

# Hands-On Control and Algorithms for Robots

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# Chapter 1

# Robot Modelling

The first step in controlling a robot is to understand its mathematical model. Every movement of a robot—at every tiny time step—is governed by its equations of motion. As a simple yet powerful example, let us consider a **two-wheel differential drive robot**.

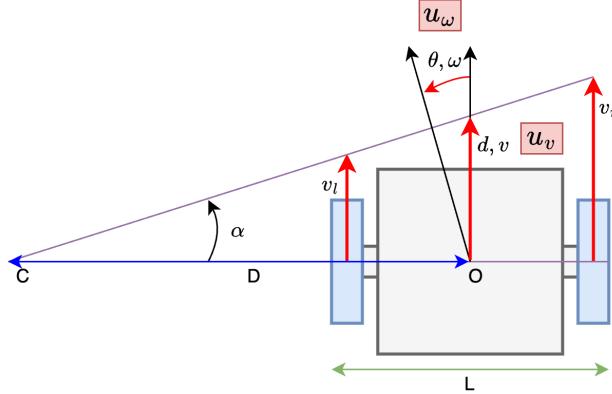


Figure 1.1: Differential drive robot geometry and notation.

Figure 1.1 shows a schematic of such a robot. The center of the robot is denoted by  $O$ , and the distance between the two driving wheels is  $L$  [m]. Each wheel produces a linear velocity, denoted  $v_l$  and  $v_r$  [m/s] for the left and right wheel respectively. If the wheels are rotating with angular velocities  $\omega_l$  and  $\omega_r$  [rad/s], then the corresponding linear velocities can be expressed as

$$v_l = \omega_l r, \quad v_r = \omega_r r,$$

where  $r$  [m] is the wheel radius.

In practical robotics, motor speed is often measured in revolutions per minute (rpm). The relationship between angular velocity in radians per second and rpm is given by

$$\omega = \text{rpm} \cdot \frac{2\pi}{60}.$$

Knowing the individual wheel velocities, we can express the **robot's linear velocity**  $v$  as the average of both wheels:

$$v = \frac{1}{2}(v_l + v_r).$$

The corresponding **linear displacement**  $d$  [m] evolves over time as

$$\frac{dd}{dt} = v.$$

Similarly, the **robot's angular rate**  $\omega$  [rad/s] and **orientation angle**  $\theta$  [rad] are related by

$$\omega = \frac{v_r - v_l}{L}, \quad \frac{d\theta}{dt} = \omega.$$

These two variables — linear and angular velocities — fully describe the planar motion of a differential drive robot.

### Control Inputs

In practice, the control inputs to the robot are the individual motor commands, denoted  $u_{\text{left}}$  and  $u_{\text{right}}$ . However, it is often more convenient to work with combined control variables: the **linear velocity command**  $u_v$  and the **angular velocity command**  $u_\omega$ , defined as

$$\begin{aligned} u_{\text{left}} &= u_v - u_\omega, \\ u_{\text{right}} &= u_v + u_\omega. \end{aligned} \tag{1.1}$$

This formulation allows independent control of forward motion and rotation, which greatly simplifies controller design and enables linear control strategies.

### Instantaneous Center of Rotation (ICR)

In the diagram, point  $C$  represents the **instantaneous center of rotation (ICR)**. This point lies on the axis connecting the two wheels, and its distance from the robot's center  $O$  depends on the ratio between left and right wheel velocities.

The distance  $D$  from  $O$  to the ICR is given by

$$D = \frac{L}{2} \cdot \frac{v_l + v_r}{v_r - v_l}.$$

When both wheels move at the same speed ( $v_l = v_r$ ), the denominator becomes zero, and thus  $D \rightarrow \infty$  — which corresponds to straight-line motion.

## Motion Formalism

The **state-space model** is commonly used to describe how a robot's trajectory evolves over time. A physical system can be represented by a continuous-time state-space model, where the state is expressed as  $x(t)$ . However, for our purposes — particularly for controller implementation in digital systems — a **discrete-time model** is more convenient, denoted as  $x(n)$ .

For a very simple case, such as a slowly moving robot with negligible inertia, the state can be represented using only two quantities: the **linear displacement**  $d$  [m] and the **orientation angle**  $\theta$  [rad]:

$$x(n) = \begin{bmatrix} d(n) \\ \theta(n) \end{bmatrix}. \quad (1.2)$$

However, a real robot — especially one that moves quickly or has non-negligible inertia — cannot be accurately modeled in this minimal form. When the motors are turned off, the robot does not stop instantaneously due to its momentum. To capture this physical behavior, we extend the state vector by including the **robot's linear velocity**  $v$  [m/s] and the **robot's angular rate**  $\omega$  [rad/s], leading to the following definition:

$$x(n) = \begin{bmatrix} d(n) \\ \theta(n) \\ v(n) \\ \omega(n) \end{bmatrix}. \quad (1.3)$$

In a similar way, we define the **control input vector**  $u(n)$  as:

$$u(n) = \begin{bmatrix} u_v(n) \\ u_\omega(n) \end{bmatrix}, \quad (1.4)$$

where  $u_v$  and  $u_\omega$  represent the commanded linear and angular control inputs, respectively. In this book, we assume that

$$\textcolor{red}{u \text{ is unit-less, in the range } \langle -1, 1 \rangle}, \quad (1.5)$$

meaning that these commands are normalized. They can later be mapped to real-world actuator quantities — for example, motor RPM, PWM duty cycle, or desired wheel velocity — depending on the specific hardware configuration.

Let us now describe the relationship between the robot's state  $x$  and control input  $u$ , starting with a simplified (inertia-free) model that does not include velocity terms:

$$\begin{aligned} d(n+1) &= d(n) + b_0 u_v(n), \\ \theta(n+1) &= \theta(n) + b_1 u_\omega(n). \end{aligned} \quad (1.6)$$

These two equations are completely independent: the first describes how the robot's position changes, while the second describes how its orientation evolves. The constant  $b_0$  expresses how strongly the linear control  $u_v$  affects forward motion — it depends on motor speed, wheel diameter, and control step length. Similarly,  $b_1$  expresses how efficiently the robot turns — it depends on the same motor characteristics and the wheelbase  $L$ .

**Higher motor speed or larger wheel radius** increases  $b_0$ , while a **larger wheelbase** decreases  $b_1$  because the robot must rotate more slowly for the same wheel speed difference.

**Practical estimation of  $b_0$  and  $b_1$ :**

If we know the maximum wheel speed at full control input  $u = 1$ , denoted  $RPM_{\max}$ , and the wheel radius  $r$  [m], along with the control sampling period  $\Delta T$  [s], we can estimate the corresponding maximum linear and angular velocities as:

$$\begin{aligned} v_{\max} &= 2\pi \frac{RPM_{\max} r}{60}, \\ \omega_{\max} &= 4\pi \frac{RPM_{\max} r}{60L}. \end{aligned} \quad (1.7)$$

From these relations, we can approximate the discrete-time model parameters as:

$$\begin{aligned} b_0 &= v_{\max} \Delta T, \\ b_1 &= \omega_{\max} \Delta T. \end{aligned} \quad (1.8)$$

This allows us to estimate a simple, inertia-free model directly from basic geometric parameters and motor specifications. Although this model is only approximate, it works well for low-speed motion or when a rough dynamic approximation is sufficient.

**Matrix form of the state-space model:**

In conventional control theory, the discrete-time state-space model is expressed compactly as:

$$x(n+1) = Ax(n) + Bu(n). \quad (1.9)$$

Rewriting the simplified dynamics from Eq. 1.6 in this formalism yields:

$$\begin{bmatrix} d(n+1) \\ \theta(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_A \begin{bmatrix} d(n) \\ \theta(n) \end{bmatrix} + \underbrace{\begin{bmatrix} b_0 & 0 \\ 0 & b_1 \end{bmatrix}}_B \begin{bmatrix} u_v(n) \\ u_\omega(n) \end{bmatrix}. \quad (1.10)$$

**Commentary:**

This simple kinematic model forms the foundation of robot motion control. Although it ignores acceleration dynamics, wheel slip, and mechanical

delays, it is extremely useful for initial controller design, algorithm prototyping, and understanding how inputs affect motion. In later chapters, we will extend this formulation to include velocity dynamics, actuator limitations, and state estimation, which together lead to more realistic and robust control architectures. The more details how state space models works can be found in chapter the **Capturing dynamics**.

## Creating a Simulator

We start by implementing a robot class that handles the robot dynamics. The implementation is intentionally simple, using the **NumPy** library for matrix operations. This choice allows us to later describe more complex dynamics (e.g., with inertia or slip effects) without changing the underlying mathematical framework.

The constructor of the **class DifferentialDriveModel** is defined as:

```
1 def __init__(self, b0, b1, width, height)
```

This constructor receives the parameters describing robot dynamics ( $b_0$ ,  $b_1$ ) and the robot's physical dimensions (`width`, `height`), useful for visualization purposes.

By calling:

```
1 def reset(self, initial_x, initial_y, initial_theta)
```

we can reset the robot's state vector and bring it to the desired initial position and orientation.

The main function, which is called repeatedly in the simulation loop, is:

```
1 def step(self, u)
```

It accepts a control input vector  $u$  and returns the current state dictionary:

```
1 self.robot_state = []
2 self.robot_state["width"] = self.width
3 self.robot_state["height"] = self.height
4
5 self.robot_state["x"] = self.x
6 self.robot_state["x_pos"] = self.x_pos
7 self.robot_state["y_pos"] = self.y_pos
8 self.robot_state["theta"] = theta
```

---

The remaining question is: **How do we convert the internal robot state  $(d, \theta)$  into Cartesian coordinates  $(x_{pos}, y_{pos})$ ?**

The robot's state space  $(d, \theta)$  is convenient for control and dynamics modeling, but it does not directly tell us where the robot is located in the plane. We therefore need a geometric transformation.

At each discrete time step, the robot travels a distance increment:

$$\Delta d(n) = d(n+1) - d(n),$$

and its current heading (orientation in the global frame) is  $\theta(n)$ . Thus, the displacement in Cartesian coordinates can be computed as:

$$\begin{aligned}\Delta x(n) &= \Delta d(n) \cos(\theta(n)), \\ \Delta y(n) &= \Delta d(n) \sin(\theta(n)).\end{aligned}$$

**Why this works:** The robot's forward motion always occurs along its heading direction  $\theta$ . By decomposing the traveled distance  $\Delta d$  into its projections on the global  $x$  and  $y$  axes (using the cosine and sine of  $\theta$ ), we obtain the actual movement in Cartesian space. This is a standard transformation between the robot's local frame and the world frame.

Integrating (in discrete simulation simply summing) these increments gives the full global position:

$$\begin{aligned}\Delta d(n) &= d(n+1) - d(n), \\ \Delta x(n) &= \Delta d(n) \cos(\theta(n)), \\ \Delta y(n) &= \Delta d(n) \sin(\theta(n)), \\ x_{pos}(n+1) &= x_{pos}(n) + \Delta x(n), \\ y_{pos}(n+1) &= y_{pos}(n) + \Delta y(n).\end{aligned}\tag{1.11}$$

Here,  $d$  and  $\theta$  are taken from the robot state vector  $x$ . This approach lets us keep a simple linear state-space model for control purposes, while still maintaining accurate global position information for visualization, mapping, and collision detection.

The complete Python implementation of the robot model is shown below:

```

1 import numpy
2
3 class DifferentialDriveModel:
4     def __init__(self, b0, b1, width, height):
5         self.A = numpy.array([[1, 0],
6                               [0, 1]])
7         self.B = numpy.array([[b0, 0],
8                               [0, b1]])
9         self.width = width
10        self.height = height
11
12        self.reset(0, 0, 0)
13
14    def reset(self, initial_x, initial_y, initial_theta):
15        # Initialize state [d, theta]
16        self.x = numpy.zeros((2, 1))
17        self.x[1, 0] = initial_theta
18        self.x_pos = initial_x
19        self.y_pos = initial_y
20
21        self.step(numpy.zeros((2, 1)))
22
23    def step(self, u):
24        # Discrete-time dynamics update
25        x_new = self.A @ self.x + self.B @ u
26
27        # Calculate position increment in Cartesian coordinates
28        dpos = x_new[0, 0] - self.x[0, 0]
29        theta = self.x[1, 0]
30
31        dx = dpos * numpy.cos(theta)
32        dy = dpos * numpy.sin(theta)
33
34        # Update global position
35        self.x_pos += dx
36        self.y_pos += dy
37
38        # Update internal state
39        self.x = numpy.array(x_new)
40
41        # Fill result values
42        self.robot_state = {
43            "width": self.width,
44            "height": self.height,
45            "x": self.x,
46            "x_pos": self.x_pos,
47            "y_pos": self.y_pos,
48            "theta": theta
49        }
50
51        return self.robot_state
52
53    def get_state(self):
54        return self.robot_state

```

Every model must be tested. It is always good practice to choose **non-trivial parameters** to avoid errors that are hidden by “nice” numbers. Here we define robot dynamics, the simulation time step, motor parameters, and robot dimensions.

From the equations for maximum velocities (Eq. 1.7) and the parameter estimation (Eq. 1.8), we compute  $b_0$  and  $b_1$  and pass them into the robot model:

```

1 # Input parameters
2 dt      = 0.001          # time step [s]
3 r_wheel = 15.0 * 0.001   # wheel radius [m]
4 l_wheels= 95.0 * 0.001   # wheel separation [m]
5 rpm_max = 750           # maximum motor speed [RPM]
6
7 width   = 2 * l_wheels
8 height  = l_wheels
9
10 v_max  = 2.0 * numpy.pi * rpm_max * r_wheel / 60.0
11 w_max  = 4.0 * numpy.pi * rpm_max * r_wheel / (60.0 * l_wheels)
12
13 b0 = v_max * dt
14 b1 = w_max * dt
15
16 # Initialize robot
17 robot = DifferentialDriveModel(b0, b1, width, height)
18 robot.reset(0, 0, 0)
```

Now, let the robot follow an **equilateral triangular trajectory**. This is a better test than a square path because 90-degree turns are a special case — they only affect one Cartesian coordinate at a time, while 120-degree turns give a more general motion pattern.

We define two possible control inputs:

- Full-speed forward motion:  $u = (1, 0)$
- In-place rotation:  $u = (0, 1)$

To move along the triangle sides, we must compute how many simulation steps correspond to a 1-meter straight motion and a  $120^\circ$  turn (converted to radians):

```

1 n_steps_forward = round(1.0 / (v_max * dt))
2 angle = 120.0 * numpy.pi / 180.0
3 n_steps_turn = round(angle / (w_max * dt))
```

We now execute three segments of motion (forward 1 m, turn 120°) to complete the triangle, recording the robot's positions for later plotting:

```

1 x_pos = []
2 y_pos = []
3
4 for j in range(3):
5     # move robot for 1m distance
6     for n in range(n_steps_forward):
7         u = numpy.array([[1.0],[0.0]])
8
9         state = robot.step(u)
10
11        x_pos.append(state["x_pos"])
12        y_pos.append(state["y_pos"])
13
14    # turn robot at 120 degrees
15    for n in range(n_steps_turn):
16        u = numpy.array([[0.0],[1.0]])
17
18        state = robot.step(u)
19
20        x_pos.append(state["x_pos"])
21        y_pos.append(state["y_pos"])
22
23 print("terminal state")
24 print("x_pos = ", round(state["x_pos"], 4))
25 print("y_pos = ", round(state["y_pos"], 4))
26 print("theta = ", round(state["theta"]*180.0/numpy.pi, 4))

```

At the end, the robot should return to approximately the same starting point, with its orientation close to zero (modulo 360°). In the test output, we observe:

```

1 terminal state
2 x_pos = -0.0094
3 y_pos = 0.0167
4 theta = 356.6842

```

**Explanation:** This small offset from the starting position is caused by **numerical integration error** due to discretization. Because the simulation uses a finite time step  $\Delta T = 1$  ms, the continuous motion is approximated by small discrete jumps. Such integration errors accumulate over time and cause the trajectory to drift slightly, especially in long simulations or when using small angular increments.

The resulting trajectory plot is shown in Figure 1.2.

If we visualize the trajectory history (e.g., using OpenCV), we can see how the position gradually drifts — a natural effect of discrete simulation and rounding errors, as illustrated in Figure 1.3. The complete implementation including visualization is available at: `examples/robot_simulator_01`.

