Chapter Two

Modeling in the Frequency Domain

Fourth Academic Year

Power, E/C, and C/C

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The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

The inverse Laplace transform is defined as

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

Laplace Transform Table: Table 2.1

Function	f(t)	F(s)
Unit impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Unit Ramp	t	$1/s^2$
Exponential	e^{-at}	$1/(s+a)$
Repeated Root	te^{-at}	$1/(s+a)^2$
Sine	$\sin(\omega t)$	$\omega/(s^2+\omega^2)$
Cosine	$\cos(\omega t)$	$s/(s^2+\omega^2)$
Polynomial	$t^n(n=1,2,3\ldots)$	$n!/s^{n+1}$
Damped sine	$e^{-at}\sin(\omega t)$	$\omega/((s+a)^2+\omega^2)$
Damped cosine	$e^{-at}\cos(\omega t)$	$(s+a)/((s+a)^2+\omega^2)$

Laplace Transform Theorem

Table 2.2 Laplace transfer theorem

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Example 2.1: Find the Laplace transform of the following function. Where u(t) is a unit step function

$$f(t) = Ae^{-at}u(t)$$

Answer

$$F(s) = \frac{A}{s+a}$$

Example 2.2: Find the inverse Laplace transform of

$$\boldsymbol{F(s) = \frac{s+3}{(s+1)(s+2)}}$$

Answer $\boldsymbol{f(t) = 2e^{-t} - e^{-2t}}$

Example 2.3: Find the inverse Laplace transform of

$$\boldsymbol{F(s) = \frac{2s+12}{s^2+2s+5}}$$

Answer $\boldsymbol{f(t) = 2e^{-t}sin\, 2t + 2e^{-t}cos\, 2t}$

Transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the a_i's, b_i's, and the form of the differential equation represent the system. Taking the Laplace transform of both sides,

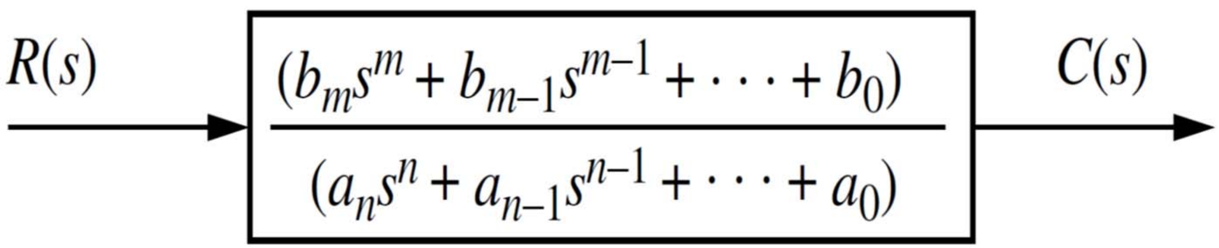
$$\begin{aligned} &a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) \\ &= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t) \end{aligned}$$

For simplicity, let assume all initial conditions are zero, thus

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

Now form the ratio of the output transform, C(s), divided by the input transform, R(s):

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}}$$



Block diagram of a transfer function

Example 2.4: Given the following differential equation, solve for y(t) if all initial conditions are zero. Use the Laplace transform

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Answer $y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$

Example 2.5: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Answer: Taking the Laplace transform of both sides, assuming zero initial conditions, lead to

$$sC(s) + 2C(s) = R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Example 2.6: Use the result of Example 2.5 to find the response, c(t) to an input, r(t) = u(t), a unit step, assuming zero initial conditions.

Answer: Since r(t) = u(t), thus, R(s) = 1/s, from table 2.1. Since the initial conditions are zero

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

Expanding by partial fraction,

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

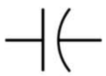
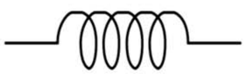
HW 2.1

Use Laplace transform for solving the following ordinary differential equation

$$2y'' + 3y' - 2y = t\mathbf{e}^{-2t}, \quad y(0) = 0 \quad y'(0) = -2$$

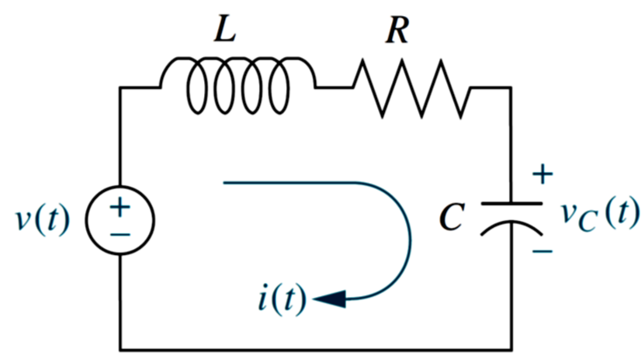
Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.

