CSC165H1: Problem Set 2

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2023-02-26

Question 1. Number theory.

(a) Prove that $\forall n \in \mathbb{N}$, $\gcd(9n+1, 10n+1) = 1$.

Proof. Let n be an arbitrary natural number, let $r = 10$, and let $s = -9$. We want to prove that for all natural numbers n, $\gcd(9n+1, 10n+1) = 1$. Using Fact 6 from the Week 4 worksheet, we can rewrite the statement to

$$\forall n \in \mathbb{N}, \exists r, s \in \mathbb{Z}, r(9n+1) + s(10n+1) = 1.$$

Substituting r and s into the equation, we get

$$\begin{aligned} 10(9n+1) - 9(10n+1) &= 90n + 10 - 90n - 9 \\ &= 1 \end{aligned}$$

By the definition of $\gcd(a, b)$, the gcd is the smallest postive integer that can be expressed as a linear combination of a, b. Since 1 is the smallest positive integer, we can conclude that $\gcd(9n+1, 10n+1) = 1$.

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(b) Prove that $\forall m, n \in \mathbb{Z}, n \mid m \wedge Prime(n) \Rightarrow n \nmid (m+1)$.

Proof. Let $m, n \in \mathbb{Z}$. Assume $n \mid m \wedge Prime(n)$. We want to prove that $n \nmid (m+1)$. We will use a proof by contradiction and assume $n \mid (m+1)$. By the definition of divisibility, let there be $k_1, k_2 \in \mathbb{Z}$, such that $k_1 n = m$ and $k_2 n = m + 1$. Substituting these equations we get

$$\begin{aligned} k_2 n &= k_1 n + 1 \\ k_2 n - k_1 n &= 1 \\ n(k_2 - k_1) &= 1 \end{aligned}$$

Then there exists integers k_1, k_2 such that $n(k_2 - k_1) = 1$ by the assumption that $n \mid (m+1)$.

However, the only scenarios when $n(k_2 - k_1) = 1$ are when $n = 1$ or $n = -1$ since n is an integer. We know that n is prime, so n has to be greater than 1 by the definition of $Prime(n)$, which rules out both $n = -1$ and $n = 1$. Then there does not exist k_1, k_2 such that $n(k_2 - k_1) = 1$, this is a contradiction.

The statement $n(k_2 - k_1) = 1$ is both True *and* False by the assumption that $n \mid (m+1)$. Thus, $n \nmid (m+1)$ is True by proof by contradiction when n is Prime and $n \mid m$.

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Question 2. Floors and ceilings.

(a) Prove that $\forall x \in \mathbb{Z}, \lceil \frac{x-1}{2} \rceil = \lfloor \frac{x}{2} \rfloor$

Proof. Let $x \in \mathbb{Z}$. We want to prove that $\lceil \frac{x-1}{2} \rceil = \lfloor \frac{x}{2} \rfloor$.

We will use a proof by cases for even and odd x terms:

Case 1. Assume x is even, such that $x = 2k$ for some integer k.

$$\begin{aligned}
LHS &= \left\lceil \frac{2k-1}{2} \right\rceil \\
&= \left\lceil k - \frac{1}{2} \right\rceil \\
&= k \qquad (1) \\
RHS &= \left\lfloor \frac{2k}{2} \right\rfloor \\
&= \lfloor k \rfloor \\
&= k \\
LHS &= RHS = k
\end{aligned}$$

(1) By the definition of ceiling, we know that the ceiling of $k - \frac{1}{2}$ is the smallest integer that is greater or equal to $k - \frac{1}{2}$, in this case, it would just be the integer k.

