Hands on Algorithms for robotics

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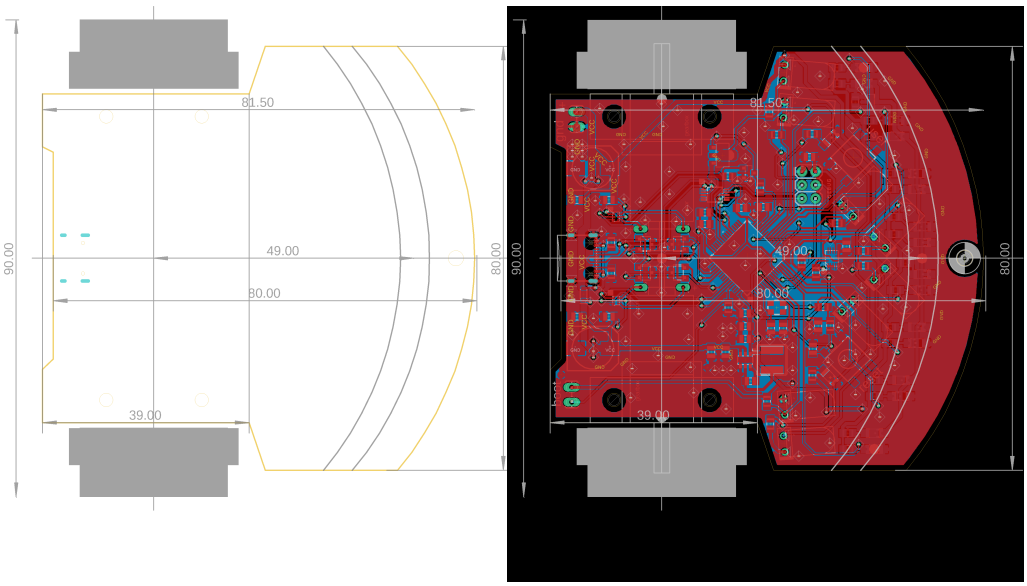
Chapter 1