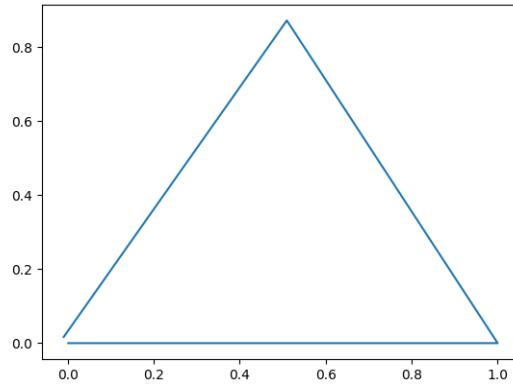


Figure 1.2: Triangular trajectory plot

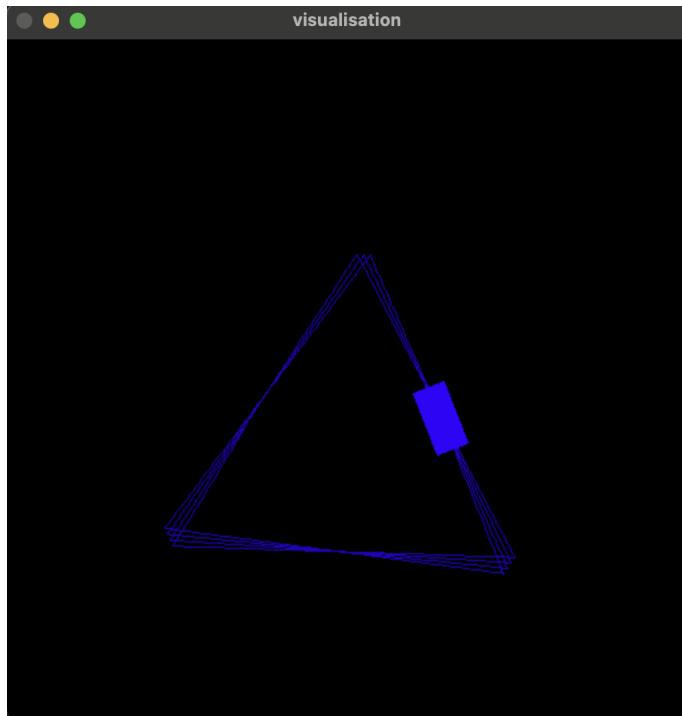


Figure 1.3: Simulator visualization

### Special cases of motion control

It is useful to analyze how the robot behaves under specific control

inputs. By assigning different values to the control vector  $u = [u_v, u_\omega]^T$ , we can immediately observe fundamental motion primitives that compose any possible trajectory:

- **Straight-line motion:**  $u = (1, 0)$

Both wheels receive identical input, hence they rotate at the same speed. There is no angular component ( $u_\omega = 0$ ), so the robot's heading  $\theta$  remains constant. As a result, the robot moves in a straight line in the direction it currently faces.

- **In-place rotation:**  $u = (0, 1)$

In this case, both wheels rotate in opposite directions with equal magnitude. This creates a pure rotational motion around the robot's center point, without any translation. The linear velocity component is zero ( $u_v = 0$ ), and the heading  $\theta$  changes at a constant rate.

- **Circular motion:**  $u = (1, 0.1)$

Here, both translational and rotational components are active. The robot moves forward while simultaneously turning with a constant angular rate. The resulting trajectory is a perfect circle, where the radius of the circle is given by:

$$R = \frac{v}{\omega} = \frac{u_v v_{\max}}{u_\omega \omega_{\max}}.$$

The larger the ratio  $u_v/u_\omega$ , the wider the circle. When  $u_\omega$  is small compared to  $u_v$ , the robot moves almost straight; when  $u_\omega$  is large, it spins tightly around a small radius.

This elegant behavior results from the **decoupled control formulation** in which the input vector  $u$  directly represents the *linear* and *angular* control actions. Because the left and right motor signals are internally computed as:

$$u_{\text{left}} = u_v - u_\omega,$$

$$u_{\text{right}} = u_v + u_\omega,$$

the two control modes — translation and rotation — are cleanly separated. This makes both mathematical modeling and controller design

straightforward, since any arbitrary trajectory can be expressed as a combination of linear motion and circular motion segments.

In fact, **the entire movement of a differential-drive robot is a continuous composition of these two basic primitives** — straight motion and circular arcs. This geometric interpretation is extremely valuable in higher-level path planning, as it allows complex paths to be represented using simple building blocks such as arcs and lines.

An example of such combined motion is shown in Figure 1.4, where the robot follows a smooth circular trajectory as a result of constant-speed turning.

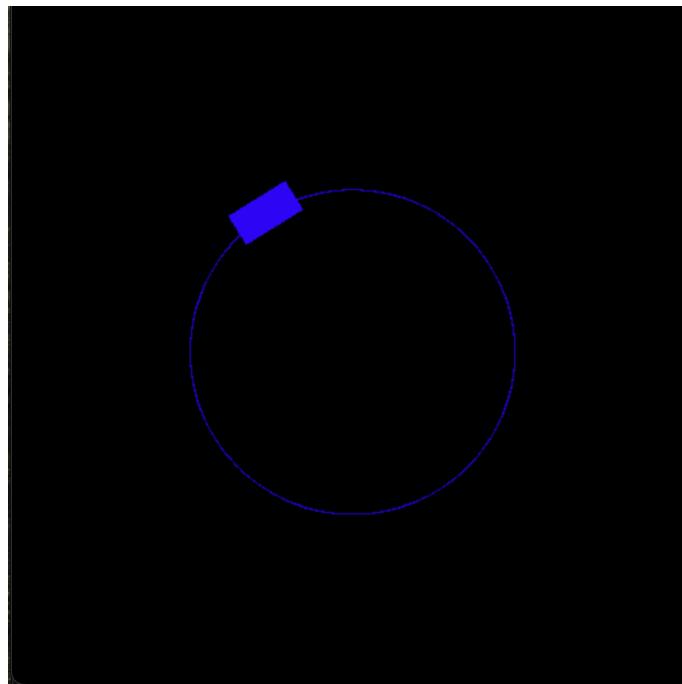


Figure 1.4: Simulator visualization of circular motion

**Summary:** In this section, we created a simple but complete simulation environment for a differential-drive robot. The reader learned how to:

- Represent the robot dynamics in discrete state-space form.
- Transform robot motion from state variables  $(d, \theta)$  into global Cartesian coordinates  $(x, y)$ .
- Implement and simulate the model in Python using NumPy.
- Test the model on a non-trivial trajectory and interpret integration errors.

Although the model ignores inertia and wheel slip, it provides a solid base for more advanced topics such as velocity control, model predictive control (MPC), and extended state estimation in later chapters.

# Chapter 2

## Capturing dynamics

### 2.1 State space models

A convenient mathematical framework for describing the dynamics of a system is the **state space model**. Such a model captures how the system's internal state evolves over time in response to inputs, and how these inputs affect the outputs.

In other words, the state space model represents the relationship between:

- the **state vector**  $x(n)$  (which describes the internal condition of the system),
- the **input vector**  $u(n)$  (the control or excitation applied to the system),
- and optionally, the **output vector**  $y(n)$  (the measured quantities).

#### System State

The system state is represented as a column vector with  $N$  elements:

$$x(n) = \begin{bmatrix} x_0(n) \\ x_1(n) \\ \vdots \\ x_{N-1}(n) \end{bmatrix} \quad (2.1)$$

The number of rows in this vector,  $N$ , is called the **system order**.

## Examples of System States

**Single-State Example:** For a simple first-order system, such as a motor where we only consider its angular velocity  $\omega(n)$ , the state vector contains a single element:

$$x(n) = [\omega(n)] \quad (2.2)$$

**Second-Order Example:** If we model a servo motor with rotational inertia, the state vector might include both the motor's velocity and position:

- Motor shaft velocity  $\omega(n)$  [rad/s]
- Motor shaft angle  $\theta(n)$  [rad]

$$x(n) = \begin{bmatrix} \omega(n) \\ \theta(n) \end{bmatrix} \quad (2.3)$$

**Extended Example:** A more accurate servo model might also include the motor current  $i(n)$ :

$$x(n) = \begin{bmatrix} i(n) \\ \omega(n) \\ \theta(n) \end{bmatrix} \quad (2.4)$$

**Mobile Robot Example:** For a robot moving in a 2D plane, the state could describe its position and orientation:

$$x(n) = \begin{bmatrix} x'(n) \\ y'(n) \\ \theta(n) \end{bmatrix} \quad (2.5)$$

## System Inputs

The system is controlled by  $M$  inputs, grouped into the **input vector**  $u(n)$ :

$$u(n) = \begin{bmatrix} u_0(n) \\ u_1(n) \\ \vdots \\ u_{M-1}(n) \end{bmatrix} \quad (2.6)$$

For a single-input system, such as a current-controlled motor:

$$u(n) = [i(n)] \quad (2.7)$$

A differential-drive robot with two independently driven wheels would have two inputs:

$$u(n) = \begin{bmatrix} i_{\text{left}}(n) \\ i_{\text{right}}(n) \end{bmatrix} \quad (2.8)$$

## Discrete-Time State Space Model

The relationship between the current state  $x(n)$ , the control input  $u(n)$ , and the next state  $x(n + 1)$  can be expressed as a **linear discrete-time state space model**:

$$x(n + 1) = A x(n) + B u(n) \quad (2.9)$$

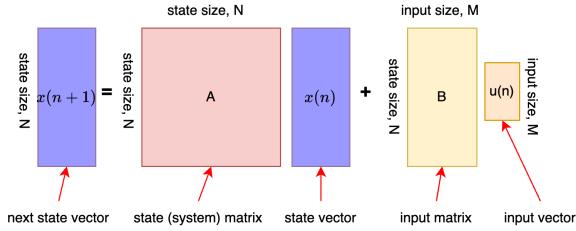


Figure 2.1: Matrix dimensions in a discrete-time state space model.

This form can describe the dynamics of any **linear system**. In some cases, the full state  $x(n)$  cannot be measured directly. Instead, we observe only the system output:

$$y(n) = C x(n) \quad (2.10)$$

where matrix  $C$  has  $K$  rows and  $N$  columns, mapping the internal state to observable outputs. In this section, we assume full-state measurements are available.

## Continuous-Time Model and Discretization

Physical systems are often described by **continuous-time state space equations**:

$$\frac{dx(t)}{dt} = \bar{A}x(t) + \bar{B}u(t) \quad (2.11)$$

The matrices  $\bar{A}$  and  $\bar{B}$  differ from their discrete-time counterparts  $A$  and  $B$ . They can be related through a **bilinear (Tustin) transformation**:

$$S_a = \left( I - \frac{\Delta T}{2} \bar{A} \right)^{-1} \quad (2.12)$$

$$S_b = I + \frac{\Delta T}{2} \bar{A} \quad (2.13)$$

$$A = S_a S_b \quad (2.14)$$

$$B = S_a \bar{B} \Delta T \quad (2.15)$$

Here,  $\Delta T$  is the sampling period. This method converts a continuous (physical) model into a discrete form suitable for digital control.

The following Python code implements this discretization process:

```

1 def c2d(a, b, c, dt):
2     i = numpy.eye(a.shape[0])
3
4     tmp_a = numpy.linalg.inv(i - (0.5*dt)*a)
5     tmp_b = i + (0.5*dt)*a
6
7     a_disc = tmp_a@tmp_b
8     b_disc = (tmp_a*dt)@b
9
10    return a_disc, b_disc, c

```

## 2.2 System Examples

### First-Order Dynamical System

In this section, we demonstrate a simple **first-order dynamical system**. Such systems appear frequently in both mechanical and electrical domains. Common real-world examples include:

- the angular velocity dynamics of a DC motor,

- the voltage response of an RC circuit,
- the thermal response of a heating element,
- or the water level in a tank with a constant inflow and proportional outflow.

### System Definition

A first-order system can be described by two key parameters:

- the **time constant**  $\tau$ , which determines how quickly the system responds (its settling time),
- the **gain**  $k$ , which defines how strongly the input  $u$  affects the output (or state)  $x$ .

In this case, both the input  $u$  and the state  $x$  are single scalar quantities. The continuous-time model is given by:

$$\frac{dx(t)}{dt} = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t) \quad (2.16)$$

Here, the system matrices have dimensions  $1 \times 1$ :

$$\bar{A} = \left[ -\frac{1}{\tau} \right], \quad \bar{B} = \left[ \frac{k}{\tau} \right] \quad (2.17)$$

### Simulation Setup

To visualize how this system behaves, we can simulate its response to a **unit step input**, where:

$$u(t) = 1 \quad (2.18)$$

We simulate for 1000 steps using a discretization step of  $\Delta t = 0.01$  seconds, which corresponds to a total simulation time of 10 seconds. The initial condition is set to  $x(0) = 0$ .

A simple (naive) implementation in Python looks like this:

```

1  x = numpy.zeros((1, 1))    # initial state
2  u = numpy.ones((1, 1))     # constant input = 1
3
4  for n in range(num_steps):
5      dx = A @ x + B @ u    # update step
6      x  = x + dx * dt      # integration

```

## Numerical Integration Methods

The update step  $x = x + dx \cdot dt$  performs **Euler integration**, also known as the **forward Euler method**. This approach is conceptually simple but can accumulate significant numerical error, especially for stiff or higher-order systems. To achieve stable and accurate results,  $\Delta t$  must be very small — which increases the number of simulation steps dramatically.

More advanced numerical solvers, such as the **Runge–Kutta methods** (commonly RK4), provide much better stability and accuracy for the same time step size. These solvers are standard in modern simulation environments and control design tools.

If we define **system forward func** as :

```

1 def forward_func(x, u):
2     dx = A@x + B@u
3     y  = C@x
4
5     return dx, y

```

The ODE solvers can be implemented straightforward. The naive Euler method is :

```

1 def ODESolverEuler(forward_func, x, u, dt):
2     dx, y  = forward_func(x, u)
3
4     return x + dx*dt, y

```

And commonly more used (and preferred for balance between accuracy and speed), Runge Kutta RK4 method is :

```

1 def ODESolverRK4(forward_func, x, u, dt):
2     k1, y1  = forward_func(x, u)
3     k1      = k1*dt
4
5     k2, y2  = forward_func(x + 0.5*k1, u + 0.5*dt)
6     k2      = k2*dt
7
8     k3, y3  = forward_func(x + 0.5*k2, u + 0.5*dt)
9     k3      = k3*dt
10
11    k4, y4  = forward_func(x + k3, u + dt)
12    k4      = k4*dt
13
14    x_new   = x + (1.0/6.0)*(1.0*k1 + 2.0*k2 + 2.0*k3 + 1.0*k4)
15    y       = (1.0/6.0)*(1.0*y1 + 2.0*y2 + 2.0*y3 + 1.0*y4)
16
17    return x_new, y

```

The entire simulation is then single for loop, calling one step of dynamical system inside solver. Every loop iteration is one small discrete simulation step.

```

1 def simulate(n, m, dt, n_steps):
2     # input and
3     u = numpy.zeros((m, 1))
4     x = numpy.zeros((n, 1))
5
6     # log results
7     t_result = []
8     u_result = []
9     x_result = []
10    y_result = []
11
12    for n in range(n_steps):
13        # here the input into system is obtained
14        # e.g. from controller, path planner, user input
15        u = obtain_control()
16
17        # solver step
18        x, y = ODESolverRK4(forward_func, x, u, dt)
19
20        # optional visualisation
21        visualise(n, u, x, y)
22
23        # log results
24        t_result.append(n*dt)
25        u_result.append(u[:, 0])
26        x_result.append(x[:, 0])
27        y_result.append(y[:, 0])
28
29        t_result = numpy.array(t_result)
30        u_result = numpy.array(u_result)
31        x_result = numpy.array(x_result)
32        y_result = numpy.array(y_result)
33
34    return t_result, u_result, x_result, y_result
35

```

## Response Analysis

Figure 2.2 shows several step responses for different values of the time constant  $\tau$  and gain  $k$ .

From these plots, we can observe that:

- Increasing  $\tau$  makes the system respond more slowly — it takes longer to reach the steady state. This behavior can be interpreted as the system having more inertia or lag.
- Increasing  $k$  scales the steady-state value of  $x(t)$ . For a step input of  $u = 1$ , the steady-state output is  $x_{ss} = k \cdot u = k$ .

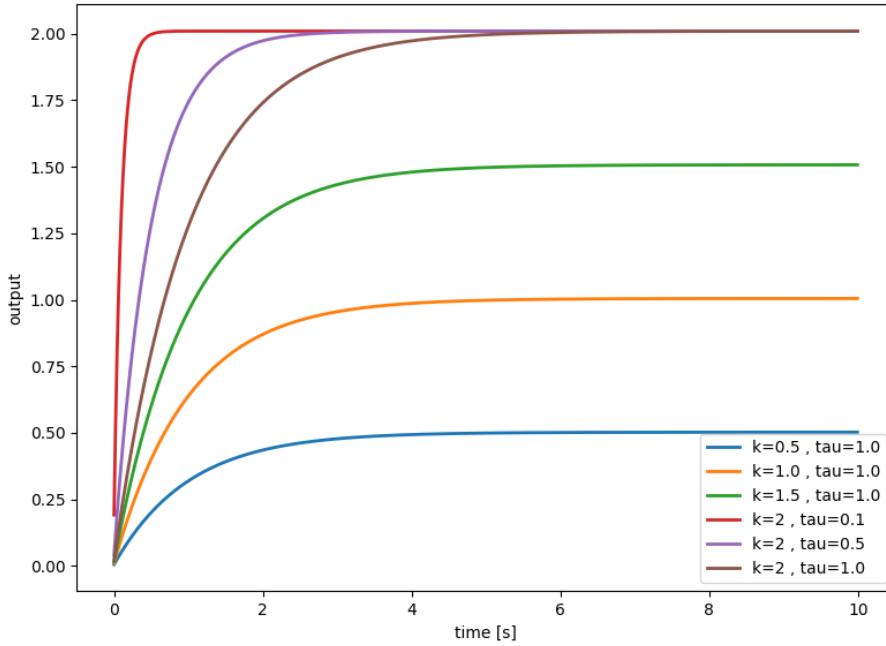


Figure 2.2: Step responses of a first-order system for different values of  $\tau$  and  $k$ .