Table 2.3 Capacitor, resistor, and inductor transfer functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Example 2.7: Find the transfer function relating the capacitor voltage, Vc(s), to the input voltage, V(s) as shown in figure below



Assuming the voltages around the loop, assuming zero initial conditions, yields the integral-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Changing variables from current to charge using i(t) = dq(t)/dt yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

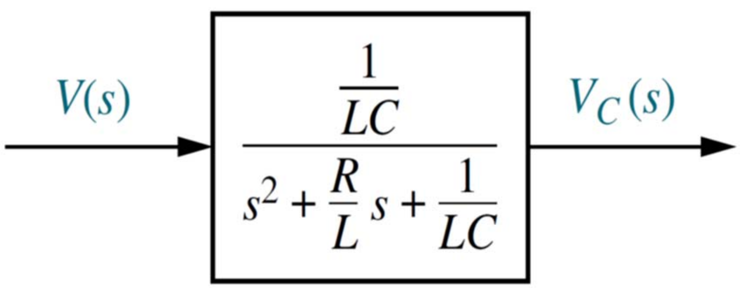
Where q(t) = CVc (t)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

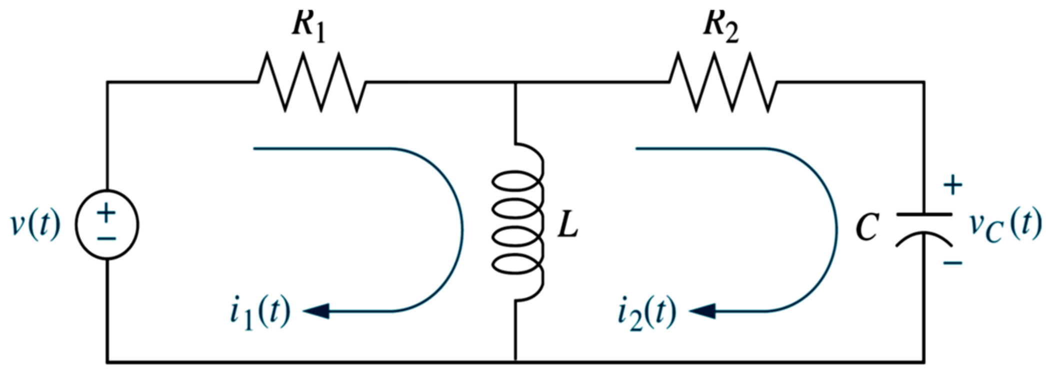
$$(LCs^2+RCs+1)V_C(s)=V(s)$$

Solving for the transfer function, Vc(s) / V(s), lea

$$\frac{V_C(s)}{V(s)}=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



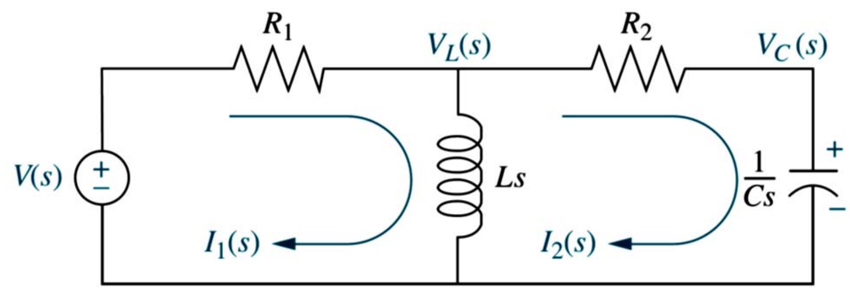
Example 2.8: Given the circuit in the following figure, find the transfer function, $I_2(s) / V(s)$.



Answer: The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, I1(s) and I2(s), flow. Around Mesh 1, where I1(s) flows,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Around Mesh 1, where $I_2(s)$ flows,



$$LsI_2(s) + R_2I_2(s) + \frac{1}{Cs}I_2(s) - LsI_1(s) = 0$$

Rearranging the simultaneous equations in $I_1(s)$ and $I_2(s)$:

$$\begin{aligned}(R_1 + Ls)I_1(s) \qquad\qquad &- LsI_2(s) = V(s)\\ -LsI_1(s) + &\left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0\end{aligned}$$

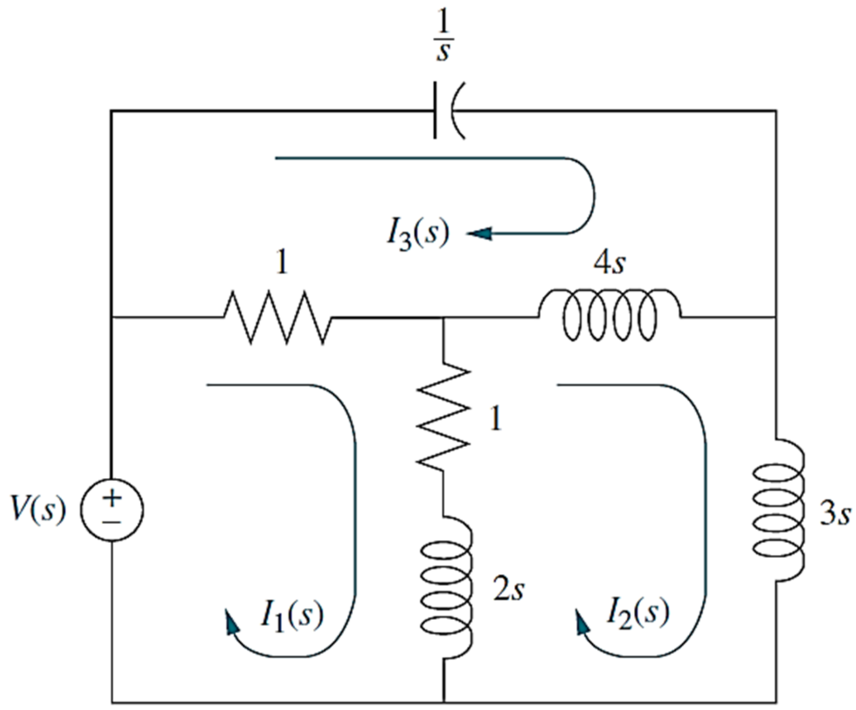
$$I_2(s) = \frac{\begin{vmatrix}(R_1 + Ls) & V(s)\\ -Ls & 0\end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

