CSC165H1: Problem Set 1

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Question 1. Propositional formulas

(a) $(\neg p \Leftrightarrow q) \Rightarrow q$

(i) Truth table.

p	q	$(\neg p \Leftrightarrow q) \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

(ii) Logically equivalent formula.

$$(\neg p \Leftrightarrow q) \Rightarrow q$$
$$((\neg p \Rightarrow q) \wedge (q \Rightarrow \neg p)) \Rightarrow q$$
$$((p \vee q) \wedge (\neg q \vee \neg p)) \Rightarrow q$$
$$\neg((p \vee q) \wedge (\neg q \vee \neg p)) \vee q$$
$$\neg(p \vee q) \vee \neg(\neg q \vee \neg p) \vee q$$

$$\boxed{(\neg p \wedge \neg q) \vee (q \wedge p) \vee q}$$

(b) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$

(i) Truth table.

p	q	r	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$
True	True	True	True
True	True	False	True
True	False	True	True
True	False	False	True
False	True	True	True
False	True	False	False
False	False	True	True
False	False	False	False

(ii) Logically equivalent formula.

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$$
$$\neg(\neg p \vee (\neg q \vee r)) \vee (\neg(\neg p \vee q) \vee r)$$
$$(p \wedge \neg(\neg q \vee r)) \vee ((p \wedge \neg q) \vee r)$$

$$\boxed{(p \wedge q \wedge \neg r) \vee ((p \wedge \neg q) \vee r)}$$

Question 2. **Translating statements.**

(a) $\forall s \in S,\ \forall c \in C,\ CS(s) \Rightarrow \neg Fail(s,c)$

(b) $\exists c \in C,\ \forall s \in S,\ CS(s) \Rightarrow Study(s,c)$

(c) $\exists s \in S,\ \forall c \in C,\ CS(s) \wedge \neg Study(s,c)$

(d) $\forall c \in C,\ \exists s_1 \in S,\ \forall s_2 \in S,\ Study(s_1,c) \wedge (Study(s_2,c) \Rightarrow s_1 = s_2)$

(e) $\forall s \in S,\ \exists c_1, c_2 \in C, \forall c_3 \in C,\ Study(s,c_1) \wedge Study(s,c_2) \wedge c_1 \neq c_2 \wedge (Study(s,c_3) \Rightarrow (c_3 = c_1 \vee c_3 = c_2))$

Question 3. **Choosing a univserse and predicates**

$$\forall x \in \mathbb{N}, P(x, 165) \vee P(x, 0) \qquad \text{(Statement 1)}$$
$$(\forall x \in \mathbb{N}, P(x, 165)) \Rightarrow (\exists x \in \mathbb{N}, P(x, 0)) \qquad \text{(Statement 2)}$$

(a) (Statement 1) is False and (Statement 2) is True when both terms of (Statement 1) are False — in this case, (Statement 2) will always be True due to the vacuous truth cases for implication operators. So, we can define the binary predicate P as the following:

$$\boxed{P(a, b) : \text{“}b \mid a,\text{” where } a, b \in \mathbb{N}}$$

Then (Statement 1) would be False since 165 does not divide all natural numbers, and 0 does not divide all natural numbers. And (Statement 2) would be True since the hypothesis is False from the first term of (Statement 1).

(b) For both statements to be True, only one term of (Statement 1) has to be True due to the OR operator, and either the hypothesis of (Statement 2) has to be False, or both terms have to be True. So, we can define the binary predicate P as the following:

$$\boxed{P(a, b) : \text{“}a \mid b,\text{” where } a, b \in \mathbb{N}}$$

Then (Statement 1) is True since all natural numbers divide 0 ($0 \cdot x = 0$), and (Statement 2) is True since the hypothesis is False (not all natural numbers divide 165).

(c)

$$\forall x \in S, \Big(\exists y \in T, P(x, y)\Big) \Rightarrow Q(x) \qquad \text{(Statement 3)}$$
$$\forall x \in S, Q(x) \Rightarrow \Big(\forall y \in T, P(x, y)\Big) \qquad \text{(Statement 4)}$$

Let's choose Q as the False predicate use it to find P. We will first define the sets S and T:

$$\boxed{S = \{x \mid x \in \mathbb{N} \text{ and } 0 \leq x \leq 10\}}$$
$$\boxed{T = \{y \mid y \in \mathbb{N} \text{ and } 5 \mid y\}}$$

If Q is False for $\forall x \in S$, then we can define a predicate P that is True for at least one $y \in T$, but False for all $y \in T$. This will make (Statement 3) False and (Statement 4) True. We will now define a binary predicate P that is True for $\forall x \in S$ and $\exists y \in T$, but False for $\forall x \in S$ and $\forall y \in T$, and a unary predicate Q that is False in the case of $\forall x \in S$:

$$\boxed{P(a, b) : \text{“}10 \mid (a \cdot b)\text{”}, \text{ where } a \in S \text{ and } b \in T}$$
$$\boxed{Q(a) : \text{“}3 \mid a,\text{” where } a \in S}$$

Thus P is True for some but not all $y \in T$ (10 divides 20, but 10 does not divide 15), and Q is False in the case of $\forall x \in S$ (3 does not divide 2), and they are both not constant functions. (Statement 3) will now be $True \Rightarrow False$, which is False, and (Statement 4) will be $False \Rightarrow True$, which is True.

Question 4. Pierre Numbers

(a)

$$PierreNumber(n) : \exists k \in \mathbb{Z},\ n = 2^{2^k} + 1,\ \text{where } n \in \mathbb{N}$$

(b)

$$\forall n \in \mathbb{N},\ PierreNumber(n) \Rightarrow (\exists k \in \mathbb{Z},\ n = 2k + 1)$$

(c)

$$\forall n \in \mathbb{N},\ \exists p \in \mathbb{N},\ p > n \land PierreNumber(p)$$

(d)

$$\forall n \in \mathbb{N},\ \exists k \in \mathbb{Z},\ n = 2^{2^k} + 1 \land k < 5 \Rightarrow Prime(n)$$