The Delaunay Constrained Triangulation : The Delaunay Stable Algorithms

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Abstract

The Delaunay triangulation is well known for its use in geometric design. A derived version of this structure, the Delaunay constrained triangulation, takes into account the triangular mesh problem in presence of rectilinear constraints.

The Delaunay constrained triangulation is very useful for CAD, topography and mapping and in finite element analysis. This technique is still developing. We present a taxonomy of this geometric structure. First we describe the different tools used to introduce the problem. Then we introduce the different approaches highlighting various points of view of the problem.

We will focus on the Delaunay stable methods. A Delaunay stable method preserves the Delaunay nature of the constrained triangulation. Each method is detailed by its algorithms, performances, and properties. For instance we show how these methods approximate the generalised Voronoï diagram of the configuration.

The Delaunay stable algorithms are used for 2.5D DEM design. The aim of this work is to demonstrate that the use of topographic constraints in a regular DEM without adding new points preserves the terrain shape. So the resulting DEM can be more easily interpreted because its realism is preserved and the mesh still owns all the Delaunay triangulation properties.

Keywords : Delaunay triangulation, Delaunay constrained triangulation, surface model, Delaunay stable algorithm, DEM application.

1. Introduction

After the presentation of the Delaunay constrained triangulation problem, we define tools to describe the working area and the algorithm behaviour classification. Then we expose the different approaches, from the basic redefinition of the problem to full preserving methods. Among them, the Delaunay stable methods are detailed with the algorithmie description.

Finally, we use these stable algorithms to improve and maintain at the lowest cost the DEM realism during the resampling process.

2. The problem

The problem of the constrained triangulation is to make appear a constraint graph described by constraint edges. Each constraint edge is then a part of triangles. We will use the Delaunay triangular structure. Let's define the constraint elements.

Definition 2.1 (The constraints field)

The constraint field C_{ont} is the set of all the constraint edges, having no intersections except with other vertices or edges at their ends.

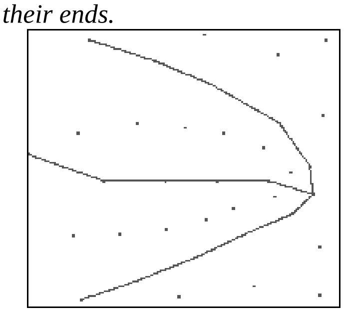

Figure 2.1 Example of data and Constraint field

Definition 2.2 (Polygon triangulation)

The triangulation of a polygon is performed by *looking at its defining elements separately as constraint edges.*

2.1 Tools and definitions

Definition 2.3 (Delaunay compliant edge)

A Delaunay compliant edge is an edge for which the insertion of its extremities in the Delaunay triangulation makes sure the appearance of the edge as a Delaunay edge.

Definition 2.4 (The constraint tube)

*The tube of a constraint edge **e** in the triangulation T(s) is the set of triangles of T which are directly crossed by the edge.*

$$t_u(e) = \{triangles\ t \in T \mid t \cap e \neq \varnothing\}$$

This area is the limit of the direct triangulation impact of the constraint incorporating methods. This helps to keep the local aspect of the Delaunay triangulation (Markovian behaviour).

Definition 2.5 (Exact verification of a constraint field)

A triangulation T verifies exactly a constraint field if each element of C_{ont} appears as element of the triangulation..

So we can define a first class of incorporating methods :

The constraint forcing method :

The principle of this method is to modify the direct neighbourhood triangulation to avoid the constraint crossing edges without modifying the constraint edge.

Definition 2.6 (Poor verification of a constraint field)

A triangulation T verifies poorly a constraint field if each element of C_{ont} appears as it is or as a partition in the triangulation.

This leads to the second class of incorporating methods.

The constraint breaking method :

The constraint breaking principle is to transform the constraint edge by partitioning it into Delaunay compliant edges. The corresponding algorithms are mainly Delaunay stable.