Using Cramer's rule (or any other method for solving simultaneous equations) to solve the above system equations for $I_2(s)$. Hence,

$$\Delta = \begin{vmatrix}(R_1 + Ls) & -Ls\\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right)\end{vmatrix}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Example 2.9: Write the mesh equations in Laplace transform



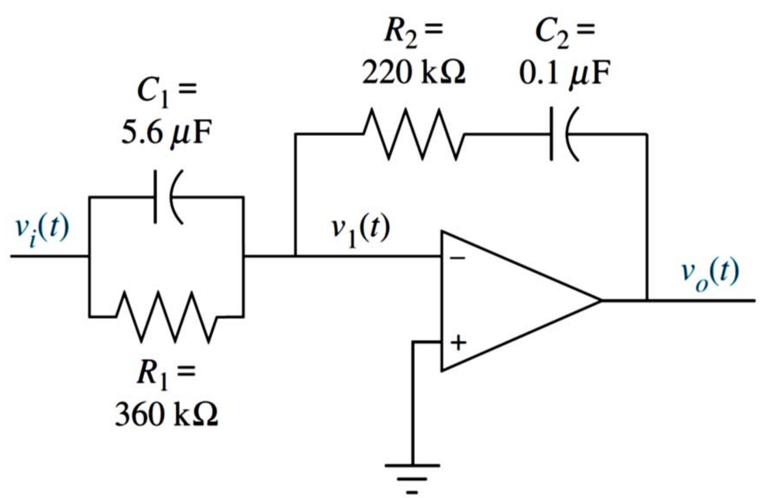
Answer

$$+(2s+2)I_1(s) - (2s+1)I_2(s) \qquad - I_3(s) = \mathrm{V}(s)$$

$$-(2s+1)I_1(s) + (9s+1)I_2(s) \qquad - 4sI_3(s) = 0$$

$$-I_1(s) \qquad - 4sI_2(s) + \left(4s+1+\tfrac{1}{s}\right)I_3(s) = 0$$

Example 2.10: Find the transfer function, Vo(s)/Vi(s), for the circuit given in the following figure. Assume large gain amplifier.



Answer

Knowing the characteristics of operational amplifies as

1. **Differential input, V2(t)-v1(t)**
2. **High input impedance, Zi = ∞ (ideal)**
3. **Low output impedance, Zo = 0 (ideal)**
4. **High constant gain amplification, A = ∞ (ideal)**

The output, vo(t), is given by : vo(t) = A(v2(t) – v1(t))

For large gain A, v1(t) ≈ 0. Since the admittances of parallel components add, Z 1(s) is the reciprocal of the sum of the admittance, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1}$$

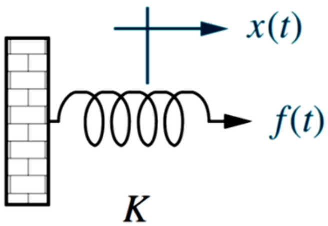
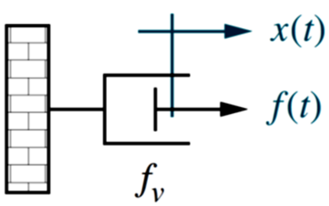
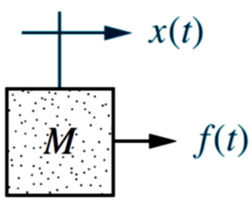
$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

And since, Vo(s)/Vi(s) = - Z2(s)/Z1(s), thus

$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s}$$

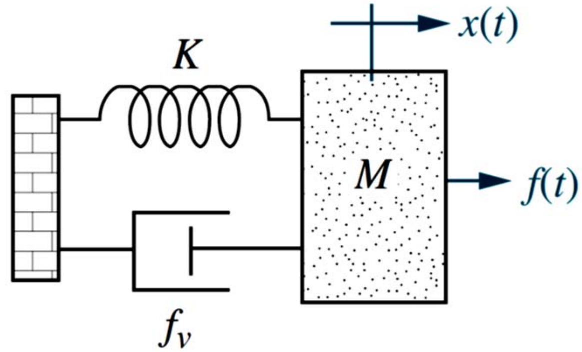
The resulting circuit is called a PID controller and can be used to improve the performance of a control system.

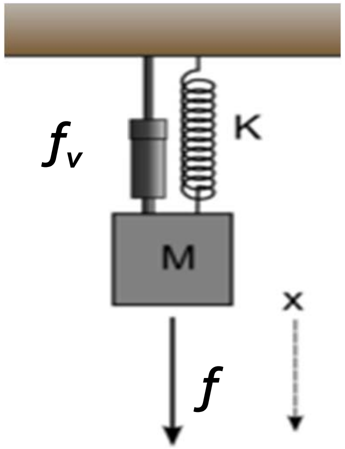
Table 2.4 Spring, damper, and mass transfer function

Component	**Force-velocity**	**Force-displacement**	**Impedence** $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t) = \mathrm{N}$ (newtons), $x(t) = \mathrm{m}$ (meters), $v(t) = \mathrm{m/s}$ (meters/second), $K = \mathrm{N/m}$ (newtons/meter), $f_v = \mathrm{N\text{-}s/m}$(newton-seconds/meter), $M = \mathrm{kg}$ (kilograms $=$ newton-seconds^2/meter).

