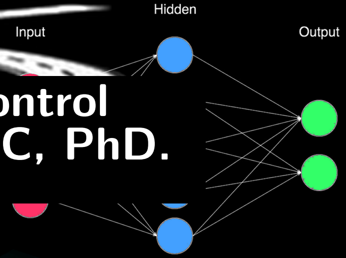
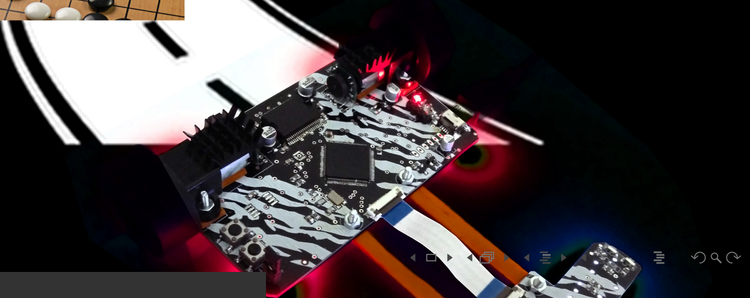
$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha\left[r_{t+1} + \lambda \max_{a} Q(s_{t+1}, a) - Q(s_t, a_t)\right]$$

(The New Action Value = The Old Value) + The Learning Rate × (The New Information − the Old Information)

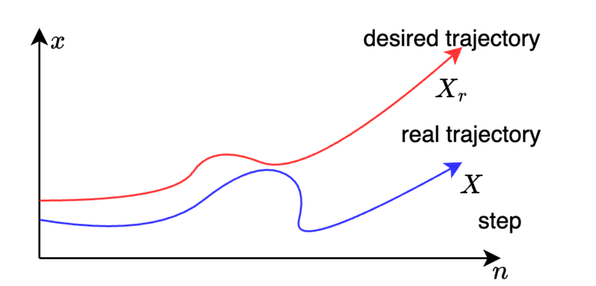


model predictive control

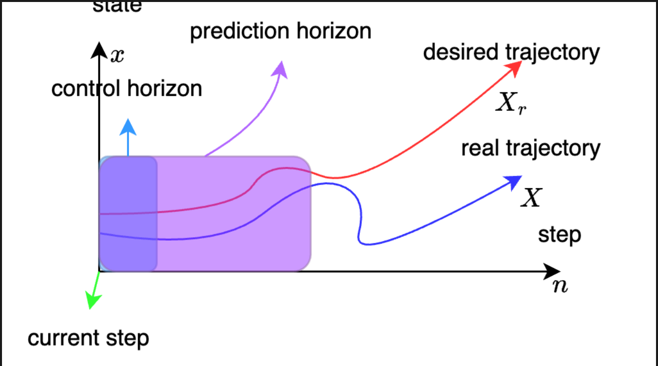
Michal CHOVANEC, PhD.



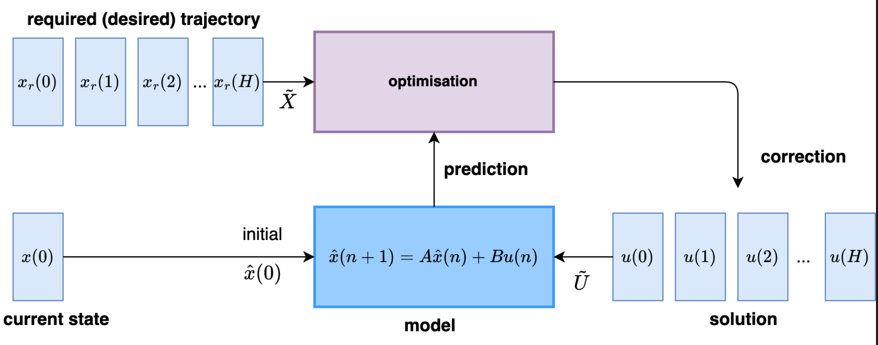
MPC - overview



MPC - overview



MPC - overview



quadratic problem formulation

loss (cost) function is quadratic, with weighting terms Q and R,

$$\mathcal{L} = \sum_{h=0}^{H-1} \left((x_r(h) - x(h))^T Q (x_r(h) - x(h)) \right) + \Delta u^T(h) R \Delta u(h),$$

$$u(n) = u(n-1) + \Delta u(n),$$

$$\text{s.t.} \quad x(n+1) = Ax(n) + Bu(n).$$

where :

