Robust Control

Lecture 8

LQR, Kalman Filter, and LQG

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Consider a linear system characterized by $\dot{x} = Ax + Bu$

where (A, B) is stabilizable. We define the cost index as

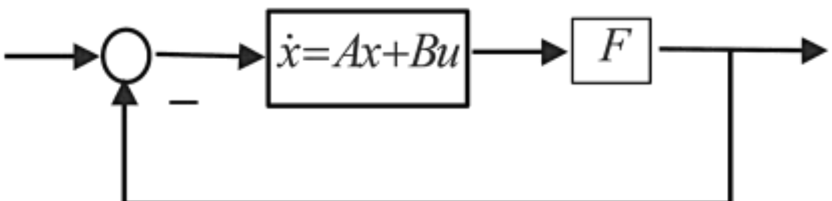
$$J(x,u,Q,R) = \int_0^\infty (x^T Q x + u^T R u)dt, \qquad Q \geq 0, \ R > 0$$

and (A, $Q^{1/2}$) is detectable. The linear quadratic regulation problem is to find a control law $u(t) = -Fx(t)$ such that $(A - BF)$ is stable and J is minimized.

The solution is given by: $F = R^{-1}B^T P$

with $$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

If we arrange the LQR control in the following block diagram:

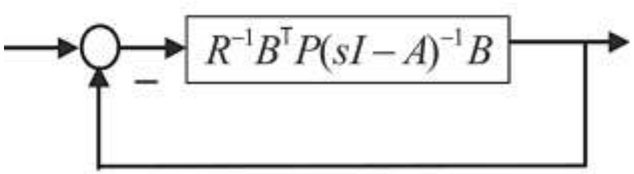


We can find its gain margin and phase margin as we have done in classical control.

It is clear that :

$$\textit{Open loop transfer function} = F(sI - A)^{-1}B = R^{-1}B^T P(sI - A)^{-1}B$$

The block diagram can be re-drawn as:



Consider the LQR control law.

The following so-called return difference equality hold:

$$R + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}Q(j\omega I - A)B = [I + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}]R[I + F(j\omega I - A)^{-1}B]$$

The following is called the return difference inequality:

$$[I + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}]R[I + F(j\omega I - A)^{-1}B] \geq R$$

In the single input case, the transfer function

Open loop transfer function $= \boldsymbol{f}(\boldsymbol{sI} - \boldsymbol{A})^{-1}\boldsymbol{b}$

is a scalar function.

Let Q $= \boldsymbol{hh^T}$**, then, the return difference equation is reduced to**

$$r + b^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1}hh^{\mathrm{T}}(j\omega I - A)b = r\,[1 + b^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1}f^{\mathrm{T}}][1 + f(j\omega I - A)^{-1}b]$$

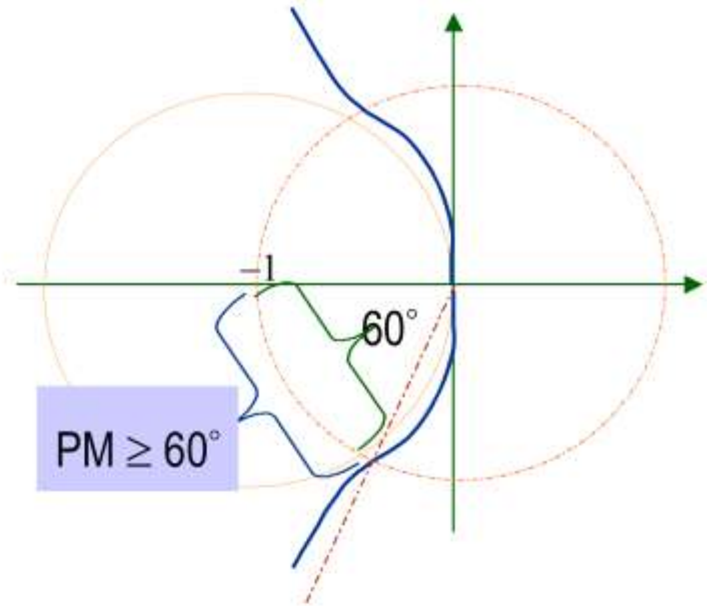
$$r + \left| h^{\mathrm{T}}(j\omega I - A)^{-1}b \right|^2 = r\left|1 + f(j\omega I - A)^{-1}b\right|^2$$

