

# Mathematical proofs

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March 2022

## 1 The inner angle of an $n$ - sided convex regular polygon

Suppose an  $n$  - sided convex regular polygon. Its 4 consecutive vertices are shown in the figures,  $N_1, N_2, N_3, \dots, N_i$  respectively. Thus  $\Delta_{N_1, N_2, S} \cong \Delta_{N_2, N_3, S}$ , i.e. such  $n$  - sided polygon consists of  $n$  congruent isosceles triangles. That is,  $|\angle N_2 N_1 S| = |\angle N_1 N_2 S| \Rightarrow |\angle N_1 N_2 S| = |\angle S N_2 N_3|$ , denoted as  $\beta, \beta'$  respectively.

An angle  $\alpha = |\angle N_1 S N_2| = \frac{360}{n}$ , since such an angle multiplied by  $n$  makes for a perfect circle of  $360^\circ$ . Likewise  $\alpha = 180^\circ - 2\beta$ .

Let  $\Phi$  be an inner angle of the polygon, such that  $\phi = 2\beta$  (shown in the figure at the vertex  $N_2$ ).

Express in terms of  $\beta$ :  $\alpha = 180 - 2\beta \iff \beta = \frac{-\alpha + 180}{2}$

Substitute  $\alpha$  for  $\alpha = \frac{360}{n}$ :  $\beta = \frac{-\frac{360}{n} + 180}{2} \iff \beta = \frac{180n - 360}{2n}$

Express in terms of  $\Phi$ :  $\Phi = 2\beta = 2\left(\frac{180n - 360}{2n}\right) = \frac{180n - 360}{n}$

The expression can be further simplified to the following form:

$$\Phi = \frac{(n - 2)\pi}{n}$$

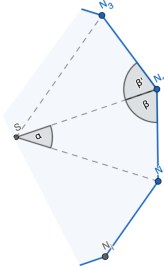


Figure 1:  $n$  - sided polygon with its 4 vertices in a plane

## 2 The number of diagonals of an $n$ -sided convex regular polygon

Suppose a geometrical locus of  $n$  points on the plane, i.e. a set of points  $A_1, A_2, \dots, A_n$  such that  $A_1, A_2, \dots, A_n$  create an  $n$ -sided convex regular polygon. The number of different abscissas in the geometrical locus is  $\binom{n}{2}$  and denoted as  $N_a$ . It implies that no abscissa in the locus is given by more than two points, i.e. each point is unique. Likewise,  $\overline{A_1 A_2} \equiv \overline{A_2 A_1}$  holds for any 2 points in the locus, thus such abscissas are counted as one. It can be inferred that the number of diagonals, denoted as  $N_D$ , is the same as *the difference of the number of abscissas and the number of sides*:

$$N_D = N_a - n$$

## 3 The similarity coefficient

Suppose a scalene triangle  $\Delta_{ABC}$ . We assume that there exists a triangle  $\Delta_{A'B'C'}$ , such that  $\Delta_{ABC} \cong \Delta_{A'B'C'}$ . It implies that there exists some constant  $c$ , such that any abscissa created in the locus of points (from the original triangle) is equal to the product of  $c$  and the corresponding abscissa of the similar triangle. Symbolically:

$\exists! c \in \mathbb{Q} : |V_1 V_2| = c \times |V'_1 V'_2| \wedge c \geq 1$ , where  $V_1, V_2$  are the vertices of the original triangle.

Then, it can be easily inferred, that for the **perimeters** of the triangles holds the following: Let  $P = \overline{AB} + \overline{BC} + \overline{AC}$ , similarly  $P' = \overline{A'B'} + \overline{B'C'} + \overline{A'C'}$ . Likewise  $\overline{AB} = c \times \overline{A'B'} \wedge \overline{BC} = c \times \overline{B'C'} \wedge \overline{AC} = c \times \overline{A'C'}$ .

$$P = c \times \overline{A'B'} + c \times \overline{B'C'} + c \times \overline{A'C'} \Rightarrow P = c \times (\overline{A'B'} + \overline{B'C'} + \overline{A'C'}) = c \times P'$$

$$P = c \times P'$$

It implies that the ratio of the lengths of the sides of the similar triangles, i.e. the ratio of their perimeters, is equal to the similarity coefficient:  $c = \frac{P}{P'}$ .

Similar methodology is applied to the similarity coefficient of the areas of the triangles:

$$\text{Let } A = \frac{\overline{AB} \times \overline{H_{AB}}}{2} \wedge A' = \frac{\overline{A'B'} \times \overline{H'_{AB}}}{2}, \text{ where } H_{AB} \text{ and } H'_{AB} \text{ are the heights of the triangles. Thus, } A = \frac{c \times \overline{A'B'} \times c \times \overline{H'_{AB}}}{2} = c^2 \times \left( \frac{\overline{A'B'} \times \overline{H'_{AB}}}{2} \right) = c^2 \times A'$$

$$A = c^2 \times A'$$

It implies, that the ratio of the areas equals to the similarity coefficient taken to the second power:  $c^2 = \frac{A}{A'}$ .

