Mathematical proofs

Michal Špano

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The inner angle of an n–*sided* convex regular polygon

Suppose an $n - sided$ convex regular polygon. Its 4 consecutive vertices are shown in the figures, $N_1, N_2, N_3, ..., N_i$ respectively. Thus $\Delta_{N_1,N_2,S} \cong \Delta_{N_2,N_3,S}$, i.e. such $n - sided$ polygon consists of n congruent isosceles triangles. That is, $|\measuredangle N_2N_1S| = |\measuredangle N_1N_2S| \Rightarrow |\measuredangle N_1N_2S| = |\measuredangle SN_2N_3|$, denoted as β, β' respectively.

An angle $\alpha = |\measuredangle N_1SN_2| = \dfrac{360}{n}$, since such an angle multiplied by n makes for a perfect circle of 360°. Likewise $\alpha = 180° - 2\beta$.

Let Φ be an inner angle of the polygon, such that $\phi = 2\beta$ (shown in the figure at the vertex N_2).

Express in terms of β: $\alpha = 180 - 2\beta \iff \beta = \dfrac{-\alpha + 180}{2}$

Substitute α for $\alpha = \dfrac{360}{n}$: $\beta = \dfrac{-\dfrac{360}{n} + 180}{2} \iff \beta = \dfrac{180n - 360}{2n}$

Express in terms of Φ: $\Phi = 2\beta = 2\left(\dfrac{180n - 360}{2n}\right) = \dfrac{180n - 360}{n}$

The expression can be further simplified to the following form:

$$\Phi = \frac{(n-2)\pi}{n}$$

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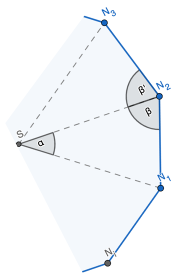

Figure 1: $n - sided$ polygon with its 4 vertices in a plane

The number of diagonals of an $n - sided$ convex regular polygon

Suppose a geometrical locus of n points on the plane, i.e. a set of points $A_1, A_2, ..., A_n$ such that $A_1, A_2, ..., A_n$ create an $n-sided$ convex regular polygon. The number of different abscissas in the geometrical locus is $\binom{n}{2}$ and denoted as N_a. It implies that no abscissa in the locus is given by more than two points, i.e. each point is unique. Likewise, $\overline{A_1 A_2} \equiv \overline{A_2 A_1}$ holds for any 2 points in the locus, thus such abscissas are counted as one. It can be inferred that the number of diagonals, denoted as N_D, is the same as *the difference of the number of abscissas and the number of sides*:

$$N_D = N_s - n$$

The similarity coefficient

Suppose a scalene triangle Δ_{ABC}. We assume that there exists a triangle $\Delta_{A'B'C'}$, such that $\Delta_{ABC} \cong \Delta_{A'B'C'}$. It implies that there exists some constant c, such that any abscissa created in the locus of points (from the original triangle) is equal to the product of c and the corresponding abscissa of the similar triangle. Symbolically:

$\exists! c \in \mathbb{Q} : |V_1 V_2| = c \times |V_1' V_2'| \wedge c \gneqq 1$, where V_1, V_2 are the vertices of the original triangle.

Then, it can be easily inferred, that for the **perimeters** of the triangles holds the following: Let $P = \overline{AB} + \overline{BC} + \overline{AC}$, similarly $P' = \overline{A'B'} + \overline{B'C'} + \overline{A'C'}$. Likewise $\overline{AB} = c \times \overline{A'B'} \wedge \overline{BC} = c \times \overline{B'C'} \wedge \overline{AC} = c \times \overline{A'C'}$.

$$P = c \times \overline{A'B'} + c \times \overline{B'C'} + c \times \overline{A'C'} \Rightarrow P = c \times \left(\overline{A'B'} + \overline{B'C'} + \overline{A'C'} \right) = c \times P'$$

$$P = c \times P'$$

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It implies that the ratio of the lengths of the sides of the similar triangles, i.e. the ratio of their perimeters, is equal to the similarity coefficient: $c = \dfrac{P}{P'}$.

