Mathematical proofs

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The inner angle of an *n-sided* convex regular polygon

Suppose an $n - sided$ convex regular polygon. Its 4 consecutive vertices are shown in the figures, $N_1, N_2, N_3, ..., N_i$ respectively. Thus $\Delta_{N_1, N_2, S} \cong \Delta_{N_2, N_3, S}$, i.e. such $n - sided$ polygon consists of n congruent isosceles triangles. That is, $|\sphericalangle N_2 N_1 S| = |\sphericalangle N_1 N_2 S| \Rightarrow |\sphericalangle N_1 N_2 S| = |\sphericalangle S N_2 N_3|$, denoted as β, β' respectively.

An angle $\alpha = |\sphericalangle N_1 S N_2| = \dfrac{360}{n}$, since such an angle multiplied by n makes for a perfect circle of 360°. Likewise $\alpha = 180° - 2\beta$.

Let ϕ be an inner angle of the polygon, such that $\phi = 2\beta$ (shown in the figure at the vertex N_2).

Given α, let us re-arrange the expression in terms of β:

$$\alpha = 180 - 2\beta \iff \beta = \frac{-\alpha + 180}{2}$$

Knowing the properties of α, we substitute α for $\alpha = 360/n$ and obtain:

$$\beta = \frac{-\dfrac{360}{n} + 180}{2} \iff \beta = \frac{180n - 360}{2n}$$

Knowing the definition of the inner angle of the polygon, namely $\phi = 2\beta$, we have that:

$$\phi = 2\beta = 2 * \left(\frac{180n - 360}{2n} \right) = \frac{180n - 360}{n}$$

The obtained expression can be further simplified to the following:

$$\phi = \frac{(n-2)\pi}{n}$$

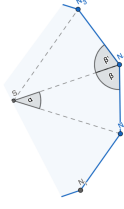$\therefore$ the inner angle of an *n-sided* convex regular polygon is $(n-2) * \pi n^{-1}$. ∎

Figure 1: *n-sided* polygon with its 4 vertices in a plane

The number of diagonals of an $n-sided$ convex regular polygon

Suppose a geometrical locus of n points on the plane, i.e. a set of points $A_1, A_2, ..., A_n$ such that $A_1, A_2, ..., A_n$ create an $n-sided$ convex regular polygon. The number of different abscissas in the geometrical locus is $\binom{n}{2}$ and denoted as N_a. It implies that no abscissa in the locus is given by more than two points, i.e. each point is unique. Likewise, $\overrightarrow{A_1 A_2} \equiv \overrightarrow{A_2 A_1}$ holds for any 2 points in the locus, thus such abscissas are counted as one. It can be inferred that the number of diagonals, denoted as N_D, is the same as *the difference of the number of abscissas and the number of sides*:

$$N_D = N_s - n$$

The similarity coefficient

Suppose a scalene triangle Δ_{ABC}. We assume that there exists a triangle $\Delta_{A'B'C'}$, such that $\Delta_{ABC} \cong \Delta_{A'B'C'}$. It implies that there exists some constant c, such that any abscissa created in the locus of points (from the original triangle) is equal to the product of c and the corresponding abscissa of the similar triangle. Symbolically:

$\exists! c \in \mathbb{Q} : |V_1 V_2| = c \times |V_1' V_2'| \wedge c \gtreqqless 1$, where V_1, V_2 are the vertices of the original triangle.

Then, it can be easily inferred, that for the **perimeters** of the triangles holds the following: Let $P = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC}$, similarly $P' = \overrightarrow{A'B'} + \overrightarrow{B'C'} + \overrightarrow{A'C'}$. Likewise $\overrightarrow{AB} = c \times \overrightarrow{A'B'} \wedge \overrightarrow{BC} = c \times \overrightarrow{B'C'} \wedge \overrightarrow{AC} = c \times \overrightarrow{A'C'}$.

$$P = c \times \overrightarrow{A'B'} + c \times \overrightarrow{B'C'} + c \times \overrightarrow{A'C'} \Rightarrow P = c \times \left(\overrightarrow{A'B'} + \overrightarrow{B'C'} + \overrightarrow{A'C'} \right) = c \times P'$$

$$\therefore P = c \times P'$$

It implies that the ratio of the lengths of the sides of the similar triangles, i.e. the ratio of their perimeters, is equal to the similarity coefficient: $c = P/P'$.

