Modeling selected computational problems as SAT-CNF and analyzing (some) structural properties of obtained formulas

Michał Mrowczyk

Department of Computer Science
AGH Kraków

M.Sc. Defense, 2017

Goals

- Reduce selected computational problems i.e. the Integer-Factorization and the OWA-Winner to SAT-CNF
- Investigate properties of the obtained formulas (using SAT solvers such as PicoSAT to solve these formulas)
  - running time of the solver
  - clauses to variables ratio
  - ...

SAT Decision Problem

- Given a boolean formula F decide (answer yes/no) whether there exists a satisfying assignment of True/False to variables (so that the formula evaluates to True).
- $x_1 \wedge (\overline{x_1} \vee x_2)$
- Setting $x_1 = 1$ and $x_2 = 1$ makes the formula above to evaluate to 1 (True), so we call it satisfiable (SAT)
- What about this one: $(\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (x_1 \vee x_2)$?
- It turns out that this one always evaluates to 0 (False), so we call it unsatisfiable (UNSAT)

SAT Decision Problem

- Given a boolean formula F decide (answer yes/no) whether there exists a satisfying assignment of True/False to variables (so that the formula evaluates to True).
- $x_1 \wedge (\overline{x_1} \vee x_2)$
- Setting $x_1 = 1$ and $x_2 = 1$ makes the formula above to evaluate to 1 (True), so we call it satisfiable (SAT)
- What about this one: $(\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (x_1 \vee x_2)$?
- It turns out that this one always evaluates to 0 (False), so we call it unsatisfiable (UNSAT)

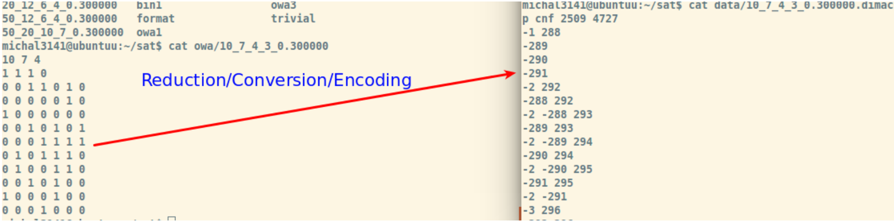
Pseudo DIMACS Format

- A way of writing (format) boolean formulas
- Example:
  $(x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_4) \land (\overline{x_3} \lor x_5) \to (1, -2, 3), (2, 4), (-3, 5)$

$$\begin{array}{ccc} 1 & -2 & 3 \\ 2 & 4 & \\ -3 & 5 & \end{array}$$

Big Picture

- We want to transform the instances of both OWA-Winner and Integer-Factorization problems into the boolean formulas
- ... And investigate properties of these formulas



OWA-Winner Problem Example

- We want to select $K = 3$ items
- We use the following OWA vector: $\alpha = (2, 1, 0)$
- What is the score of $\{a_1, a_2, a_6\}$?

	$u(a_1)$	$u(a_2)$	$u(a_3)$	$u(a_4)$	$u(a_5)$	$u(a_6)$
3 agents	5	4	3	0	2	1
2 agents	4	0	2	3	1	5
1 agent	0	3	2	4	5	1

$\text{score} = 3 \cdot (2 \cdot 5 + 1 \cdot 4 + 0 \cdot 1) + 2 \cdot (2 \cdot 5 + 1 \cdot 4 + 0 \cdot 0) + 1 \cdot (2 \cdot 3 + 1 \cdot 1 + 0 \cdot 0) = 42 + 28 + 7 = 77$

OWA-Winner Problem

- The notation used in this presentation:
  - α_k is an OWA vector
  - u_{i,a_j} is a utility function value that i-th voter assigns to the j-th item/candidate
  - $x_{i,j,k}$ is 1 if the i-th voter views the j-th item/candidate on the k-th position from items taken into the solution
- The goal of the OWA-Winner problem is to determine a comitee (set of items/candidates) of cardinality K maximizing the following expression:
  - $$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{K}\alpha_k u_{i,a_j} x_{i,j,k}$$
- We convert the OWA-Winner problem instances to ILP and then to SAT-CNF