Definition 2.7(Delaunay stable incorporating method)

A constraint incorporating method in a Delaunay triangulation is said to be stable if the resulting triangulation is still a Delaunay triangulation.

Proposition 2.1 (Delaunay unstable method)

A constraint incorporating method in a Delaunay triangulation is said to be Delaunay unstable if the result is a triangulation respecting no longer the empty circle criterion.

2.2 Semantics

The difference between the stable and unstable methods can be translated in their naming manner.

It has to be stressed that the Delaunay Constrained Triangulation (DCT) is different from the Constrained Delaunay Triangulation (CDT). The CDT are produced by Delaunay unstable methods. On the contrary, DCT are the result of a Delaunay stable algorithm.

3. Delaunay triangulation under constraint

The principle is to redefine the building criterion of the Delaunay triangulation. So, we define the constrained empty circle criterion taking into account the graph of visibility of the configuration. The constraint field is exactly verified because the edge integrity is preserved.

Definition 3.1 (mutual visibility)

Two vertices Vi and Vj are mutually visible if no constraint edge crosses their linking segment.

Definition 3.2 (Constraint empty circle criterion)

A triangle t(vi,vj,vk) of T respects the constraint empty circle if and only if there is no other vertex v of T such :

- *v is contained in the circumscribed circle to t.*
- *v is not visible from the three vertices vi,vj,vk at the same time.*

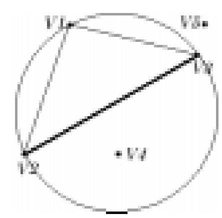

Figure 3.1: The constraint empty circle criterion (v2,v3 is a constraint edge)

Definition 3.3 (The Delaunay triangulation under constraint)

A triangulation is a Delaunay triangulation under constraint if all the triangles respect the constrained empty circle criteria.

So the defined triangulation contains the constraint graph as part of itself. The constraint field is exactly verified.

The Voronoï diagram is redefined too and it has been proved that the duality between the constrained Voronoï diagram and the Delaunay triangulation under constraint still exists.

Definition 3.4 (Constraint Euclidean distance)

If d(v,v') is the euclidean distance from v to v' the constrained euclidean distance by the constraint field is defined by:

$$d_{C_{ont}}(v,v') = \begin{cases} d_{C_{ont}}(v,v') \text{ if } v \text{ and } v' \text{ are mutually visible} \\ \infty \end{cases}$$

Definition 3.5 (Constrained Voronoï diagram)

The constrained Voronoï diagram of C_{ont} is defined as the set of the cVOR cells. The plan is partitioned into constrained Voronoï cells where each area is defined by:

$$cVOR(v_i) = \left\{ v \in \Re^2 \middle| \begin{array}{l} d_{c_{ont}}(v,v_i) < \infty \text{ and} \\ d_{c_{ont}}(v,v_i) < d_{c_{ont}}(v,v_j)\ \forall v_j \in V \end{array} \right\}$$

4. The unstable methods

First we compute the Delaunay triangulation of the vertices and the constraint extremities. Then we incorporate the missing constraint edges. The main principle is to retriangulate the constraint edge's tube while preserving the edge integrity. So, the resulting triangulation verifies exactly the constraint field but is no more of Delaunay type.

Both sides of the edge are processed separately. Lots oft methods are proposed using the same theorem guaranteeing an existing solution to the problem.

Theorem 4.1(triangulation without internal points) [5]

For each area whose boundary is a simple non crossed polygonal lines, there exists a triangulation without internal points.

Algorithms are based on edge swapping on each side of the constraint edge. Basic methods test every solution while elegant methods swap the edges at random, so exploiting the finite size of the problem to converge to a solution.