To sum up, such a methodology is also applicable to derive the similarity coefficient of the volume of the tetrahedron. Moreover, it is also applicable to derive the similarity coefficient of any planar polygon or solid figure (a scalene triangle per the given example is just an exemplary instance). Still, the following holds:  $P = c \times P' \wedge A = c^2 \times A' \wedge V = c^3 \times V'$  based on the principles of **congruence**.

## 4 Function bounded above and/or below

Suppose a function  $f$  which has a finite domain, say  $D_f$ . A function is bounded above if:  $\exists! a \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a$ . That is to say, that every function value is smaller or equal to some constant, say  $a$ . Similarly, a function is bounded below if:  $\exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \geq b$ . That implies, that every function value is greater or equal to some constant, say  $b$ . Lastly, a function is bounded if both of the previous statements hold, i.e.:  $\exists! a \in \mathbb{R}, \exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a \wedge f(x) \geq b$ . Hence, some bounded function  $f$  has the following range of values:  $[b; a]$ , where  $\{a, b\} \in \mathbb{R}$ . For example, the function  $f(x) = \sin(x)$  is bounded, such that  $[-1; 1]$  is its range of values.

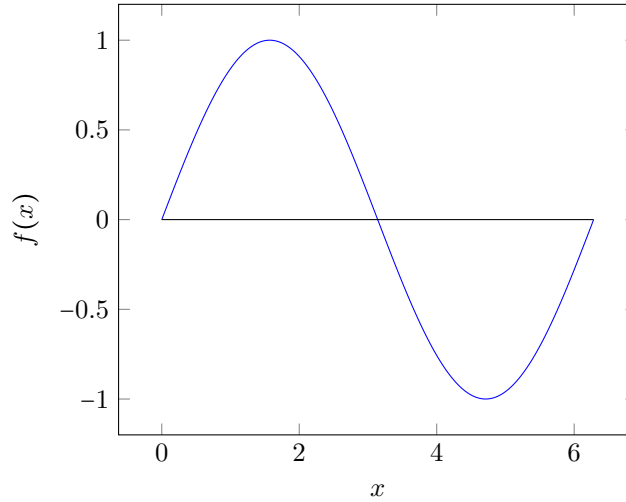


Figure 2:  $\sin(x)$ , where  $x \in [0; 2\pi]$

## 5 The standard rules of logarithms

The following rules of logarithms hold for  $x, y, a, s$ , such that:

$$\forall x, y \in \mathbb{R}^+, \forall a > 0 \wedge a \neq 1, \forall s \in \mathbb{R}$$

1.  **$\log_a 1 = 0$**

For any  $a$  holds the following  $a^0 = 1$ . Therefore  $\log_a 1 = 0$ .

2.  **$\log_a a = 1$**

For any  $a$  holds the following  $a^1 = a$ . Therefore  $\log_a a = 1$ .

3.  **$a^{\log_a x} = x$**

Let  $l = \log_a x$ . Therefore  $a^l = x$ . Combining the two rules, we obtain the following:  $a^{\log_a x} = x$ .

4.  **$\log_a (x \times y) = \log_a x + \log_a y$**

Suppose some  $x, y$ , where  $x = a^{\log_a x}$ ,  $y = a^{\log_a y}$ . If we compute the product of them, we obtain that  $x \times y = a^{\log_a x} \times a^{\log_a y}$ . Furthermore, we have that  $a^{\log_a x + \log_a y} = x \times y$ . According to the previous rule, we obtain that  $\log_a (x \times y) = \log_a x + \log_a y$ .

5.  **$\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y$**

Similarly, suppose some  $x, y$ , where  $x = a^{\log_a x}$ ,  $y = a^{\log_a y}$ . If we compute the quotient of them, we obtain that  $\frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}}$ . Furthermore, we obtain that  $a^{\log_a x - \log_a y} = \frac{x}{y}$ . According to the third rule, we have that  $\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y$ .

6.  **$\log_a x^s = s \times \log_a x$**

Suppose some  $x$ , where  $x = a^{\log_a x}$ . Power both sides of the equation by  $s$ , we obtain that  $x^s = (a^{\log_a x})^s$ . Therefore  $a^{s \times \log_a x} = x^s$ . Having applied the third rule, it can be inferred that  $\log_a x^s = s \times \log_a x$ .

7.  **$\log_a x = \frac{\log_b x}{\log_b a}$**

Suppose some  $x$ , where  $x = a^{\log_a x}$ . Then, for some log with the base  $b$  and the argument  $x$  holds  $\log_b x = \log_b a^{\log_a x}$ . Applying the sixth rule, we obtain that  $\log_b x = \log_b a \times \log_a x$ . Divide the equation by  $\log_b a$ , thus obtaining  $\log_a x = \frac{\log_b x}{\log_b a}$ .