Hardware description



(a) mechanical dimensions (b) overall PCB layout

Figure 1.1: Robot PCB

Figure 1.2: Robot photo

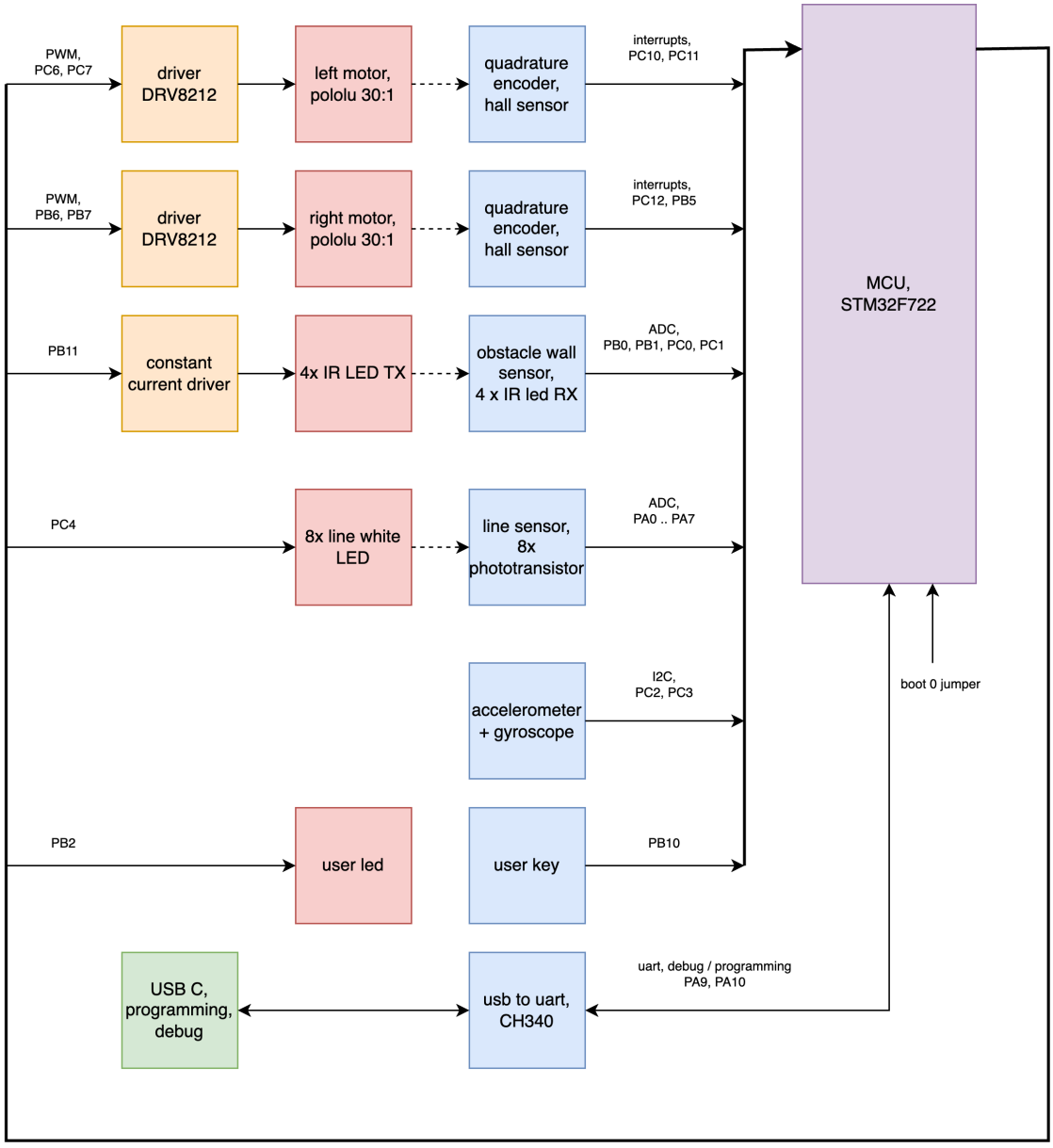


Figure 1.3: block diagram of DC motor version

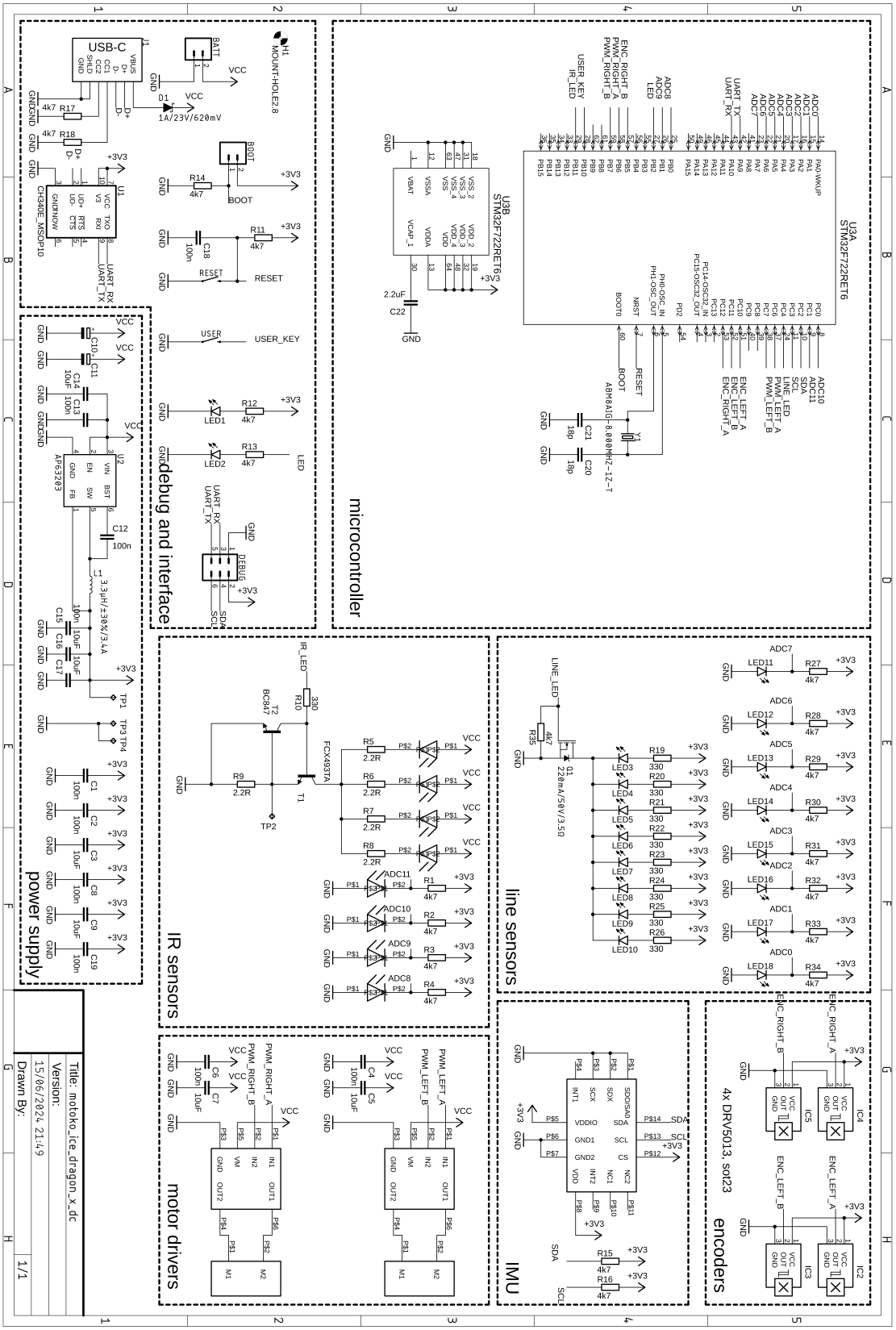


Figure 1.4: schematic diagram of DC motor version

pin number	pin name	function	peripheral	AF number
14	PA0	line sensor ADC0, right	ADC1	analog in
15	PA1	line sensor ADC1	ADC1	analog in
16	PA2	line sensor ADC2	ADC1	analog in
17	PA3	line sensor ADC3	ADC1	analog in
20	PA4	line sensor ADC4	ADC1	analog in
21	PA5	line sensor ADC5	ADC1	analog in
22	PA6	line sensor ADC6	ADC1	analog in
23	PA7	line sensor ADC7, left	ADC1	analog in
25	PB0	IR sensor ADC8, front left	ADC1	analog in
26	PB1	IR sensor ADC9, left	ADC1	analog in
8	PC0	IR sensor ADC10, right	ADC1	analog in
9	PC1	IR sensor ADC11, front right	ADC1	analog in
51	PC10	encoder left A	GPIOC	
52	PC11	encoder left B	GPIOC	
53	PC12	encoder right A	GPIOC	
57	PB5	encoder right B	GPIOB	
37	PC6	PWM left A	TIM3_CH1	AF2
38	PC7	PWM left B	TIM3_CH2	AF2
58	PB6	PWM right A	TIM4_CH1	AF2
59	PB7	PWM right B	TIM4_CH2	AF2
10	PC2	IMU I2C, SDA	GPIOC	
11	PC3	IMU I2C, SCL	GPIOC	
42	PA9	uart TX	UART1	AF7
43	PA10	uart RX	UART1	AF7
27	PB2	user LED	GPIOB	
28	PB10	user Key	GPIOB	
29	PB11	IR LED control	GPIOB	
24	PC4	line LED control	GPIOC	
7	NRST	reset		
60	BOOT0	firmware bootloader control		

Table 1.1: pin mapping and function

Chapter 2

System identification

2.1 State space models

System state is column vector, with N rows

$$x(n) = \begin{bmatrix} x_0(n) \\ x_1(n) \\ ... \\ x_{N-1}(n) \end{bmatrix} \tag{2.1}$$

For single output system this vector contains single value, e.g. the motor velocity $\omega(n)$

$$x(n) = \begin{bmatrix} \omega(n) \end{bmatrix} \tag{2.2}$$

If we have 2nd order system, e.g. servo with inertia, the state will contain two elements :

- motor shaft velocity $\omega(n)$, measured in rad/s
- motor shaft angle $\theta(n)$, measured in rad

$$x(n) = \begin{bmatrix} \omega(n) \\ \theta(n) \end{bmatrix} \tag{2.3}$$

More accurate servo model observes also motor current $i(n)$, which gives us three element state vector

$$x(n) = \begin{bmatrix} i(n) \\ \omega(n) \\ \theta(n) \end{bmatrix} \tag{2.4}$$

Robot moving in 2D plane have usualy state containing robot position x', y', and robot orientation θ

$$x(n) = \begin{bmatrix} x'(n) \\ y'(n) \\ \theta(n) \end{bmatrix} \tag{2.5}$$

The number of rows in state vector $x(n)$ is called **system order**.

The dynamical system is controlled with M inputs, stacked in column input vector $u(n)$

$$u(n) = \begin{bmatrix} u_0(n) \\ u_1(n) \\ ... \\ u_{M-1}(n) \end{bmatrix} \tag{2.6}$$

The system with single input, e.g. motor with current control, input vector contains single value

$$u(n) = \begin{bmatrix} i(n) \end{bmatrix} \tag{2.7}$$

Differential drive robot where are controlled two independent motors have two inputs

$$u(n) = \begin{bmatrix} i_{left}(n) \\ i_{right}(n) \end{bmatrix} \tag{2.8}$$

Relation between control input $u(n)$, current state $x(n)$ and next state $x(n+1)$ can be modeled using **linear state space model**

$$x(n+1) = Ax(n) + Bu(n) \tag{2.9}$$

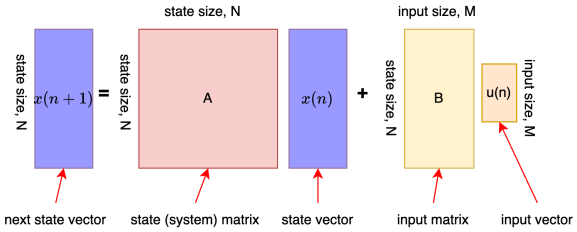


Figure 2.1: State space model matrices shapes

This model can capture dynamics of any linear system. Sometimes the state $x(n)$ can't be meassured directly, and we observe only system output $y(n) = Cx(n)$. Where matrix C have k-rows and n-columns. In our case we consider we have access to full state $x(n)$. There is of course the continuous form of state space model

$$\frac{dx}{dt} = A^c x(t) + B^c u(t) \tag{2.10}$$

note : the martices A^c and B^c differs from discrete system matrices A and B. They are related with bilinear transform as

$$\begin{aligned} S_a &= (I - \frac{\Delta T}{2} A^c)^{-1} \\ S_b &= I + \frac{\Delta T}{2} A^c \\ A &= S_a S_b \qquad (2.11) \\ B &= S_a B^c \Delta T \qquad (2.12) \end{aligned}$$

where ΔT is sampling period. This is mostly used discretisation, from continuous (e.g. physical) model into discrete form. Following pythoncode convert continous system into discrete system. If system contains output matrix C discretisation doesn't effects its values.

```
def c2d(a, b, c, dt):
    i = numpy.eye(a.shape[0])

    tmp_a = numpy.linalg.inv(i - (0.5*dt)*a)
    tmp_b = i + (0.5*dt)*a

    a_disc  = tmp_a@tmp_b
```

```
8       b_disc  = (tmp_a*dt)@b
9
10      return a_disc, b_disc, c
```

Process of system identification is finding matrices A, B (and C). From this model can be sythetised controller, or system able to plan multiple steps ahead.