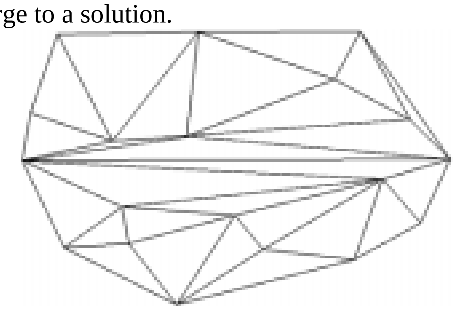

Figure 4.1 The Delaunay unstable forcing method

5. The stable methods

The Delaunay stable methods are based on the breaking method building new Delaunay compliant edges. The resulting triangulation is Delaunay type but verifies poorly the constraint field.

5.1 Densification

This method states that the constraint doesn't appear in the triangulation, because its sampling doesn't fit the neighbourhood. This method presented in [8] analyses the constraint tube to compute an adapted sampling distance to discretise the constraint edge.

Proposition 5.1 (Sampling distance)

Let d(v,e) be the distance between a tube vertex and the constraint edge.

So, the best sampling distance for this set is :

$$P(e,T) = 2 * \min_{\forall v_i \in t_u(e)} d(v_i, e)$$

Theorem 5.1 (Edge incorporation by densification)

The partition of a constraint edge with the sampling distance P(e,T) makes the edge Delaunay compliant.

Proof : The constraint edge doesn't appear in the triangulation because it doesn't fulfill the empty circle criterion . The new edges respect it because the circles, whose diameter they are, contain no other vertices. So we are sure that the partitioned edges are Delaunay compliant. ❑

5.2 Dichotomy

this method uses the classic principle of splitting the constraint edges until all the new edges are Delaunay compliant.

Theorem 5.2 (Edge incorporation by dichotomy)

It always exists an edge partition by dichotomy leading to Delaunay compliant edges.

Proof : the convergence is guaranteed by the densification method. There is a step from which all the edge sizes are below the densification distance which has been defined previously. So they are Delaunay compliant. ❑

5.3 The perpendicular projection

Each vertex of the tube is orthogonally projected on the constraint edge.

Theorem 5.3 (Incorporation by orthogonal tube projection)

The discretisation of a constraint edge by inserting all the orthogonal projections of the tube vertices makes it Delaunay compliant.

(short) **Proof** : The insertion of the orthogonal projection on the constraint edge disturb the empty circle criterion for the tube triangles. So, step by step, from the start to the end of the constraint edge we split it into Delaunay compliant edges.❑

Proposition 5.2 (Arc cost)

The cost of arcs for the incorporation of an edge with the orthogonal projections is directly related to the edge tube configuration.

$$\text{cost} = Card(t_u(e)) + 1$$

5.4 The intersection incorporation

We split the constraint edge by inserting all the intersections between the tube and its corresponding constraint edge.

Theorem 5.4 (Tube-constraint intersections incorporation)

The partition of a constraint edge by inserting all the intersections between the edge and its tube triangulation makes it Delaunay compliant.

Proof : Each intersecting edge belongs to two Delaunay circles. So, inserting the intersection point disturbs the Delaunay criterion and produces a Delaunay compliant edge.❑

Proposition 5.3(Arcs cost)

The cost in arcs of inserting the tube-edge intersection method depends on the tube configuration:

$$\text{cost} = Card(t_u(e))$$

5.5 The impact on Voronoï diagram

Stable methods produce Delaunay triangulations. So these DCT still have the Voronoï diagram as dual diagram.

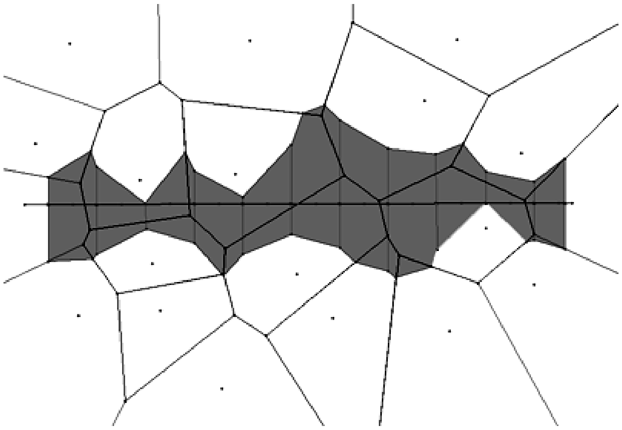

Figure 5.1 Impact of the densification method on the Voronoï diagram

In Figure 5.1, we notice the trace of the new Voronoï diagram after incorporation over the original one. Its typical shape shows the approximation of the generalised Voronoï diagram corresponding to the constraint edge. The quality of the approximation depends on the partition distance (Theorem 5.5).