ILP Formulation of the OWA-Winner

[2] Source: Finding a Collective Set of Items: From Proportional Multirepresentation to Group Recommendation

$$\text{maximize} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{K} \alpha_k u_{i,a_j} x_{i,j,k}$$

subject to :

$$(a): \sum_{i=1}^{m} y_i = K$$

$$(b): x_{i,j,k} \leq y_j \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(c): \sum_{j=1}^{m} x_{i,j,k} = 1 \qquad , i \in [n]; k \in [K]$$

$$(d): \sum_{k=1}^{K} x_{i,j,k} \leq 1 \qquad , i \in [n]; j \in [m]$$

$$(e): \sum_{j=1}^{m} u_{i,a_j} x_{i,j,k} \geq \sum_{j=1}^{m} u_{i,a_j} x_{i,j,(k+1)} \qquad , i \in [n]; k \in [K-1]$$

$$(f): x_{i,j,k} \in \{0,1\} \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(g): y_j \in \{0,1\} \qquad , j \in [m]$$

SAT Formulation of the OWA-Winner

$$(a): CNF(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{K}\alpha_k u_{i,a_j} x_{i,j,k} \geq L) \qquad L \in \mathbb{N}$$

$$(b): CNF(_{=K}(\{y_j | j \in [m]\}))$$

$$(c): (\overline{x_{i,j,k}}, y_j) \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(d): CNF(_{=1}(\{x_{i,j,k} | j \in [m]\})) \qquad , i \in [n]; k \in [K]$$

$$(e): CNF(_{\leq 1}(\{x_{i,j,k} | k \in [K]\})) \qquad , i \in [n]; j \in [m]$$

$$(f): CNF(\sum_{j=1}^{m} u_{i,a_j} x_{i,j,k} \geq \sum_{j=1}^{m} u_{i,a_j} x_{i,j,(k+1)}) \qquad , i \in [n]; k \in [K-1]$$

$$(g): x_{i,j,k} \in \{0,1\} \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(h): y_j \in \{0,1\} \qquad , j \in [m]$$

SAT Formulation - Binary Utility and OWA Vector

Both the utility function and the OWA vector are binary. In addition to this the OWA vector is nonincreasing.

$$(a): CNF(_{\geq L}(\{x_{i,j,k} | i \in [n], j \in [m], k \in [K], \alpha_k u_{i,a_j} > 0\}))$$

$$(b): CNF(_{=K}(\{y_j | j \in [m]\}))$$

$$(c): (\overline{x_{i,j,k}}, y_j) \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(d): CNF(_{=1}(\{x_{i,j,k} | j \in [m]\})) \qquad , i \in [n]; k \in [K]$$

$$(e): CNF(_{\leq 1}(\{x_{i,j,k} | k \in [K]\})) \qquad , i \in [n]; j \in [m]$$

$$(f): x_{i,j,k} \in \{0,1\} \qquad , i \in [n]; j \in [m]; k \in [K]$$

$$(g): y_j \in \{0,1\} \qquad , j \in [m]$$

General Results

- Tests were conducted by applying the PicoSAT solver to general and binary OWA-Winner models
- Bigger OWA-Winner problem instances can be solved by using the binary OWA model (because the binary OWA model is more efficient, but at the same time more restricting)
- As K gets bigger and bigger, the produced boolean formulas are becoming larger and larger very rapidly

Example: General OWA-Winner Instance

The optimization run below lasted for about 10 minutes.

p cnf 26199 124996

Produced boolean formula is available at: owa3.dimacs

```
n=6, m=12, K=5
alpha=3 3 3 2 1
u=
3 4 2 1 0 1 2 0 3 5 0 4
4 2 1 3 0 2 3 0 1 3 1 5
1 2 3 0 4 5 2 3 1 1 2 0
0 1 3 4 1 5 1 0 2 1 3 1
1 0 1 3 5 1 2 1 5 3 1 0
2 3 4 2 3 3 1 4 2 2 4 1
ILPModel::maximize for 125 lb=0 ub=250
ILPModel::maximize for 188 lb=126 ub=250
ILPModel::maximize for 219 lb=189 ub=250
ILPModel::maximize for 203 lb=189 ub=218
ILPModel::maximize for 195 lb=189 ub=202
ILPModel::maximize for 199 lb=196 ub=202
ILPModel::maximize for 201 lb=200 ub=202
ILPModel::maximize for 202 lb=202 ub=202
202
y=[0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0]
```