2.2 First order identification

Goal is to find parameters of DC motor. Motor velocity model, can be approximated as 1st linear order system. Real system is of course non-linear. The photo of example system is on fig 2.2. Control loop hardware consists of motor driver (H-driver, DRV8212), motor (Pololu HP 1:30 gear), and quadrature magnetic encoder (2x DRV5013), fig 2.3

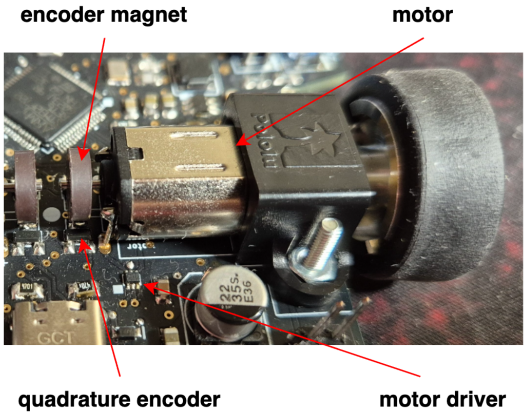


Figure 2.2: Wheel drive hardware detail

Input signal $u(n)$ can be generated arbitrarily. Mostly used is square wave, to capture high frequency dynamics. Algorithm is from observed motor velocity $x(n)$ and given input $u(n)$ estimating model parameters.

Before real motor testing, we focus on simulation example. Our simulated motor will have following parameters

- sampling frequency 4kHz, $\Delta T = 1/4000$
- maximum control input $u_{max} = 2$

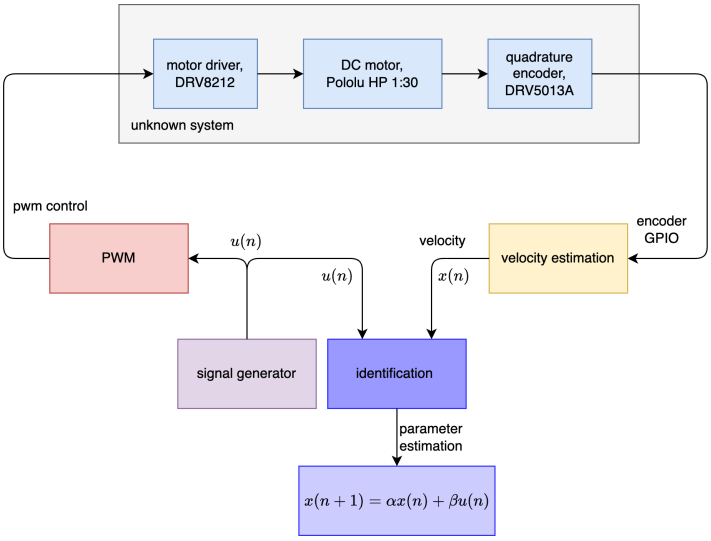


Figure 2.3: Motor identification process

- motor constant $k = 17$
- motor time constat $\tau = 29$ milliseconds

Motor state is single variable, angular velocity $\omega(t)$. Model have two parameters α, β. First order continuos model becomes

$$\frac{d\omega(t)}{dt} = \alpha\omega(t) + \beta u(t) \tag{2.13}$$

$$\alpha = -\frac{1}{\tau} \tag{2.14}$$

$$\beta = \frac{k}{\tau} \tag{2.15}$$

with given parameters k and τ becomes $\alpha = -34.48$ and $\beta = 586.2$.

Unit step response is obtained by setting $u(n) = 1$, and solving using any differential equation solver. The simplest is Euler method, which works well for small dt and low order systems. Much accurate is commonly used Runge-Kutta 4 method.

Our goal is to generate input $u(t)$ and by observing system output $x(t)$ estimate parameters k and τ.

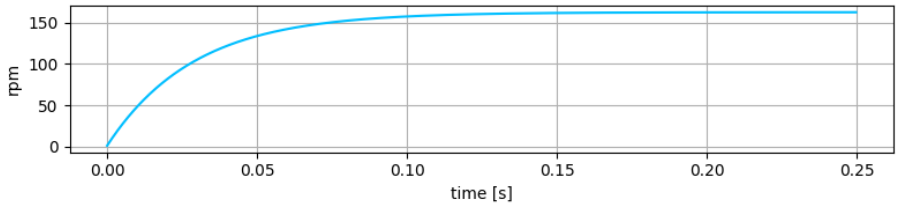


Figure 2.4: Motor step response

We start with simplest identification method, which works only for 1st order non-oscilating systems. Method have two main steps

1. steady state value estimation
2. time constant estimation

Steady state estimation

We drive motor on different input levels, e.g. ranging from 10% ... 100%. Observing steady state velocity the motor constant k can be estimated.

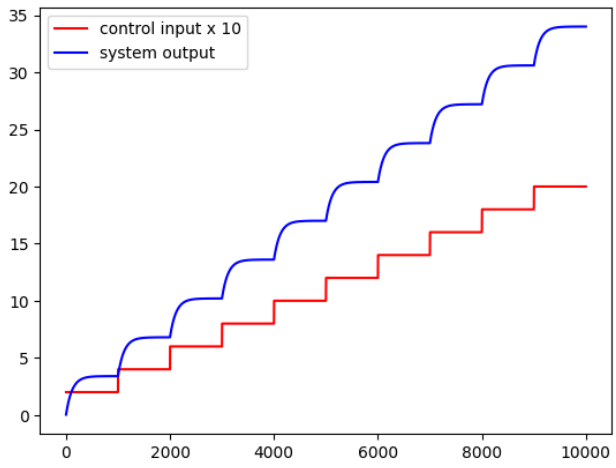


Figure 2.5: Motor constant estimation

Algorithm for estimating motor constant :

1. choose constant motor input $u \in (0, u_{max}\rangle$

2. run motor, and wait for velocity steady state

3. measure velocity

4. estimate $\hat{k} = x/u$

5. repeats to estimate average $\hat{k}$

In following code we choose 10 different input levels. After setting, we wait 500steps to steady state, then we measure output velocity. Finally, we average results.

```
# input levels
u_values = u_max*numpy.array([0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0])

# reset dynamical system into zero state
ds.reset()

for j in range(len(u_values)):

    x_mean = []
    for i in range(1000):

        # convert scalar u to column vector
        u    = u_values[j]
        u    = numpy.array([[u]])

        # compute dynamical system one step
        x, _ = ds.forward_state(u)

        # after steady state, store x
        if i > 500:
            x_mean.append(x[0][0])

    x_mean = numpy.array(x_mean)
    x_mean = x_mean.mean()

    k_est = x_mean/u_values[j]

    print(u_values[j], k_est)

k_mean = k_est.mean()

# average k estimate
print("k_mean = ", k_mean)
```

In our example the resulted $\hat{k}$ is 16.995, which very close to real value 17. Accuracy depends on velocity estimation noise (encoder noise) and how many samples we average.