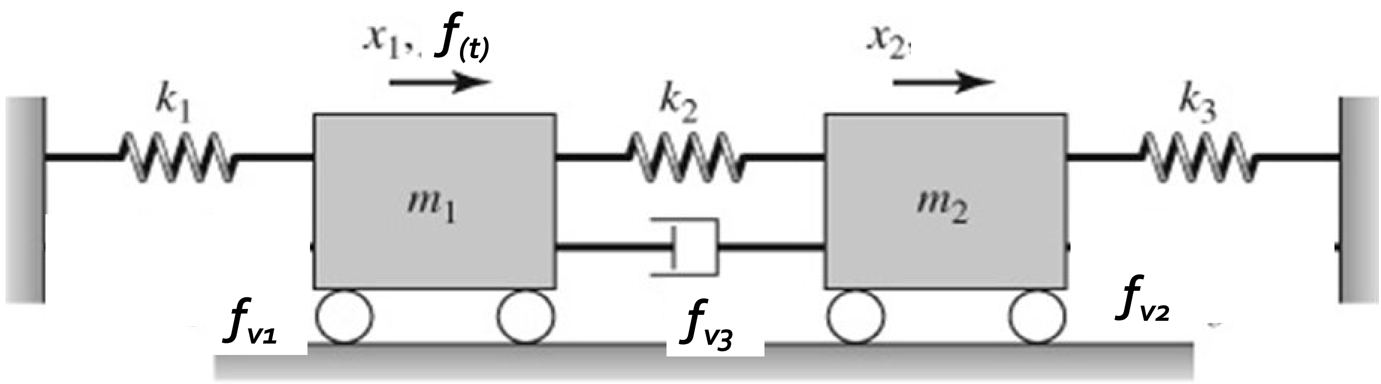
Example 2.11 Find the transfer function X(s)/F(s) for the following system.





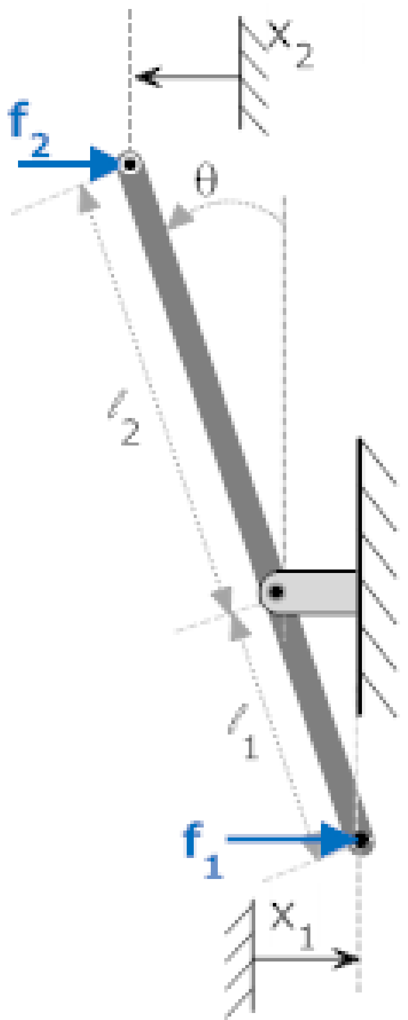
Answer (in the class)

Example 2.12 Find the transfer function $X_2(s)/F(s)$ for the following system.



Answer (in the class)

Lever



Geometric Relationships

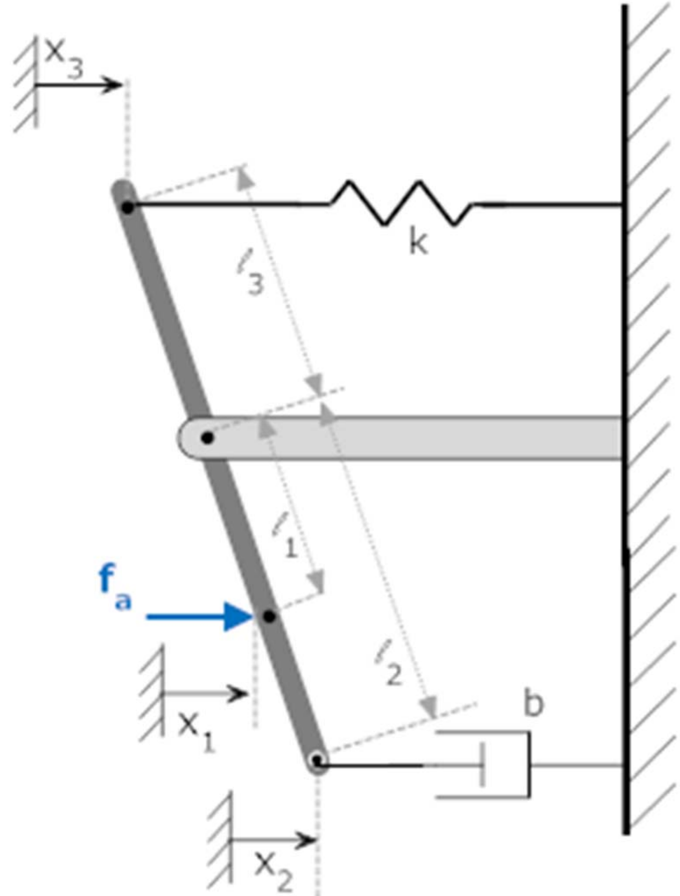
$$x_1 = \ell_1 \sin(\theta)$$

$$x_2 = \ell_2 \sin(\theta)$$

$$\frac{x_1}{x_2} = \frac{\ell_1}{\ell_2} = L, \quad \text{also } \frac{v_1}{v_2} = \frac{\ell_1}{\ell_2} = L$$