- H is prediction horizon steps
- A is matrix, $n \times n$
- B is matrix, $n \times m$
- Q is matrix, $n \times n$
- R is matrix, $m \times m$
- ${}_\Delta u$ is controller output
- where n is system orders, and m system inputs count

unrolling sequence

rewrite into form where initial conditions **depends only** on $x(n)$ and $u(n-1)$

$$
\begin{aligned}
x(n+1) &= Ax(n) + Bu(n) \\
&= Ax(n) + B(u(n-1) +_\Delta u(n)) \\
x(n+2) &= Ax(n+1) + B(u(n) +_\Delta u(n+1)) \\
&= A^2x(n) + (AB + B)u(n-1) + (AB + B)_\Delta u(n) + B_\Delta u(u) \\
x(n+3) &= Ax(n+2) + B(u(n+1) +_\Delta u(n+2)) \\
&= A^3x(n) + (A^2 + AB + B)u(n-1) \\
&+ (A^2B + AB + B)_\Delta u(n) \\
&+ (AB + B)_\Delta u(n+1) + B_\Delta u(n+2) \\
&\ldots
\end{aligned}
$$

matrix formulation

rewrite into matrix form

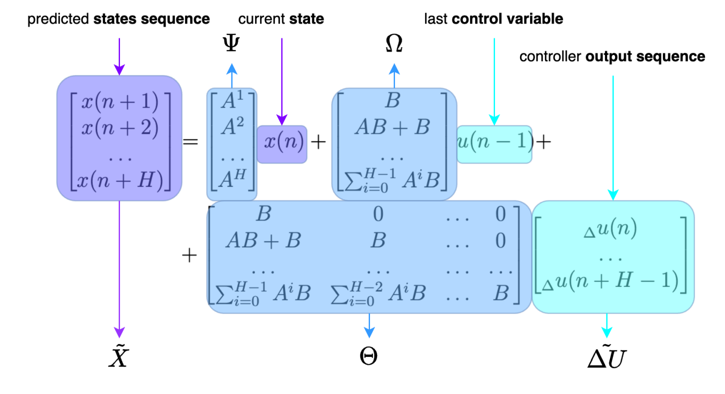
$$\begin{bmatrix} x(n+1) \\ x(n+2) \\ \dots \\ x(n+H) \end{bmatrix} = \begin{bmatrix} A^1 \\ A^2 \\ \dots \\ A^H \end{bmatrix} x(n) + \begin{bmatrix} B \\ AB \\ \dots \\ \sum_{i=0}^{H-1} A^i B \end{bmatrix} u(n-1) +$$

$$+ \begin{bmatrix} B & 0 & \dots & 0 \\ AB+B & B & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{H-1} A^i B & \sum_{i=0}^{H-2} A^i B & \dots & B \end{bmatrix} \begin{bmatrix} {}_{\Delta} u(n) \\ \dots \\ {}_{\Delta} u(n+H-1) \end{bmatrix}$$

in compaxt form

$$\tilde{X} = \Psi x(n) + \Omega u(n-1) + \Theta_{\Delta} \tilde{U}$$

matrix formulation



$$\tilde{X} = \Psi x(n) + \Omega u(n-1) + \Theta \tilde{{}_{\Delta}U}$$

n-th step input to output projection

projection how **current step** effects **1st** output step

projection how **current step** effects **2nd** output step

$$
\begin{bmatrix} x(n+1) \\ x(n+2) \\ \dots \\ x(n+H) \end{bmatrix} = \begin{bmatrix} A^1 \\ A^2 \\ \dots \\ A^H \end{bmatrix} x(n) + \begin{bmatrix} B \\ AB+B \\ \dots \\ \sum_{i=0}^{H-1} A^i B \end{bmatrix} u(n-1) +
$$

$$
+ \begin{bmatrix} B & 0 & \dots & 0 \\ AB+B & B & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{H-1} A^i B & \sum_{i=0}^{H-2} A^i B & \dots & B \end{bmatrix} \begin{bmatrix} {}_{\Delta}u(n) \\ \dots \\ {}_{\Delta}u(n+H-1) \end{bmatrix}
$$

projection of controll ${}_\triangle u(n)$ to output sequence

causality : **2nd input step can't effect 1st output step**

$$
\begin{bmatrix} x(n+1) \\ x(n+2) \\ \dots \\ x(n+H) \end{bmatrix} = \begin{bmatrix} A^1 \\ A^2 \\ \dots \\ A^H \end{bmatrix} x(n) + \begin{bmatrix} B \\ AB+B \\ \dots \\ \sum_{i=0}^{H-1} A^i B \end{bmatrix} u(n-1) +
$$

$$
+ \begin{bmatrix} B & 0 & \dots & 0 \\ AB+B & B & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{H-1} A^i B & \sum_{i=0}^{H-2} A^i B & \dots & B \end{bmatrix} \begin{bmatrix} {}_\triangle u(n) \\ \dots \\ {}_\triangle u(n+H-1) \end{bmatrix}
$$

projection how **1st input step** effects **whole output**

projection how **2nd input step** effects **whole output**

last input step effects only **last output step**

put into quadratic loss (cost) function

$$\mathcal{L} = {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + (\tilde{X}_r - \tilde{X})^T \tilde{Q}(\tilde{X}_r - \tilde{X})$$

after substitution

$$S = \tilde{X}_r - \Psi\tilde{X} - \Omega u(n-1)$$

we get

$$\begin{aligned}\mathcal{L} &= {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + (S - \Theta_{\Delta}\tilde{U})^T \tilde{Q}(S - \Theta_{\Delta}\tilde{U}) \\ &= {}_{\Delta}\tilde{U}^T \tilde{R}_{\Delta}\tilde{U} + S^T\tilde{Q}S - S^T\tilde{Q}\Theta_{\Delta}\tilde{U} - {}_{\Delta}\tilde{U}^T\Theta^T\tilde{Q}S + {}_{\Delta}\tilde{U}^T\Theta^T\tilde{Q}\Theta_{\Delta}\tilde{U}\end{aligned}$$

