Understanding the Tangent Function

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1 Definition

To define the *tangent* function, one must understand the basic principles of deriving *sine* and *cosine* using a so-called **unit circle**.

A **unit circle** is the circle of radius 1 centred at the origin $O(0,0)$ in the **Cartesian coordinate system**.

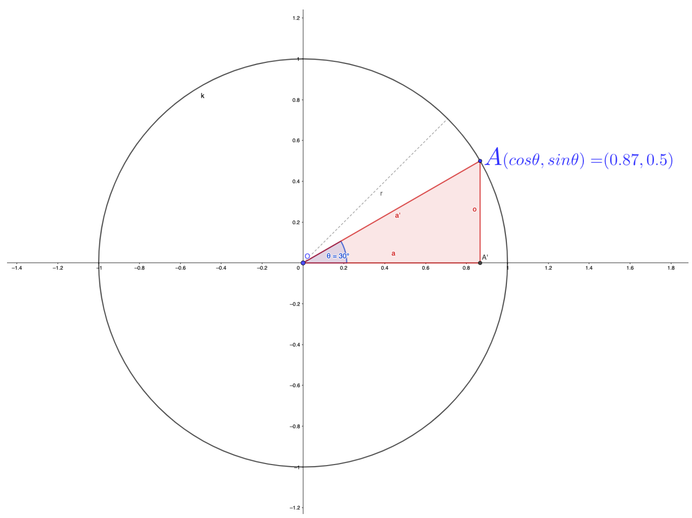

Figure 1: Unit circle

Then for a point $A = [x, y]$ moved along the circumference k by an angle θ is valid:

$$A = [cos(\theta), sin(\theta)]$$

Note: an applet is available (link) or scan its ***QR code*** *under* ***Resources****.*

A right-angled triangle consists of the **hypotenuse** (the side opposing the right angle) and 2 other adjacent sides called **legs**. The definition of tangent states that it is the quotient of the opposite side and the adjacent side to a certain angle θ, symbolically:

$$tan(\theta) = \frac{opposite}{adjacent}$$

Which can be further transformed into:

$$tan(\theta) = \frac{y}{x} ... tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$$

2 Properties

Now, let's address the properties of $f : y = tan(x)$

Graph:

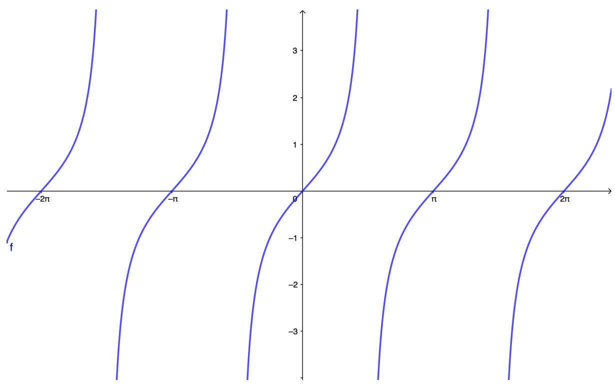

Figure 2: $y = tan(x)$

Properties:

1. **Domain**

$$f : y = tan(x) ... y = \frac{sin(x)}{cos(x)} \Leftrightarrow cos(x) \neq 0$$

Let $f : y = cos(x)$, solve for $x \in \mathbb{R}$.

$$x \in \left\{ \frac{\pi}{2}; \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z} \right\} \Leftrightarrow x \in \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}$$

Through additional steps, we obtain the following result:

$$x \in \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$$

In other words, the **Domain** of $y = tan(x)$ is all real numbers except every x for which is $sin(x) = 0$ defined. Symbolically:

$$D_f = \mathbb{R} - \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$$

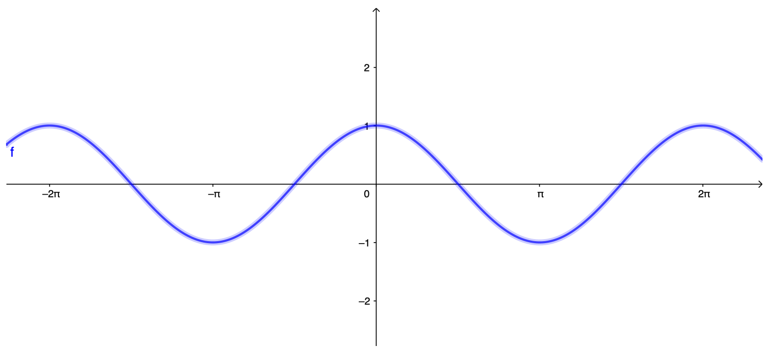

Figure 3: $y = cos(x)$

2. **Range of values**

   As shown in the figure above, the **Range** of $f : y = tan(x)$:

$$H_f = \mathbb{R}$$

3. **Values in quadrants**

I.	*II.*	*III*	*IV*
$\left(0; \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}; \pi\right)$	$\left(\pi; \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}; 2\pi\right)$
+	-	+	-