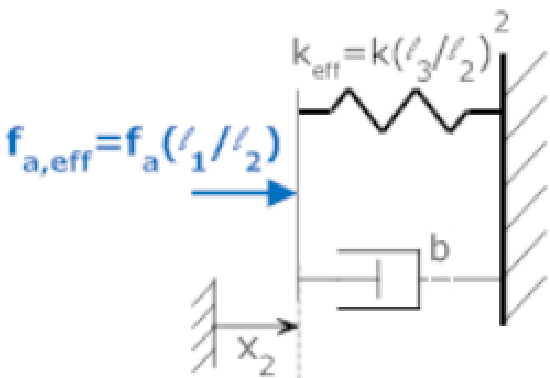
L is the ratio of lever arms.

Example 2.13 Develop an equivalent mechanical model at the position X_2.



Answer

Equivalent System



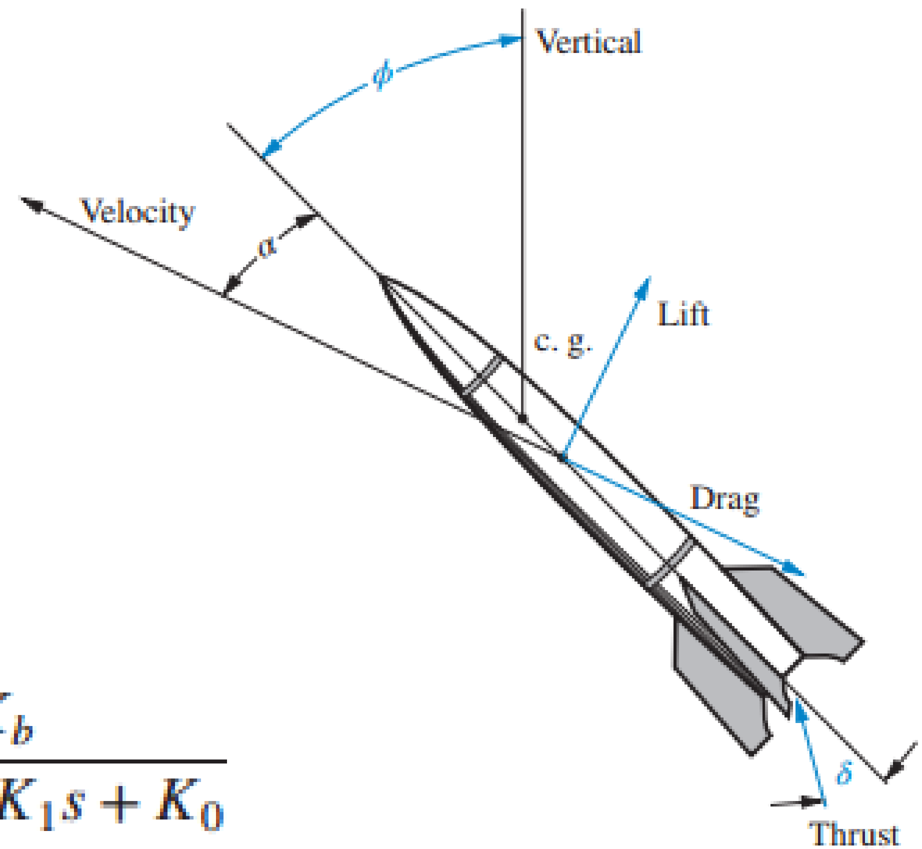
Equations of Motion

$$b\dot{x}_2 + k_{eff}x_2 = f_{a,eff}$$

$$b\dot{x}_2 + k\left(\frac{\ell_3}{\ell_2}\right)^2 x_2 = \left(\frac{\ell_1}{\ell_2}\right)f_a$$