Case 2. Assume x is odd, such that $x = 2k + 1$ for some integer k. This case is symmetrical to the even case, since when one side is odd, the other side is even, but here is the proof anyway.

$$\begin{aligned}
LHS &= \left\lceil \frac{2k+1-1}{2} \right\rceil \\
&= \lceil k \rceil \\
&= k \\
RHS &= \left\lfloor \frac{2k+1}{2} \right\rfloor \\
&= \left\lfloor k + \frac{1}{2} \right\rfloor \qquad (2) \\
&= k \\
LHS &= RHS = k
\end{aligned}$$

(2) By the definition of floor, we know that the floor of $k + \frac{1}{2}$ is the largest integer less than or or equal to $k + \frac{1}{2}$, which is the integer k in this case.

Thus, we have shown the LHS is equal to the RHS for both even and odd integers, such that $\lceil \frac{x-1}{2} \rceil = \lfloor \frac{x}{2} \rfloor$ is True for all $x \in \mathbb{Z}$.

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(b) (i) Prove that $\forall x \in \mathbb{R}, \lceil x-1 \rceil = \lceil x \rceil - 1$.

Proof. Let $x \in \mathbb{R}$. We want to show that for all real numbers x, $\lceil x-1 \rceil = \lceil x \rceil - 1$.

We will use a proof by cases for integer and non-integer terms.

Case 1. Assume x is an integer.

By the definition of ceiling, we know that the ceiling of x is the smallest integer bigger or equal to x. In other words, when x is an integer, $\lceil x \rceil$ is just x. Then we have $\lceil x-1 \rceil = x-1$ and $\lceil x \rceil - 1 = x - 1$. The left hand side and the right hand side are equal, thus the statement is True when x is an integer.

Case 2. Assume x is not an integer.

By the definition of ceiling, we can remove the case where $\lceil x-1 \rceil = x-1$ since $x-1$ is not an integer, then we know that $x-1 < \lceil x-1 \rceil$, and $\lceil x-1 \rceil$ is the smallest integer that satisfies the inequality.

Let $n = \lceil x-1 \rceil$. We can now express the ceiling of $x-1$ as an inequality:

$$\begin{aligned} x-1 &< \lceil x-1 \rceil \\ x-1 &< n \\ x &< n+1 \end{aligned}$$

Now we have $x < n+1$, where $n+1$ is the smallest integer that satisfies the inequality, and by the definition of ceiling, $\lceil x \rceil = n+1$.

Substituting values into the left and right hand side of the given statement we have

$$\begin{aligned} LHS &= \lceil x-1 \rceil && \text{(By } \lceil x-1 \rceil = n\text{)} \\ &= n \\ RHS &= \lceil x \rceil - 1 && \text{(By } \lceil x \rceil = n+1\text{)} \\ &= (n+1) - 1 \\ &= n \\ LHS &= RHS = n \end{aligned}$$

Thus we have proven the LHS is equal to the RHS, such that $\lceil x-1 \rceil = \lceil x \rceil - 1$ for all $x \in \mathbb{R}$.

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(ii) Prove or disprove that $\forall x, y \in \mathbb{R}, \lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$.

Proof. We will disprove the statement by proving the negation of the statement: $\exists x, y \in \mathbb{R}, \lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$.

Let $x = 1$. Let $y = 0.5$. We want to show that $\lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$. Then the statement becomes

$$\begin{aligned} LHS &= \lceil 0.5 \cdot 1 \rceil \\ &= 1 \\ RHS &= \lceil 1 \rceil \cdot \lfloor 0.5 \rfloor \\ &= 1 \cdot 0 \\ &= 0 \\ LHS &\neq RHS \end{aligned}$$

Thus we have proven that there exists a pair of real numbers x, y such that $\lceil xy \rceil \neq \lceil x \rceil \cdot \lfloor y \rfloor$.

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Question 3. Induction.

(a) Prove that for all natural numbers n, $9 \mid 11^n - 2^n$.

Proof. Let $P(n)$ be the statement $9 \mid 11^n - 2^n$.