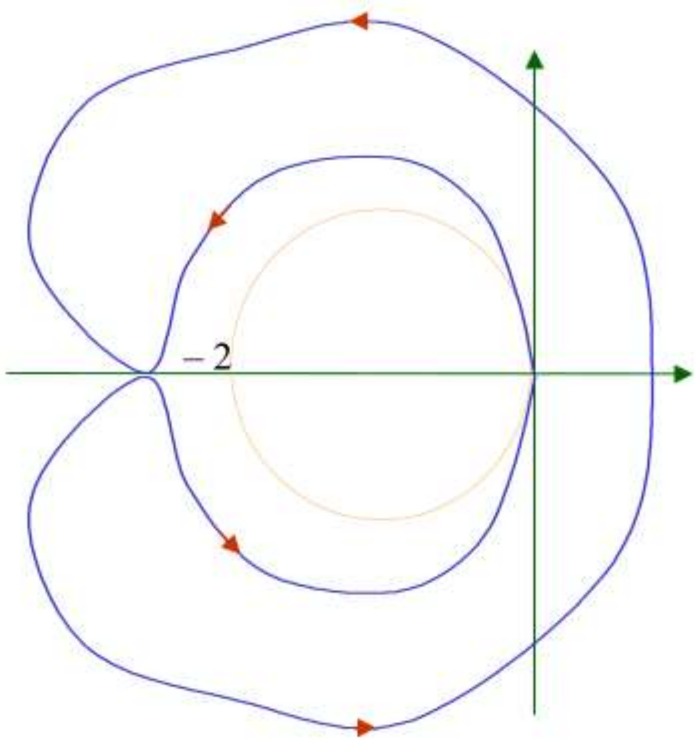
$$r\left|1 + f(j\omega I - A)^{-1}b\right|^2 \geq r$$

$$\left|1 + f(j\omega I - A)^{-1}b\right|^2 \geq 1$$ Return Difference Inequality

Graphically, $\left|1+f(j\omega I-A)^{-1}b\right|^2 \geq 1 \;\Rightarrow\; \left|f(j\omega I-A)^{-1}b-(-1+j0)\right| \geq 1$

implies that





Clearly, the phase margin resulting from the LQR design is at least 60 degrees.

The gain margin is from $[\,0.5, \infty\,)$.

Consider a given plant characterized by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u}$$

Solving the LQR problem which minimizes the following cost function:

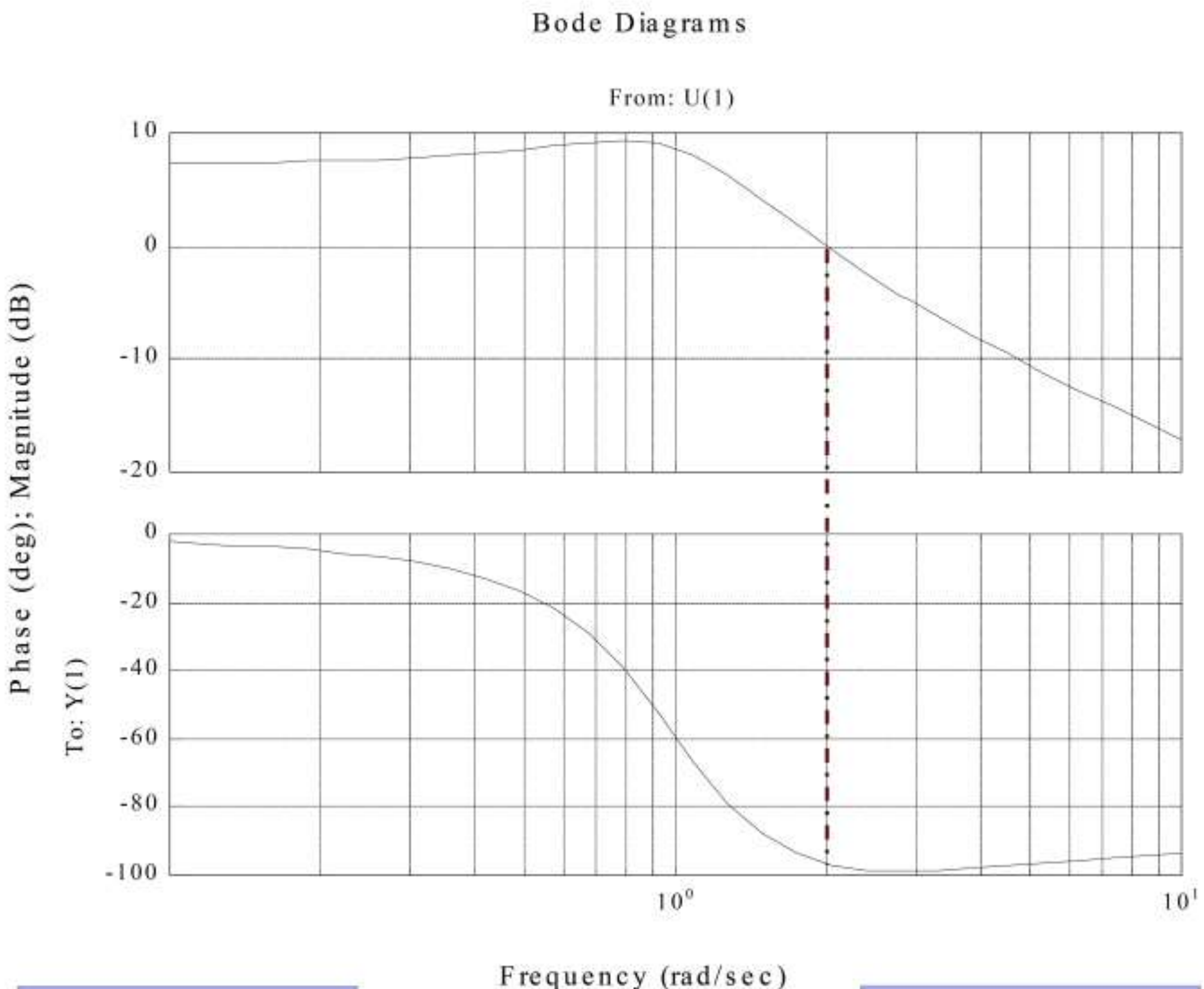
$$J(x,u,Q,R) = \int_0^\infty (x^T Q x + u^T R u)dt, \quad \text{with} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 0.1$$

we obtain,

$$P = \begin{bmatrix} 0.6872 & 0.2317 \\ 0.2317 & 0.1373 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 2.3166 & 1.3734 \end{bmatrix}$$

which results the closed-loop eigenvalues at $-\mathbf{1.1867} \pm \boldsymbol{j}\mathbf{1.3814}$.

Clearly, the closed-loop system is asymptotically stable.


Bode Diagrams
From: U(1)
Phase (deg); Magnitude (dB)
To: Y(1)
10
0
-10
-20
0
-20
-40
-60
-80
-100
10^0
10^1
Frequency (rad/sec)

PM = 84°

GM = ∞

Let f (x,t) be the probability density function (p.d.f.) associated with a random process X(t).

If the p.d.f. is independent of time t, i.e., f (x,t) = f (x), then the corresponding random process is said to be stationary.