Time constant estimation

Second parameter is time constant τ. Most textbooks uses measuring time at which system reaches 63.2% of x_{max}. Where x_{max} is motor velocity on choosen u, a.k.a. steady state velocity at u. This method suffers to noise, and precise timing. Here we presents more accurate method, briging system into natural frequency resonance. Algorithm is as follow :

1. choose u_{set}, and set $u_{in} = u_{set}$
2. repeat n_{steps}, each step duration is ΔT
   a) $if\ u_{in} > 0\ and\ x(n) > 0.632ku_{in}$
   increment $n_{HalfPeriods}$, and set $u_{in} = -u_{set}$
   b) $else\ if\ u_{in} < 0\ and\ x(n) < 0.632ku_{in}$
   increment $n_{HalfPeriods}$, and set $u_{in} = u_{set}$

The time constant $\hat{\tau}$ can be now estimated by

$$\hat{\tau} = \frac{2}{\pi} \frac{n_{steps}}{n_{HalfPeriods}} \Delta T \tag{2.16}$$

Where ΔT is sampling period, ratio $n_{steps} : n_{HalfPeriods}$ is estimating system frequency. In core, this method is nothing more than period duration estimation. The resulted time plot is on fig. 2.6. Result for our simulated motor is $\hat{\tau} = 27.44[ms]$, which is good estimation of real value $29[ms]$.

Using equations 2.14 and 2.15 we can estimate $\hat{\alpha}$ and $\hat{\beta}$ as $\hat{\alpha} = -36.443$ and $\hat{\beta} = 619.372$

Real motor identification

For implementation in real system, the code must runs in real time. With relative high sampling frequency 4kHz, the code is written in c++. Interface is universal, with few key functions.

- timer.delay_ms(unsigned int time) - waits given milliseconds
- float x = motor_control.get_right_velocity() - return meassured wheel velocity

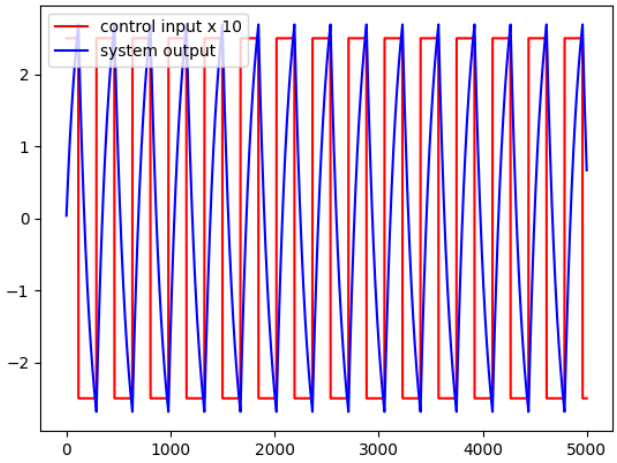


Figure 2.6: Motor time constant estimation

- motor_control.set_right_torque(float x) - sets the PWM value, depends on sign the rotation direction can be controlled

This functions needs to be customised for user application.

First we estimate motor constant k :

```
//1, estimate motor constant k, on different input values
uint32_t n_steps = 500;

float u_values[10] = {0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0};

float k_mean     = 0.0;
float x_var_mean = 0.0;

for (unsigned int j = 0; j < 10; j++)
{
    //run motor with desired input and wait for steady state
    float u_in = u_values[j];

    motor_control.set_right_torque(u_in*MOTOR_CONTROL_MAX_TORQUE);
    timer.delay_ms(500);

    //estimate average velocity
    float x_mean = 0.0;
    for (unsigned int i = 0; i < n_steps; i++)
    {
        float x = motor_control.get_right_velocity();
        x_mean+= x;
        timer.delay_ms(1);
    }
    x_mean = x_mean/n_steps;

    //estimate average variance - encoder noise
    float x_var = 0.0;
    for (unsigned int i = 0; i < n_steps; i++)
    {
        float x = motor_control.get_right_velocity();
        x_var+= (x - x_mean)*(x - x_mean);
        timer.delay_ms(1);
    }
    x_var = x_var/n_steps;

    float k = x_mean/u_in;
    k_mean+= x_mean/u_in;
    x_var_mean+= x_var;

    terminal << "u_in   = " << u_in << "\n";
    terminal << "k      = " << k << "\n";
    terminal << "x_mean = " << x_mean << "\n";
    terminal << "x_var  = " << x_var << "\n";
    terminal << "\n\n";
}

//print summary results
k_mean     = k_mean/10.0;
x_var_mean = x_var_mean/10.0;

terminal << "k          = " << k_mean << "\n";
terminal << "x_var_mean = " << x_var_mean << "\n";
```

Second we estimate motor time constant τ :

```
//2, estimate time constant by oscilating motor
float u_in = 0.25;
uint32_t periods = 0;

n_steps = 2000;

for (unsigned int i = 0; i < n_steps; i++)
{
    motor_control.set_right_torque(u_in*MOTOR_CONTROL_MAX_TORQUE);
    float x = motor_control.get_right_velocity()*60.0/(2.0*PI);

    if (u_in > 0.0)
    {
        if (x > 0.632*k_mean*u_in)
        {
            u_in = -u_in;
            periods++;
        }
    }
    else
    {
        if (x < 0.632*k_mean*u_in)
        {
            u_in = -u_in;
            periods++;
        }
    }

    timer.delay_ms(1);
}

float t_period = (2*n_steps/periods)/PI;
terminal << "periods = " << periods << "\n";
terminal << "tau     = " << t_period << "[ms]\n";
terminal << "\n\n";
```

Average motor constant is $k = 80.056$, and time constant is $\tau = 9.230[ms]$.
Terminal output :

```
...

u_in   =  0.800
k      =  87.194
x_mean =  69.755
x_var  =  19.236


u_in   =  0.900
k      =  86.440
x_mean =  77.796
x_var  =  18.124


u_in   =  1.000
k      =  85.750
x_mean =  85.750
x_var  =  7.554


k          =  80.056
x_var_mean =  170.903


periods = 136
tau     =  9.230[ms]
```

Chapter 3

Motion control

3.1 Motor velocity controller

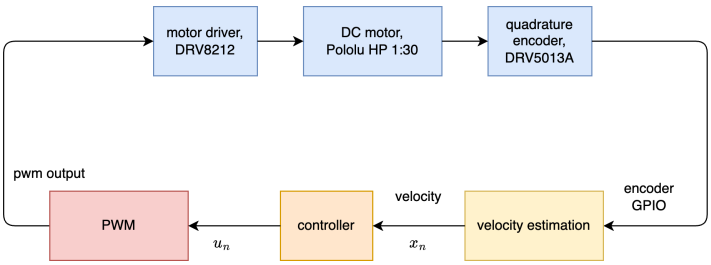


Figure 3.1: Velocity control overview

3.2 PID control

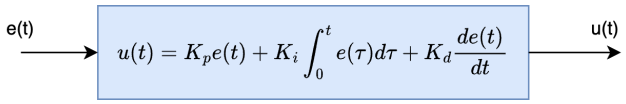


Figure 3.2: Textbook continuous PID controller

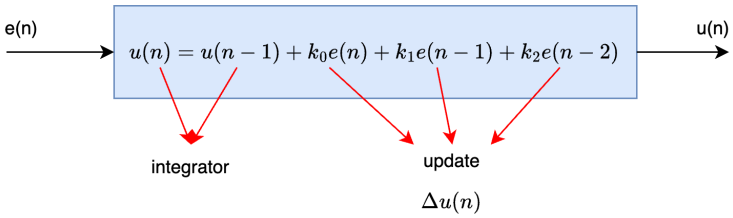


Figure 3.3: Discrete PID controller

$$
\begin{aligned}
k_0 &= K_p + K_i \Delta T + \frac{K_d}{\Delta T} \\
k_1 &= -K_p - 2\frac{K_d}{\Delta T} \\
k_2 &= \frac{K_d}{\Delta T}
\end{aligned}
\tag{3.1}
$$

