# Understanding the Tangent Function

## Michal Špano

### 1 Definition

To define the *tangent* function, one must understand the basic principles of deriving *sine* and *cosine* using a so-called **unit circle**.

A unit circle is the circle of radius 1 centred at the origin O(0,0) in the Cartesian coordinate system.

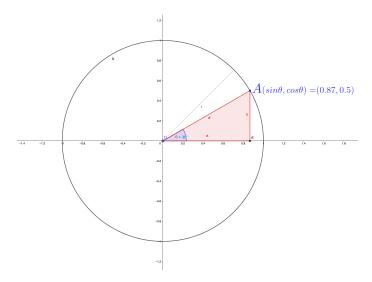


Figure 1: Unit circle

Then for a point A = [x, y] moved along the circumference k by an angle  $\theta$  is valid:

$$A = [cos(\theta), sin(\theta)]$$

Note: an applet is available (link) or scan its **QR** code under **Resources**. A right-angled triangle consists of the **hypotenuse** (the side opposing the right angle) and 2 other adjacent sides called **legs**. The definition of tangent states that it is the quotient of the opposite side and the adjacent side to a

certain angle  $\theta$ , symbolically:

$$tan(\theta) = \frac{opposite}{adjacent}$$

Which can be further transformed into:

$$tan(\theta) = \frac{y}{x}...tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$$

## 2 Properties

Now, let's address the properties of f: y = tan(x)Graph:

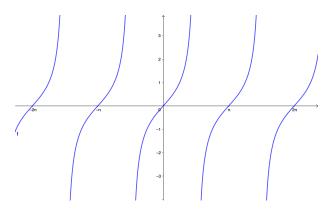


Figure 2: y = tan(x)

#### **Properties:**

#### 1. Domain

$$f: y = tan(x)...y = \frac{sin(x)}{cos(x)} \Leftrightarrow cos(x) \neq 0$$

Let f: y = cos(x), solve for  $x \in \mathbb{R}$ .

$$x \in \left\{\frac{\pi}{2}; \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\} \Leftrightarrow x \in \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$$

Through additional steps, we obtain the following result:

$$x \in \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$$

In other words, the **Domain** of y = tan(x) is all real numbers except every x for which is sin(x) = 0 defined. Symbolically:

$$D_f = \mathbb{R} - \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$$

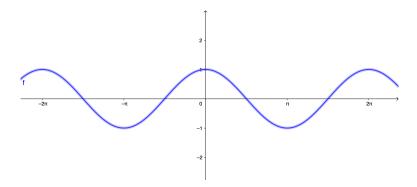


Figure 3: y = cos(x)

#### 2. Range of values

As shown in the figure above, the **Range** of f: y = tan(x):

$$H_f = \mathbb{R}$$

#### 3. Values in quadrants

I.	II.	III	IV
$\left(0;\frac{\pi}{2}\right)$	$\left(\frac{\pi}{2};\pi\right)$	$\left(\pi; \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2};2\pi\right)$
+	-	+	-

#### 4. Periodicity

Suppose that the period of y = tan(x) is  $2\pi$ :

$$tan(x+2k\pi) = \frac{sin(x+2k\pi)}{cos(x+2k\pi)} = \frac{sin(x)}{cos(x)} = tan(x)$$

However,  $2\pi$  is not the smallest possible period. Let us consider a case where period is  $\pi$ .

$$tan(x+\pi) = \frac{sin(x+\pi)}{cos(x+\pi)} = \frac{-sin(x)}{-cos(x)} = tan(x)$$

**Explanation**: The *period* of f: y = sin(x) and f: y = cos(x) is  $2\pi$ , any  $x \in D_f$  shifted by the period of  $\pi$  will produce a **function value** for which is valid:  $\forall x \in D_f \subset \mathbb{R}: f(x+\pi) = -f(x)$ .

Let A' be a point shifted along the circumference k by an angle  $\theta$ ,  $A'_1$  be a point shifted along the same circumference by an angle  $\theta + \pi$ . Their coordinates are depicted using a **unit circle** and are symmetrical through O[0,0] (point symmetry). Notice:

$$A'_{x} = -A'_{x_{1}} \wedge A'_{y} = -A'_{y_{1}}$$

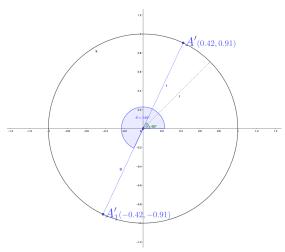


Figure 4: Periodicity of tangent

Note: an applet is available (link) or scan its QR code under Resources. Conclusion Function tangent is periodic with the smallest possible period  $\pi$ .

#### 5. Parity

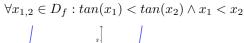
$$f^{+}(x): y = tan(x)$$
 
$$f^{-}(-x): y = tan(-x) = \frac{sin(-x)}{cos(-x)} = \frac{-sin(x)}{cos(x)} = -\frac{sin(x)}{cos(x)} = -tan(x),$$

Based on the parity of sine (odd), cosine (even)  $\Rightarrow sin(-x) = -sin(x) \land cos(-x) = cos(x)$ .

 $f(-x) = -f(x) \Rightarrow \text{Tangent is an odd function}.$ 

#### 6. Monotonicity

The tangent function is **increasing** on  $\mathbb{R}$  between 2 consecutive vertical **asymptotes**, where every asymptote is defined as:  $x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$ , that is:



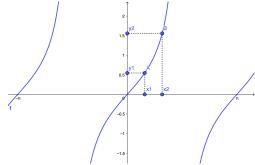


Figure 5: Monotonicity of tangent

### 7. Graph

 ${\bf Tangentoid}$  is the plot of the tangent function.

### 8. Important function values of tangent

x:	0°	30°	$45^{\circ}$	60°	90°
tan(x):	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Ø

# 3 Resources



Applet 1, Applet 2

# List of Figures

1	Unit circle	1
2	$y = tan(x) \dots \dots$	2
3	$y = cos(x) \dots \dots$	3
4	Periodicity of tangent	4
5	Monotonicity of tangent	4