.

Translation. $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^n \leq c \cdot n!$

Proof. Let $n \in \mathbb{N}$. We want to prove that $n^n \notin \mathcal{O}(n!)$. We will use a proof by contradiction.

Let c, n_0 be arbitrary positive real numbers. Assume that $n^n \in \mathcal{O}(n!)$, such that if $n \geq n_0$ then

$$\begin{aligned} n^n &\leq c \cdot n! \\ \frac{n^n}{n!} &\leq c \end{aligned}$$

Then there would exist a c that is always greater than $\frac{n^n}{n!}$ by the assumption that $n^n \in \mathcal{O}(n!)$.

We know that n^n is larger than $n!$ since if we line up all values, the values of n^n are all greater than $n!$ after the first term, so the ratio grows to infinity:

$$\begin{aligned} n^n &= n \cdot n \cdot \ldots \cdot n \cdot n && (n \text{ terms}) \\ n! &= n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 && (n \text{ terms}) \end{aligned}$$

But we know that there does not exist a largest natural number, since the set is infinite. Thus, there cannot exist a c that always satisfies this inequality, creating a contradiction.

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(b) Prove that if $a, b \in \mathbb{R}$ and $b > 0$, then $(n+a)^b \in \Theta(n^b)$.

Proof. Let $a, b \in \mathbb{R}$. Assume that $b > 0$. We want to prove that $(n+a)^b \in \Theta(n^b)$.

Expanding on the definition of $\Theta(n^b)$, we have

$$\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 n^b \leq (n+a)^b \leq c_2 n^b$$

Let $c_1 = (\frac{1}{2})^b$, $c_2 = (\frac{3}{2})^b$, $n_0 = 2|a|$. Assume $n \geq n_0$. We want to show that $c_1 n^b \leq (n+a)^b \leq c_2 n^b$. Solving for the left side we get

$$\begin{aligned} (n+a)^b &= \left(\frac{n}{2} + \frac{n}{2} + a\right)^b \\ &\geq \left(\frac{n}{2} + |a| + a\right)^b && (\text{Since } n > 2|a|) \\ &\geq \left(\frac{n}{2}\right)^b && (\text{Since } |a| + a \geq 0) \\ &\geq \left(\frac{1}{2}\right)^b \cdot n^b \\ &\geq c_1 n^b \end{aligned}$$

Solving for the right side we get

$$\begin{aligned} (n+a)^b &\leq \left(n + \frac{n}{2}\right)^b && (\text{Since } n \geq 2|a|) \\ &\leq \left(\frac{3n}{2}\right)^b \\ &\leq \left(\frac{3}{2}\right)^b \cdot n^b \\ &\leq c_2 n^b \end{aligned}$$

Thus we have proven $c_1 n^b \leq (n+a)^b \leq c_2 n^b$ assuming $n \geq n_0$ and $b > 0$.

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Question 4. More asymptotic notation.

(a) Prove or disprove that: if $f : \mathbb{N} \to \mathbb{R}^{\geq 0}, k \in \mathbb{R}^+$ and $f(n) \in \mathcal{O}(n^k)$, then $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$.

Proof. Let $k \in \mathbb{R}^+$. We want to show that if $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $f(n) \in \mathcal{O}(n^k)$, then $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$. Let n_0 be an arbitrary positive real number. Let $c = k + 1$. Let $n_0 = c$. Let $n \in \mathbb{N}$. Then by our assumption and the definition of Big-Oh, if $n \geq n_0$, we have

$$\begin{aligned}
f(n) &\in \mathcal{O}(n^k) \\
f(n) &\leq c \cdot n^k \\
\log_2(f(n)) &\leq \log_2(c \cdot n^k) \\
\log_2(f(n)) &\leq \log_2(c) + \log_2(n^k) && \text{(Using log identity)} \\
\log_2(f(n)) &\leq \log_2(n) + \log_2(n^k) && \text{(Since } n \geq c\text{)} \\
\log_2(f(n)) &\leq log_2(n) + k \cdot \log_2(n) \\
\log_2(f(n)) &\leq (k+1) \cdot \log_2(n) \\
\log_2(f(n)) &\leq c \cdot \log_2(n) \\
\log_2(f(n)) &\in \mathcal{O}(\log_2(n))
\end{aligned}$$

Thus we have proven the $\log_2(f(n)) \in \mathcal{O} \log_2 n$ by the assumption that $f(n) \in \mathcal{O}(n^k)$.

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(b) Prove that: if $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, f_1 \in \mathcal{O}(g_1)$, and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(max(g_1, g_2))$. Here, $(f_1 + f_2)(n) = f_1(n) + f_2(n)$ and $\max(g_1, g_2)(n) = \max(g_1(n), g_2(n))$.

Proof. Assume that $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_1 \in \mathcal{O}(g_1)$, and $f_2 \in \mathcal{O}(g_2)$. We want to show that $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$.

Using the definition of Big-Oh, we can express the hypothesis as $\exists c_1, c_2, n_1, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, (n \geq n_1 \Rightarrow f_1 \leq c_1 \cdot g_1) \wedge (n \geq n_2 \Rightarrow f_2 \leq c_2 \cdot g_2)$.

Let $c_3 = c_1 + c_2$, let $n_3 = \max(n_1, n_2)$. Then by our assumptions, we know that $f_1 \leq c_1 \cdot g_1$ and $f_2 \leq c_2 \cdot g_2$. Adding these together we have

$$\begin{aligned}
f_1 + f_2 &\leq c_1 \cdot g_1 + c_2 \cdot g_2 \\
f_1 + f_2 &\leq c_1 \cdot \max(g_1, g_2) + c_2 \cdot \max(g_1, g_2) && (\max(g_1, g_2) \geq g_1, \text{ same for } g_2) \\
f_1 + f_2 &\leq (c_1 + c_2) \cdot \max(g_1, g_2) \\
f_1 + f_2 &\leq c_3 \cdot \max(g_1, g_2) \\
f_1 + f_2 &\in \mathcal{O}(\max(g_1, g_2)) && (n \geq n_3)
\end{aligned}$$

Thus we have proven $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$ assuming that $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$.

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