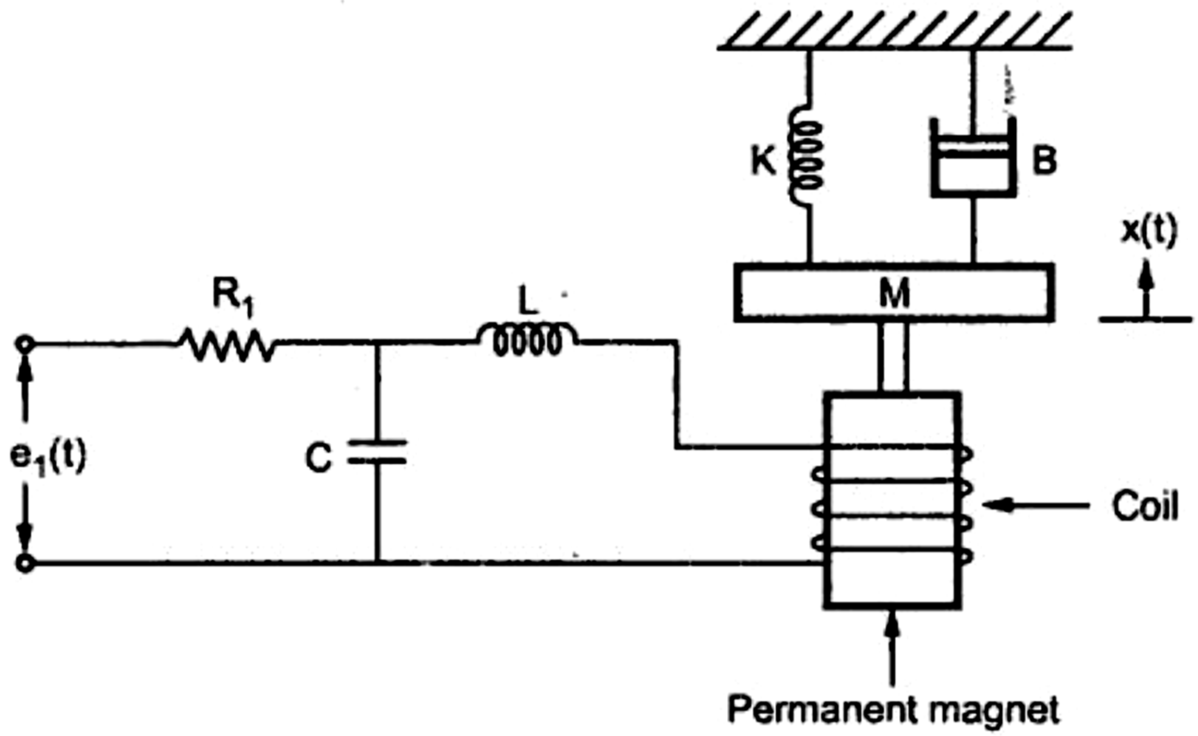
$$bv_2 + k\left(\frac{\ell_3}{\ell_2}\right)^2 x_2 = \left(\frac{\ell_1}{\ell_2}\right)f_a$$

A missile in flight is subject to four forces: thrust, lift, drag, and gravity. The missile flies at an angle of attack, α, from its longitudinal axis, creating lift. For steering, the body angle from vertical, ϕ, is controlled by rotating the engine at the tail. The transfer function relating the body angle, ϕ, to the angular displacement, δ, of the engine is of the form



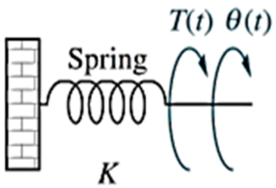
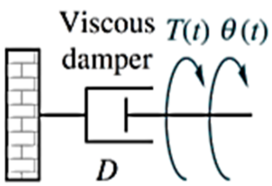
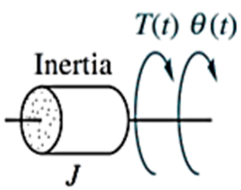
$$\frac{\Phi(s)}{\delta(s)} = \frac{K_a s + K_b}{K_3 s^3 + K_2 s^2 + K_1 s + K_0}$$

Example 2.14 Consider the system shown, R, L, and C are electrical parameters while K, M, and B are mechanical parameters as shown. Find the T.F. X(s)/E1(s) for the system where e1(t) is input voltage and x(t) is the output displacement.

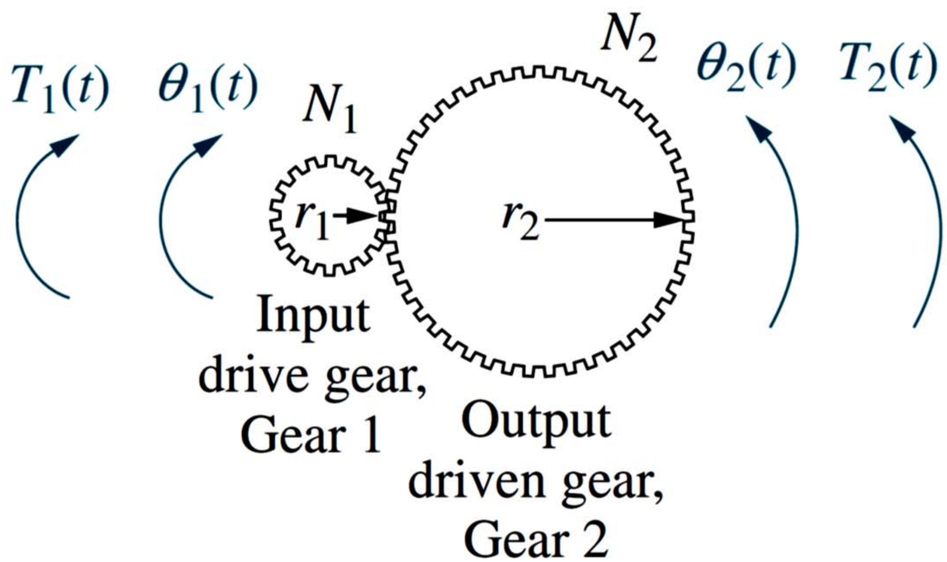


Answer (in the class)

Table 2.5 Torque-angular velocity, torque-angular displacement

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).