Similar methodology is applied to the similarity coefficient of the areas of the triangles:

Let $A = \dfrac{\overline{AB} \times \overline{H_{AB}}}{2} \wedge A' = \dfrac{\overline{A'B'} \times \overline{H'_{AB}}}{2}$, where H_{AB} and H'_{AB} are the heights of the triangles. Thus, $A = \dfrac{c \times \overline{A'B'} c \times \overline{H'_{AB}}}{2} = c^2 \times \left(\dfrac{\overline{A'B'} \times \overline{H'_{AB}}}{2} \right) = c^2 \times A'$

$$A = c^2 \times A'$$

It implies, that the ratio of the areas equals to the similarity coefficient taken to the second power: $c^2 = \dfrac{A}{A'}$.

To sum up, such a methodology is also applicable to derive the similarity coefficient of the volume of the tetrahedron. Moreover, it is also applicable to derive the similarity coefficient of any planar polygon or solid figure (a scalene triangle per the given example is just an exemplary instance). Still, the following holds: $P = c \times P' \wedge A = c^2 \times A' \wedge V = c^3 \times V'$ based on the principles of **congruence**.

Function bounded above and/or below

Suppose a function f which has a finite domain, say D_f. A function is bounded above if: $\exists! a \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a$. That is to say, that every function value is smaller or equal to some constant, say a. Similarly, a function is bounded below if: $\exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \geq b$. That implies, that every function value is greater or equal to some constant, say b. Lastly, a function is bounded if both of the previous statements hold, i.e.: $\exists! a \in \mathbb{R}, \exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a \wedge f(x) \geq b$. Hence, some bounded function f has the following range of values: $[b; a]$, where $\{a, b\} \in \mathbb{R}$. For example, the function $f(x) = sin(x)$ is bounded, such that $[-1; 1]$ is its range of values.

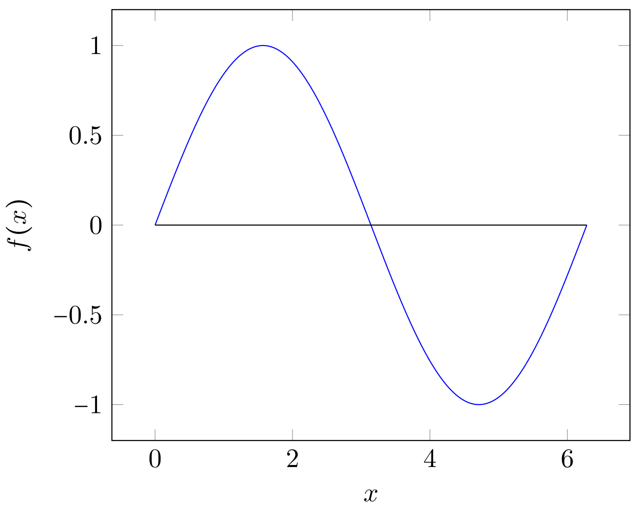

Figure 2: $sin(x)$, where $x \in [0; 2\pi]$

The standard rules of logarithms

The following rules of logarithms hold for x, y, a, s, such that:

$$\forall x, y \in \mathbb{R}^{+}, \forall a > 0 \wedge a \neq 1, \forall s \in \mathbb{R}$$

1. $\boldsymbol{\log_a 1 = 0}$

   For any a holds the following $a^0 = 1$. Therefore $log_a 1 = 0$.

2. $\boldsymbol{\log_a a = 1}$

   For any a holds the following $a^1 = a$. Therefore $log_a a = 1$.

3. $\boldsymbol{a^{\log_a x} = x}$

   Let $l = \log_a x$. Therefore $a^l = x$. Combining the two rules, we obtain the following: $a^{\log_a x} = x$.

4. $\boldsymbol{\log_a (x \times y) = \log_a x + \log_a y}$

   Suppose some x, y, where $x = a^{\log_a x}$, $y = a^{\log_a y}$. If we compute the product of them, we obtain that $x \times y = a^{\log_a x} \times a^{\log_a y}$. Furthermore, we have that $a^{\log_a x + \log_a y} = x \times y$. According to the previous rule, we obtain that $\log_a (x \times y) = \log_a x + \log_a y$.

