Due Friday 3 March, *before* 1:00pm

## General instructions for Problem Sets

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Solutions must be typeset and submitted as a PDF with the correct filename. **Handwritten submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set2.pdf**.
- Each problem set may be completed in groups of up to three—**except for Problem Set 0**. If you are working in a group for this problem set, please consult https://github.com/MarkUsProject/Markus/wiki/Student-Guide for a brief explanation of how to create a group on MarkUs.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with one or more partner(s), you must form a group on MarkUs, and make one submission per group.
- Your submitted file(s) should not be larger than 19MB. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF; in that case, you should look into PDF compression tools to make your PDF smaller, but please make sure that your PDF is still legible before submitting!
- Submissions must be made *before* the due date on MarkUs. You may use *grace credits* to extend the deadline; please see the course syllabus for details on using grace credits.
- MarkUs may be slow when many students try to submit right before a deadline. **Aim to submit your work at least one hour before the deadline. It is your responsibility to meet the deadline.** You can submit your work more than once (and you are encouraged to do so); the most recent version submitted within the deadline (or within the late submission period) is the version that will be marked.
- The work you submit must be that of your group; you may not use or copy from the work of other groups, or external sources like websites or textbooks. Please see the section on Academic Integrity in the course syllabus for further details.

## Additional instructions

- You may **not** define your own propositional operators, predicates, or sets for this problem set, unless the question asks for it explicitly.
- For every question that asks you for a proof, remember to write detailed proofs based on the logical structure of the statement being proven, including complete proof headers to introduce variables and assumptions, and deductions with justification for each step (other than basic algebra and arithmetic). In particular, clearly identify every fact and definition required to justify your deductions.

- For proofs involving *divisibility*, *primes*, or the *floor or ceiling* functions, you **may** use (without proof) all definitions and facts provided during lectures, in the course notes, or on worksheets. You may **not** use external facts from other sources. (Remember, the point of these problems is to get you to practice writing precise proofs, NOT to dig up facts about the concepts involved.)

  You *may* make statements about concrete numbers, without proof. For example, "7 is odd" or "$8 \mid 24$' or "15 is not prime".

- For all other concepts, you may use *definitions* that were covered in lectures, the course notes, or worksheets, but you may **not** use any external facts about these definitions, unless they are explicitly stated as part of the question.

**1. [6 marks] Number theory.**

**(a)** [**2 marks**] Prove that $\forall n \in \mathbb{Z}, \gcd(9n+1, 10n+1) = 1$.

**HINT**: Use some of the facts from the Week 4 worksheets for a very short proof.

**(b)** [**4 marks**]
Prove the following statement:

$$\forall m, n \in \mathbb{Z}, n \mid m \land Prime(n) \Rightarrow n \nmid (m+1)$$

**2. [6 marks] Floors and ceilings.** For all proofs in this question, you may use the following fact from a worksheet:

$$\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1, \tag{1}$$

and its generalization to the ceiling function:

$$\forall x \in \mathbb{R}, 0 \leq \lceil x \rceil - x < 1. \tag{2}$$

**(a)** [**2 marks**] Prove that

$$\forall x \in \mathbb{Z}, \left\lceil \frac{x-1}{2} \right\rceil = \left\lfloor \frac{x}{2} \right\rfloor$$

**(b)** [**4 marks**] Prove or disprove each of the following. In each case, first write down in symbolic notation the exact statement you are attempting to prove (either the original statement or its negation).

i. $\forall x \in \mathbb{R}, \lceil x - 1 \rceil = \lceil x \rceil - 1$

ii. $\forall x, y \in \mathbb{R}, \lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$

**3. [8 marks] Induction.**

**(a)** [**3 marks**] Prove that for all natural numbers $n$, $9 \mid 11^n - 2^n$.

**(b)** [**5 marks**] Recall the definition of **Pierre Numbers** from Problem Set 1:

> A natural number $p$ is said to be a "Pierre Number" when it can be expressed as $2^{2^k} + 1$ for some integer $k$.

Consider the sequence of Pierre numbers $p_n = 2^{2^n} + 1$, for $n \in \mathbb{N}$. Prove that for all $n \in \mathbb{N}$,

$$p_n = \prod_{i=0}^{n-1} p_i + 2$$

(**Hint**: You may find it easier to expand the product.)