- The system always approaches the steady-state value **exponentially**, following the general solution:

$$x(t) = k u \left( 1 - e^{-t/\tau} \right) \quad (2.19)$$

Thus, the parameters  $\tau$  and  $k$  completely determine how the system responds to changes in input:  $\tau$  controls the speed, while  $k$  controls the magnitude.

### Practical Insight

Understanding this basic model is important because many complex systems can be approximated by combinations of such first-order dynamics. For example, the motor's current loop, the temperature of an actuator, or even the airflow in a ventilation system can each be modeled as a first-order

process. This simplicity makes first-order models a cornerstone of control theory and system identification. The example of this 1st order system is located at examples/01\_dynamical\_system/first\_order.py

## 2.3 First Order Identification

The process of system identification involves finding the matrices  $A$ ,  $B$  (and optionally  $C$ ). From this model, a controller can be synthesized, or the system can be used for multi-step prediction and planning.

The goal in this section is to identify the parameters of a DC motor. The motor velocity model can be approximated as a first-order linear system, although the real system is, of course, nonlinear. A photo of the example setup is shown in Fig. 2.3.

The control loop hardware consists of a motor driver (H-bridge DRV8212), a DC motor (Pololu HP 1:30 gear), and a quadrature magnetic encoder ( $2 \times$  DRV5013), as illustrated in Fig. 2.4.

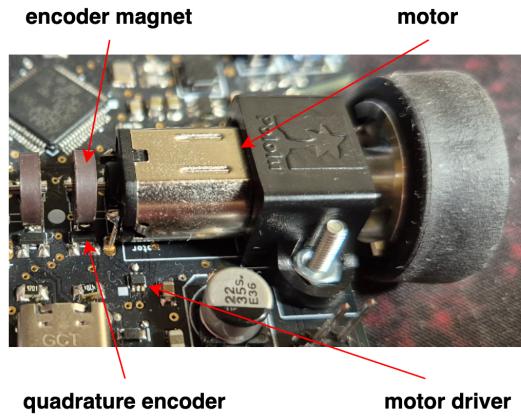


Figure 2.3: Wheel drive hardware detail.

The input signal  $u(n)$  can be generated arbitrarily. A square wave is often used to capture higher-frequency dynamics. The identification algorithm estimates the model parameters based on the observed motor velocity  $x(n)$  and the known input  $u(n)$ .

Before testing the real motor, we first demonstrate the approach using a simulation example. The **simulated motor** has the following parameters - chosen arbitrarily :

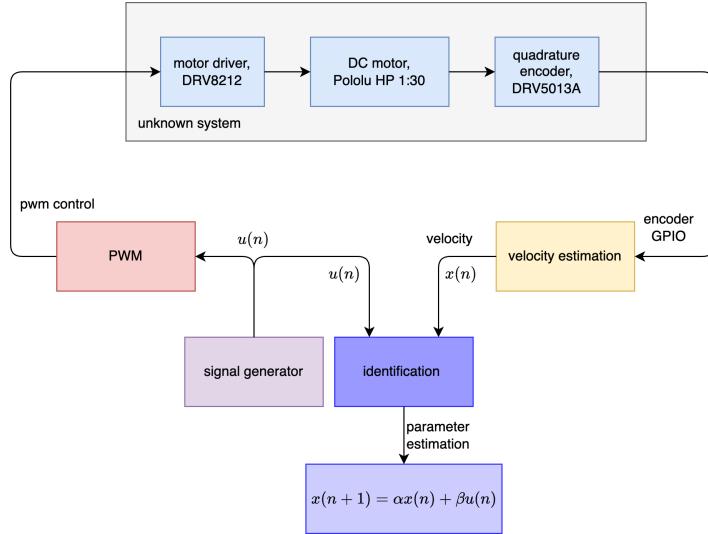


Figure 2.4: Motor identification process.

- sampling frequency: 4 kHz,  $\Delta T = 1/4000$
- maximum control input:  $u_{\max} = 2$
- motor constant:  $k = 17$
- motor time constant:  $\tau = 29$  milliseconds

The motor state is a single variable—the angular velocity  $\omega(t)$ . The model has two parameters,  $\alpha$  and  $\beta$ . The first-order continuous model is defined as:

$$\frac{d\omega(t)}{dt} = \alpha \omega(t) + \beta u(t) \quad (2.20)$$

$$\alpha = -\frac{1}{\tau} \quad (2.21)$$

$$\beta = \frac{k}{\tau} \quad (2.22)$$

With the given parameters  $k$  and  $\tau$ , we obtain  $\alpha = -34.48$  and  $\beta = 586.2$ .

The unit step response is obtained by setting  $u(n) = 1$  and solving the differential equation using any ODE solver. The simplest method is

the Euler approach, which works well for small  $\Delta T$  and low-order systems. For higher accuracy, the commonly used fourth-order Runge–Kutta (RK4) method is preferred.

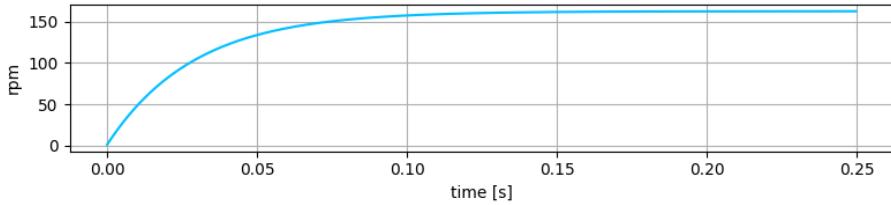


Figure 2.5: Motor step response.

Our goal is to generate input  $u(t)$  and, by observing the system output  $x(t)$ , estimate the parameters  $k$  and  $\tau$ .

We start with the simplest identification method, suitable for first-order, non-oscillating systems. This method has two main steps:

1. steady-state value estimation
2. time constant estimation

## Steady-State Estimation

We drive the motor with different constant input levels, ranging from 10% to 100%. By observing the steady-state velocity, the motor constant  $k$  can be estimated.

Algorithm for estimating the motor constant:

1. choose a constant motor input  $u \in (0, u_{\max}]$
2. run the motor and wait until steady-state velocity is reached
3. measure the velocity
4. estimate  $\hat{k} = x/u$
5. repeat and average the resulting  $\hat{k}$

In the following code, we use 10 different input levels. After setting each input, we wait 500 steps to reach steady state, measure the velocity, and then compute the average.

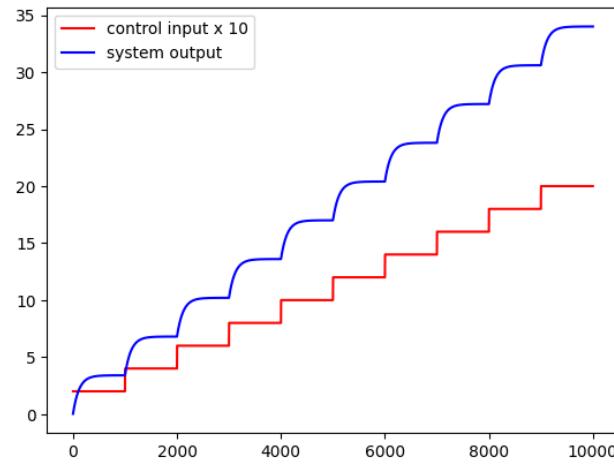


Figure 2.6: Motor constant estimation.

```

1 # input levels
2 u_values = u_max*numpy.array([0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0])
3
4 # reset dynamical system into zero state
5 ds.reset()
6
7 for j in range(len(u_values)):
8
9     x_mean = []
10    for i in range(1000):
11
12        # convert scalar u to column vector
13        u      = u_values[j]
14        u      = numpy.array([[u]])
15
16        # compute dynamical system one step
17        x, _ = ds.forward_state(u)
18
19        # after steady state, store x
20        if i > 500:
21            x_mean.append(x[0][0])
22
23    x_mean = numpy.array(x_mean)
24    x_mean = x_mean.mean()
25
26    k_est = x_mean/u_values[j]
27
28    print(u_values[j], k_est)
29
30    k_mean = k_est.mean()
31
32 # average k estimate
33 print("k_mean = ", k_mean)

```

In our example, the resulting  $\hat{k}$  is 16.995, which is very close to the true value 17. Accuracy depends on encoder noise and the number of samples used for averaging.

## Time Constant Estimation

The second parameter is the time constant  $\tau$ . Most textbooks estimate  $\tau$  by measuring the time at which the system reaches 63.2% of its final value  $x_{\max}$ . Here,  $x_{\max}$  is the steady-state velocity for a given  $u$ . However, this method is sensitive to noise and precise timing.

We present a more accurate approach, which excites the system into a natural oscillatory response. The algorithm is as follows:

1. choose  $u_{\text{set}}$  and set  $u_{\text{in}} = u_{\text{set}}$
2. repeat for  $n_{\text{steps}}$ , each step of duration  $\Delta T$ 
  - a) if  $u_{\text{in}} > 0$  and  $x(n) > 0.632 k u_{\text{in}}$ :  
increment  $n_{\text{HalfPeriods}}$  and set  $u_{\text{in}} = -u_{\text{set}}$
  - b) else if  $u_{\text{in}} < 0$  and  $x(n) < 0.632 k u_{\text{in}}$ :  
increment  $n_{\text{HalfPeriods}}$  and set  $u_{\text{in}} = u_{\text{set}}$

The time constant  $\hat{\tau}$  is then estimated as:

$$\hat{\tau} = \frac{2}{\pi} \frac{n_{\text{steps}}}{n_{\text{HalfPeriods}}} \Delta T \quad (2.23)$$

Here,  $\Delta T$  is the sampling period, and the ratio  $n_{\text{steps}}/n_{\text{HalfPeriods}}$  estimates the system's oscillation period. Essentially, this method performs period-duration estimation. The resulting time plot is shown in Fig. 2.7. For our simulated motor, the result is  $\hat{\tau} = 27.44$  ms, which is a good estimate of the real value, 29 ms.

Using equations 2.21 and 2.22, we can estimate  $\hat{\alpha}$  and  $\hat{\beta}$  as  $\hat{\alpha} = -36.443$  and  $\hat{\beta} = 619.372$ .

## Real Motor Identification

For implementation on real hardware, the code must run in real time. With a relatively high sampling frequency of 4 kHz, the software is written in C++. The interface is universal, consisting of a few key functions:

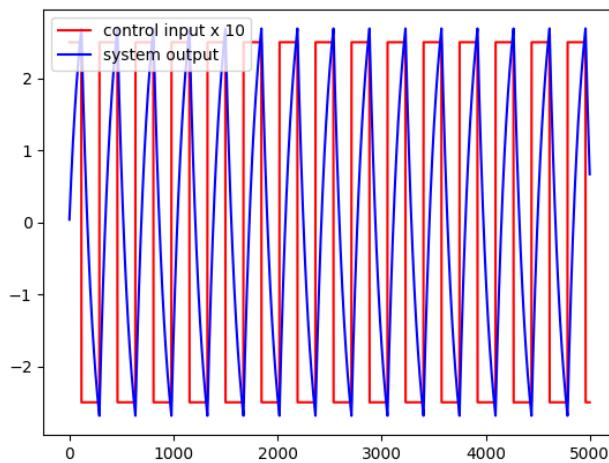


Figure 2.7: Motor time constant estimation.

- `timer.delay_ms(unsigned int time)` — waits for the given time in milliseconds
- `float x = motor_control.get_right_velocity()` — returns the measured wheel velocity
- `motor_control.set_right_torque(float x)` — sets the PWM torque command; the sign controls rotation direction

These functions must be adapted for the user's specific hardware.

First, we estimate the motor constant  $k$ :

```

1 //1, estimate motor constant k, on different input values
2 uint32_t n_steps = 500;
3
4 float u_values[10] = {0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0};
5
6 float k_mean      = 0.0;
7 float x_var_mean = 0.0;
8
9 for (unsigned int j = 0; j < 10; j++)
10 {
11     //run motor with desired input and wait for steady state
12     float u_in = u_values[j];
13
14     motor_control.set_right_torque(u_in*MOTOR_CONTROL_MAX_TORQUE);
15     timer.delay_ms(500);
16
17     //estimate average velocity
18     float x_mean = 0.0;
19     for (unsigned int i = 0; i < n_steps; i++)
20     {
21         float x = motor_control.get_right_velocity();
22         x_mean+= x;
23         timer.delay_ms(1);
24     }
25     x_mean = x_mean/n_steps;
26
27     //estimate average variance - encoder noise
28     float x_var = 0.0;
29     for (unsigned int i = 0; i < n_steps; i++)
30     {
31         float x = motor_control.get_right_velocity();
32         x_var+= (x - x_mean)*(x - x_mean);
33         timer.delay_ms(1);
34     }
35     x_var = x_var/n_steps;
36
37     float k = x_mean/u_in;
38     k_mean+= x_mean/u_in;
39     x_var_mean+= x_var;
40
41     terminal << "u_in    = " << u_in << "\n";
42     terminal << "k      = " << k << "\n";
43     terminal << "x_mean = " << x_mean << "\n";
44     terminal << "x_var  = " << x_var << "\n";
45     terminal << "\n\n";
46 }
47
48 //print summary results
49 k_mean      = k_mean/10.0;
50 x_var_mean = x_var_mean/10.0;
51
52 terminal << "k      = " << k_mean << "\n";
53 terminal << "x_var_mean = " << x_var_mean << "\n";

```

Next, we estimate the motor time constant  $\tau$ :

```

1 //2, estimate time constant by oscillating motor
2 float u_in = 0.25;
3 uint32_t periods = 0;
4
5 n_steps = 2000;
6
7 for (unsigned int i = 0; i < n_steps; i++)
8 {
9     motor_control.set_right_torque(u_in*MOTOR_CONTROL_MAX_TORQUE);
10    float x = motor_control.get_right_velocity()*60.0/(2.0*PI);
11
12    if (u_in > 0.0)
13    {
14        if (x > 0.632*k_mean*u_in)
15        {
16            u_in = -u_in;
17            periods++;
18        }
19    }
20    else
21    {
22        if (x < 0.632*k_mean*u_in)
23        {
24            u_in = -u_in;
25            periods++;
26        }
27    }
28
29    timer.delay_ms(1);
30 }
31
32 float t_period = (2*n_steps/periods)/PI;
33 terminal << "periods = " << periods << "\n";
34 terminal << "tau      = " << t_period << "[ms]\n";
35 terminal << "\n\n";

```

The average motor constant is  $k = 80.056$ , and the time constant is  $\tau = 9.230$  ms. Terminal output:

```

1 ...
2
3 u_in      =  0.800
4 k          =  87.194
5 x_mean   =  69.755
6 x_var    =  19.236
7
8
9 u_in      =  0.900
10 k         =  86.440
11 x_mean  =  77.796
12 x_var   =  18.124
13
14
15 u_in      =  1.000
16 k          =  85.750
17 x_mean   =  85.750
18 x_var    =  7.554
19
20
21 k          =  80.056
22 x_var_mean = 170.903
23
24
25 periods  = 136
26 tau       =  9.230[ms]
```

```

1 u_in      =  0.200
2 k          = 295.103
3 rpm       = 563
4 x_mean   =  59.020
5 x_var    =  21.035
6
7
8 u_in      =  0.300
9 k          = 289.379
10 rpm      = 829
11 x_mean  =  86.813
12 x_var   =  21.276
13
14
15 u_in      =  0.400
16 k          = 268.297
17 rpm       = 1024
18 x_mean   =  107.319
19 x_var    =  69.277
20
21
22 u_in      =  0.500
23 k          = 247.716
24 rpm       = 1182
25 x_mean   =  123.858
26 x_var    =  55.126
```

```

27
28
29 u_in    =  0.600
30 k       = 225.210
31 rpm     = 1290
32 x_mean = 135.126
33 x_var   = 29.614
34
35
36 u_in    =  0.700
37 k       = 204.766
38 rpm     = 1368
39 x_mean = 143.336
40 x_var   = 113.756
41
42
43 u_in    =  0.800
44 k       = 187.124
45 rpm     = 1429
46 x_mean = 149.699
47 x_var   = 446.506
48
49
50 u_in    =  0.900
51 k       = 177.842
52 rpm     = 1528
53 x_mean = 160.058
54 x_var   = 76.448
55
56
57 u_in    =  1.000
58 k       = 158.996
59 rpm     = 1518
60 x_mean = 158.996
61 x_var   = 274.646
62
63
64 k           = 205.443
65 x_var_mean = 110.768
66
67
68 periods = 240
69 tau      = 7.957[ms]

```

## Summary of Identification Results

The identification results from both simulation and real hardware show that the proposed methods provide accurate and robust estimates of the motor parameters.