4. **Periodicity**

   Suppose that the *period* of $y = tan(x)$ is 2π:

$$tan(x + 2k\pi) = \frac{sin(x + 2k\pi)}{cos(x + 2k\pi)} = \frac{sin(x)}{cos(x)} = tan(x)$$

However, 2π is not the smallest possible period. Let us consider a case where *period* is π.

$$tan(x + \pi) = \frac{sin(x + \pi)}{cos(x + \pi)} = \frac{-sin(x)}{-cos(x)} = tan(x)$$

Explanation: The *period* of $f : y = sin(x)$ and $f : y = cos(x)$ is 2π, any $x \in D_f$ shifted by the period of π will produce a **function value** for which is valid: $\forall x \in D_f \subset \mathbb{R} : f(x + \pi) = -f(x)$.

Let A' be a point shifted along the circumference k by an angle θ, A'_1 be a point shifted along the same circumference by an angle $\theta + \pi$. Their coordinates are depicted using a **unit circle** and are symmetrical through $O[0, 0]$ (point symmetry). Notice:

$$A'_x = -A'_{x_1} \wedge A'_y = -A'_{y_1}$$

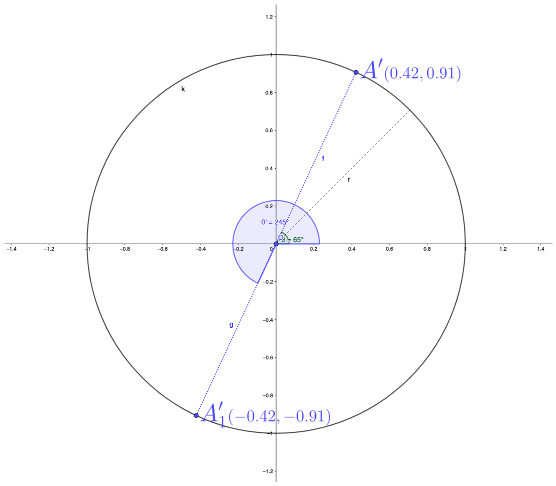

Figure 4: Periodicity of tangent

Note: an applet is available (link) or scan its ***QR code*** *under* ***Resources****.*

Conclusion Function **tangent** is **periodic** with the smallest possible period π.

5. **Parity**

$$f^{+}(x) : y = tan(x)$$

$$f^{-}(-x) : y = tan(-x) = \frac{sin(-x)}{cos(-x)} = \frac{-sin(x)}{cos(x)} = -\frac{sin(x)}{cos(x)} = -tan(x),$$

Based on the parity of **sine** (odd), **cosine** (even) $\Rightarrow sin(-x) = -sin(x) \wedge cos(-x) = cos(x)$.

$f(-x) = -f(x) \Rightarrow$ Tangent is an **odd function**.

6. **Monotonicity**

The tangent function is **increasing** on $\mathbb{R}$ between 2 consecutive vertical **asymptotes**, where every asymptote is defined as: $x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$, that is:

$$\forall x_{1,2} \in D_f : tan(x_1) < tan(x_2) \wedge x_1 < x_2$$

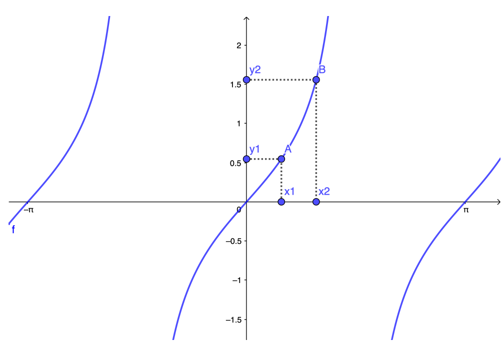

Figure 5: Monotonicity of tangent

7. **Graph**

   Tangentoid is the plot of the tangent function.

8. **Important function values of tangent**

$x:$	$0°$	$30°$	$45°$	$60°$	$90°$
$tan(x):$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\emptyset$

3 Resources

Applet 1, Applet 2

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