For this type of random processes, we define:

1) **Mean (Exception):-** $m = E[X] = \int_{-\infty}^{\infty} x.f(x)dx$
2) **Moment (j**th **order moment):-** $E[X^j] = \int_{-\infty}^{\infty} x^j.f(x)dx$
3) **Variance :-** $\sigma^2 = E[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2.f(x)dx$
4) **Covariance of two random process:-** $\mathrm{con}(\mathrm{v},\mathrm{w}) = E[(v-E[v])(w-E[w])]$

Two random processes v and w are said to be independent if

$$E[vw] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} vw\, f(v,w)dvdw = 0$$

where $f(v,w)$ is the joint point p.d.f of v and w.

Autocorrelation function: is used to describe the time domain property of random process. Given a random process v, its autocorrelation function is defined as

$$R_x(t_1,t_2) = E[v(t_1)v(t_2)]$$

and when v is stationary $R_x(t_1,t_2) = R_x(t_1-t_2) = R_x(\tau) = R_x(t,t+\tau) = E[v(t)v(t+\tau)]$

Power spectrum of a random process: is the Fourier transform of its autocorrelation function:-

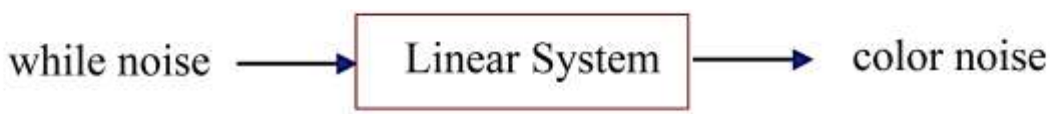
$$S_x(w) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} R_x(\tau) e^{j\omega t} d\tau$$

White Noise: is a random process with a constant power spectrum, and an autocorrelation function:

$$R_x(\tau) = q.\delta(\tau)$$

which implies that a white noise has an infinite power and thus it is non existent in our real life.

However, many noises (or the so-called color noises, or noises with finite energy and finite frequency components) can be modeled as the outputs of linear systems with an injection of white noise into their inputs, i.e., any color noise can be generated by a white noise



Gaussian Process v, is also known as normal process has a p.d.f.

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v-\mu)^2}{2\sigma^2}}$$

where μ = mean , σ^2 = variance

Consider a LTI system characterized by

$$\dot{x} = Ax + Bu + v(t)$$ **v is the input noise**

$$y = Cx + w(t)$$ **w is the measurement noise**

Assume:

1) **(A, C) is observable**
2) **v(t) and w(t) are independent white noise with the following properties**

$$E[v(t)] = 0\,, \qquad E[w(t)] = 0$$

$$E[v(t)v^T(\tau)] = Q\delta(t-\tau), \qquad Q = Q^T \geq 0$$

$$E[w(t)w^T(\tau)] = R\delta(t-\tau), \qquad R = R^T > 0$$

3) **$(A, Q^{1/2})$ is stabilizable (to guarantee closed-loop stability)**

The problem of Kalman Filter is to design a state estimator to estimate the state $x(t)$ by $\hat{x}(t)$ such that the estimation error covariance is minimized, i.e., the following index is minimized

$$J_e = E[\{x(t) - \hat{x}(t)\}^T \{x(t) - \hat{x}(t)\}]$$

Kalman filter is a state observe with a specially selected observer gain (or Kalman filter gain).

It has the dynamic equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \qquad \hat{x}(0) \text{ is given}$$

$$\hat{y} = C\hat{x}$$

with the Kalman filter K_e being given as

$$K_e = P_e C^{\mathrm{T}} R^{-1}$$

where P_e is the positive definite solution of the following Riccati equation,

$$P_e A^{\mathrm{T}} + AP_e - P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

Let $\mathrm{e} = x - \hat{x}$, therefore, the Kalman filter has the following properties:

$$\lim_{t\to\infty} E\left[e(t)\right] = \lim_{t\to\infty} E\left[x(t) - \hat{x}(t)\right] = 0$$

$$\lim_{t\to\infty} J_e = \lim_{t\to\infty} E[e^{\mathrm{T}}(t)\, e(t)] = \text{trace } P_e$$

Recall the optimal regulator problem:

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \text{ given}$$

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt, \quad Q = Q^T \geq 0 \text{ and } R = R^T > 0$$

The LQR problem is to find a state feedback law $u = -Fx$ such that J is minimized.