The purpose of the Python simulation was to demonstrate the principle of the proposed identification method and to show how well it fits our imaginary motor model. Table 2.1 presents a direct comparison between

the ground truth parameters and the estimated values obtained from the simulation.

Table 2.1: Comparison between ground truth and estimated parameters in simulation

Parameter	Symbol	Ground Truth	Estimated
Motor constant	$k$	17.000	16.995
Time constant	$\tau$	29 ms	27.44 ms

The results clearly show that the estimation method performs very well for a simple first-order system, with minimal deviation from the true parameter values. This confirms the correctness of the identification algorithm in a noise-free simulation environment.

For the real hardware experiment, the results demonstrate consistent values of both  $\tau$  and  $k$  across multiple input levels  $u_{in}$ , which indicates that the method is robust even under real-world conditions. On the physical motor, the identified parameters were  $k = 80.056$  and  $\tau = 9.230$  ms, which are consistent with the expected values for a small, high-speed DC drive.

This experiment demonstrates how a simple first-order model can successfully capture the **essential dynamics of a real motor** and serve as a foundation for controller design. In practice, the identification accuracy depends on factors such as encoder resolution, sampling frequency, and PWM control precision. Despite nonlinearities such as friction, saturation, and supply voltage variations, the identified model provides a reliable linear approximation—suitable for designing velocity or torque controllers in robotic drive systems.

Additionally, we also estimated the velocity variance  $x_{var}$ , which provides valuable information about measurement noise. This value will be further utilized in the **design of a Kalman filter** for sensor fusion and state estimation in the following chapter.

# Chapter 3

## Linear Quadratic Regulator

The LQR controller is a standard baseline for regulating systems with multiple inputs and multiple outputs. The controller requires knowledge of the system dynamics (state-space model), and two weighting matrices in the cost function, which are design parameters chosen by the engineer.

LQR gained its position as a baseline method because it is:

- mathematically well-defined with a unique optimal solution,
- robust for a wide class of linear systems,
- able to naturally balance performance vs. control effort,
- easy to tune once the engineer understands the role of  $Q$  and  $R$ ,
- computationally efficient (solved offline),
- superior to PID for multivariable, coupled, high-dimensional systems.

Why not PID? PID is fantastic for **single-input single-output** systems. But as soon as the dynamics are coupled (e.g. drones, robots, manipulators, cars), hand-tuning dozens of gains quickly becomes impossible. LQR solves all gains at once optimally — which is why it is the industry default for linear robotics control.

Assume we would like to control a 2D system: a moving sphere with inertia.

This system has two inputs, applied forces  $u_x(n)$  and  $u_y(n)$ . The state is 4-dimensional: position and velocity in both spatial dimensions. As shown in Fig. 3.1, the state vector is  $x(n)$ , and we explicitly mark the velocity term as  $v(n)$  (which is part of the state). The control input  $u(n)$  must counteract velocity and inertia to steer the system into the desired state  $x_r(n)$ .

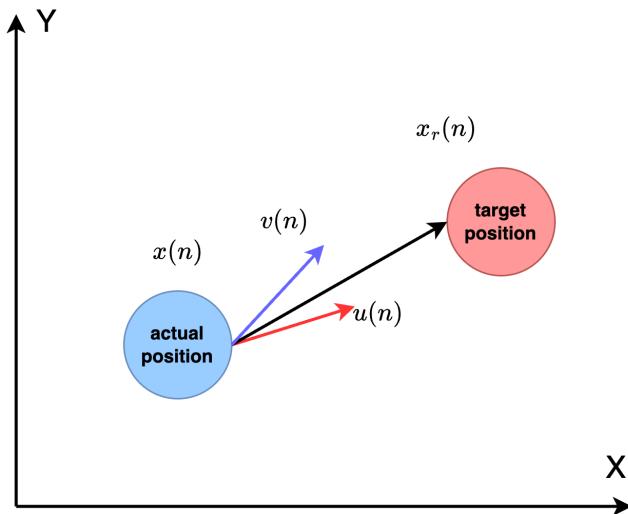


Figure 3.1: 2D position control

This is a system with multiple outputs and multiple inputs:

$$x(n) = \begin{bmatrix} p_x(n) \\ p_y(n) \\ v_x(n) \\ v_y(n) \end{bmatrix} \quad (3.1)$$

$$u(n) = \begin{bmatrix} u_x(n) \\ u_y(n) \end{bmatrix} \quad (3.2)$$

The ordering of state variables is arbitrary; here we place positions first, then velocities.

The intuition behind LQR is simple:

**Can we find a linear mapping from the error between the current state and desired state into an optimal control action?**

We define the tracking error:

$$e(n) = x_r(n) - x(n) \quad (3.3)$$

The simplest general mapping from error space to control space is matrix multiplication:

$$u(n) = K e(n) \quad (3.4)$$

The matrix  $K$  is the gain matrix. It assigns weights to each state variable and uses a linear combination to compute the optimal action  $u$ .

### 3.1 Derivation of LQR

We now derive the matrix  $K$ . We begin by defining notation and assumptions used throughout this derivation.

- $n$  — discrete **time step**.
- $N$  — number of **state variables** (state dimension).
- $M$  — number of **control inputs** (input dimension).
- $x(n)$  — **system state**, column vector of size  $N \times 1$ .
- $u(n)$  — **control input**, column vector of size  $M \times 1$ .
- $A$  — **system matrix**,  $N \times N$ .
- $B$  — **input matrix**,  $N \times M$ .
- $Q$  — positive semidefinite **state weighting matrix**,  $N \times N$ .
- $R$  — positive semidefinite **input weighting matrix**,  $M \times M$ .

The controller drives the system toward the desired state  $x_r(n)$  by minimizing the error  $e(n) = x_r(n) - x(n)$ .

The discrete-time system dynamics are:

$$x(n+1) = Ax(n) + Bu(n) \quad (3.5)$$

## Quadratic Cost Function

The standard discrete-time LQR problem minimizes the infinite-horizon quadratic cost

$$\mathcal{L} = \sum_{n=0}^{\infty} [x(n)^T Q x(n) + u(n)^T R u(n)] \quad (3.6)$$

This cost penalizes:

- **state deviation** through  $Q$  (how much we care about errors),
- **control effort** through  $R$  (how expensive the control is).

Intuitively:

- Large  $Q$  entries  $\Rightarrow$  state errors are very expensive.
- Large  $R$  entries  $\Rightarrow$  control effort is expensive (fuel, power, torque limits).

Fig. 3.2 interprets the quadratic loss. For a 4-D state and 2-D control input,  $Q$  is  $4 \times 4$ ,  $R$  is  $2 \times 2$ . Multiplying a vector by its transpose forms a quadratic (parabolic) penalty — the same idea as MSE loss.

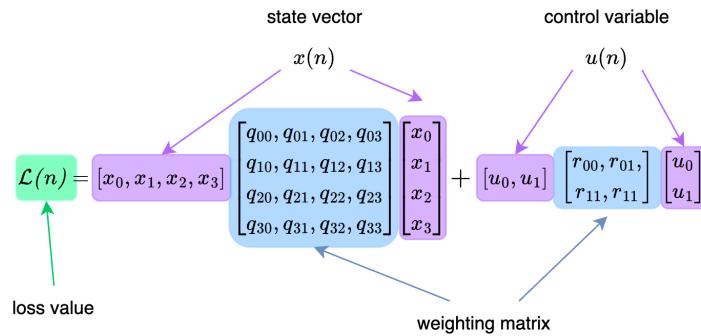


Figure 3.2: Quadratic loss meaning

Choosing  $Q$  and  $R$  is a design choice. Most often they are diagonal, meaning each state variable is weighted independently.

## Solving the LQR Problem

The optimal controller is obtained by solving the **discrete algebraic Riccati equation (DARE)**. The solution yields the optimal gain matrix:

$$u(n) = -Kx(n) \quad (3.7)$$

This steers the system to the origin  $x = 0$ . To track non-zero references, we rewrite:

$$u(n) = K(x_r(n) - x(n)) \quad (3.8)$$

(Here the minus sign is absorbed into the error definition.)

Fig. 3.3 shows the flow diagram of basic LQR controller. Fig. 3.4 shows the complete LQR algorithm.

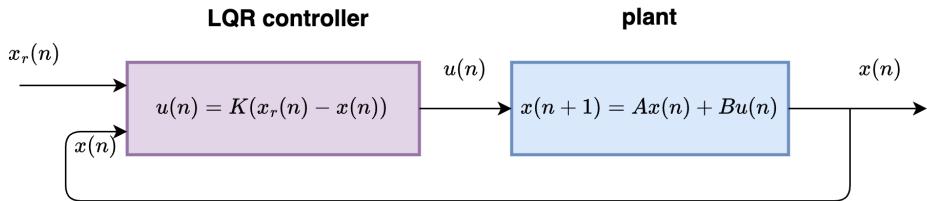


Figure 3.3: LQR block diagram

### 1. Offline (precompute):

- a) Given  $A$ ,  $B$ ,  $Q$ ,  $R$ , solve the discrete algebraic Riccati equation (DARE) for  $P$ :

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q.$$

- b) Compute the optimal gain matrix  $K$  :

$$K = (R + B^\top PB)^{-1}B^\top PA$$

- c) **Store**  $K$  and the nominal design matrices — these are used online.
2. **Online (every control time-step  $n$ ):**

- Measure or estimate current state  $x(n)$  and set reference  $x_r(n)$ .
- Compute tracking error:

$$e(n) = x_r(n) - x(n).$$

- Compute control action:

$$u(n) = K e(n).$$

- Apply  $u(n)$  to the plant.

*Notes:*  $K$  is constant for linear time-invariant designs; computing it offline makes the real-time loop very cheap.

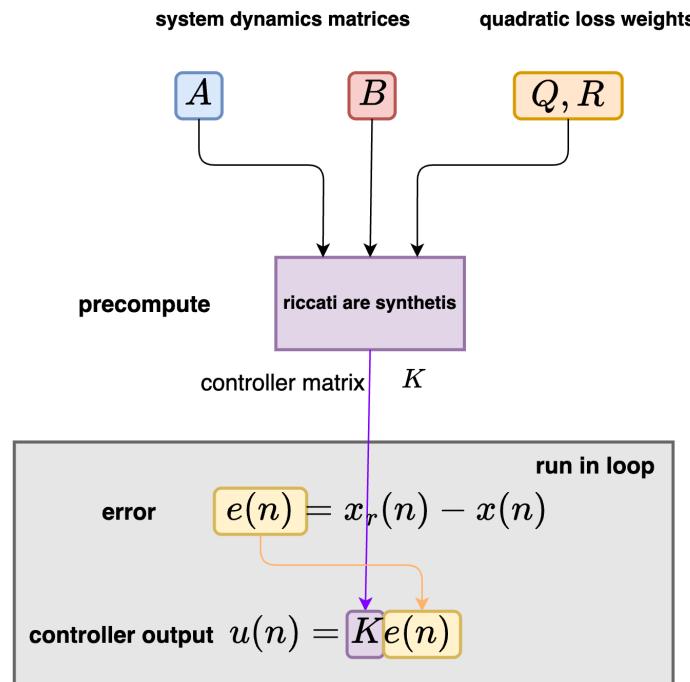


Figure 3.4: LQR algorithm

## 3.2 Adding Integral Action

The basic LQR has no integral term; thus it cannot remove steady-state error under constant disturbance. Integral action is necessary when:

- the plant has no natural integrator,
- external disturbances are constant or slowly varying,
- we need zero steady-state error for positions.

The integral term accumulates error over time:

**if error persists, push harder**

The structure of extended LQR with integral action is in the Fig. 3.5.

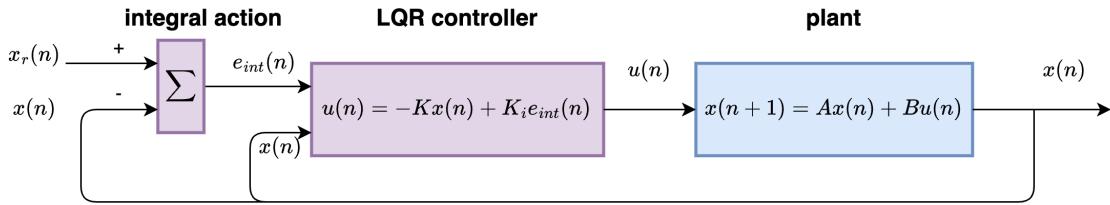


Figure 3.5: LQR with integral action block diagram

We augment the system with integral states:

$$\tilde{A} = \begin{bmatrix} A & 0 \\ I & I \end{bmatrix} \quad (3.9)$$

The cost matrix must be augmented as well:

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \quad (3.10)$$

Solving DARE for the augmented system yields:

$$\tilde{K} = \begin{bmatrix} K & K_i \end{bmatrix} \quad (3.11)$$

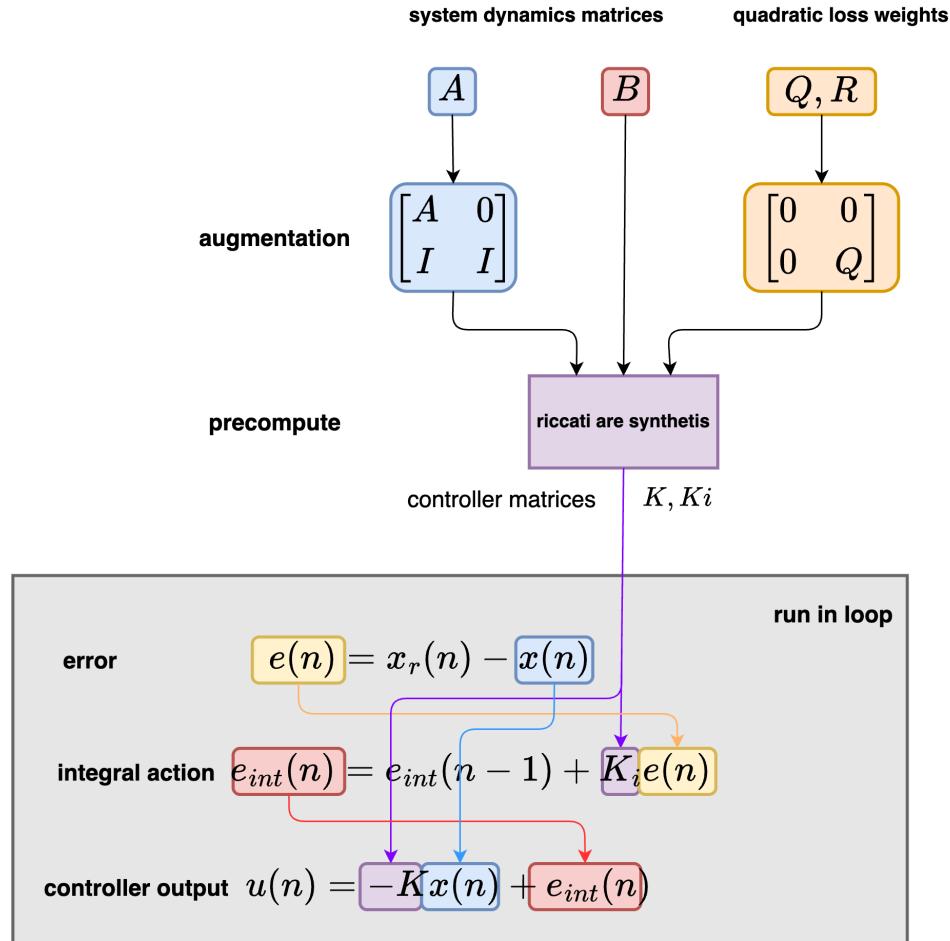


Figure 3.6: LQR algorithm with integral action

Fig. 3.6 shows the algorithmic structure.

The implemented integral loop:

$$e(n) = x_r(n) - x(n) \quad (3.12)$$

$$e_{int}(n) = e_{int}(n-1) + K_i e(n) \quad (3.13)$$

$$u(n) = -Kx(n) + e_{int}(n) \quad (3.14)$$

Note: the operator  $K$  transforms state  $x(n)$  and operator  $K_i$  is related with error signal  $e(n)$ .

### 3.3 Adding Antiwindup

Real actuators are limited. A motor might accept only  $[-12, 12]$  volts. But the controller may compute  $u = 20V$  because of the accumulated integral term.

What happens?

- Actuator outputs only 12V.
- Integrator keeps increasing (windup).
- Controller thinks it is applying 20V (internal model), but the plant sees only 12V.
- As soon as the error becomes small, the huge integrator causes overshoot.

This leads to oscillations and long settling time — not due to bad  $K$  or bad model, but due to ignoring actuator limits.

#### Antiwindup correction

The Fig. 3.7 shows full LQR block diagram, with both, integral action and antiwindup. Antiwindup section is marked by orange color. Complete algorithm is demonstrated in Fig. 3.8.