5. $\boldsymbol{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y}$

   Similarly, suppose some x, y, where $x = a^{\log_a x}$, $y = a^{\log_a y}$. If we compute the quotient of them, we obtain that $\dfrac{x}{y} = \dfrac{a^{\log_a x}}{a^{\log_a y}}$. Furthermore, we obtain that $a^{\log_a x - \log_a y} = \dfrac{x}{y}$. According to the third rule, we have that $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

6. $\boldsymbol{\log_a x^s = s \times \log_a x}$

   Suppose some x, where $x = a^{\log_a x}$. Power both sides of the equation by s, we obtain that $x^s = (a^{\log_a x})^s$. Therefore $a^{s \times \log_a x} = x^s$. Having applied the third rule, it can be inferred that $\log_a x^s = s \times \log_a x$.

7. $\boldsymbol{\log_a x = \dfrac{\log_b x}{\log_b a}}$

   Suppose some x, where $x = a^{\log_a x}$. Then, for some log with the base b and the argument x holds $\log_b x = \log_b a^{\log_a x}$. Applying the sixth rule, we obtain that $\log_b x = \log_b a \times \log_a x$. Divide the equation by $\log_b a$, thus obtaining $\log_a x = \dfrac{\log_b x}{\log_b a}$.

The sum of n terms of an arithmetic sequence

Suppose an arithmetic sequence (also called a progression or a series) $\{a_n\}_{n=1}^{\infty}$. That is, a sequence of n terms, where:

$$\exists! d \in \mathbb{R} - \{0\}, \forall i, j \in \mathbb{N} : d = a_i - a_j \land i - j = 1$$

That implies, that there exists exactly one constant, say d, such that the difference between every 2 consecutive terms is d. We label the sum of the first n terms of the sequence as $S_n = \sum_{i=1}^{n} a_i$. More explicitly, we have $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. Similarly, if we reverse the sequence, we obtain $S_n = a_n + a_{n-1} + \cdots + a_2 + a_1$. Add the two sums, we obtain:

$$2 \times S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_{n-1} + a_2) + (a_n + a_1)$$

Observe the similarity between the sums. E. g. $(1+4)+(2+3)+\cdots+(n+k) = (k+n)+\cdots+(3+2)+(4+1)$ Thus, it can be infered that $2 \times S_n = n \times (a_n + a_1)$. Lastly, we have that:

$$S_n = \frac{n \times (a_n + a_1)}{2}$$

Conclusion: The sum of n terms of an arithmetic sequence is $\dfrac{n}{2}(a_n + a_1)$.

The sum of n terms of an arithmetic sequence (proof)

Suppose you are given a simple arithmetic sequence, where the difference between every 2 consecutive terms d is 1. We assume the following (using the notation from the previous section):

$$S_n = 1 + 2 + \cdots + (n-1) + n = \sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

Formulate a statement, say $S(n)$, which is to be proven.

$$S(n) : \forall n \in \mathbb{N} : 1 + 2 + \cdots + n = \frac{n}{2}(n+1)$$

First and foremost, we let $n = 1$: $S(1) : 1 = \frac{1}{2}(1+1)$. That is true. Secondly, assume that $S(k)$ holds for some arbitrary $k \in \mathbb{N}$. That is our **induction hypothesis** (also called the induction assumption). We obtain:

$$S(k) : 1 + 2 + \cdots + k = \frac{k}{2}(k+1)$$

Thirdly, $S(k) \implies S(k^+)$, so we let $n = k + 1$. Steps of the proof:

$$S(k+1) : 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1) : S(k) + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1) : \frac{k}{2}(k+1) + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1) : \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$S(k+1) : \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

We obtain that $S(k+1)$ holds and so must $S(k)$ (our induction hypothesis). That suffices to prove $S(n)$. Finally, we have that $S(n)$ holds for all $n \in \mathbb{N}$, and thus **the proof is complete**.