The solution to the above problem is given by:

$$F = R^{-1} B^T P \qquad \text{with} \qquad PA + A^T P - PBR^{-1} B^T P + Q = 0, \qquad P = P^T > 0$$

and the optimal value of J is given by $J = x_0^T P x_0$. Note that x_o is arbitrary.

Let consider a special case when x_o is a random vector with:

$$E[x_0] = 0, \qquad E\left[x_0 x_0^T\right] = I$$

Then ,

$$E[J] = E[x_0^T P x_0] = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_{0i} x_{0j}\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} E[x_{0i} x_{0j}] = \sum_{i=1}^{n} p_{ii} = \operatorname{trace} P$$

Linear Quadratic Regulator

$$F = R^{-1}B^{T}P$$

$$PA + A^{T}P - PBR^{-1}B^{T}P + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P$$

Kalman Filter

$$K_e = P_e C^{T} R^{-1}$$

$$P_e A^{T} + AP_e - P_e C^{T} R^{-1} C P_e + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P_e$$

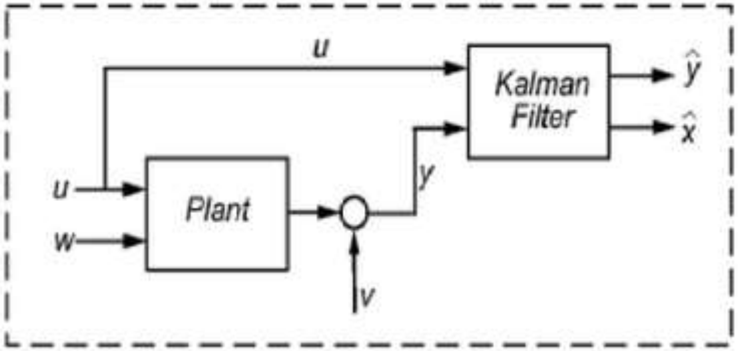
These two problems are equivalent (or dual) if we let

$$A^{T} \longleftrightarrow A$$

$$B^{T} \longleftrightarrow C$$

$$F^{T} \longleftrightarrow K_e$$

$$P \longleftrightarrow P_e$$



- **It is very often in control system design for a real life problem that one cannot measure all the state variables of the given plant.**
- **Thus, the linear quadratic regulator, although it has a very impressive gain and phase margins (GM = ∞ and PM = 60 degrees), is impractical as it utilizes all state variables in the feedback, i.e., u = – F x.**
- **In most of practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback.**

The natural questions one would ask:

- **Can we recover or estimate the state variables of the plant through the partially measurable information? The answer is yes. The solution is Kalman filter.**
- **Can we replace x the control law in LQR, i.e., u = – F x, by the estimated state to carry out a meaningful control system design? The answer is yes. The solution is called LQG.**
- **Do we still have impressive properties associated with LQG? The answer is no. Any solution? Yes. It is called loop transfer recovery (LTR) technique.**

Consider a given plant characterized by

$$\dot{x} = Ax + Bu + v(t) \qquad v \text{ is the input noise}$$

$$y = Cx + w(t) \qquad w \text{ is the measurement noise}$$

where v(t) and w(t) are the white with zero means. v(t), w(t), x(o) are independent, and

$$E[v(t)v^{\mathrm{T}}(\tau)] = Q_e\delta(t-\tau),\ Q_e \geq 0,\ E[w(t)w^{\mathrm{T}}(\tau)] = R_e\delta(t-\tau),\ R_e > 0,\quad E[x(0)] = x_0$$

The performance index has to be modified as follows:

$$J = \lim_{T\to\infty}\frac{1}{T}E\left[\int_0^T (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)dt\right], \qquad Q \geq 0,\ R > 0$$

The Linear Quadratic Gaussian (LQG) control is to design a control law that only requires the measurable information such that when it is applied to the given plant, the overall system is stable and the performance index is minimized.