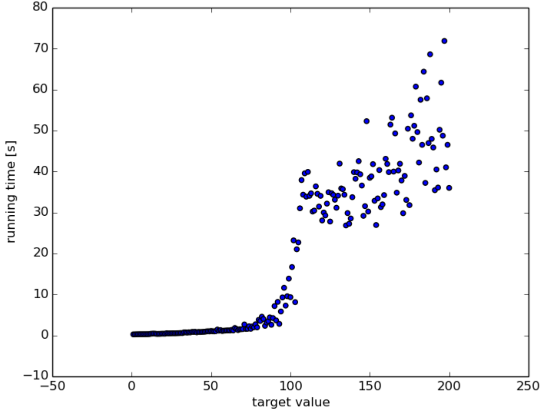
Example: Binary (Approval) OWA-Winner Instance

To solve this one below to optimality it took roughly 90 minutes on my machine... (there are $\binom{20}{10} = 184756$ possible comitees to check when using pure brute-force)

```
n=50, m=20, K=10
alpha=1 1 1 1 1 1 1 0 0 0
u=
0 0 1 0 1 1 1 1 0 0 0 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0 1 0 1 1 0 0 0 1 1 0
0 1 0 0 0 1 0 1 0 0 1 0 0 0 0 1 1 0 0 0
0 0 0 0 1 0 1 0 0 0 1 0 0 1 1 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 1 0
0 1 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0
0 1 0 1 0 0 1 1 0 0 0 0 0 0 1 1 1 0 0 1
1 0 1 1 0 0 1 0 0 0 0 1 1 0 1 0 1 0 1 0
1 0 0 0 1 1 0 0 1 0 1 0 0 0 0 0 0 0 1 0
0 1 0 0 1 0 1 0 1 0 0 0 0 0 0 1 0 1 0 1
0 0 1 0 1 0 0 1 1 0 0 0 0 1 0 1 1 0 1 1
1 1 0 0 0 1 0 0 0 1 1 0 0 0 1 0 0 0 0 0
0 0 0 0 0 1 0 0 1 1 0 0 0 0 1 0 0 0 0 0
1 1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 0 1 0 1 0 0 0 0 1 1 1 1 0 0 1 0 1 1 0
0 0 1 1 1 1 0 0 0 0 0 1 0 1 1 1 0 0 0 0
0 0 0 0 1 0 1 1 0 0 0 1 0 0 1 0 1 0 0 0
0 0 0 0 0 1 1 0 1 0 1 1 0 0 1 0 0 0 0 0
```

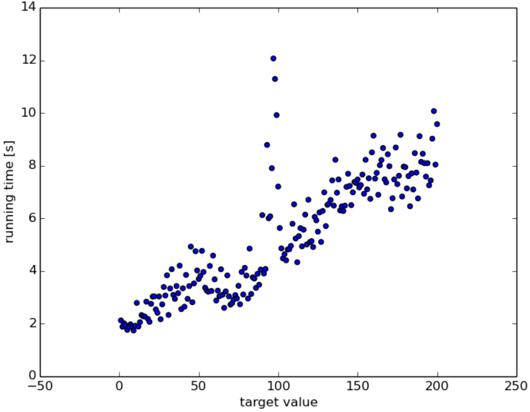
Running Time vs Solution Quality (target value)

kBestOWAApprovalWinner$(50, 12, 6, \mu, 4, 0.3, v)$; Unsatisfiability for target value=108



Running Time vs Solution Quality (target value) 2

kBestOWAApprovalWinner$(100, 24, 10, \mu, 1, 0.3, v)$; Unsatisfiability for target value=101

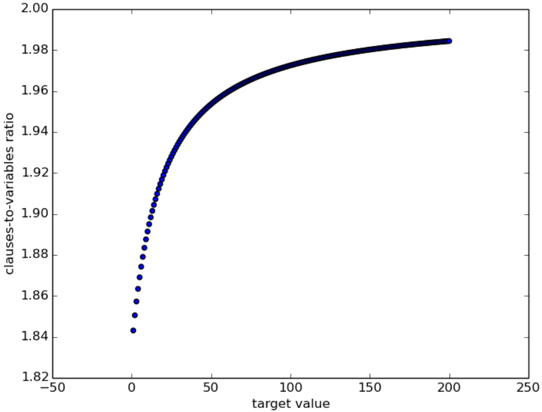


Clauses to Variables Ratio

- clauses-to-variables-ratio $cv = \frac{\#clauses}{\#variables}$
- Example: $F \equiv (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2})$. $cv = 2$
- Randomly generated SAT-CNF formulas with $cv > 4.26$ are mostly UNSAT, but formulas with $cv < 4.26$ are mostly SAT
- Studying cv for various formulas generated for both the OWA-Winner and Integer Factorization instances