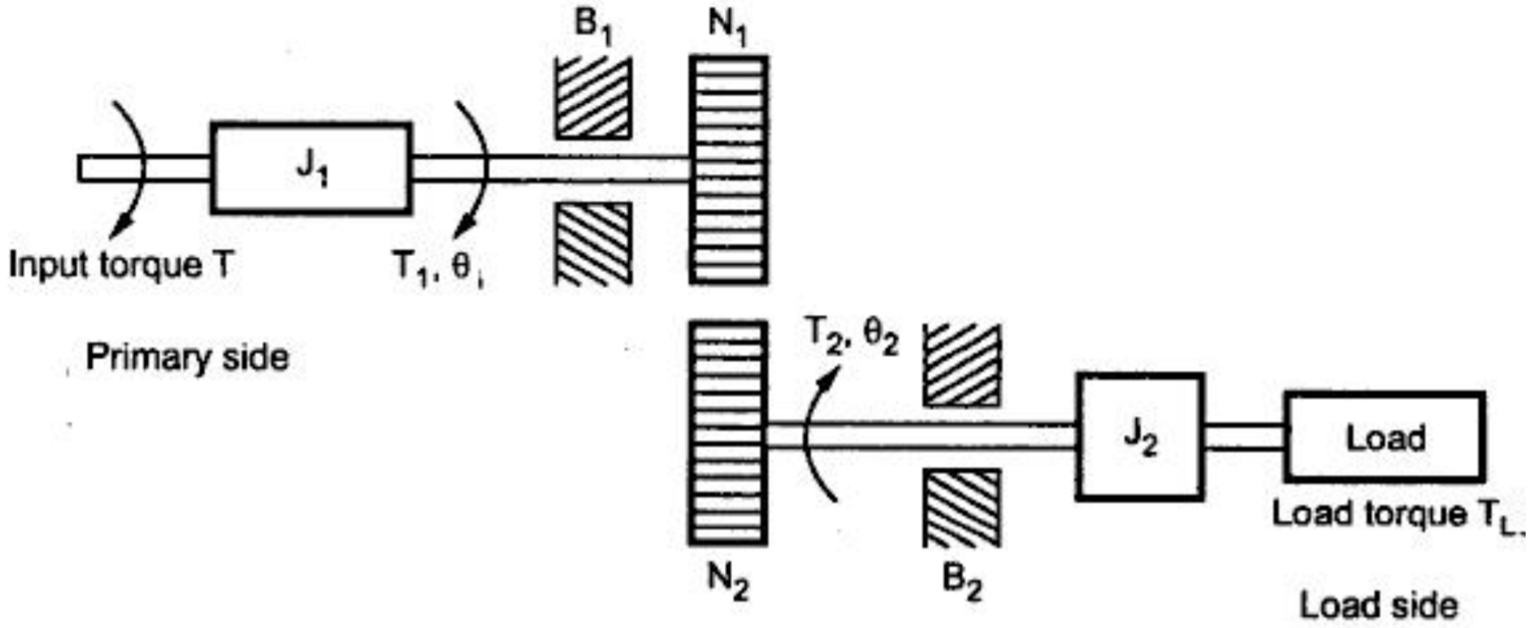
$$r_1\theta_1 = r_2\theta_2$$

$$\boxed{T_1\theta_1 = T_2\theta_2}$$

$$\boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$

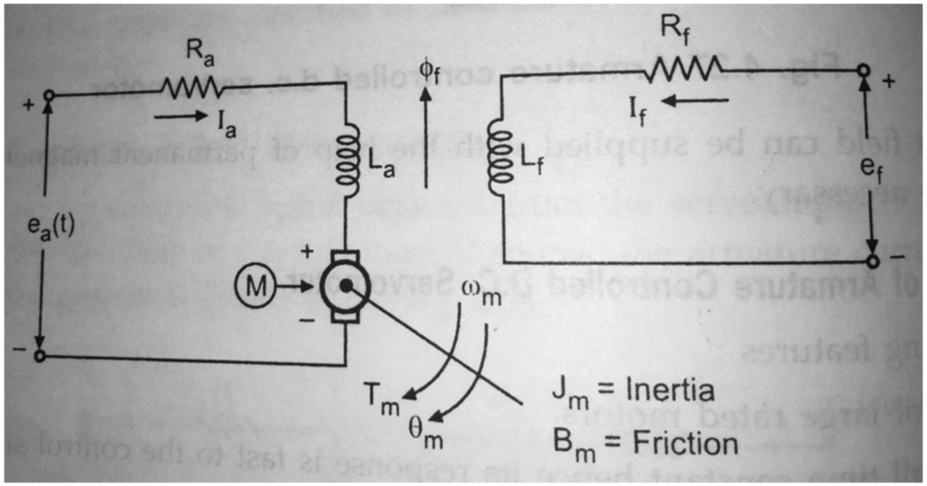
$$\boxed{\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}}$$

Example 2.15: Write the input torque equation to the following mechanical system.