Base case. Let $n = 0$, then $9 \mid 11^0 - 2^0$, which evaluates to $9 \mid 0$. By the definition of divisibility, any number divides 0 since $9 \cdot 0 = 0$, so the base case is True.

Inductive step. Let $k \in \mathbb{N}$. Assume that $P(k) : 9 \mid 11^k - 2^k$ is True. We want to prove that $P(k+1)$: $9 \mid 11^{k+1} - 2^{k+1}$ is True. Then by the definition of divisibility, there exists a $d_1 \in \mathbb{Z}$ such that $9d_1 = 11^k - 2^k$. Let $d_2 = 11^k + d_1$. We will start off with the right hand side.

$$\begin{aligned}
11^{k+1} - 2^{k+1} &= 11 \cdot 11^k - 2 \cdot 2^k \\
&= 11 \cdot 11^k - 11^k \cdot 2 + 11^k \cdot 2 - 2 \cdot 2^k \\
&= 11^k(11-2) + 2(11^k - 2^k) \\
&= 11^k(9) + 2(9d_1) && \text{(By induction hypothesis)} \\
&= 9(11^k + d_1) \\
&= 9d_2
\end{aligned}$$

Then there exists an integer d_2 such that $9d_2 = 11^{k+1} - 2^{k+1}$ by the assumption of $P(k)$. Using the definition of divisibility, we get $9 \mid 11^{k+1} - 2^{k+1}$. Thus $9 \mid 11^n - 2^n$ is proven by induction for all $n \in \mathbb{N}$.

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(b) Consider $p_n = 2^{2^n} + 1$ for $n \in \mathbb{N}$. Prove that for all $n \in \mathbb{N}$,

$$p_n = \left(\prod_{i=0}^{n-1} p_i\right) + 2$$

Proof. Let $P(n)$ be the statement $p_n = (\prod_{i=0}^{n-1} p_i) + 2$.

Base case. Let $n = 0$, then $n - 1 = -1$. From the definition of product notation, the right side product will equal 1 since the upper limit is smaller than the lower limit, so the right hand side will be $1 + 2 = 3$. The left hand side will be $2^{2^0} + 1 = 3$. Both sides are 3, so the base case is True.

Inductive step. Let $k \in \mathbb{N}$. Assume $P(k)$ is True, such that $p_k = \left(\prod_{i=0}^{k-1} p_i\right) + 2$. We want to prove that $P(k+1)$: $p_{k+1} = \left(\prod_{i=0}^{k} p_i\right) + 2$ is True. Using the difference of squares we can expand $p_{k+1} = 2^{2^k+1} + 1 = (2^{2^k} - 1)(2^{2^k} + 1) + 2$, and then use the assumption that $P(k)$ is True.

$$\begin{aligned}
p_{k+1} &= 2^{2^{k+1}} + 1 \\
2^{2^{k+1}} + 1 &= (2^{2^k} - 1)(2^{2^k} + 1) + 2 && \text{(By difference of squares)} \\
&= (2^{2^k} - 1)p_k + 2 && \text{(By definition of } p_k\text{)} \\
&= (2^{2^k} + 1 - 2)p_k + 2 \\
&= (p_k - 2)p_k + 2 && \text{(By definition of } p_k\text{)} \\
&= \left[(\prod_{i=0}^{k-1} p_i) + 2 - 2\right] p_k + 2 && \text{(By induction hypothesis)} \\
&= (p_0 \times p_1 \times \ldots \times p_{k-2} \times p_{k-1})p_k + 2 \\
&= (p_0 \times p_1 \times \ldots \times p_{k-2} \times p_{k-1} \times p_k) + 2 \\
&= \left(\prod_{i=0}^{k} p_i\right) + 2
\end{aligned}$$

$P(k+1)$ is True by the assumption that $P(k)$ is True. Thus $P(n) : p_n = \left(\prod_{i=0}^{n-1} p_i\right) + 2$ is proven True for all $n \in \mathbb{N}$ by induction.

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