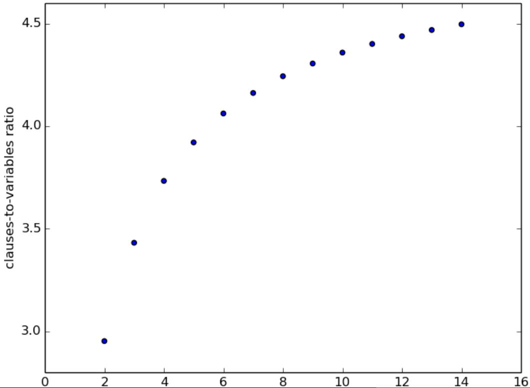
Clauses to Variables for OWA-Winner Instances

kBestOWAApprovalWinner$(50, 12, 6, \mu, 4, 0.3, v)$



Clauses to Variables for Integer Factorization Instances

All n-bit integers Factorization problems corresponds to the formulas with the same cv ratio $\rightarrow$ we consider how cv varies depending on number of bits. We show it tends to $\frac{39}{8}$.



Summary

- Reduction from the OWA-Winner to SAT-CNF was presented with some (but not all) quirks
- In addition to the general OWA-Winner, we prepared a more efficient reduction for the OWA-Winner with binary utility and OWA-vector (+ nonincreasing OWA vector)
- Running Time and Clauses to Variables Ratio were considered as measures of hardness for selected SAT-CNF instances

Thank you

Thank you!

Boolean Cardinality Constraints

- Given a set of boolean variables $\{x_1, x_2, ..., x_n\}$ we want to ensure that exactly (at least, at most,...) $k \leq n$ of them are set to True
- At least one of the variables is set to True ?
- Easy: $x_1 \vee x_2 \vee ... \vee x_n$
- At most one of the variables is set to True ?
- Idea: Ensure that for every pair of variables at least one of the variables is False !
- How to model generic 'at least k of' and 'at most k' of constraints ?

Encoding at most k of Constraint

Below is the encoding that requires additional $k * n$ variables: s to ensure that at most k of n variables from x sequence are chosen:

$$
\begin{array}{l}
(\neg x_1 \lor s_{1,1}) \\
(\neg s_{1,j}) \qquad \text{for } 1 < j \le k \\
\left.\begin{array}{l}
(\neg x_i \lor s_{i,1}) \\
(\neg s_{i-1,1} \lor s_{i,1}) \\
\left.\begin{array}{l}
(\neg x_i \lor \neg s_{i-1,j-1} \lor s_{i,j}) \\
(\neg s_{i-1,j} \lor s_{i,j})
\end{array}\right\} \quad \text{for } 1 < j \le k \\
(\neg x_i \lor \neg s_{i-1,k})
\end{array}\right\} \text{for } 1 < i < n \\
(\neg x_n \lor \neg s_{n-1,k})
\end{array}
$$

[1] Source: Towards an Optimal CNF Encoding of Boolean Cardinality Constraints

Implementation of at most k of Constraint

Below is my implementation of at most *k* of encoding proposed by Sinz:

```
def at_most_k_of(self, x, k):
    n = len(x)
    if n <= 1:
        return

    s = [[self.add_var() for j in xrange(k)] for i in xrange(n-1)]
    self.add_clause([~x[0], s[0][0]])
    for j in xrange(1, k):
        self.add_clause([~s[0][j]])
    for i in xrange(1, n-1):
        self.add_clause([~x[i], s[i][0]])
        self.add_clause([~s[i-1][0], s[i][0]])
        for j in xrange(1, k):
            self.add_clause([~x[i], ~s[i-1][j-1], s[i][j]])
            self.add_clause([~s[i-1][j], s[i][j]])
        self.add_clause([~x[i], ~s[i-1][k-1]])
    self.add_clause([~x[n-1], ~s[n-2][k-1]])
    return s
```

For Further Reading I

Handbook of Satisfiability, Biere, A., Heule, M., Van Maaren, H., Walsh, T, February 2009

- Towards an Optimal CNF Encoding of Boolean Cardinality Constraints - Carsten Sinz
- Finding a Collective Set of Items: From Proportional Multirepresentation to Group Recommendation ∗ Piotr Skowron University of Warsaw Warsaw, Poland Piotr Faliszewski AGH University Krakow, Poland J´er^ome Lang Universit´e Paris-Dauphine Paris, France