PID control - P only controller

- target value : **1000rpm**
- P-only control causes **steady state error**

$$u(n+1) = u(n-1) + k_p e(n) - k_p e(n-1) \tag{3.2}$$
$$k_p = 0.01$$

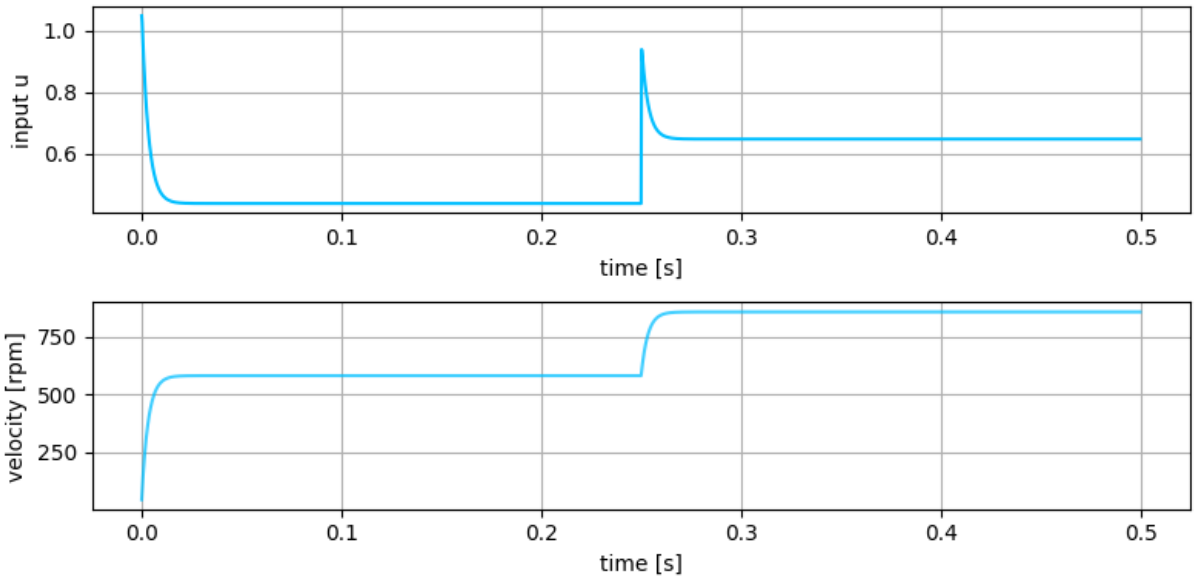


Figure 3.4: P-only controller

PID control - PI

too high I term

- target value : **1000rpm**
- PI control **removes steady state error**
- too high I term causes **oscilations and overshot**

$$u(n+1) = u(n-1) + (k_p + k_i\Delta T)e(n) - k_p e(n-1) \tag{3.3}$$
$$k_p = 0.01$$
$$k_i\Delta T = 0.005$$

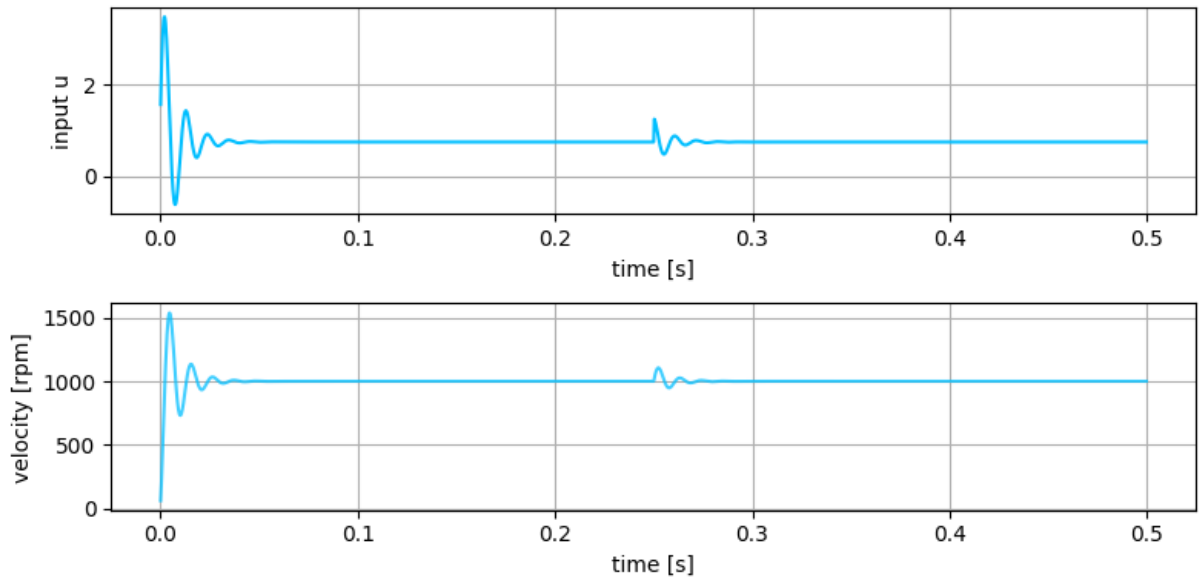


Figure 3.5: PI controller

correct I term

- target value : **1000rpm**
- correct tunned PI controller for 1st order system

- **no overshot, no steady state error**

$$u(n+1) = u(n-1) + (k_p + k_i\Delta T)e(n) - k_p e(n-1) \tag{3.4}$$
$$k_p = 0.01$$
$$k_i\Delta T = 0.0002$$

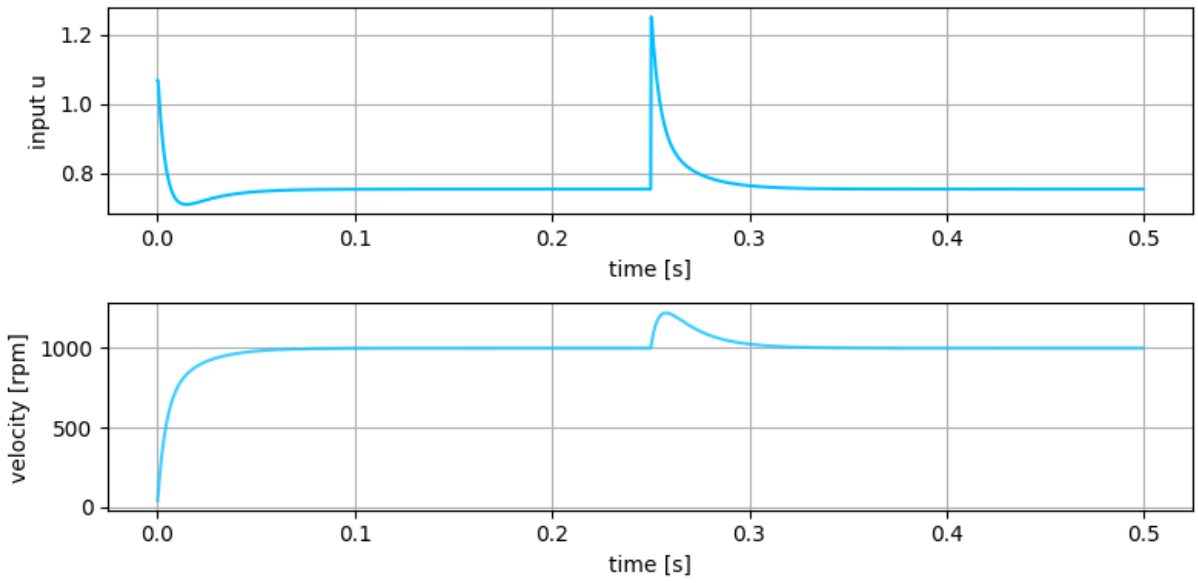


Figure 3.6: PI controller with correct parameters

Complete discrete PID algorithm

1. calculate u-change candidate :

