# Knight's Tours and Circuits on the 3 × n Chessboard

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# Knight’s Tours and Circuits on the 3xn Chessboard
(Classroom Notes)

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In order to introduce some fun and mathematics into highschool one can study the Knight’s Tours and Circuits on the 3xn chessboard. The reason for the 3xn board rather than the 8x8 board is that the tours and circuits on the 3xn board are easier to find and n being a variable one can go ahead to conjecture and prove theorems.

First one must make the students comfortable with the board and the knight move. Figure 1 shows the board and the numbering system on the board that will be used for reference. In Figure 2, if a knight is positioned on an infinite board at the x, then a knight can move in an L-shaped move ie. one square forward, backward or to either side and two squares at a right angle to the first direction. These squares are indicated in Figure 2 with an o.

Numbering of the Squares on a 3xn Boards

| | | | | |
|---|---|---|---|---|
| 1 | 4 | 7 | 10 | |
| 2 | 5 | 8 | 11 | ******* |
| 3 | 6 | 9 | 12 | |

FIGURE 1

Possible Knight Moves

| | | | | |
|---|---|---|---|---|
| | o | | o | |
| o | | | | o |
| | | x | | |
| o | | | | o |
| | o | | o | |

FIGURE 2

Once the students understand how to move the knight about the 3xn board, they can be introduced informally and then formally to the ideas of tours and circuits. We will use the terminology of Eggleton and Eid[1]. One can, of course, immediately move to graph theory concepts and replace the board by vertices and edges or give the board's squares coordinate labelling but I will assume that the student's may not need or like such an approach.

**Definition 1** A knight's tour on a 3xn board is a sequence of 3n squares representing a sequence of knight's moves so that no square is visited more than once and no square is missed ie. adjacent squares in the sequence must be such that a knight could move from one to the other.

**Definition 2** A knight's circuit on a 3xn board is a knight's tour with the extra proviso that the knight must be able to move from the last square to the first square.

The best thing to do now is to show them some examples. For a 3x4 board the following is a knight's tour but not a knight's circuit. (11 4 3 8 1 6 7 2 9 13 5 12). Of course, one would show this to the class on the 3x4 part of a chessboard. At the same time, the instructor could show that a circuit or tour could be traversed in either direction and that although a tour has a definite start and end, a circuit could be listed starting anywhere. Another idea that should come out is that if there is no tour then there is no circuit.

At this time it is best to let the students, in groups, start experimenting with different size small boards to see if there are other circuits or tours. You can let them look for the other tour on the 3x4 board (1 8 3 4 11 6 7 2 9 10 5 12). Quite quickly they will discover the following facts. That there are no circuits or tours for 3x1, 3x2 and 3x3 boards. There is also no circuit on the 3x4 board. These facts or lemmas can be recorded with a "proof" in a journal as recommended by Fellows and Koblitz[2]. In fact, the backtrack searching that must be done for the bigger boards gets to be quite intimidating. So the students can be called together for the following tips.

Consider circuits. In particular consider square 2 on a circuit. There is no choice for the squares the knight is on immediately before moving to 2 or immediately after moving from 2. They are 7 and 9. So a circuit must contain the sequence 7 2 9 or 9 2 7. In the same way you can show that a circuit must contain 8 1 6 or 6 1 8 and also 4 3 8 and 8 3 4.

At this point a bright student (or stooge) notices that the last two subsequences both contain 8. From this you get that there must be a subsequence 4 3 8 1 6 or its reverse in any circuit. Now if one notices the positions of 4 and 6 and tries to extend the sequence in both directions then the knight could not both get to 4 from 11 and leave 6 for a 11. So either the knight got to 4 from 9 or it leaves 6 for 7. These are isomorphic subsequences when considering a circuit though. It is best to talk fast and wave your arms quickly when going through this argument. If the students have been taught geometry by mirrors[3], these last observation will be easy for the students. So let the class pick one, say 9 precedes 4 and look at it. This hooks up with the 7 2 9 sequence and forces 11 to follow 6. It is time to record this result.

So now the students have a lemma telling them that in any circuit they may assume that there is a subsequence 7 2 9 4 3 8 1 6 11 or its reverse. One should point out to the students that the sequence may be split when writing it down because of the arbitrary starting point. Further, that on a circuit the exact same thing happens on the extreme right hand side of the board. Also one should point out to the students that this same analysis works for tours on the side(s) of the board not containing the start or the end. At this point, the students will be eager to tackle some of the bigger boards with these tools. An important point to tell the class about the backtrack search is that none of these squares on the side to do with the subsequence mentioned can be filled in except as part of the subsequence. This severely cuts down on the search space. Perhaps one should tell the class to concentrate only on circuits at this time. They should realize that on a small 3xn board, the left and right subsequences of a circuit overlap and there are many constraints on the would-be circuit. Hopefully, they will have learned to this (and other) problems backwards and forwards and inside out. The students may also start to look for symmetry of various kinds in their problems.

What should they have found out? It should have been easy to rule out circuits for the 3x5 and 3x6 boards and that the 3x7 board is the first board big enough that the forced sequence of moves for both sides of the board are possible. The subsequences are joined back to front but can not be completed to a circuit but a tour does exist. The last square is one column over too far to be a tour. Again, the 3x8 board has no knight’s circuit and this is very easy to show. However, if the students continue their search they will find a knight’s tour on the 3x8 board. We will record this information in the following lemmas.