finding minima

find derivative with respect to ${}_{\Delta}\tilde{U}$

$$\frac{\partial \mathcal{L}}{\partial {}_{\Delta}\tilde{U}} :$$

$$\frac{\partial {}_{\Delta}\tilde{U}^{T} \tilde{R} {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = 2\tilde{R} {}_{\Delta}\tilde{U}$$

$$\frac{\partial S^{T} \tilde{Q} S}{\partial {}_{\Delta}\tilde{U}} = 0$$

$$\frac{\partial - S^{T} \tilde{Q} \Theta {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = -\Theta^{T} \tilde{Q} S$$

$$\frac{\partial - {}_{\Delta}\tilde{U}^{T} \Theta^{T} \tilde{Q} S}{\partial {}_{\Delta}\tilde{U}} = -\Theta^{T} \tilde{Q} S$$

$$\frac{\partial {}_{\Delta}\tilde{U}^{T} \Theta^{T} \tilde{Q} \Theta {}_{\Delta}\tilde{U}}{\partial {}_{\Delta}\tilde{U}} = 2\Theta^{T} \tilde{Q} \Theta {}_{\Delta}\tilde{U}$$

finding minima

put derivate equal to zero, and solve

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial {}_{\Delta}\tilde{U}} &= 2\tilde{R}{}_{\Delta}\tilde{U} - 2\Theta^T \tilde{Q} S + 2\Theta^T \tilde{Q} \Theta {}_{\Delta}\tilde{U} \\
0 &= 2\tilde{R}{}_{\Delta}\tilde{U} - 2\Theta^T \tilde{Q} S + 2\Theta^T \tilde{Q} \Theta {}_{\Delta}\tilde{U} \\
(\tilde{R} + \Theta^T \tilde{Q} \Theta){}_{\Delta}\tilde{U} &= \Theta^T \tilde{Q} S
\end{aligned}$$

and obtain **analytical solution** for model predictive controll

$${}_{\Delta}\tilde{U} = (\tilde{R} + \Theta^T \tilde{Q} \Theta)^{-1} \Theta^T \tilde{Q} S$$

full algorithm

given matrices :

$$\tilde{Q}, \tilde{R}, \Theta, \Phi, \Omega$$

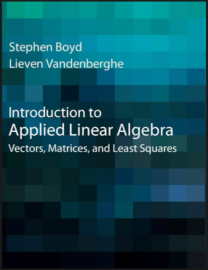
initialization (precompute) :

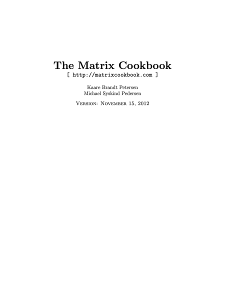
$$\Xi = (\tilde{R} + \Theta^T \tilde{Q} \Theta)^{-1} \Theta^T \tilde{Q}$$

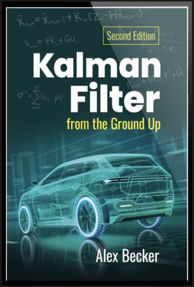
in loop :

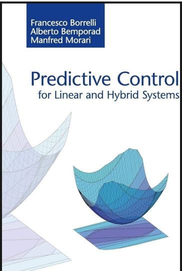
$$\begin{aligned} E(n) &= X_r(n) - \Phi x(n) - \Omega u(n-1) \\ {}_\Delta u(n) &= \Xi E(n) \\ u(n) &= u(n-1) +_\Delta u(n) \end{aligned}$$

recommended reading

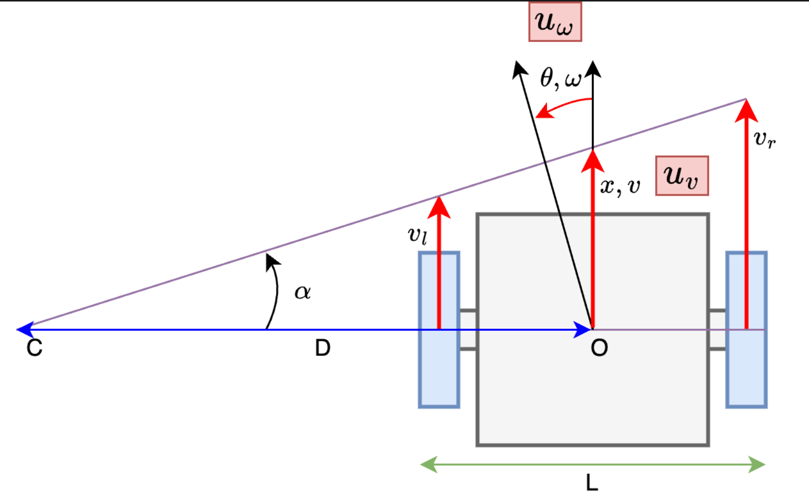








robot trajectory tracking



robot trajectory tracking