$$\Delta\hat{u}(n) = k_0 e(n) + k_1 e(n) + k_2 e(n)$$

2. clip maximum allowed u-change, to avoid u-kick :

$$\Delta u(n) = clip(\Delta\hat{u}(n), -du_{min}, du_{max})$$

3. clip maximum allowed u value, to avoid saturation / windup :

$$u(n) = clip(u(n-1) + \Delta u(n), -u_{min}, u_{max})$$

Full Python code example for discrete PID controller :

```
class PID:

def __init__(self, kp, ki, kd, antiwindup = 10**10, du_max=10**10):
    self.k0 = kp + ki + kd
    self.k1 = -kp -2.0*kd
    self.k2 = kd

    self.e0 = 0.0
    self.e1 = 0.0
    self.e2 = 0.0

    self.antiwindup = antiwindup
    self.du_max     = du_max

def forward(self, xr, x, u_prev):
    # error compute
    self.e2 = self.e1
    self.e1 = self.e0
    self.e0 = xr - x

    # u-change computing
    du = self.k0*self.e0 + self.k1*self.e1 + self.k2*self.e2

    # kick clipping, maximum output value change limitation
    du  = numpy.clip(du, -self.du_max , self.du_max)

    # antiwindup, maximum output value limitation
    u   = numpy.clip(u_prev + du,  -self.antiwindup, self.antiwindup)

    return u
```

3.3 Optimal control

Optimal control is solution of following optimisation problem

$$\min_{x(.),u(.)} \sum_{n=0}^{\infty} x^T(n)Qx(n) + u^T(n)Ru(n) \tag{3.5}$$
$$s.t.\ x(n+1) = Ax(n) + Bu(n)$$

the cost function is quadratic

$$J = \sum_{n=0}^{\infty} x^T(n)Qx(n) + u^T(n)Ru(n) \tag{3.6}$$

where Q and R are positive semidefinite square matrices, with shapes $N \times N$ and $M \times M$ respectively. The system order is N and system have M inputs. Matrix Q penalises magnitude of $x(n)$ elements, matrix R penalises control input $u(n)$ elements. Solution is obtained by solving algebraic Riccati equation, and results in feedback gain $u(n) = -Kx(n)$. This controller is called **Linear Quadratic Regulator - LQR**. This matrix can fully manipulate with system poles and guarantee system stability. However, given structure have steady state error - brings system always into zero state. To eliminate this issue, we introduce integral action term by augmenting system matrices. Resulted controller have two matrices : feedback gain K and integral action gain K_i matrix. The modified LQR controller have form on following fig 3.7 with equations 3.7.

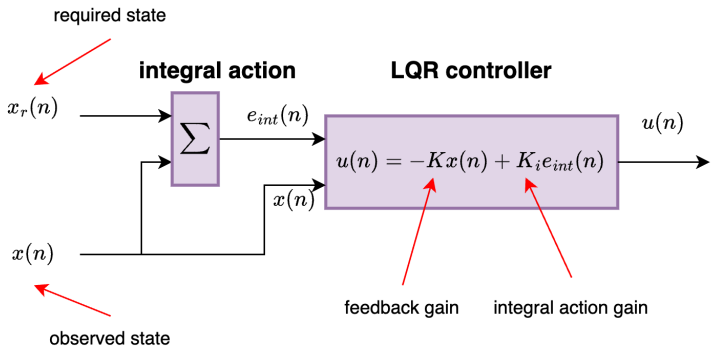


Figure 3.7: LQR controller with integral action

$$u(n) = -Kx(n) + K_i e_{int}(n) \tag{3.7}$$
$$e_{int}(n) = e_{int}(n-1) + x_r(n) - x(n)$$

where :

- $x(n)$ is observerd state, column vector, with shape $(N, 1)$
- $x_r(n)$ is required state, column vector, with shape $(N, 1)$
- $u(n)$ is controller output, column vector, with shape $(M, 1)$
- K is feedback gain matrix, with shape (M, N)
- K_i is integral gain matrix, with shape (M, N)

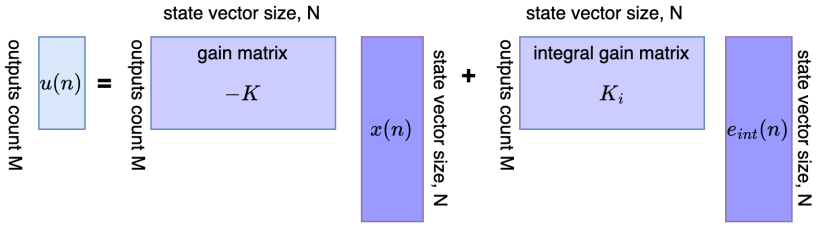


Figure 3.8: Matrices shapes in LQR controller

3.4 Model predictive control

Previous controllers were focused only on single step in required state $x_r(n)$. Model predictive control solves more generic problem, where required input is not single a state $x_r(n)$ but whole sequence $\tilde{X}_r$, defined as stacked state vectors $x_r(n)$, eq. 3.8. This is called trajectory tracking problem and $\tilde{X}_r$ is required trajectory with length of H-steps. H is called prediction horizon. Controller uses state space model of controlled system to predict future states, defined as 3.13. The control sequence $u(n)$ is optimised in such way the controlled system follows the required trajectory $\tilde{X}_r$.

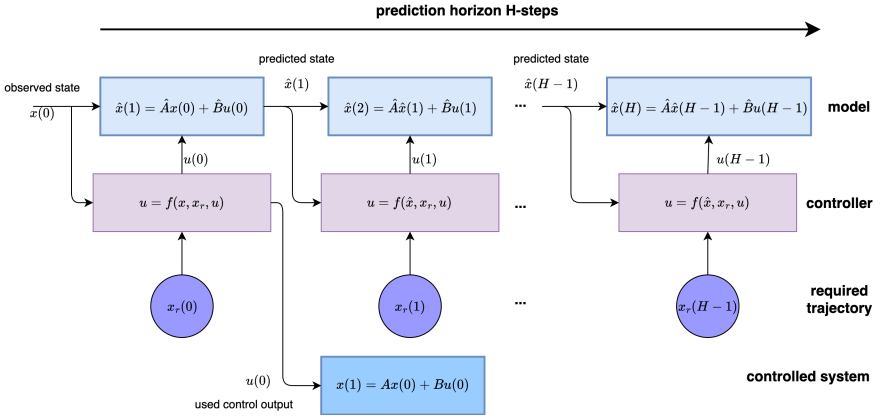


Figure 3.9: Principle of model predictive control

$$\tilde{X}_r = \begin{bmatrix} x_r(n) \\ x_r(n+1) \\ \dots \\ x_r(n+H-1) \end{bmatrix} \tag{3.8}$$