**Lemma 1** There are no knight’s tours on the 3xn boards when n=1, 2, 3, 5 or 6.

**Lemma 2** There are no knight’s circuit’s on the 3xn boards when n=1, 2, 3, 4, 5, 6, 7, 8, or 9.

The students may have difficulty finding their first circuits so they may need help setting up the sequence of moves on the left side of the board and the corresponding sequence of moves on the right side of the board. They then need to do a backtrack search to connect up the ends of the two sequences. If the students have the knowledge and resources, the search may be computerized. One way or the other the students should record the following.

**Lemma 3** There are knight’s circuits on the 3x10 and 3x12 board.

Proof: The following are knight’s circuits.
(17 22 27 20 25 24 19 26 21 14 7 2 9 4 3 8 1 6 11 16 15 10 5 12 13 18 23)

(27 22 17 10 5 12 13 20 15 16 21 14 7 2 9 4 3 8 1 6 11 18 23 28 35 30 31 36 29 34 33 26 19 24 25 32) □

With further examples and hints the students may guess that it is impossible to have a circuit on a 3x2n board. With much constructive leadership, the class may slowly discover the following theorem and proof.

**Theorem 1** There are no knight's circuits on 3xn boards if n is odd.

Proof If one colours the 3xn board in a checkerboard pattern with two colours, there is one more square of one colour, say red than squares of the other colour, say black. It is a well-known fact in chess that a knight can jump only from one colour to the other colour on a checkerboard coloured square. Therefore, if you examine any circuit the number of red squares in the circuit equals the number of black squares in the circuit. There is a parity problem in any circuit on a 3xn board. □

Note that on a Knight's tour the last and first squares may be the same colour so that there may be one more square of one colour than squares of the other colour. So that tours on a 3xn board are possible.

It is now time to tell the students that it is getting too tedious to find a 3x14 knight's circuit. Perhaps it is best to tell them to use the old mathematician's trick of using old solutions to get a new solution ie. use the 3x10 knight's circuit and one of the 3x4 knight's tours to get a 3x14 knight's circuit. The idea is to put them side by side with the tours marked on them and try to break the circuit somewhere near the edge to include the little tour and then jump back to the big circuit to finish it. The instructor may try to do it with the wrong 3x4 (the start and end are at opposite ends) at first so that the students get the idea a bit better. The idea is shown in Figure 3. The 3x10 circuit can be set by symmetry so that the circuit has the subsequence a,b,c and the tour starts and finishes in the bottom two rows of the first column marked by x's. Now the students must realize that the 3x10 circuit could detour from square b to the corner square of the left side of the 3x4 board, do the tour then get back on the circuit by going from the last square of the tour to square c and then finish the circuit.

Connecting a Circuit to a 3x4 Tour

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | b | | | | |
| ****** | a | | | x | | | |
| | | c | | x | | | |

FIGURE 3

The students now realize they can get 3x18 circuit from a 3x14 circuit and they can also get a 3x16 circuit from a 3x12 circuit. The following theorem could be formally proved by mathematical induction but why spoil their fun.

**Theorem 2** Knight’s tours exist only on 3xn boards where n is even and greater than 8.

At this point, one can collect the facts the students have found out about tours. Of course half the cases are already done as circuits.

**Lemma 4** There are knight’s tours on the 3x7, 3x8, 3x9 and 3x10 boards.

Proof: The following are knight’s tours.
(17 12 5 10 15 20 13 18 19 14 21 16 11 6 1 8 3 4 9 2 7)

(11 6 1 8 3 4 9 2 7 14 21 22 17 24 19 18 23 16 15 10 5 12 13 20)

(17 22 27 20 25 24 19 26 21 14 7 2 9 4 3 8 1 6 11 16 15 10 5 12 13 18 23)

(21 14 7 2 9 4 3 8 1 6 11 16 15 10 5 12 17 22 29 24 25 30 23 28 27 20 13 18 19 26) □

Somewhat trickier is the error prone task of conclusively proving that 3x5 and 3x6 tours do not exist. The long subsequence need not exist in these tours, so the backtrack has no big tricks to speed it up. Probably best to keep it simple and beat these subcases into submission.

**Lemma 5** There are no knight’s tours on the 3x5 and 3x6 boards.

Finally the students may be encouraged to apply the same trick of extending circuits to extending tours. Figure 4 shows the set you need. The x notes the last square of the tour and the o marks the beginning of a tour. Note that all the tours given in Lemma 6 end in the second from the right column in the middle row. The knight can then jump to the corner square of the 3x4 and continue that tour to end in the opposite corner. Then this tour can be extended because from that corner, the knight can jump to the corner of the next 3x4, this can continue for ever. In this way all cases are covered. The final theorem can now be entered in the 3xn theorem book.

Extending a Tour with Two 3x4 Tours

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | o | | | | o | | | |
| *** | x | | | | | | | | | |
| | | | | | | x | | | | x |

FIGURE 4

**Theorem 3** All 3xn boards have a knight's tour for $n \geq 7$ and $n=4$.

I almost forgot the homework. The following problem has a nice double parity argument. This is difficult for highschool students but give it to them as a challenge. I am sure that the readers of the Bulletin of the ICA will not need to have a solution given to them.

**Problem 1** Show that there are no knight's circuits on any 4xn board.

## References

[1] R. B. Eggleton and A Eid, *Knight's Circuits and Tours*, Ars Combin. 17A (1984) 145-167.

[2] M. R. Fellows and N. Koblitz, *Combinatorially Based Cryptography for Children (and Adults)*, Congr. Numer. 99 (1994) 9-41.

[3] E. Woodward and T. Hamel, *Geometric Constructions and Investigations with MIRA*, 1992

19 January 1995

Dear Ralph:

I would like to submit the following paper to the classroom notes section of the Bulletin of ICA. Thank you.

Sincerely

John van Rees, FICA