We present the required principles to define the generalised Voronoï diagram. So we can check the link between the DCT related Voronoï diagram and the generalised Voronoï diagram.

Definition 5.1 (Objects)

Points, open segments and open polygons are considered as simple elements. An object is a set of simple elements.

Definition 5.2 (Generalised Voronoï diagram)

The generalised Voronoï diagram is the nearest neighbourhood cell partition of a set of objects.

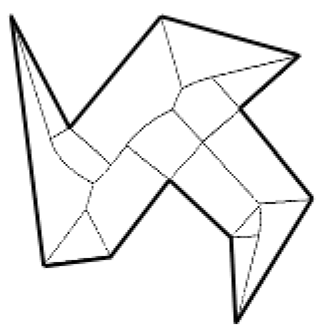

Figure 5.2 : Sample of the generalised Voronoï diagram for a polygon interior.

The "classic" Voronoï diagram is called the punctual Voronoï diagram dealing with vertices. In Figure 5.2 we can see that the generalised Voronoï diagram is made of arcs and parabola sections.

Theorem 5.5 (Convergence of the punctual diagram to the generalised Voronoï diagram)[1]

Let S be a set of objects and S(h) be a discretisation of S. The punctual Voronoï diagram of S(h) converges to the generalised Voronoï diagram of S when the discretisation step decreases to 0.

Theorem 5.6 (The Voronoï diagram associated to the Delaunay stable methods)

The Voronoï diagram corresponding to the triangulation made by Delaunay stable methods is a punctual approximation of the generalised Voronoï diagram of the configuration.

5.6 Performance analysis

The performance analysis is conducted in two ways. First we use subjective criteria to compare the methods. Then we evaluate quantitatively the gains over ten different configurations.

Definition 5.3 (Certitude of an incorporation method)

the method certitude evaluates how this method progresses at each step toward the solution : i.e. the appearance of the constraint in the Delaunay triangulation.

The densification is a reliable method but it costs a lot. The orthogonal projection or intersection insertion methods are also reliable and improve the arcs cost because it is directly related to the tube configuration. The dichotomy method offers the lowest cost of arcs but we can not predict the final cost.

Method	Certitude	Arcs cost
Densification	+	-
Orthogonal	+	+/-
Intersections	+	+/-
Dichotomy	-	+

Table 1 : Certitude/Cost

Method	**Arcs Cost**
densification	f(distance(t_u(e),e)
dichotomy	-
orthogonal projection	= Card(t_u(e))+1
intersections	= Card(t_u(e))

Table 2 : Bound of the arcs cost for the Delaunay stable methods

The following table shows the average cost of new arcs computed over ten different configuration.

		Gain			
Method	**nb arcs**	**dens.**	**ortho.**	**inters.**	**dicho.**
dens.	104,00		*-209,52%*	*-219,02%*	*-336,97%*
ortho.	33,60	**67,69%**		*-3,07%*	*-41,18%*
inters.	32,60	**68,65%**	**2,98%**		*-36,97%*
dicho.	23,80	**77,12%**	**29,17%**	**26,99%**	

Tableau 1: Average arcs gains over 10 various configurations

5.7 Example

We present the behaviour of the different Delaunay stable algorithms on the same configuration. In the following figure we have:

1. The original configuration.
2. The Densification method.
3. The dichotomy method.

4. The orthogonal projection method.
5. The intersection method.

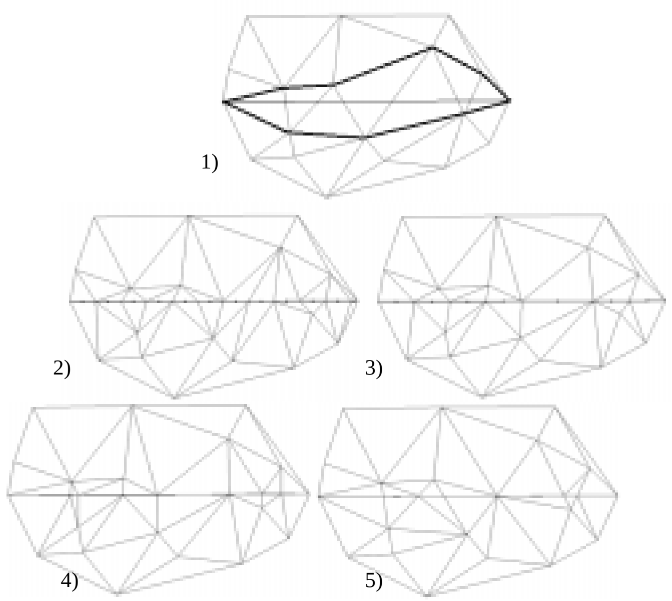

Figure 5.3

The bold line in the first vignette outlines the constraint edge tube boundary. Figure 5.1 is the associated Voronoï diagram to this configuration for the densification method.

6. Application for DEM

6.1 The DEM design

Rippa in [7] shows that the Delaunay triangulation minimises the flexion energy of the mesh. So the Delaunay triangulation provides the best approximating surface reconstruction. This property is very useful for terrain surface from a set of scattered data. Moreover, the duality of the Delaunay triangulation with the Voronoï diagram offers a lot of new perspectives for exploiting the DEM For instance in [8], the Voronoï cells are used to extend the ground roughness measured at different points. So the 2D Delaunay triangulation is used to build a 2.5D surface.

Those properties lead us to look for a constrained Delaunay triangulation preserving its nature and so its properties. So we used the developed algorithms to design the DEM and improve its realism.

The constraints lines describe topographic lines (ridges, valleys) which help to sketch the final DEM

6.2 The resampling problem

The problem of regular DEM is that their sampling method misses topographic features whose size is below the sampling rate distance. We want to improve those DEM by using a triangular Design and incorporating the missed topographic constraints before resampling.

Definition 6.1 (topographic link)

A topographic link is a constraint edge linking two points which belong to the same terrain feature and are mutually visible (for instance two points in the same river).

A topographic link doesn't costs anything because the information needed along the edge for its incorporation is interpolated from the height of its extremities.

6.3 Application

In this example we take a regular mesh over the terrain. Its sampling rate has missed the valley. So when resampling is performed to a better scale, the valley is no longer seen. It is quite a problem for hydrologic computation or when piloting a vehicle.

Figure 8.1 presents the three strategies we developed to improve the DEM realism. The first one is the classic method resampling the grid to make another grid. The second adds topographic links and in the last we correct (move) a few points before adding topographic links.

The Figure 8.2 shows the results of these strategies. We can verify the appearance of the main valley whereas the secondary valley is still missing. In the best method both valleys are well described for a very low cost (1.25% of the whole DEM points are modified).

7. Conclusion and outlooks

We have shown that we have the choice between different philosophies for constraining the Delaunay triangulation. This is summarised in Table 3 .

We have especially described Delaunay stable methods which preserve the Delaunay nature of the resulting triangulation and keep the duality with the Voronoï diagram. The cost of these methods can be evaluated and bounded.

So we are no more limited by the design algorithm to exploit the mesh information.

We have presented a practical application for the DEM resampling. It shows that for a very low cost the Delaunay constrained triangulation can improve the DEM realism and preserves it along the resampling process.