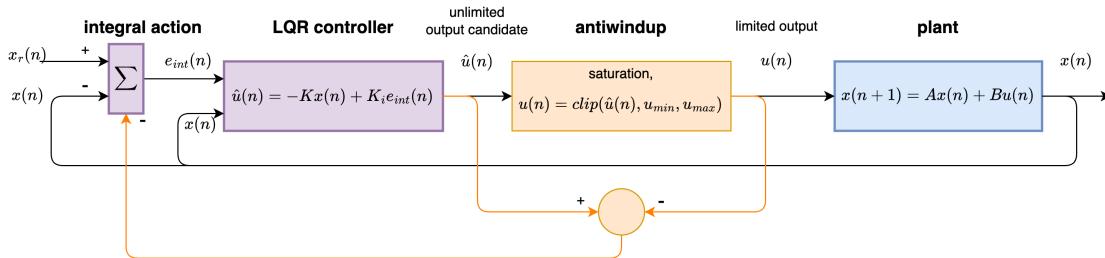


Figure 3.7: LQR with integral action and antiwindup block diagram

First compute the tentative integral update:

$$\hat{e}_{int}(n) = e_{int}(n-1) + K_i e(n)$$

Compute the tentative (unsaturated) control:

$$\hat{u}(n) = -Kx(n) + \hat{e}_{int}(n)$$

Apply actuator saturation:

$$u(n) = \text{clip}(\hat{u}(n), u_{min}, u_{max})$$

Correct the integrator:

$$e_{int}(n) = \hat{e}_{int}(n) - (\hat{u}(n) - u(n))$$

The correction term zeroes out whenever the actuator is not saturated.

This works cleanly because the integral term lives in the **action space** ( $K_i e$ ), so saturation and projection are consistent and simple.

## 3.4 Practical Implementation

Below is a complete Python implementation of the LQR controller with integral action and antiwindup.

The constructor precomputes control matrices. The method **solve** augments matrices and solves DARE. The real-time loop calls **forward**, which requires:

- desired state  $x_r$ ,
- current state  $x$ ,
- current integral state.

The output is control  $u$  and updated integrator.

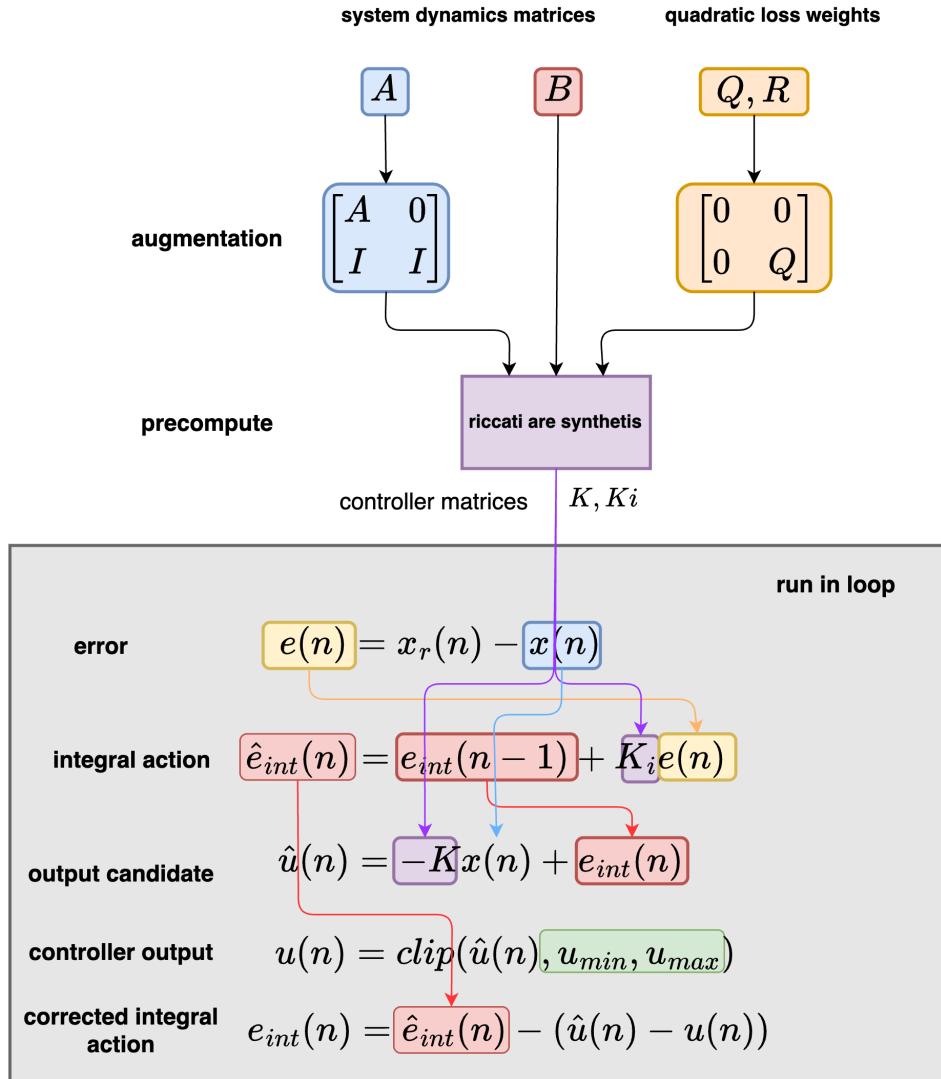


Figure 3.8: LQR algorithm with integral action and antiwindup

```

1 import numpy
2 import scipy
3
4 class LQRIDiscrete:
5
6     def __init__(self, a, b, q, r, antiwindup = 10**10):
7         self.k, self.ki = self.solve(a, b, q, r)
8
9         self.antiwindup = antiwindup
10
11    def forward(self, xr, x, integral_action):
12        # integral action
13        error = xr - x
14
15        integral_action_new = integral_action + self.ki@error
16
17        #LQR control law
18        u_new = -self.k@x + integral_action
19
20        #conditional antiwindup
21        u = numpy.clip(u_new, -self.antiwindup, self.antiwindup)
22
23        integral_action_new = integral_action_new - (u_new - u)
24
25        return u, integral_action_new
26
27    def solve(self, a, b, q, r):
28        n = a.shape[0]    # system order
29        m = b.shape[1]    # system inputs
30
31        # augmented system
32        a_aug = numpy.block([
33            [a, numpy.zeros((n, n))],
34            [numpy.eye(n), numpy.eye(n)]
35        ])
36
37        b_aug = numpy.vstack([b, numpy.zeros((n, m))])
38
39        # augmented cost
40        q_aug = numpy.block([
41            [numpy.zeros((n, n)), numpy.zeros((n, n))],
42            [numpy.zeros((n, n)), q]
43        ])
44
45        p = scipy.linalg.solve_discrete_are(a_aug, b_aug, q_aug, r)
46        k_aug = numpy.linalg.inv(r) @ (b_aug.T @ p)
47
48        #truncated small elements
49        k_aug[numpy.abs(k_aug) < 10**-10] = 0
50
51        k = k_aug[:, :n]
52        ki = k_aug[:, n:]
53
54        return k, ki

```

## 3.5 Summary

The Linear Quadratic Regulator is one of the most widely used controllers in modern control engineering. Its strength comes from combining a mathematically optimal formulation with practical usability. Once a state-space model is available, LQR provides a systematic way to compute all feedback gains at once, even for large multivariable systems.

### Applications

LQR is a standard tool in fields where:

- accurate state-space models exist,
- systems have multiple coupled states and inputs,
- fast, reliable, and stable control is required.

Typical applications include:

- robotics and manipulators,
- quadrotors, drones, and aerospace vehicles,
- autonomous driving and vehicle dynamics control,
- inverted pendulums and balancing systems,
- power systems and energy control,
- any linearized model in engineering simulations.

### Advantages

LQR is popular because it offers:

- **Uniqueness:** the solution to the Riccati equation is unique and globally optimal for the linear-quadratic problem.
- **Multivariable optimality:** all gains are solved simultaneously, avoiding the combinatorial tuning of PID.

- **Stability guarantees:** the resulting  $K$  stabilizes the system whenever the standard LQR conditions are met.
- **Clear tuning philosophy:**  $Q$  shapes state penalties,  $R$  shapes effort penalties.
- **Computational efficiency:** Riccati equations are solved once offline; online computation is a single matrix multiplication.
- **Smooth control actions:** the quadratic penalty naturally avoids aggressive or discontinuous inputs.

## Limitations

Despite its strengths, LQR has several important limitations:

- **Requires a linear state-space model.** Nonlinear systems must be linearized or approximated.
- **Sensitive to modelling errors.** Large unmodelled dynamics or unmodelled delays degrade performance.
- **No inherent integral action.** Steady-state errors require explicit integrator augmentation.
- **No built-in actuator constraints.** Saturation must be handled externally (antiwindup, MPC, etc.).
- **Not optimal for non-quadratic cost functions.** The controller's optimality holds only for quadratic penalties.

## Final Remarks

LQR remains a gold standard for linear control because it is:

- theoretically elegant,
- computationally cheap,
- practically effective,
- easy to integrate with integral action and antiwindup,

- naturally suited to multivariable systems.

For systems with good linear models and strong coupling, LQR often provides a performance baseline that is difficult to surpass using manual PID tuning.

## Chapter 4

# Motor Control Example

In this chapter we put everything together. So far we have introduced the mathematical framework of **state-space models**, the **LQR controller** (including integral action and anti-windup), robot kinematics, and the **Kalman filter**. All these pieces allow us to close a real control loop.

Our goal here is simple and practical: **control the velocity of a DC motor**.

We first build a simulation based on real motor measurements. Inside this simulation we will tune the LQR controller, test the effect of noise and the Kalman filter, and demonstrate the important concepts of **windup** and **anti-windup**. After tuning, we will transfer everything to real hardware and compare the results.

The controller receives as input:

- the (noisy) measured motor velocity,
- the desired velocity.

Its output is the motor torque command (scaled into the PWM range).

The **measured motor** has the following parameters:

- sampling time step:  $dt = 0.0005\text{ s}$  (sampling frequency 2 kHz)
- maximal usable torque command:  $u_{\max} = 1.0$
- motor constant:  $k = 205.443$
- motor time constant:  $\tau = 7.957\text{ ms}$
- encoder noise variance:  $enc_{\text{var}} = 29$

This leads to a very simple **first-order system** with one input. Although matrices are technically not required, we keep the  $1 \times 1$  matrix formulation for consistency with the rest of the book.

The chosen sampling frequency reflects the motor dynamics: the time constant is roughly 8 ms, and the sampling period is 0.5 ms. The ratio  $\frac{0.008}{0.0005} = 16$  fits the common rule of thumb that the sampling frequency should be **5–20 times faster** than the fastest time constant in the system.

The motor driver accepts a control signal  $u \in \langle -1, 1 \rangle$ , which is internally converted to a PWM value (e.g. in the range 0 . . . 255 with sign indicating rotation direction).

## 4.1 Motor Model

We now construct the continuous-time state-space model of the motor and plot its unit step response.

For a first-order motor with time constant  $\tau$  and motor constant  $k$ , the continuous system matrices are:

$$\bar{A} = \left[ -\frac{1}{\tau} \right], \quad \bar{B} = \left[ \frac{k}{\tau} \right].$$

The resulting dynamics are

$$\frac{dx}{dt} = \bar{A}x(t) + \bar{B}u(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t).$$

With our numerical values, this becomes

$$\frac{dx}{dt} = [-125.67] x(t) + [25819.15] u(t).$$

The following code builds the continuous system and evaluates its unit-step response:

```

1 # continuous time dynamics
2 mat_a = numpy.zeros((1, 1))
3 mat_b = numpy.zeros((1, 1))
4 mat_c = numpy.eye(1)
5
6 tau = tau * 0.001
7 mat_a[0][0] = -1.0 / tau
8 mat_b[0][0] = k * (1.0 / tau)
9
10 print("continuous model")
11 print("a = ", mat_a)
12 print("b = ", mat_b)
13
14 # create dynamical system
15 ds = LibsControl.DynamicalSystem(mat_a, mat_b, mat_c, dt)
16
17 # compute step response
18 t_result = []
19 u_result = []
20 x_result = []
21
22 for n in range(500):
23
24     # unit step as input
25     if n > 100 and n < 400:

```

```
26     u = numpy.ones((1, 1))
27 else:
28     u = numpy.zeros((1, 1))
29
30 x, _ = ds.forward_state(u)
31
32 t_result.append(n*dt)
33 u_result.append(u[:, 0])
34 x_result.append(x[:, 0])
35
36 t_result = numpy.array(t_result)
37 x_result = numpy.array(x_result)
38
39 #plot result
40 LibsControl.plot_response(t_result, u_result, x_result*60.0/(2.0*numpy.pi),
    ↪ "plots/step_response.png", ["control u"], ["rpm"])
```

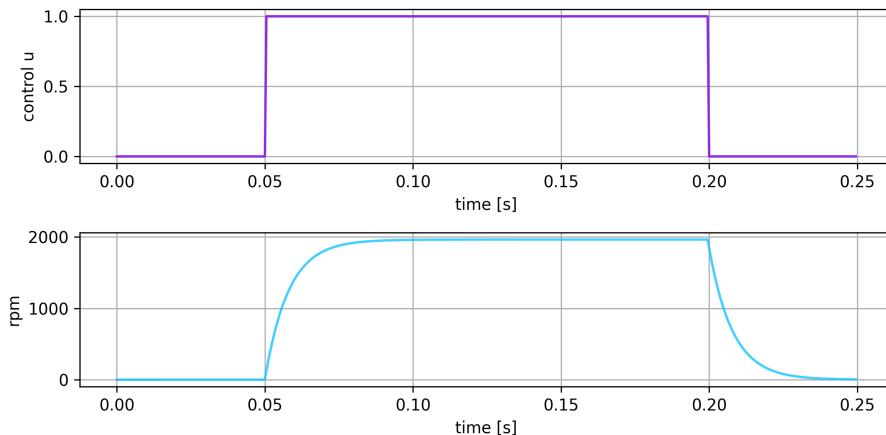


Figure 4.1: Simulated motor step response

## 4.2 Controller Design

We discretise the model with sampling time  $dt = 0.0005$  s. From the continuous matrices we obtain:

$$A = \begin{bmatrix} 0.939 \end{bmatrix}, \quad B = \begin{bmatrix} 12.516 \end{bmatrix}.$$

Thus the discrete dynamics are

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ x(n+1) &= \begin{bmatrix} 0.939 \end{bmatrix} x(n) + \begin{bmatrix} 12.516 \end{bmatrix} u(n) \end{aligned}$$

To design the LQR we choose the quadratic loss weights

$$Q = \begin{bmatrix} 1 \end{bmatrix}, \quad R = \begin{bmatrix} 4 \cdot 10^7 \end{bmatrix}.$$

From these we compute the LQR gain and the integral gain. The complete LQRI algorithm with anti-windup is shown in Fig. 4.2.

```

1 # create loss weighting matrices
2 q = numpy.array([[1.0]])
3 r = numpy.array([[4*(10**7)]])
4
5 # compute LQRI controller
6 lqr = LibsControl.LQRIDiscrete(a_disc, b_disc, q, r)
7
8 print("k = ", lqr.k)
9 print("ki = ", lqr.ki)
```

Terminal output:

```

1 continuous model
2 a = [[-125.67550584]]
3 b = [[25819.15294709]]
4
5 discrete model
6 a = [[0.9390764]]
7 b = [[12.51632753]]
8
9 controller
10 k = [[0.00216876]]
11 ki = [[0.00016025]]
```

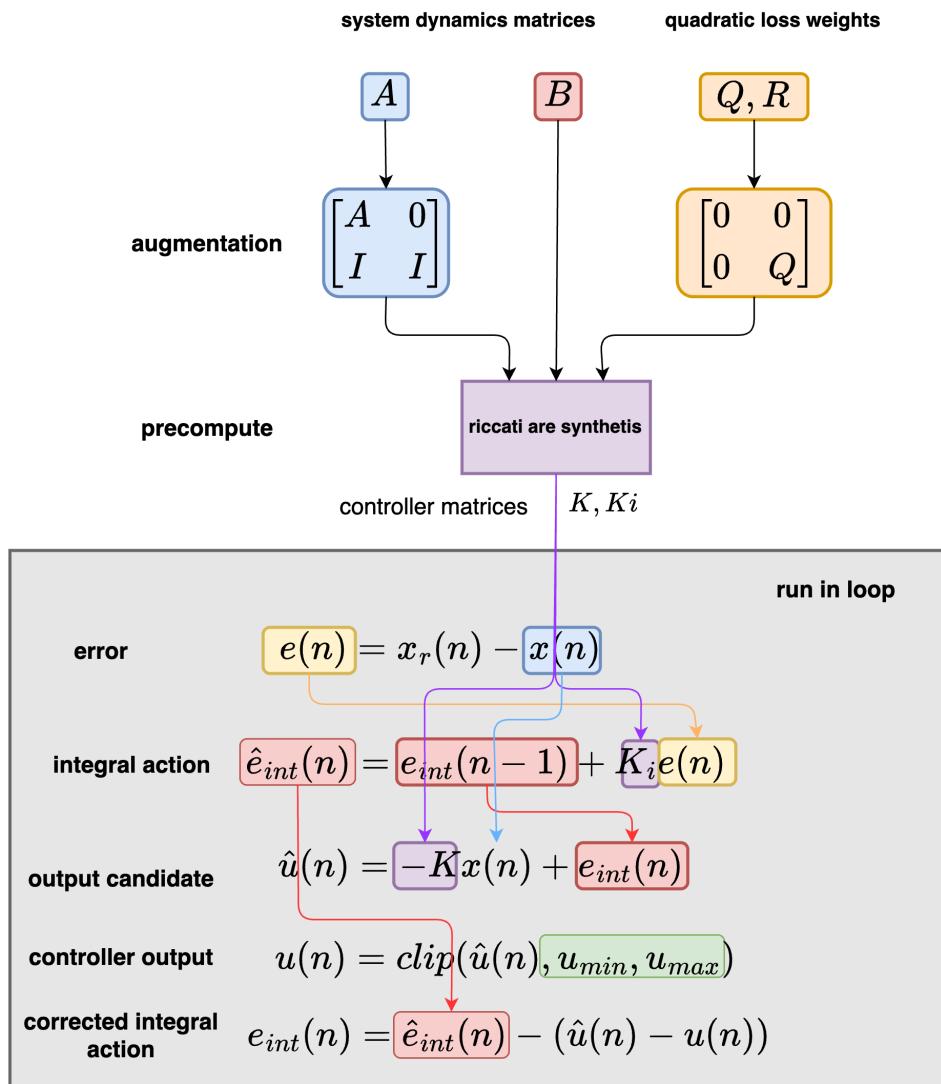


Figure 4.2: LQR algorithm with integral action and antiwindup

## 4.3 Closed-Loop Simulation

Finally, we run the closed-loop control simulation for 2000 discrete steps (corresponding to 1s of simulated time). The desired velocity is set to 1000 rpm, which must be converted into rad/s for the controller. We initialise the integral action to zero, and we also reset the dynamical system state to zero by calling `ds.reset()`.