Similar methodology is applied to the similarity coefficient of the areas of the triangles:

Let $A = \dfrac{\overrightarrow{AB} \times \overrightarrow{H_{AB}}}{2} \wedge A' = \dfrac{\overrightarrow{A'B'} \times \overrightarrow{H'_{AB}}}{2}$, where H_{AB} and H'_{AB} are the heights

of the triangles. Thus, $A = \dfrac{c \times \overrightarrow{A'B'} c \times \overrightarrow{H'_{AB}}}{2} = c^2 \times \left(\dfrac{\overrightarrow{A'B'} \times \overrightarrow{H'_{AB}}}{2} \right) = c^2 \times A'$

$$\therefore A = c^2 \times A'$$

It implies, that the ratio of the areas equals to the similarity coefficient taken to the second power: $c^2 = A/A'$.

To sum up, such a methodology is also applicable to derive the similarity coefficient of the volume of the tetrahedron. Moreover, it is also applicable to derive the similarity coefficient of any planar polygon or solid figure (a scalene triangle per the given example is just an exemplary instance). Still, the following holds: $P = c \times P' \wedge A = c^2 \times A' \wedge V = c^3 \times V'$ based on the principles of **congruence**.

Function bounded above and/or below

Suppose a function f which has a finite domain, say D_f. A function is bounded above if: $\exists! a \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a$. That is to say, that every function value is smaller or equal to some constant, say a. Similarly, a function is bounded below if: $\exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \geq b$. That implies, that every function value is greater or equal to some constant, say b. Lastly, a function is bounded if both of the previous statements hold, i.e.: $\exists! a \in \mathbb{R}, \exists! b \in \mathbb{R}, \forall x \in D_f \subset \mathbb{R} : f(x) \leq a \wedge f(x) \geq b$. Hence, some bounded function f has the following range of values: $[b; a]$, where $\{a, b\} \in \mathbb{R}$. For example, the function $f(x) = sin(x)$ is bounded, such that $[-1; 1]$ is its range of values.
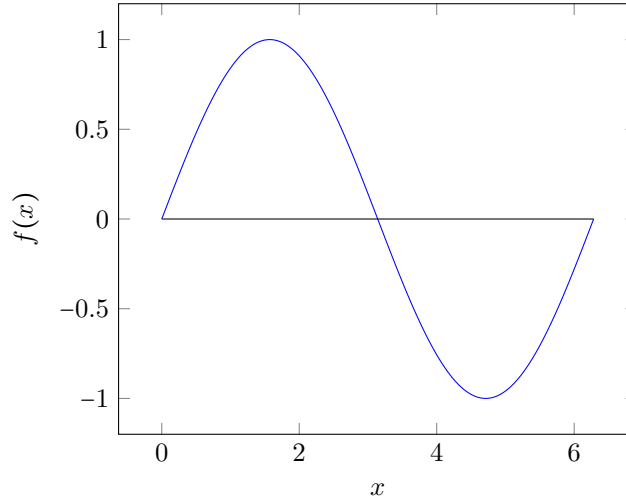

Figure 2: $sin(x)$, where $x \in [0; 2\pi]$

The standard rules of logarithms

The following rules of logarithms hold for x, y, a, s, such that:

$$\forall x, y \in \mathbb{R}^+, \forall a > 0 \wedge a \neq 1, \forall s \in \mathbb{R}$$

1. $\boldsymbol{\log_a 1 = 0}$

   For any a holds the following $a^0 = 1$. Therefore $log_a 1 = 0$.

2. $\boldsymbol{\log_a a = 1}$

   For any a holds the following $a^1 = a$. Therefore $log_a a = 1$.

3. $\boldsymbol{a^{\log_a x} = x}$

   Let $l = \log_a x$. Therefore $a^l = x$. Combining the two rules, we obtain the following: $a^{\log_a x} = x$.

4. $\boldsymbol{\log_a (x \times y) = \log_a x + \log_a y}$

   Suppose some x, y, where $x = a^{\log_a x}$, $y = a^{\log_a y}$. If we compute the product of them, we obtain that $x \times y = a^{\log_a x} \times a^{\log_a y}$. Furthermore, we have that $a^{\log_a x + \log_a y} = x \times y$. According to the previous rule, we obtain that $\log_a (x \times y) = \log_a x + \log_a y$.

5. $\boldsymbol{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y}$

   Similarly, suppose some x, y, where $x = a^{\log_a x}$, $y = a^{\log_a y}$. If we compute the quotient of them, we obtain that $\dfrac{x}{y} = \dfrac{a^{\log_a x}}{a^{\log_a y}}$. Furthermore, we obtain that $a^{\log_a x - \log_a y} = \dfrac{x}{y}$. According to the third rule, we have that $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

6. $\boldsymbol{\log_a x^s = s \times \log_a x}$

   Suppose some x, where $x = a^{\log_a x}$. Power both sides of the equation by s, we obtain that $x^s = (a^{\log_a x})^s$. Therefore $a^{s \times \log_a x} = x^s$. Having applied the third rule, it can be inferred that $\log_a x^s = s \times \log_a x$.