Step 1: Design an LQR control law $u = -Fx$ which solves the following problem:

$$\dot{x} = Ax + Bu \qquad J(x,u,Q,R) = \int_0^\infty (x^T Q x + u^T R u)dt, \qquad Q \geq 0,\ R > 0$$

i.e., compute

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \quad P > 0, \qquad F = R^{-1}B^T P.$$

Step 2: Design a Kalman filter for the given, i.e.

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

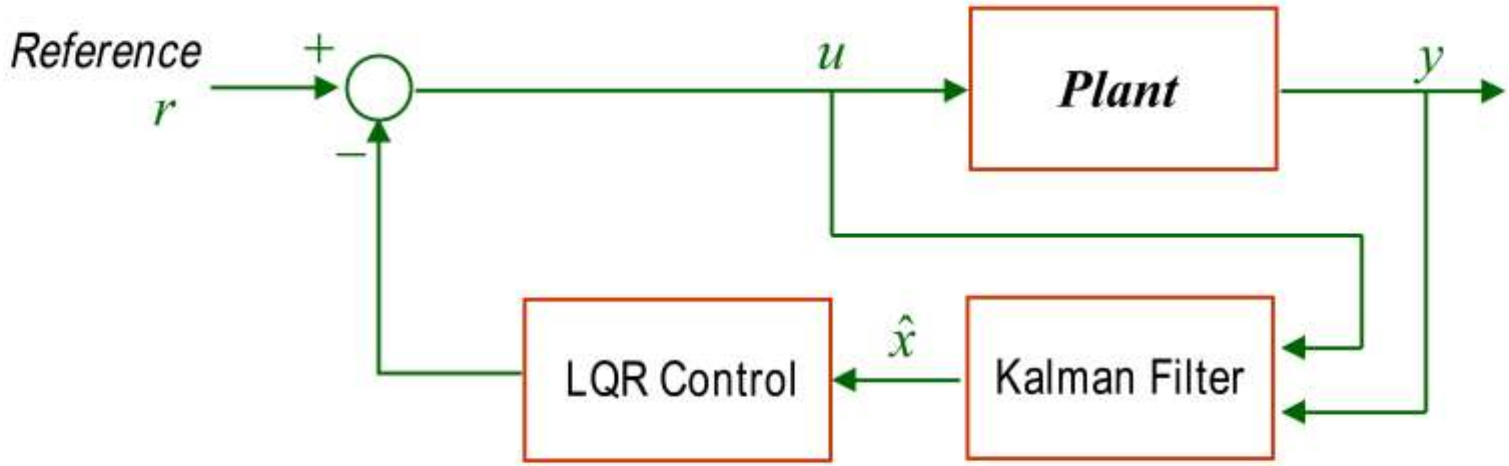
where

$$K_e = P_e C^T R^{-1}, \qquad P_e A^T + AP_e - P_e C^T R_e^{-1} C P_e + Q_e = 0, \quad P_e > 0.$$