In the main simulation loop, the reference remains at zero for the first 200 steps. Then it switches to 1000 rpm until step number 1500, after which it returns to zero. This allows us to observe the transient behaviour both when accelerating and when decelerating.

The computation steps inside the loop are as follows:

- obtain the current state  $x(n)$  (this effectively represents reading the sensor value, e.g. `x = ds.read()`),
- compute the controller output and updated integral term using `u, integral_action = lqr.forward(xr, x, integral_action)`,
- apply the controller output to the plant: `x, y = ds.forward_state(u)`, which returns the new state  $x(n+1)$ ,
- log the results for later plotting (optional but very useful for analysis).

```

1 # process simulation
2 n = mat_a.shape[0]
3 m = mat_b.shape[1]
4
5 n_max = 2000
6
7 # required output: 1000 rpm
8 rpm_req = 1000
9 xr = numpy.zeros((n, 1))
10
11 # convert rpm to rad/s
12 xr_req_value = (rpm_req / 60.0) * (2.0 * numpy.pi)
13
14 # initial integral action
15 integral_action = numpy.zeros((m, 1))
16
17 # result log
18 t_result = []
19 u_result = []
20 xr_result = []
21 x_result = []
22
23 # initial motor state

```

```

24 ds.reset()
25
26 for n in range(n_max):
27
28     # reference step input
29     if n > 200 and n < 1500:
30         xr[0][0] = xr_req_value
31     else:
32         xr[0][0] = 0.0
33
34     # plant current state
35     x = ds.x
36
37     # compute controller output
38     u, integral_action = lqr.forward(xr, x, integral_action)
39
40     # propagate plant dynamics
41     x, y = ds.forward_state(u)
42
43     # store results for plotting
44     t_result.append(n * dt)
45     u_result.append(u[:, 0].copy())
46     xr_result.append(xr[:, 0].copy())
47     x_result.append(x[:, 0].copy())

```

After the simulation finishes, we plot the results as shown in Fig. 4.3. The horizontal axis represents time in seconds. The top graph shows the controller output  $u(n)$ , i.e. the control input applied to the motor. The lower graph compares the **required velocity**  $x_r(n)$  with the **actual motor velocity**  $x(n)$ . For improved readability, the velocity is converted from rad/s back to rpm.

```

1 t_result = numpy.array(t_result)
2 xr_result = numpy.array(xr_result)
3 x_result = numpy.array(x_result)
4 u_result = numpy.array(u_result)
5
6 # convert rad/s to rpm
7 xr_result[:, 0] *= 60.0 / (2.0 * numpy.pi)
8 x_result[:, 0] *= 60.0 / (2.0 * numpy.pi)
9
10 # plot results
11 LibsControl.plot_cl_response(
12     t_result,
13     u_result,
14     xr_result,
15     x_result,
16     "plots/lqr_result.png",
17     ["input u"],
18     ["velocity [rpm]"]
19 )

```

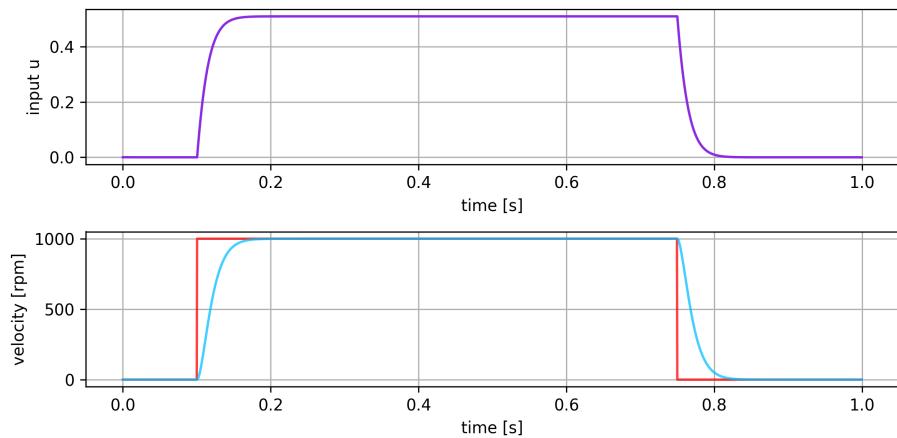


Figure 4.3: LQR controller closed-loop response

## 4.4 Ablation Testing

In the previous section we selected the  $Q$  and  $R$  weighting matrices somewhat arbitrarily. Here, we analyse how different values of  $R$  influence the closed-loop behaviour. Since the considered system is relatively simple, we keep  $Q$  fixed and vary only  $R$ , which directly penalises the control input  $u$ .

We evaluate the following values of  $R$ :

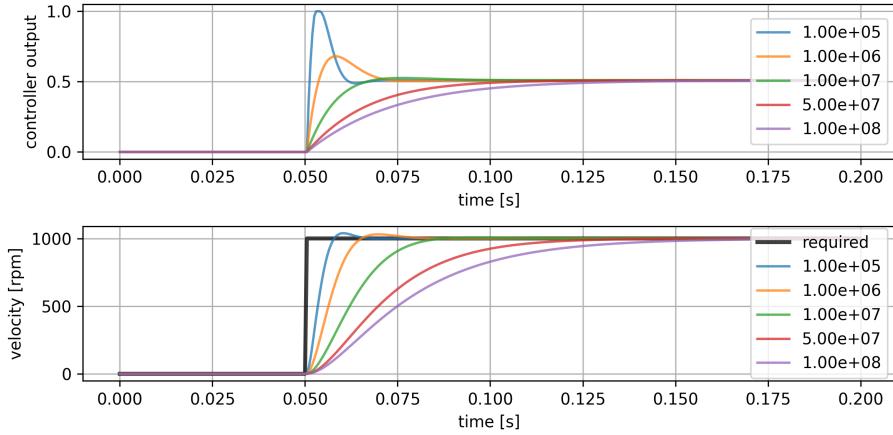
$$R \in \{0.01(10^7), 0.1(10^7), 1(10^7), 5(10^7), 10(10^7)\}.$$

To better emphasise differences in the transient response, we shorten the simulation to 400 steps (approximately 0.2 s). The resulting plot is shown in Fig. 4.4.

In the top plot, the controller output is displayed. We can observe that **smaller values of  $R$  produce larger control actions**, which is expected: the cost function penalises large  $u$  values less severely. In the bottom plot, the corresponding velocity response is shown.

A small  $R$  value results in a very fast response with short settling time, but at the cost of overshoot and extremely large control effort. Conversely, a **large  $R$  value leads to a slower response and a very long settling time**, but with almost no overshoot. Between these extremes lies a balanced configuration, offering a reasonable trade-off between response speed, overshoot, and control effort.

It is also important to note that extremely large controller kicks (produced when  $R$  is very small) are typically undesirable in practice, as they may induce excessive mechanical stress on the system.

Figure 4.4: Influence of the  $R$  weighting value

## 4.5 Antiwindup Behaviour

The second experiment evaluates whether the implemented **antiwindup** mechanism properly handles actuator saturation. We consider the following testing scenarios:

- no antiwindup and no saturation — a textbook controller,
- no antiwindup on the integral term, but with controller output clipping,
- full antiwindup: both integral correction and output clipping.

To push the system close to its saturation limits, we set the desired velocity to 1900 rpm, which is near the maximum achievable motor speed. The corresponding results are presented in Fig. 4.5.

The **blue line** shows the textbook controller, where no clipping and no antiwindup are applied. This controller **produces very large control inputs** (well beyond our limit of  $u_{\max} = 1$ ) and therefore the plant reaches steady state artificially quickly. This is, of course, unrealistic — in practice the actuator cannot exceed  $u_{\max}$ . To enforce this practical constraint, we clip the controller output to the range  $\langle -u_{\max}, u_{\max} \rangle$ . The resulting behaviour is shown by the **orange line**.

However, while the output is properly clipped, the integrator is still unaware of the saturation. It continues accumulating error, causing a large

integral term. As soon as the system reaches the reference value, the controller output continues to rise due to the accumulated integral action. This classic **integral windup** effect causes a long settling time, clearly visible in the lower plot as the **orange** response overshoots and slowly decays.

The solution is the full antiwindup mechanism, which both clips the output and compensates the integrator. The corresponding response is shown by the **green line**. Here, the controller respects the actuator limits and avoids integral windup, achieving a stable and well-behaved response.

Finally, note in the top plot that the controller correctly saturates at  $u_{\max}$  without producing unrealistic peaks. Given the high requested speed, the resulting settling time is acceptable and significantly more stable than in the non-antiwindup case.

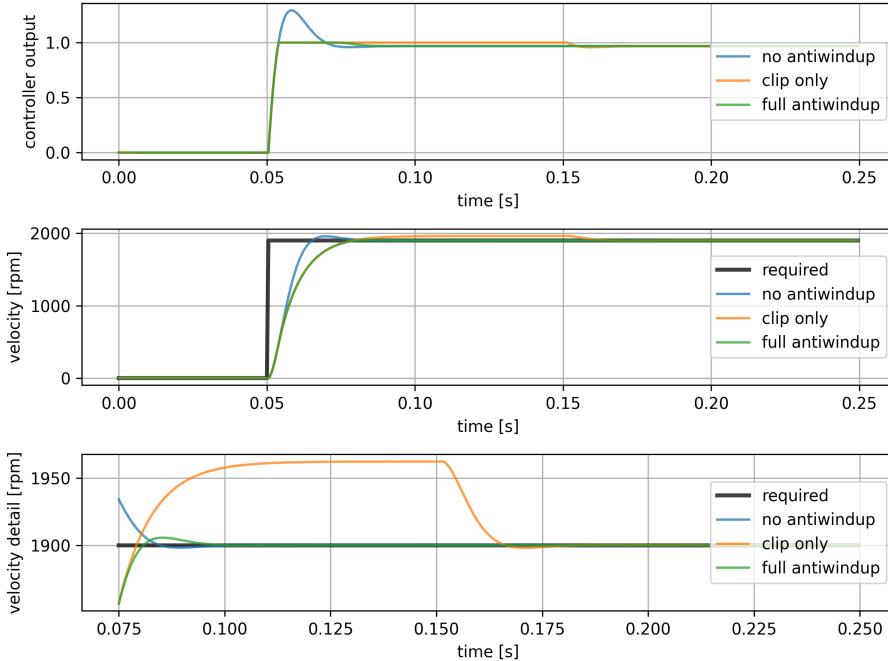


Figure 4.5: Comparison of antiwindup strategies under actuator saturation

## 4.6 Kalman Filtering

To reduce encoder noise we use a steady-state Kalman filter. For simple systems one could use an exponential moving average (EMA) based only on the noise variance. However, the Kalman filter exploits the known system dynamics ( $A, B, C$ ), producing a much faster and more accurate estimate than EMA.

## 4.7 Real Motor Testing

Finally, we deploy the controller to a real robot using C++ on an STM32 microcontroller. The hardware details are not critical: we only require a function to measure wheel velocity (rad/s) and a function to send the motor command. A periodic timer maintains the control loop timing, and a UART output streams the measured data for plotting and analysis.

## Chapter 5

# Model Predictive Control

We begin by defining the control problem. The robot must navigate along a desired trajectory  $X_r$ . The robot's motion is constrained by its own dynamics, typically expressed as a discrete-time state-space model.

The Linear Quadratic Regulator (LQR) optimal control law can **only regulate** the system to a desired state  $x_r$ , which corresponds to a **single point on the trajectory**. In contrast, Model Predictive Control (MPC) leverages the knowledge of **future desired states**  $x_r$ . At its input, the controller receives the entire reference trajectory  $X_r$ , covering several future time steps defined by the prediction horizon.

Figure 5.1 illustrates the difference between LQR and MPC for a robot trajectory tracking task.

MPC consists of two main components:

- a model of the system dynamics,
- and an optimizer.

The system dynamics model uses the current state  $x(n)$  to predict the future states  $\hat{x}(n+1), \hat{x}(n+2), \dots, \hat{x}(n+H_p)$  up to the prediction horizon  $H_p$ . The controller takes as input the **desired trajectory**  $X_r$ , the **current state**  $x(n)$ , and the **system model**, and performs **optimization** to compute an optimal sequence of control actions  $U$ . The working principle demonstrates figure 5.2. Note in general model can be imperfect, and the predicted states as well.

This optimization incrementally adjusts the control signal so that the predicted system trajectory  $X$  follows the reference  $X_r$  as closely as possible.

Such **predictive behavior** allows the controller to, for example, initiate braking or turning earlier, resulting in smoother motion and overall higher control performance.

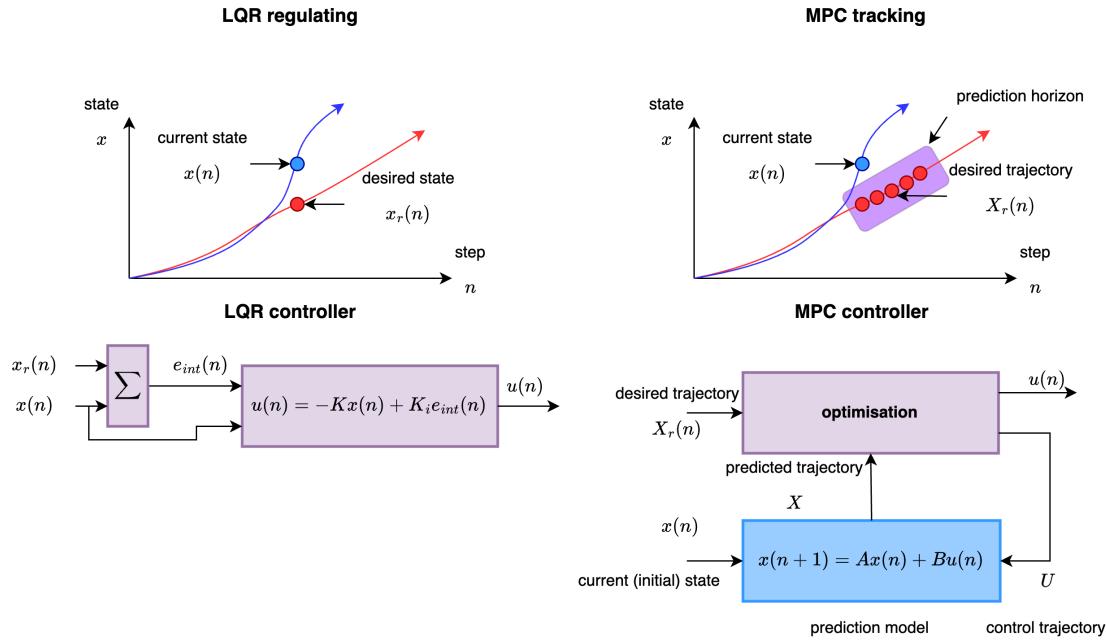


Figure 5.1: Comparison of LQR and MPC for robot trajectory tracking.

