Super-factorials - Advanced methods of factorials

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0. **Standard** factorial

Suppose $n!$ that implies:

$$n! = n \times (n-1) \times (n-2) \ldots 2 \times 1$$

More generally:

$$n! = \prod_{k=1}^{n} n$$

1. **Double** factorial

The parity of n affects the series. Suppose $n!!$, then:

$$n!! = \begin{cases} n \times (n-2) \ldots 4 \times 2 | even \\ n \times (n-2) \ldots 3 \times 1 | odd \end{cases}$$

Similarly expressed as:

$$n!! = \begin{cases} \prod_{k=1}^{\frac{n}{2}} (2 \times k) | even \\ \prod_{k=1}^{\frac{n+1}{2}} (2 \times k - 1) | odd \end{cases}$$

2. **Subfactorial**

A subfactorial of n, denoted as $!n$, expresses the number of de-arrangements of a set with n terms.

The following holds:

$$!n = n! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} \times \cdots \times (-1)^n \times \frac{1}{n!}\right)$$

3. **Primorial**

Suppose a **primorial** of n, denoted as $n\#$, which express the series of prime numbers p:

$$n\# = \prod_{p \leq n} p$$

4. **Super** factorial (the *Sloane* definition)

Suppose a **super** factorial of n, denoted as $sf(n)$, which expresses the series of all $k!$, such that $k \leq n$. That is:

$$sf(n) = n! \times (n-1)! \times (n-2)! \times \cdots \times 2! \times 1!$$

Which is similarly expressed in the form:

$$sf(n) = \prod_{k=1}^{n} k!$$

5. **Exponential** factorial (also know as the **hyperpower**)

As the name suggests, the **exponential** factorial of n, denoted as $n\$$, expresses n raised to the power of $(n-1)$ that is raised to the power of $(n-2)$ and so forth. This relation is also defined with a **recurrent sequence**, such that $a_n = n^{a_{n-1}} \wedge a_0 = 1$. For instance, the **exponential** factorial of 5 is $5\$ = 5^{4^{3^{2^{1}}}}$.

6. **Hyper** factorial

A **hyper** factorial of n, denoted as $H(n)$, expresses the following series:

$$H(n) = n^n \times (n-1)^{n-1} \times (n-2)^{n-2} \times \cdots \times 2^2 \times 1^1$$

Which is similarly expressed in the following form:

$$H(n) = \prod_{k=0}^{n} (n-k)^{n-k}$$