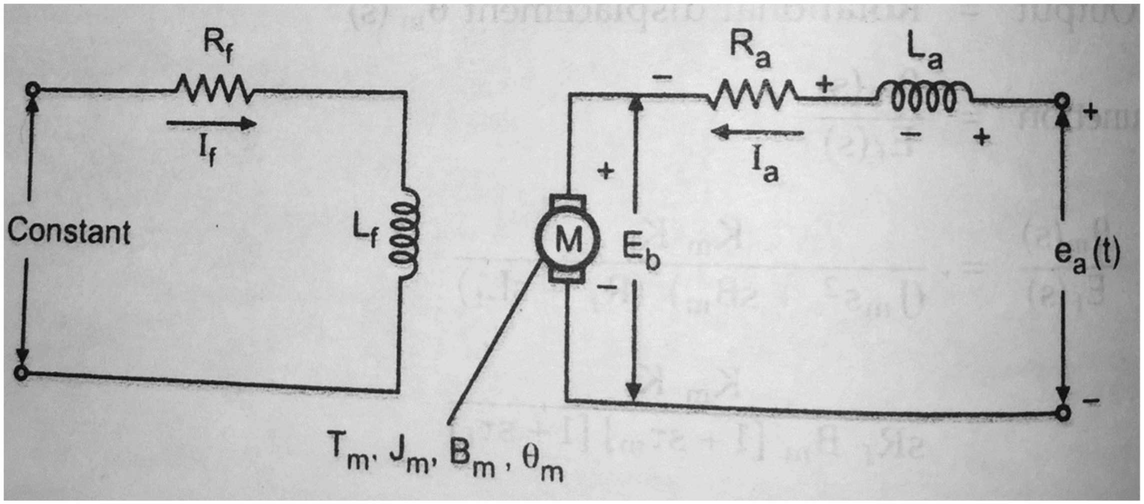


Answer (in the class)

Example 2.16: Find the T.F. $\theta_m(s)/Ef(s)$ of the following field controlled D.C. motor.

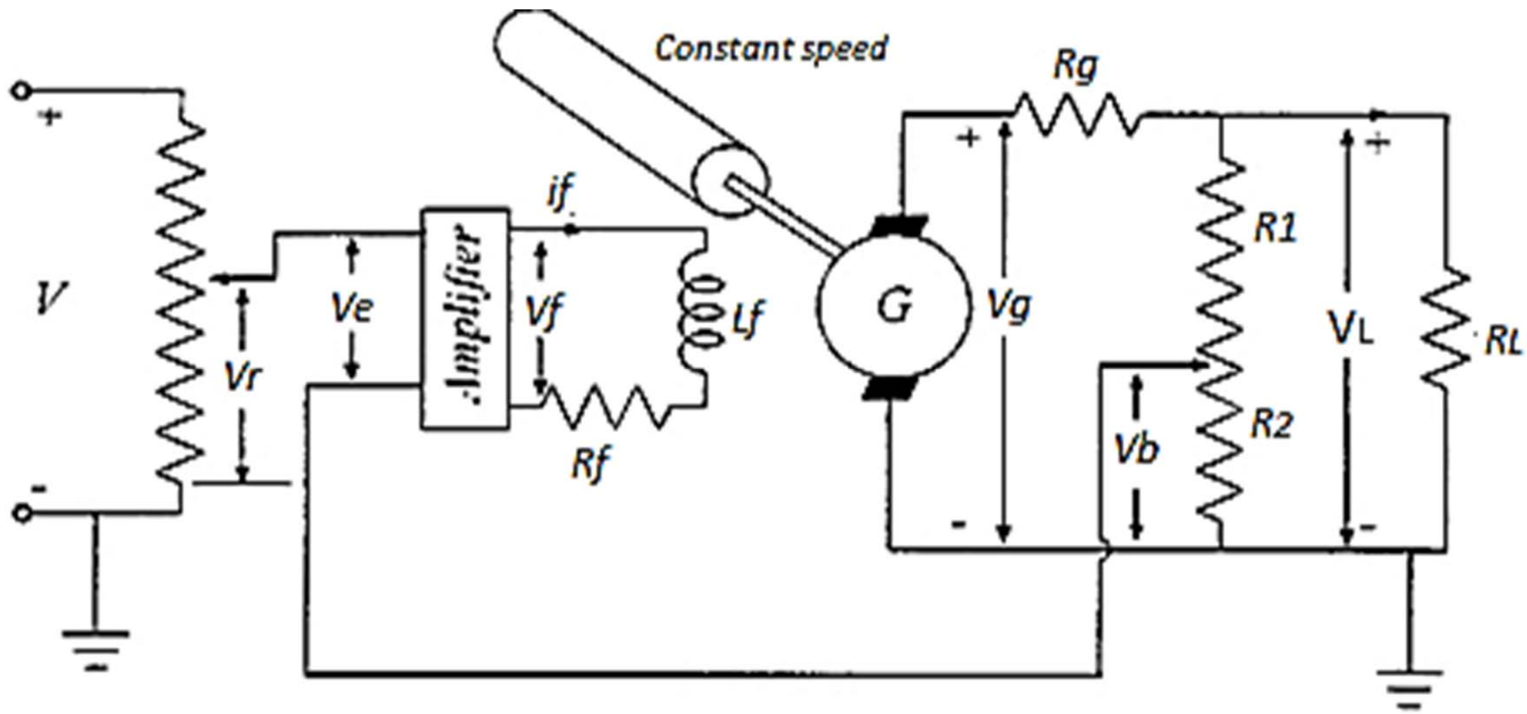


Example 2.17: Find the T.F. $\theta_m(s)/Ef(s)$ of the following armature controlled D.C. motor.



Answers (in the class)

<u>HW 2.2</u> A voltage control system is given in the following figure; draw the block diagram and the overall transfer function, $\frac{V_L(s)}{V_r(s)}$. Use K_a and K_g as constants for amplifier and generator, respectively.



End of Chapter Two!