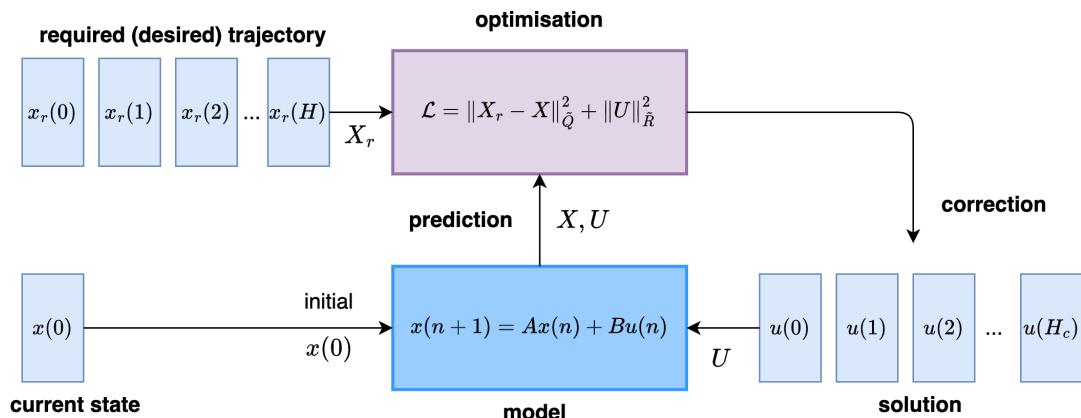


Figure 5.2: MPC algorithm working principle

In following steps, we derivate **special case** of unconstrained MPC. This allow us to completely avoid optimizer, and give us **analytical solution**, which is capable to run in real time (less than 10ms) in almost all nowadays hardware.

We begin by defining the notation and assumptions used throughout this derivation.

- $n$  — discrete **time step**.
- $N$  — number of **state variables** (state dimension).
- $M$  — number of **control inputs** (input dimension).
- $x(n)$  — **system state**, column vector of size  $N \times 1$ .
- $u(n)$  — **control input**, column vector of size  $M \times 1$ .
- $A$  — **system matrix**,  $N \times N$ .
- $B$  — **input matrix**,  $N \times M$ .
- $Q$  — positive semidefinite **state weighting matrix**,  $N \times N$ .
- $R$  — positive semidefinite **input weighting matrix**,  $M \times M$ .

The discrete-time linear system dynamics are

$$x(n+1) = Ax(n) + Bu(n). \quad (5.1)$$

For the MPC formulation, we define:

- $H_p$  — **prediction horizon** (number of future steps predicted),
- $H_c$  — **control horizon** (number of future control actions to optimize),

with  $H_p > H_c$ .

---

## 5.1 Stacked system formulation

We define the stacked vector of predicted future states - **states trajectory**

$$X = \begin{bmatrix} x(n+1) \\ x(n+2) \\ \vdots \\ x(n+H_p) \end{bmatrix}, \quad (5.2)$$

of total dimension  $H_p N \times 1$ .

Similarly, we define the stacked vector of **future control inputs**

$$U = \begin{bmatrix} u(n) \\ u(n+1) \\ \vdots \\ u(n+H_c - 1) \end{bmatrix}, \quad (5.3)$$

of size  $H_c M \times 1$ .

The **quadratic cost function** is

$$\mathcal{L}(U) = (X_r - X)^\top \tilde{Q} (X_r - X) + U^\top \tilde{R} U, \quad (5.4)$$

subject to the system dynamics constraint

$$x(n+1) = Ax(n) + Bu(n). \quad (5.5)$$

The weighting matrices  $\tilde{Q}$  and  $\tilde{R}$  are block-diagonal matrices constructed from  $Q$  and  $R$ :

$$\tilde{Q} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}. \quad (5.6)$$

Their dimensions are  $\tilde{Q} \in \mathbb{R}^{H_p N \times H_p N}$  and  $\tilde{R} \in \mathbb{R}^{H_c M \times H_c M}$ .

---

## 5.2 Prediction model formalism

We express the predicted future states as a function of the current state and future control sequence:

$$\begin{aligned} \begin{bmatrix} x(n+1) \\ x(n+2) \\ \vdots \\ x(n+H_p) \end{bmatrix} &= \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^{H_p} \end{bmatrix} x(n) \\ &+ \begin{bmatrix} A^0 B & 0 & 0 & \cdots & 0 \\ A^1 B & A^0 B & 0 & \cdots & 0 \\ A^2 B & A^1 B & A^0 B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A^{H_p-1} B & A^{H_p-2} B & \dots & A^{H_p-H_c} B & A^{H_p-H_c-1} B \end{bmatrix} \begin{bmatrix} u(n) \\ u(n+1) \\ \vdots \\ u(n+H_c-1) \end{bmatrix} \end{aligned}$$

This compactly becomes

$$X = \Psi x(n) + \Theta U, \quad (5.7)$$

where

$$\Psi = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^{H_p} \end{bmatrix}, \quad \Theta_{ij} = \begin{cases} A^{i-j} B, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases} \quad (5.8)$$

$\Psi$  has dimensions  $H_p N \times N$  and  $\Theta$  has dimensions  $H_p N \times H_c M$ .

---

## 5.3 Understanding the Prediction Model

Let us break down what the recent expressions mean on an **intuitive level**. The purpose of formulating the problem in a matrix framework is to express the objective loss function in terms of the current state  $x(n)$ , the desired trajectory  $X_r(n)$ , and the sequence of control inputs  $U$ . The following figure 5.3 illustrates how the matrices  $\Psi$  and  $\Theta$

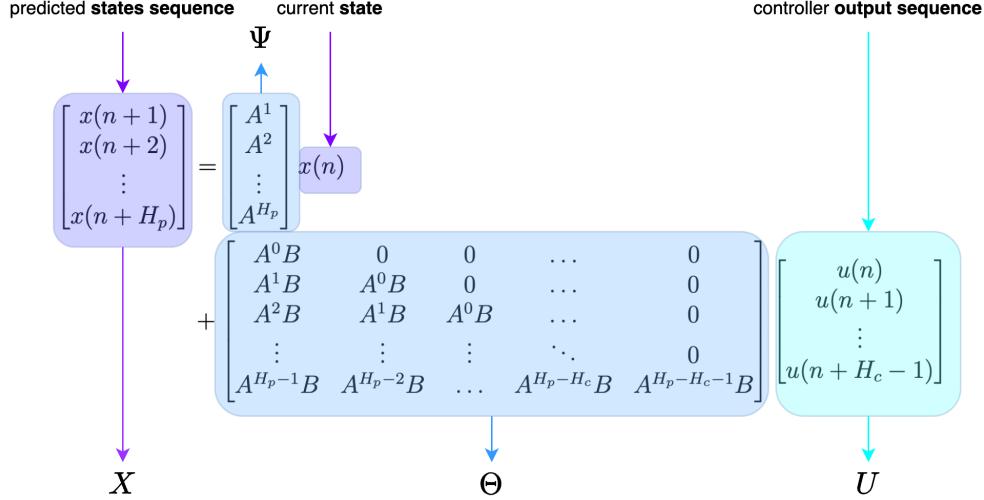


Figure 5.3: Prediction model representation.

are constructed, and how they relate the current state  $x(n)$  and the control sequence  $U$  to the predicted future states.

Let us first question what the terms inside the  $\Psi$  and  $\Theta$  matrices actually mean.

---

### Projection of the Current State

Assume we select the **first row** from  $\Psi$  and  $\Theta$ , as shown in figure~5.4. This single row tells us **how the current state  $x(n)$  is projected into the next state  $x(n+1)$** . As we can see, this follows directly from the linear dynamic model

$$x_a(n+1) = Ax(n),$$

where  $x_a$  represents the component of the next state due solely to the system dynamics.

Similarly, the input  $u(n)$  is projected via the first row of the matrix  $\Theta$ , as expected from the linear state-space model:

$$x_b(n+1) = Bu(n),$$

where  $x_b$  represents the component due to control inputs. Together, these two components reproduce the well-known discrete-time linear dynamical

system:

$$x(n+1) = Ax(n) + Bu(n).$$

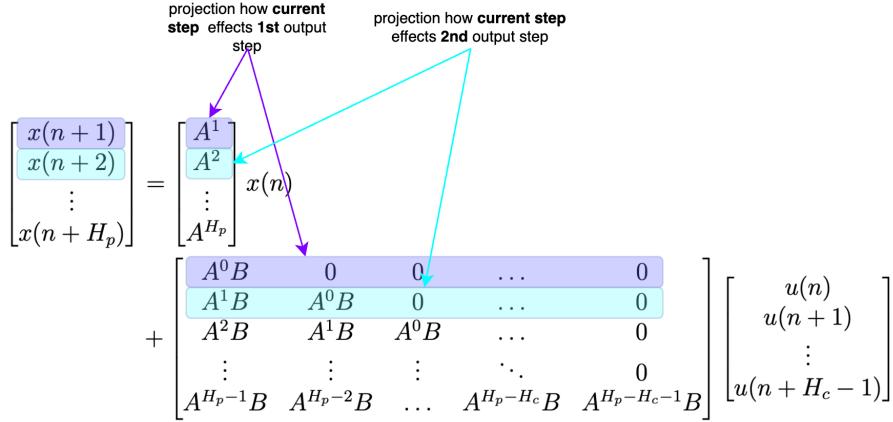


Figure 5.4: Effect of  $\Psi$  and  $\Theta$  matrix terms.

---

### Extending the Prediction Horizon

Next, let us consider the **second row** of matrix  $\Psi$ , which describes how the **current state**  $x(n)$  is projected further into the future, producing the  $A$ -term for  $x_a(n+2)$ . The current state  $x(n)$  must be propagated twice to obtain  $x_a(n+2)$ :

$$\begin{aligned} x_a(n+1) &= Ax(n), \\ x_a(n+2) &= Ax_a(n+1) = A^2x(n). \end{aligned}$$

This is why the **second row of matrix  $\Psi$  contains  $A^2$** . In general, projecting the current state  $x(n)$   $h$  steps into the future is expressed as:

$$x_a(n+h) = A^h x(n).$$

---

### Projection of Control Inputs

Now, let us apply the same reasoning to the **second row** of the matrix  $\Theta$  and the future control input  $u(n+1)$ . This describes how **future control inputs** are projected into future states through the  $B$ -terms:

$$\begin{aligned} x_b(n+1) &= Bu(n), \\ x_b(n+2) &= ABu(n) + Bu(n+1). \end{aligned}$$

This expression directly shows how to construct the matrix  $\Theta$ .

In summary,  $\Psi$  projects the current state  $x(n)$  into future states, whereas  $\Theta$  projects the sequence of future control inputs  $U$  into their effect on future states.

### Intuitive View of the $\Theta$ Matrix

Another way to understand  $\Theta$  is to look at how each individual control input  $u$  in the sequence  $U$  affects future states. This is demonstrated in figure 5.5.

Consider the **first control input**  $u(n)$ , marked in purple in the figure. This term is projected into all future states  $x(n+1), x(n+2), \dots$ , which makes intuitive sense — the first control action influences the entire future trajectory.

Now look at the **second control input**  $u(n+1)$ , marked in cyan. In the second column of  $\Theta$ , the first term is zero, reflecting causality — the input  $u(n+1)$  cannot affect any past state such as  $x(n+1)$ .

Finally, examine the **last control input**  $u(n+H_c-1)$ , shown in blue. The last column of  $\Theta$  contains zeros except for the last few entries, meaning that the last control input only affects the final predicted state(s).

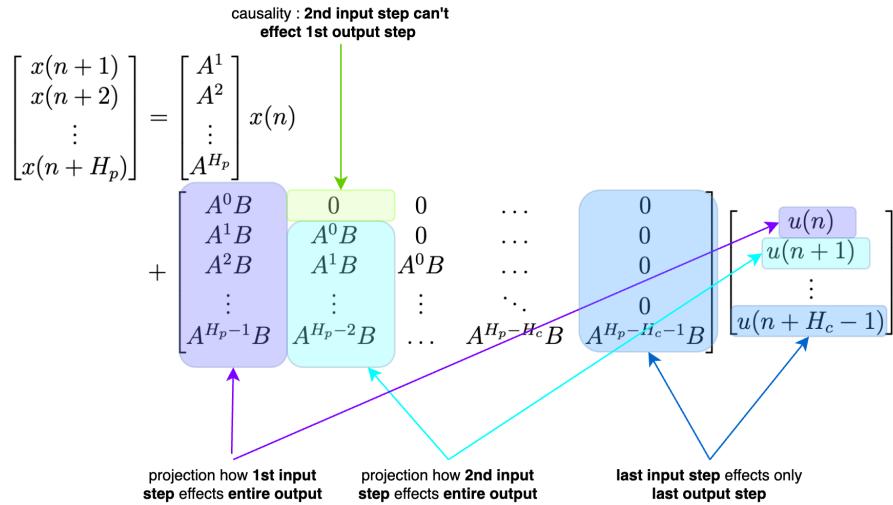


Figure 5.5: Effect of  $\Theta$  matrix terms on predicted states.

This intuitive view helps us understand how the prediction model aggregates both system dynamics and control inputs into a single compact matrix formulation, enabling MPC to efficiently predict and optimize future behavior.

---

## 5.4 Optimization problem

Substitute the predicted state into the cost function:

$$\mathcal{L}(U) = (X_r - \Psi x(n) - \Theta U)^\top \tilde{Q} (X_r - \Psi x(n) - \Theta U) + U^\top \tilde{R} U. \quad (5.9)$$

Define the state–reference residual

$$S = X_r - \Psi x(n), \quad (5.10)$$

to obtain

$$\mathcal{L}(U) = U^\top \tilde{R} U + (S - \Theta U)^\top \tilde{Q} (S - \Theta U). \quad (5.11)$$

Expanding and collecting terms:

$$\mathcal{L}(U) = U^\top (\tilde{R} + \Theta^\top \tilde{Q} \Theta) U - 2U^\top \Theta^\top \tilde{Q} S + S^\top \tilde{Q} S. \quad (5.12)$$

The derivative terms are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial U} : \\ \frac{\partial U^\top \tilde{R} U}{\partial U} &= 2\tilde{R} U \\ \frac{\partial U^\top \Theta^\top \tilde{Q} \Theta U}{\partial U} &= 2\Theta^\top \tilde{Q} \Theta U \\ \frac{\partial -2U^\top \Theta^\top \tilde{Q} S}{\partial U} &= -2\Theta^\top \tilde{Q} S \\ \frac{\partial S^\top \tilde{Q} S}{\partial U} &= 0 \end{aligned} \quad (5.13)$$


---

## 5.5 Analytical solution

Taking the derivative with respect to  $U$  and setting it to zero:

$$\frac{\partial \mathcal{L}}{\partial U} = 2(\tilde{R} + \Theta^\top \tilde{Q} \Theta) U - 2\Theta^\top \tilde{Q} S = 0. \quad (5.14)$$

Hence, the **optimal control sequence for the unconstrained MPC problem is**

$$U^* = (\tilde{R} + \Theta^\top \tilde{Q} \Theta)^{-1} \Theta^\top \tilde{Q} S, \quad (5.15)$$

which is equivalent to

$$U^* = (\tilde{R} + \Theta^\top \tilde{Q} \Theta)^{-1} \Theta^\top \tilde{Q} (X_r - \Psi x(n)). \quad (5.16)$$

## 5.6 Remarks

- The matrices  $\tilde{Q}$  and  $\tilde{R}$  must be symmetric and positive semidefinite to guarantee convexity of the optimization problem.
  - If  $H_c < H_p$ , the remaining predicted inputs beyond  $u(n + H_c - 1)$  are assumed to be zero.
  - Actuator saturation can be applied by clipping  $u(n)$  to its physical limits.
  - The presented derivation corresponds to the *unconstrained* MPC case; if hard constraints on inputs or states are required, the same quadratic structure can be solved using a quadratic programming (QP) solver.
- 

## 5.7 Algorithm implementation

Since all matrices are constant for a given system, we can precompute

$$\Sigma = (\tilde{R} + \Theta^\top \tilde{Q} \Theta)^{-1} \Theta^\top \tilde{Q}. \quad (5.17)$$

At each control step:

$$E(n) = X_r(n) - \Psi x(n), \quad (5.18)$$

$$U(n) = \Sigma E(n), \quad (5.19)$$

$$u(n) = \text{first } M \text{ elements of } U(n). \quad (5.20)$$

The control signal  $u(n)$  is then applied to the plant, and the process repeats at the next sampling instant. The overall idea of algorithm is in the following figure~5.6.