Model predictive control is solution of following optimisation problem

$$\min_{x(.),\Delta u(.)} \sum_{h=0}^{H} (x_r^T(h) - x^T(h))Q(x_r(h) - x(h)) + {}_{\Delta}u^T(h)R_{\Delta}u(h) \tag{3.9}$$

$$s.t.\ x(n+1) = Ax(n) + Bu(n)$$
$$u(n) = u(n-1) + {}_{\Delta}u(n)$$

where :

- H is prediction horizon steps
- A is matrix, $n \times n$
- B is matrix, $n \times m$
- Q is matrix, $n \times n$
- R is matrix, $m \times m$
- ${}_{\Delta}u$ is controller output
- n is system orders, and m system inputs count

If we consider unconstrained model precitive control problem, there is **analytical solution**. Solution consists on following main steps :

1. formulating problem in matrix form, such the current state is $x(n)$, current controll value is $u(n-1)$ and input is only control sequence ${}_{\Delta}u(n)$, see eq. 3.12
2. putting matrix form formulation into quadratic loss function, see eq 3.15
3. finding minima of quadratic loss function : derivate with respect to ${}_{\Delta}\tilde{U}$, put derivative equal to zero and solve for ${}_{\Delta}\tilde{U}$, see eq. 3.18, 3.19 and 3.20.
4. simplifying solution by precomputing computational expensive parts, see eq. 3.21 and 4.1

Step 1 : Unwrap into form where only $x(n)$ and $u(n-1)$ are used to

predict all other x

$$\begin{aligned}
x(n+1) &= Ax(n) + Bu(n) \\
&= Ax(n) + B(u(n-1) + {}_{\Delta}u(n)) \\
x(n+2) &= Ax(n+1) + B(u(n) + {}_{\Delta}u(n+1)) \\
&= A^2x(n) + (AB+B)u(n-1) + (AB+B)_{\Delta}u(n) + B_{\Delta}u(n) \\
x(n+3) &= Ax(n+2) + B(u(n+1) + {}_{\Delta}u(n+2)) \\
&= A^3x(n) + (A^2+AB+B)u(n-1) \\
&\quad + (A^2B+AB+B)_{\Delta}u(n) \\
&\quad + (AB+B)_{\Delta}u(n+1) + B_{\Delta}u(n+2) \\
&\quad ...
\end{aligned} \tag{3.10}$$

Rewrite into matrix form

$$\begin{aligned}
\begin{bmatrix} x(n+1) \\ x(n+2) \\ \dots \\ x(n+H) \end{bmatrix} &= \begin{bmatrix} A^1 \\ A^2 \\ \dots \\ A^H \end{bmatrix} x(n) + \begin{bmatrix} B \\ AB+B \\ \dots \\ \sum_{i=0}^{H-1} A^iB \end{bmatrix} u(n-1) + \\
&+ \begin{bmatrix} B & 0 & \dots & 0 \\ AB+B & B & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{H-1} A^iB & \sum_{i=0}^{H-2} A^iB & \dots & B \end{bmatrix} \begin{bmatrix} {}_{\Delta}u(n) \\ \dots \\ {}_{\Delta}u(n+H-1) \end{bmatrix}
\end{aligned} \tag{3.11}$$

in compact matrix form

$$\tilde{X} = \Psi x(n) + \Omega u(n-1) + \Theta_{\Delta}\tilde{U} \tag{3.12}$$

where predicted trajectory $\tilde{X}$ is defined as

$$\tilde{X} = \begin{bmatrix} x(n+1) \\ x_r(n+2) \\ ... \\ x_r(n+H) \end{bmatrix} \tag{3.13}$$

and stacked controll sequence ${}_{\Delta}\tilde{U}$ is defined as

$$ {}_{\Delta}\tilde{U} = \begin{bmatrix} {}_{\Delta}u(n) \\ {}_{\Delta}u(n+1) \\ \dots \\ {}_{\Delta}u(n+H-1) \end{bmatrix} \tag{3.14} $$

Explonation of matrix form:

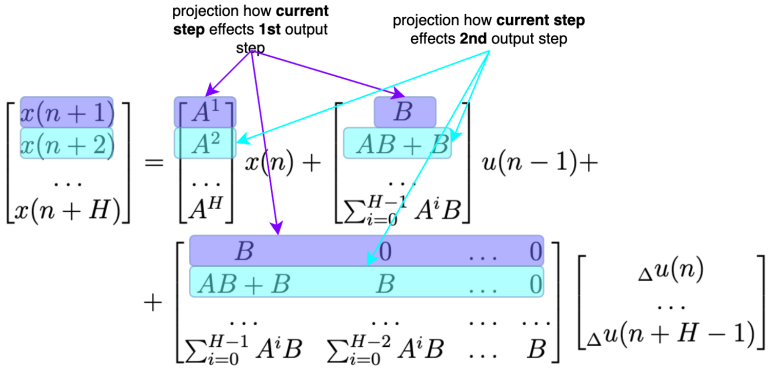


Figure 3.10: Effect of current state $x(n)$ and input $u(n-1)$ to all output steps

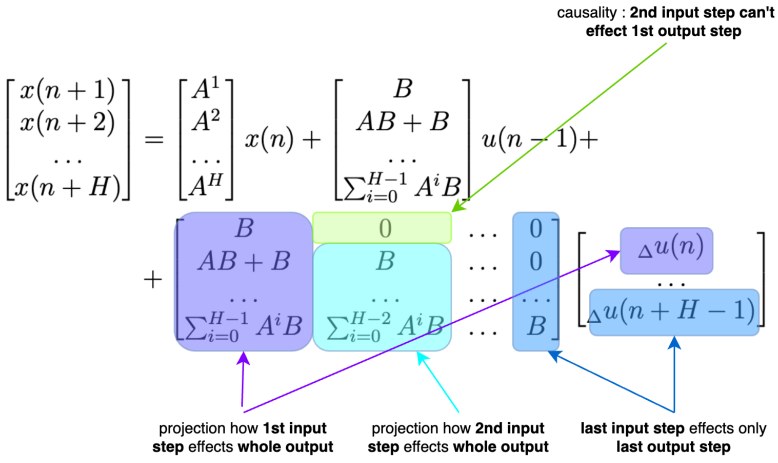


Figure 3.11: Effect of n-th input step ${}_{\Delta}u(n)$ on output

Step 2 : Formulate quadratic loss (cost) function

$$J = {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + (\tilde{X_r} - \tilde{X})^T \tilde{Q}(\tilde{X_r} - \tilde{X})^T \tag{3.15}$$

after substitution

$$S = \tilde{X_r} - \Psi\tilde{X} - \Omega u(n-1)$$

we get

$$\begin{aligned} J &= {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + (S - \Theta_{\Delta}\tilde{U})^T \tilde{Q}(S - \Theta_{\Delta}\tilde{U}) \\ &= {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + S^T\tilde{Q}S - S^T\tilde{Q}\Theta_{\Delta}\tilde{U} - {}_{\Delta}\tilde{U}^T\Theta^T\tilde{Q}S + {}_{\Delta}\tilde{U}^T\Theta^T\tilde{Q}\Theta_{\Delta}\tilde{U} \end{aligned} \tag{3.16}$$