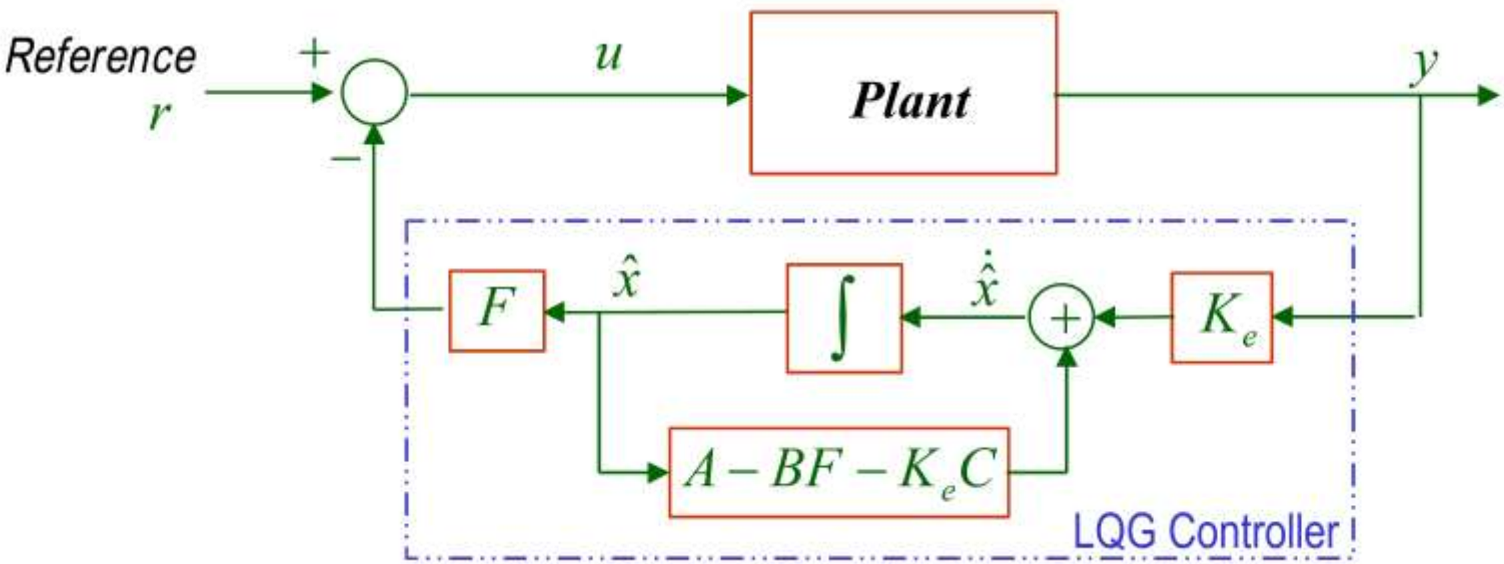
Step 3: The LQG control law is given by $u = -F\,\hat{x}$, i.e.

$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u + K_e(y - C\,\hat{x}) \\ u = -F\,\hat{x} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} + K_e\,y \\ u = -F\,\hat{x} \end{cases}}$$

Block Diagram Implementation of LQG Control Law



More Detailed Block Diagram



Recall the plant
$$\dot{x} = Ax + Bu + v(t)$$
$$y = Cx + w(t)$$

and the controller
$$\dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} + K_e\, y$$
$$u = -F\,\hat{x} + r$$

We define a new variable $e = x - \hat{x}$ and thus

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax - BF\hat{x} + Br + v(t) - A\hat{x} + BF\hat{x} + K_e C\hat{x} - K_e Cx - K_e w(t)$$
$$= A(x - \hat{x}) - K_e C(x - \hat{x}) + Br + v(t) - K_e w(t) = (A - K_e C)e + Br + v(t) - K_e w(t)$$

and
$$\dot{x} = Ax + Bu + v(t) = Ax - BF\hat{x} + Br + v(t) = Ax - BF(x - e) + Br + v(t)$$
$$= (A - BF)x + BFe + Br + v(t)$$

Clearly, the closed-loop system is characterized by the following state space equation,

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - K_e C \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r + \tilde{v}, \quad \tilde{v} = \begin{pmatrix} v \\ v - K_e w \end{pmatrix}$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + w$$

The closed-loop poles are given by $\lambda(A - BF) \cup \lambda(A - K_e C)$, which are stable.

Is an LQG Controller Robust?

- **It is now well-known that the linear quadratic regulator (LQR) has very impressive robustness properties, including guaranteed infinite gain margins and 60 degrees phase margins in all channels.**
- **The result is only valid, however, for the full state feedback case.**
- **If observers or Kalman filters (i.e., LQG regulators) are used in implementation, no guaranteed robustness properties hold.**
- **Still worse, the closed-loop system may become unstable if you do not design the observer of Kalman filter properly.**

End of Lecture 8!