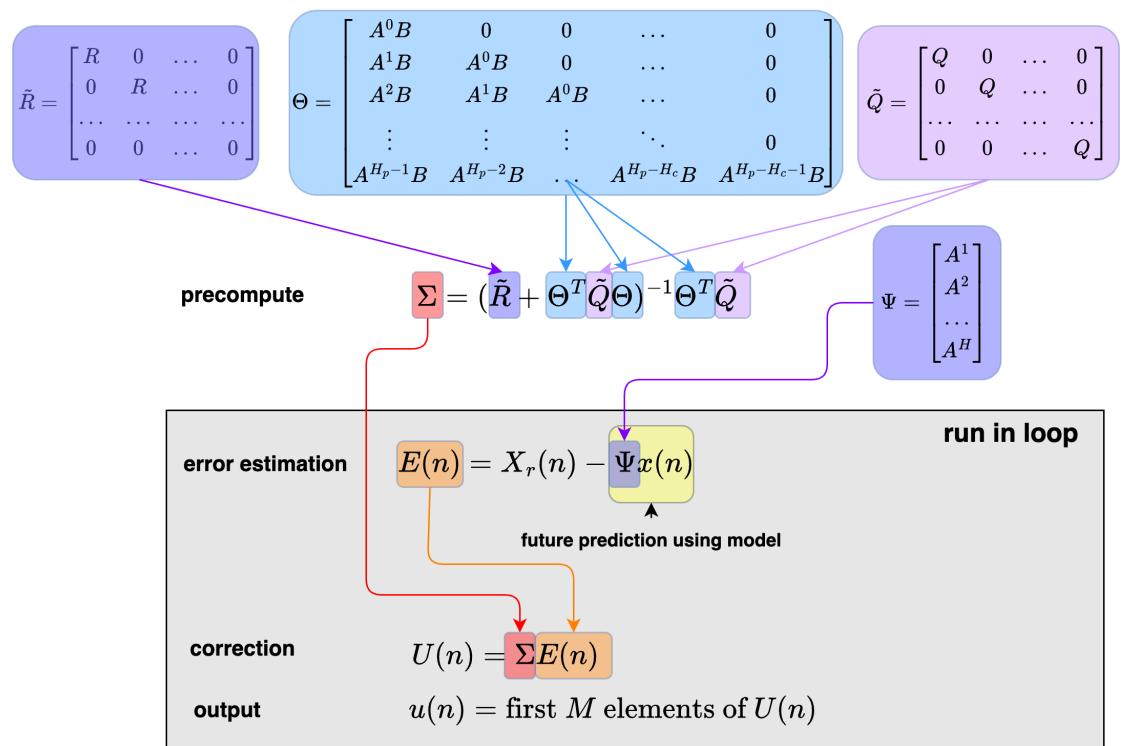


Figure 5.6: Block diagram of unconstrained MPC algorithm

## Python code example

First we construct all matrices, precompute  $\Sigma$  and  $\Theta$ . This is costly operation, hewever it can be done once in constructor :

```

1 """
2 A: (n_x, n_x)
3 B: (n_x, n_u)
4 Q: (n_x, n_x) (state cost)
5 R: (n_u, n_u) (input cost)
6 prediction_horizon : Hp, how many future states
7 control_horizon     : Hc, how many future inputs we optimize;
8                         typically <= Hp
9 """
10 def __init__(self, A, B, Q, R, prediction_horizon=16, control_horizon=4, u_max=1e10):
11
12     self.A      = A
13     self.B      = B
14     self.nx     = A.shape[0]
15     self.nu     = B.shape[1]
16     self.Hp     = prediction_horizon
17     self.Hc     = control_horizon
18     self.u_max  = u_max
19
20     # 1, build Phi and Theta
21     self.Phi, self.Theta = self._init_matrices(A, B, self.Hp, self.Hc)
22
23     # 2, build augmented tilde Q and tilde R, block-diagonal
24     self.Q_aug = numpy.kron(numpy.eye(self.Hp), Q)
25     self.R_aug = numpy.kron(numpy.eye(self.Hc), R)
26
27     # Precompute solver matrices: G and Sigma
28     G = self.Theta.T @ self.Q_aug @ self.Theta + self.R_aug
29
30     # use solve later for stability; but precompute factorization if desired
31     # here we compute Sigma by solving G Sigma^T = Theta^T Q_aug (do via solve)
32     # Sigma has shape (n_u*Hc, n_x*Hp)
33     # Solve H @ Sigma = Theta.T @ Q_aug
34     # Sigma = numpy.linalg.solve(H, Theta.T @ Q_aug)
35     self.Sigma = numpy.linalg.solve(G, self.Theta.T @ self.Q_aug)
36     self.Sigma0 = self.Sigma[:self.nu, :]
37
38
39 def _init_matrices(self, A, B, Hp, Hc):
40     nx = A.shape[0]
41     nu = B.shape[1]
42     # precompute A powers: A^0 ... A^Hp
43     A_pows = [numpy.eye(nx)]
44     for i in range(1, Hp + 1):
45         A_pows.append(A_pows[-1] @ A)
46
47     # Phi: (nx*Hp, nx) stacked [A; A^2; ...; A^Hp]
48     Phi = numpy.zeros((nx * Hp, nx))
49     for i in range(Hp):
50         Phi[i * nx:(i + 1) * nx, :] = A_pows[i + 1] # A^(i+1)
51
52     # Theta: (nx*Hp, nu*Hc) where block (i,j) is A^(i-j) B for i>=j, else 0

```

```

53     Theta = numpy.zeros((nx * Hp, nu * Hc))
54     for i in range(Hp):
55         for j in range(Hc):
56             if i >= j:
57                 # A^{i-j} B
58                 Theta[i * nx:(i + 1) * nx, j * nu:(j + 1) * nu] = A_pows[i - j] @ B
59             else:
60                 # remains zero
61                 pass
62
63     return Phi, Theta

```

Main control loop is straightforward. We precomputed almost everything, in fast running loop we have to perform only two matrix multiplications. The shape of  $X_r$  numpy matrix is  $(NH_p, 1)$ , shape of matrix  $x$  is  $N, 1$ , function returns column vector  $u$  of shape  $(M, 1)$ :

```

1 def forward_traj(self, Xr, x):
2     # residual
3     s = Xr - self.Phi @ x
4
5     # compute only first control
6     u0 = self.Sigma0 @ s
7     u0 = numpy.clip(u0, -self.u_max, self.u_max)
8
9     return u0

```

---

## 5.8 Tracking Problem Example

We consider a simple 2D moving sphere with inertia. The goal is to follow a given reference trajectory  $X_r$ , as shown in figure 5.7. We control two input forces,  $u_x$  and  $u_y$ .

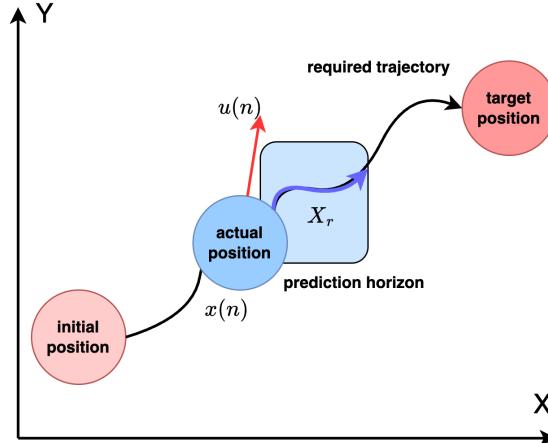


Figure 5.7: Trajectory tracking example.

The system behaves as **two independent servos with first-order inertia**, characterized by a time constant  $\tau$  and an amplification factor  $k$ . We assume a continuous state-space model  $\dot{x} = \bar{A}x + \bar{B}u$ , which we later discretize to design a controller. This allows us to demonstrate the complete process, from a physical model to a working MPC controller.

We arbitrarily set the parameters as:

- discretization step:  $dt = 0.01$  s,
- time constant:  $\tau = 0.5$  s,
- amplification:  $k = 0.3$ .

The state vector  $x$  contains position and velocity in both axes:

$$x = \begin{bmatrix} x_{\text{pos}} \\ y_{\text{pos}} \\ x_{\text{vel}} \\ y_{\text{vel}} \end{bmatrix}.$$

The continuous system matrices  $\bar{A}$  and  $\bar{B}$  are:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k}{\tau} & 0 \\ 0 & \frac{k}{\tau} \end{bmatrix}. \quad (5.21)$$

After discretization for time step  $dt$ , we obtain discrete-time matrices  $A$  and  $B$ , which are used only for controller synthesis. The physical system itself is simulated in continuous time using a Runge–Kutta 4 (RK4) ODE solver, to stay close to realistic behavior.

$$A = \begin{bmatrix} 1 & 0 & 0.00990099 & 0 \\ 0 & 1 & 0 & 0.00990099 \\ 0 & 0 & 0.98019802 & 0 \\ 0 & 0 & 0 & 0.98019802 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 5.9406 \times 10^{-3} & 0 \\ 0 & 5.9406 \times 10^{-3} \end{bmatrix}. \quad (5.22)$$

### Simulation Setup

The complete simulation workflow is illustrated in figure 5.8.

We start with a continuous-time linear dynamic model, used directly by the simulator including visualization and the RK4 solver. For controller synthesis, the model is discretized using a bilinear (Tustin) transformation for time step  $dt$ .

**The controller requires weighting matrices  $Q$  and  $R$ , and the prediction and control horizons  $H_p$  and  $H_c$ .** For realism, we also include a maximum control limit  $u_{\max}$  to represent actuator saturation — since **real actuators are always limited**.

In our example we set:

- state weighting:  $Q = \text{diag}(10000, 10000, 0, 0)$ ,
- control weighting:  $R = \text{diag}(1, 1)$ ,
- prediction horizon:  $H_p = 64$ ,
- control horizon:  $H_c = 4$ ,
- output clamping :  $u_{\max} = 10$ .

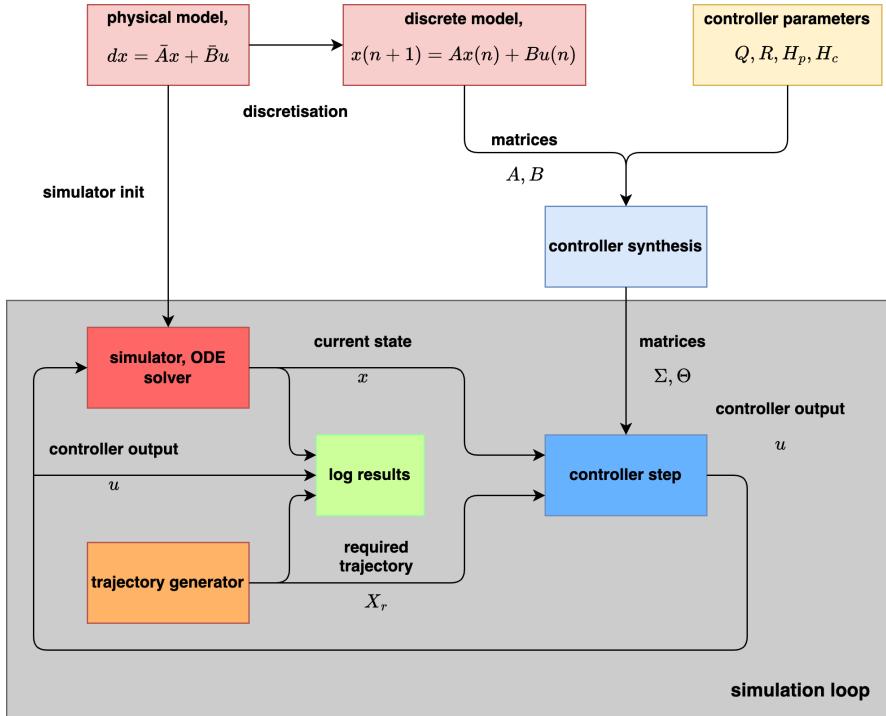


Figure 5.8: Simulation flow chart for MPC tracking.

These parameters fully define the MPC controller.

### Main Simulation Loop

The main loop executes the simulation in the following steps:

1. The required trajectory  $X_r$  is generated procedurally, assuming the current time step and a total horizon length  $H_p$ .
2. The current state  $x(n)$  is read from the simulator.
3. The controller computes the optimal control input  $u(n)$ .
4. The physical simulator performs one RK4 integration step, producing the next state  $x(n+1)$ .
5. Data for  $X_r$ ,  $x$ , and  $u$  are stored for later analysis and plotting.

### Simulation Results

The first result, shown in figure 5.9, presents a step response for an LQR controller used as a baseline. The controller output is plotted on first chart, labeled as **input X**. Note the output is saturated, and clamped into  $u_{max}$  value. In middle chart is plotted servo position labeled as **position X**, **red color** is plotted required value  $x_r$ , **cyan color** is plotted real observed system output. For completeness we plot also servo velocity on bottom chart, labeled as **velocity X**. This velocity is not controlled - matrix  $Q$  is giving zero weights for velocity terms, however, natural behaviour is, as soon position is at setpoint, velocity is zero. The controller begins acting only after the reference  $x_r$  changes from 0 to 1. LQR control is purely reactive — it acts based on the current error between  $x(n)$  and the instantaneous reference  $x_r(n)$ .

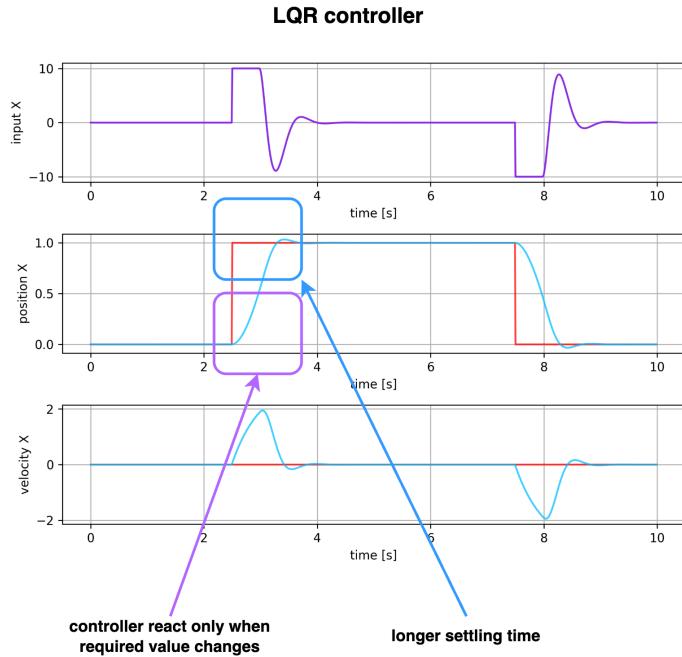


Figure 5.9: Step response for LQR control.

The MPC controller, on the other hand, receives not only the current reference but the **entire sequence** of future reference states  $x_r$  over the prediction horizon  $H_p = 64$ . This allows MPC to **anticipate** future motion and adjust its control signal proactively rather than reactively. The result

is smoother control effort, reduced overshoot, and more accurate trajectory tracking.

This behavior is shown in figure 5.10, where **MPC begins acting before the reference step occurs** — the control anticipates the upcoming change.

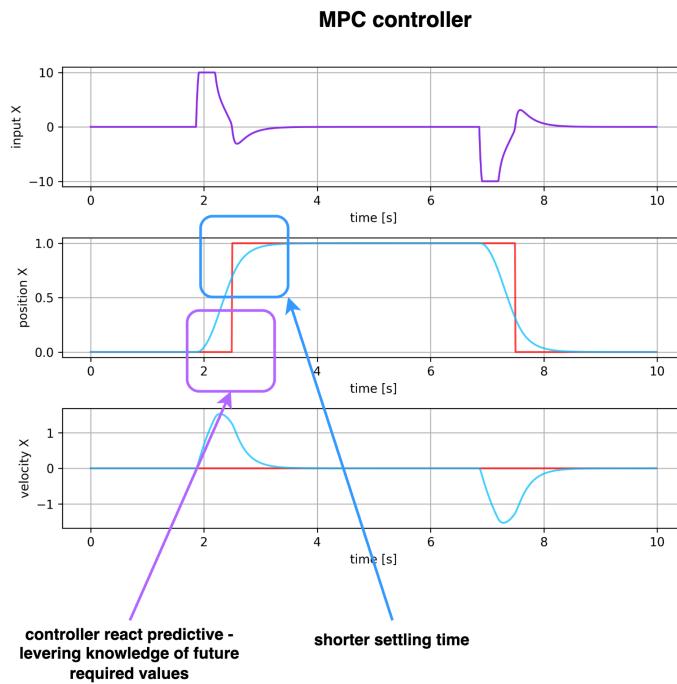


Figure 5.10: Step response for MPC control.

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### Discussion and Benefits

In summary, the example highlights several advantages of Model Predictive Control:

- **Predictive behavior:** MPC optimizes future control inputs using knowledge of the reference trajectory, rather than reacting only to current error.
- **Smooth actuation:** Because control inputs are planned ahead, actuator signals are smoother and exhibit less oscillation.

- **Constraint handling:** MPC naturally incorporates input or state constraints (such as actuator limits or safety zones) into the optimization.
- **Improved tracking performance:** MPC reduces steady-state error and overshoot while maintaining robustness against model imperfections.

Thus, even for a simple system, MPC demonstrates superior performance and provides a solid foundation for more complex multi-variable or nonlinear systems.