Step 3 : find derivative with respect to ${}_{\Delta}\tilde{U}$

$$\frac{\partial J}{\partial {}_{\Delta}\tilde{U}} : \tag{3.17}$$
$$\frac{\partial {}_{\Delta}\tilde{U}^T \tilde{R}\, {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = 2\tilde{R}\, {}_{\Delta}\tilde{U}$$
$$\frac{\partial S^T \tilde{Q} S}{\partial {}_{\Delta}\tilde{U}} = 0$$
$$\frac{\partial -S^T \tilde{Q} \Theta\, {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = -\Theta^T \tilde{Q} S$$
$$\frac{\partial -{}_{\Delta}\tilde{U}^T \Theta^T \tilde{Q} S}{\partial {}_{\Delta}\tilde{U}} = -\Theta^T \tilde{Q} S$$
$$\frac{\partial {}_{\Delta}\tilde{U}^T \Theta^T \tilde{Q} \Theta\, {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = 2\Theta^T \tilde{Q} \Theta\, {}_{\Delta}\tilde{U}$$

put derivate equal to zero, and solve

$$\frac{\partial J}{\partial {}_{\Delta}\tilde{U}} = 2\tilde{R}\, {}_{\Delta}\tilde{U} - 2\Theta^T \tilde{Q} S + 2\Theta^T \tilde{Q} \Theta\, {}_{\Delta}\tilde{U} \tag{3.18}$$
$$0 = 2\tilde{R}\, {}_{\Delta}\tilde{U} - 2\Theta^T \tilde{Q} S + 2\Theta^T \tilde{Q} \Theta\, {}_{\Delta}\tilde{U}$$
$$(\tilde{R} + \Theta^T \tilde{Q} \Theta)\, {}_{\Delta}\tilde{U} = \Theta^T \tilde{Q} S$$

and **obtain solution for unconstrained model predictive controll**

$${}_{\Delta}\tilde{U} = (\tilde{R} + \Theta^T \tilde{Q} \Theta)^{-1} \Theta^T \tilde{Q} S \tag{3.19}$$
$$\tag{3.20}$$

Step 4 : model predictive control - full algorithm
given matrices : $\tilde{Q}, \tilde{R}, \Theta, \Phi, \Omega$
initialization (precompute) :

$$\Xi = (\tilde{R} + \Theta^T \tilde{Q} \Theta)^{-1} \Theta^T \tilde{Q} \tag{3.21}$$

in loop process controller step :

$$E(n) = X_r(n) - \Phi x(n) - \Omega u(n-1) \tag{3.22}$$
$${}_{\Delta}u(n) = \Xi E(n)$$
$$u(n) = u(n-1) +_{\Delta} u(n)$$

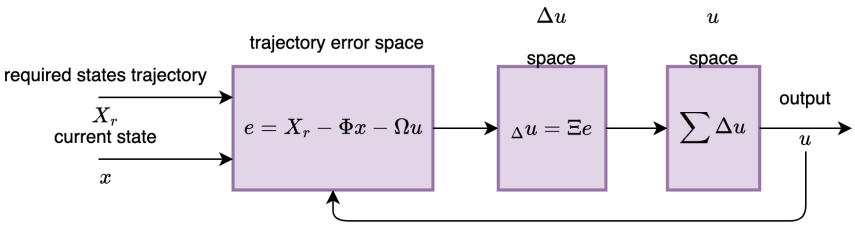


Figure 3.12: Block diagram of unconstrained MPC

Chapter 4

Reinforcement learning

Reinforcement Learning (RL) is a branch of machine learning concerned with how agents take actions in an environment to maximize cumulative reward. Unlike supervised learning, where the agent learns from a labeled dataset, RL focuses on trial-and-error interactions, as shown on figure 4.1. The agent observes the state of the environment, selects actions according to a policy, and receives feedback in the form of rewards. Over time, the agent aims to learn an optimal policy that maximizes the total long-term reward by balancing exploration (trying new actions) and exploitation (choosing known high-reward actions). Key concepts in RL include states, actions, rewards, policies, and value functions. RL has been successfully applied in areas such as robotics, game playing, and autonomous systems, where decision-making under uncertainty is crucial.

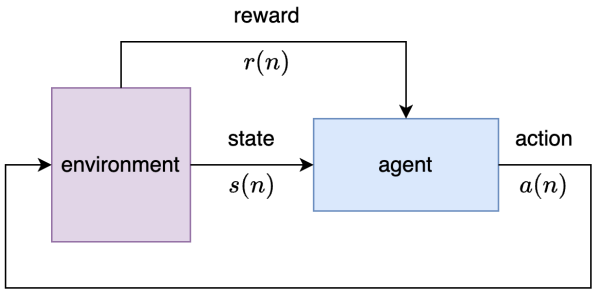


Figure 4.1: Reinforcement learning principle

4.1 Terminology

State

Represents the current status or configuration of a system or environment. It contains all the relevant information necessary to make decisions about future actions or predictions. The state encodes the minimum amount of information required to describe the system's present situation without needing to know the full history of past events.

$$s(n) \in \mathbb{S} \tag{4.1}$$

Example of states:

- *Robot state*: This may include position (x, y), velocity components v_x, v_y, and the robot's orientation θ. Together, this forms a 5-element state vector: $[x, y, v_x, v_y, \theta]$.
- *Image-based state*: A screenshot from a game represented as an RGB image of shape $(3, \text{height}, \text{width})$. To capture temporal information, multiple frames can be stacked, e.g., four frames. In this case, the state has a shape of $(4 \times 3, \text{height}, \text{width})$.

Policy

A policy defines the mechanism by which an agent selects actions based on its perception of the environment (i.e., its current state). It represents the agent's behavior, mapping states to actions and guiding how the agent interacts with the environment.

A policy can be deterministic, denoted as

$$a = \pi(s)$$

where the action a is determined directly by the state s, or it can be stochastic, defined as

$$a \sim \pi(\cdot|s)$$

where the action a is drawn from a probability distribution conditioned on the state s.

Reward

A reward is a scalar feedback signal that measures the quality of the outcome of an agent's action, guiding the agent towards behaviors that maximize long-term cumulative reward. The agent's objective is to maximize the expected cumulative reward over time.

Objective of Reinforcement Learning

The formal objective of reinforcement learning is to maximize the expected return, which is the sum of discounted rewards over time

$$J(\theta) = \mathbb{E}_{\pi_\theta}\left[\sum_{n=0}^{\infty} \gamma^n r(n)\right] \tag{4.2}$$

where:

- π_θ is the policy parameterized by θ,
- $r(n)$ is the reward at time step n,
- $\gamma \in [0, 1)$ is the discount factor, which balances immediate and future rewards.

Value

The value function represents the expected cumulative reward that an agent can obtain from a particular state or state-action pair. There are two main types of value functions:

- **State Value Function** $V^\pi(s)$: The expected return starting from state s and following policy π.
- **Action Value Function** $Q^\pi(s, a)$: The expected return starting from state s, taking action a, and then following policy π.

Value functions help the agent determine how beneficial it is to be in a certain state or take a particular action, guiding the learning process.

4.2 Actor-Critic agent

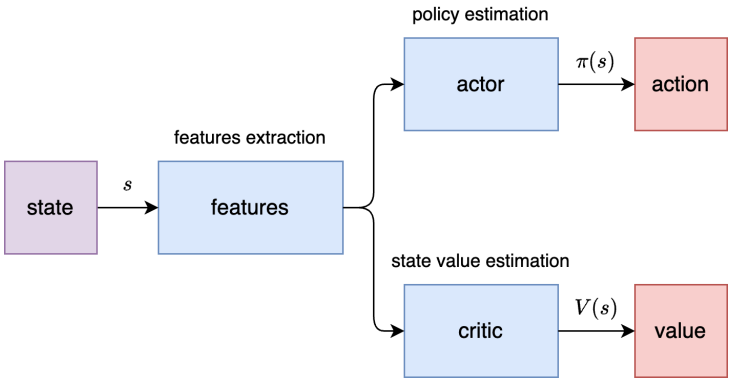


Figure 4.2: Actor critic basic structure