8. References

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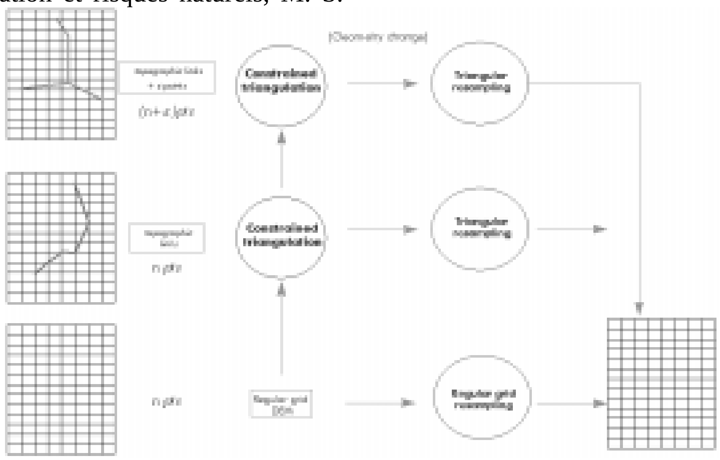

Figure 8.1 The different strategies for DEM resampling

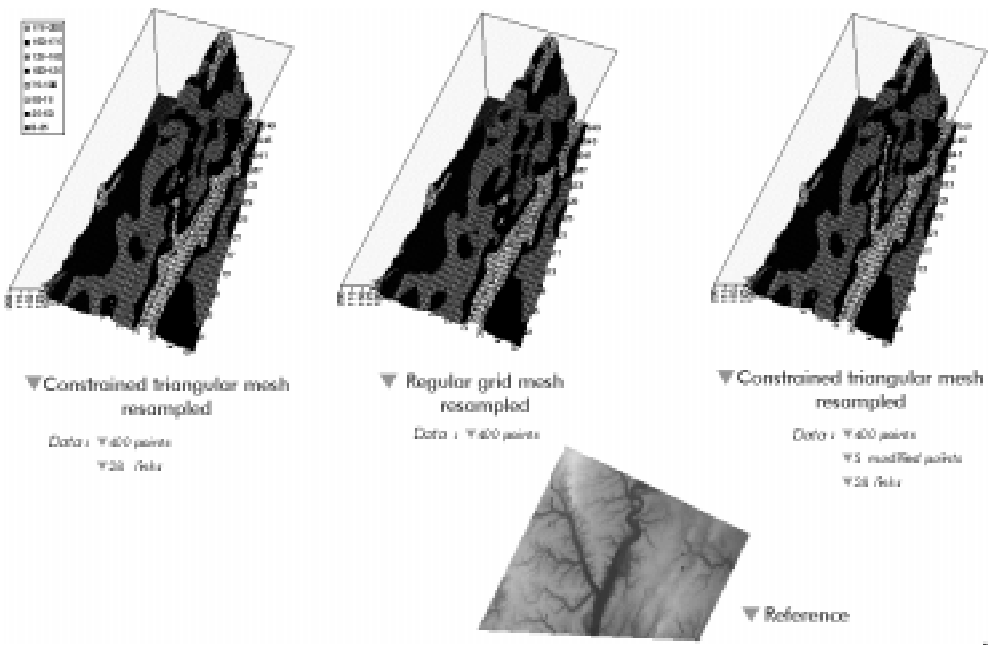

Figure 8.2 The result of the resampling process using different strategies

Triangulation			Dual diagram	Elements
Name	Delaunay Nature	Verification of C_{ont}		
Punctual Delaunay Triangulation	o	-	punctual Voronoï	points
Delaunay Triangulation under constraint	n	exact	constrained Voronoï	points, arcs
-	-	-	generalised Voronoï	points, arcs, polygons
constrained Delaunay Triangulation (unstable) - CDT	n	exact	-	points, arcs
Delaunay constrained Triangulation (stable) - DCT	o	poor	punctual approximation of the generalised Voronoï diagram	points, arcs

Table 3 Summary of the different approaches of the constrained Delaunay