Mathematical proofs

Michal Špano

March 2022

1 The inner angle of an $n-sided$ convex regular polygon

Suppose an $n-sided$ convex regular polygon. Its 4 consecutive vertices are shown in the figures, $N_1, N_2, N_3, ..., N_i$ respectively. Thus $\Delta_{N_1,N_2,S} \cong \Delta_{N_2,N_3,S}$, i.e. such $n-sided$ polygon consists of n congruent isosceles triangles. That is, $|\measuredangle N_2N_1S| = |\measuredangle N_1N_2S| \Rightarrow |\measuredangle N_1N_2S| = |\measuredangle SN_2N_3|$, denoted as β, β' respectively.

An angle $\alpha = |\measuredangle N_1SN_2| = \dfrac{360}{n}$, since such an angle multiplied by n makes for a perfect circle of $360°$. Likewise $\alpha = 180° - 2\beta$.

Let Φ be an inner angle of the polygon, such that $\phi = 2\beta$ (shown in the figure at the vertex N_2).

Express in terms of β: $\alpha = 180 - 2\beta \iff \beta = \dfrac{-\alpha + 180}{2}$

Substitute α for $\alpha = \dfrac{360}{n}$: $\beta = \dfrac{-\dfrac{360}{n} + 180}{2} \iff \beta = \dfrac{180n - 360}{2n}$

Express in terms of Φ: $\Phi = 2\beta = 2\left(\dfrac{180n - 360}{2n}\right) = \dfrac{180n - 360}{n}$

The expression can be further simplified to the following form:

$$\Phi = \frac{(n-2)\pi}{n}$$

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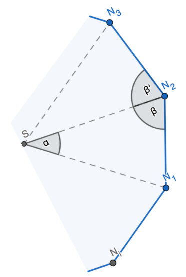

Figure 1: $n-sided$ polygon with its 4 vertices in a plane

2 The number of diagonals of an $n-sided$ convex regular polygon

Suppose a geometrical locus of n points on the plane, i.e. a set of points $A_1, A_2, ..., A_n$ such that $A_1, A_2, ..., A_n$ create an $n-sided$ convex regular polygon. The number of different abscissas in the geometrical locus is $\binom{n}{2}$ and denoted as N_a. It implies that no abscissa in the locus is given by more than two points, i.e. each point is unique. Likewise, $\overleftrightarrow{A_1A_2} \equiv \overleftrightarrow{A_2A_1}$ holds for any 2 points in the locus, thus such abscissas are counted as one. It can be inferred that the number of diagonals, denoted as N_D, is the same as *the difference of the number of abscissas and the number of sides*:

$$N_D = N_s - n$$

3 The similarity coefficient

Suppose a scalene triangle Δ_{ABC}. We assume that there exists a triangle $\Delta_{A'B'C'}$, such that $\Delta_{ABC} \cong \Delta_{A'B'C'}$. It implies that there exists some constant c, such that any abscissa created in the locus of points (from the original triangle) is equal to the product of c and the corresponding abscissa of the similar triangle. Symbolically:

$\exists! c \in \mathbb{Q} : |V_1V_2| = c \times |V_1'V_2'| \wedge c \gneqq 1$, where V_1, V_2 are the vertices of the original triangle.

Then, it can be easily inferred, that for the **perimeters** of the triangles holds the following: Let $P = \overleftrightarrow{AB} + \overleftrightarrow{BC} + \overleftrightarrow{AC}$, similarly $P' = \overleftrightarrow{A'B'} + \overleftrightarrow{B'C'} + \overleftrightarrow{A'C'}$. Likewise $\overleftrightarrow{AB} = c \times \overleftrightarrow{A'B'} \wedge \overleftrightarrow{BC} = c \times \overleftrightarrow{B'C'} \wedge \overleftrightarrow{AC} = c \times \overleftrightarrow{A'C'}$.

$$P = c \times \overleftrightarrow{A'B'} + c \times \overleftrightarrow{B'C'} + c \times \overleftrightarrow{A'C'} \Rightarrow P = c \times \left(\overleftrightarrow{A'B'} + \overleftrightarrow{B'C'} + \overleftrightarrow{A'C'}\right) = c \times P'$$

$$P = c \times P'$$

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It implies that the ratio of the lengths of the sides of the similar triangles, i.e. the ratio of their perimeters, is equal to the similarity coefficient: $c = \dfrac{P}{P'}$.

Similar methodology is applied to the similarity coefficient of the areas of the triangles:

Let $A = \dfrac{\overleftrightarrow{AB} \times \overleftrightarrow{H_{AB}}}{2} \wedge A' = \dfrac{\overleftrightarrow{A'B'} \times \overleftrightarrow{H'_{AB}}}{2}$, where H_{AB} and H'_{AB} are the heights of the triangles. Thus, $A = \dfrac{c \times \overleftrightarrow{A'B'} c \times \overleftrightarrow{H'_{AB}}}{2} = c^2 \times \left(\dfrac{\overleftrightarrow{A'B'} \times \overleftrightarrow{H'_{AB}}}{2}\right) = c^2 \times A'$

$$A = c^2 \times A'$$

It implies, that the ratio of the areas equals to the similarity coefficient taken to the second power: $c^2 = \dfrac{A}{A'}$.

To sum up, such a methodology is also applicable to derive the similarity coefficient of the volume of the tetrahedron. Moreover, it is also applicable to derive the similarity coefficient of any planar polygon or solid figure (a scalene triangle per the given example is just an exemplary instance). Still, the following holds: $P = c \times P' \wedge A = c^2 \times A' \wedge V = c^3 \times V'$ based on the principles of **congruence**.

4 Function bounded above and/or below

Suppose a function f which has a finite domain, say D_f. A function is bounded above if: $\exists! a \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a$. That is to say, that every function value is smaller or equal to some constant, say a. Similarly, a function is bounded below if: $\exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \geq b$. That implies, that every function value is greater or equal to some constant, say b. Lastly, a function is bounded if both of the previous statements hold, i.e.: $\exists! a \in \mathbb{R}, \exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a \wedge f(x) \geq b$. Hence, some bounded function f has the following range of values: $[b; a]$, where $\{a, b\} \in \mathbb{R}$. For example, the function $f(x) = sin(x)$ is bounded, such that $[-1; 1]$ is its range of values.

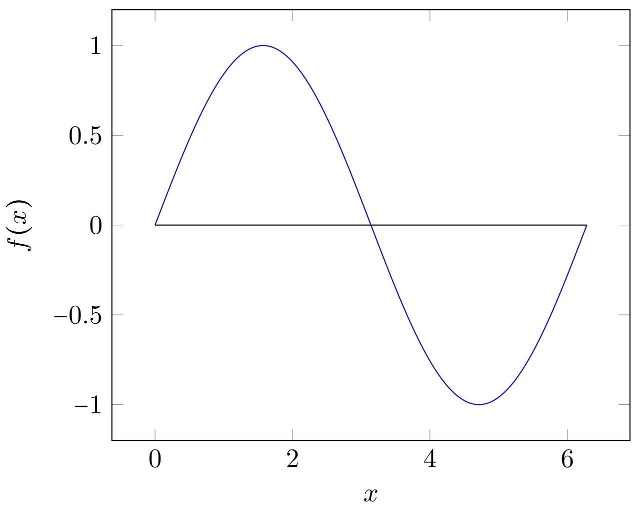

Figure 2: $sin(x)$, where $x \in [0; 2pi]$