7. $\boldsymbol{\log_a x = \dfrac{\log_b x}{\log_b a}}$

   Suppose some x, where $x = a^{\log_a x}$. Then, for some log with the base b and the argument x holds $\log_b x = \log_b a^{\log_a x}$. Applying the sixth rule, we obtain that $\log_b x = \log_b a \times \log_a x$. Divide the equation by $\log_b a$, thus obtaining $\log_a x = \dfrac{\log_b x}{\log_b a}$.

The sum of n terms of an arithmetic sequence

Suppose an arithmetic sequence (also called a progression or a series) $\{a_n\}_{n=1}^{\infty}$. That is, a sequence of n terms, where:

$$\exists!d \in \mathbb{R} - \{0\}, \forall i, j \in \mathbb{N} : d = a_i - a_j \land i - j = 1$$

That implies, that there exists exactly one constant, say d, such that the difference between every 2 consecutive terms is d. We label the sum of the first n terms of the sequence as $S_n = \sum_{i=1}^{n} a_i$. More explicitly, we have $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. Similarly, if we reverse the sequence, we obtain $S_n = a_n + a_{n-1} + \cdots + a_2 + a_1$. Add the two sums, we obtain:

$$2 \times S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_{n-1} + a_2) + (a_n + a_1)$$

Observe the similarity between the sums. E. g. $(1+4)+(2+3)+\cdots+(n+k) = (k+n)+\cdots+(3+2)+(4+1)$ Thus, it can be inferred that $2 \times S_n = n \times (a_n + a_1)$. Lastly, we have that:

$$S_n = \frac{n \times (a_n + a_1)}{2}$$

Conclusion: The sum of n terms of an arithmetic sequence is $\frac{n}{2}(a_n + a_1)$.

The sum of n terms of an arithmetic sequence (proof)

Suppose you are given a simple arithmetic sequence, where the difference between every 2 consecutive terms d is 1. We assume the following (using the notation from the previous section):

$$S_n = 1 + 2 + \cdots + (n-1) + n = \sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

Formulate a statement, say $S(n)$, which is to be proven.

$$S(n) : \forall n \in \mathbb{N} : 1 + 2 + \cdots + n = \frac{n}{2}(n+1)$$

First and foremost, we let $n = 1$: $S(1) : 1 = \frac{1}{2}(1 + 1)$. That is true. Secondly, assume that $S(k)$ holds for some arbitrary $k \in \mathbb{N}$. That is our **induction hypothesis** (also called the induction assumption). We obtain:

$$S(k) : 1 + 2 + \cdots + k = \frac{k}{2}(k+1)$$

Thirdly, $S(k) \implies S(k^+)$, so we let $n = k + 1$. Steps of the proof:

$$S(k+1) : 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1) : S(k) + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1): \frac{k}{2}(k+1) + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$S(k+1): \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$S(k+1): \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

We obtain that $S(k+1)$ holds and so must $S(k)$ (our induction hypothesis). That suffices to prove $S(n)$. Finally, we have that $S(n)$ holds for all $n \in \mathbb{N}$, and thus **the proof is complete**. ∎

Quantifiers in mathematical logic (1)

Task 1. Show that $\neg \forall x [P(x)] \iff \exists [\neg P(x)]$.

Solution: Let's write the definition of the universal quantifier, where A is a set of some n finite elements, namely, $\{a_1, a_2, \ldots, a_n\}$: $\neg \forall x \in A : P(x) \iff \neg\big(P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n)\big)$. If we apply the the De Morgan's law, we obtain $\neg \forall x \in A : P(x) \iff \neg P(a_1) \vee \neg P(a_2) \vee \cdots \vee \neg P(a_n)$. This simply states, that there's at least one element in the set A for which the statement $P(x)$ is false. We may factor the negation sign out of the sequence of predicates, thus obtaining $\neg \forall x \in A : P(x) \iff \neg\big(P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)\big)$. Were we to apply the definition of the existential quantifier, we receive that $\neg \forall x \in A : P(x) \iff \exists x \in A : \neg P(x)$, since $\exists x \in A : P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n)$ which completes the proof - we've shown that the two mathematical expressions are equivalent. ∎

Note: the same analogy is used to show the negation of the existential quantifier. For brevity, it's left to the reader to prove it.

Quantifiers in mathematical logic (2)

Task 2. Negate the following expression: $\forall x \exists y [P(x,y) \wedge Q(x)]$.

Solution: the resulting expression is broken down into 4 consecutive steps. A brief description is adduced for each step. Observe the steps one by one until the resulting expression:

$$\neg\big(\forall x \exists y [P(x,y) \wedge Q(x)]\big) \text{ - the initial expression is negated} \tag{1}$$

$$\exists x \neg\big(\exists y [P(x,y) \wedge Q(x)]\big) \text{ - the outer quantifier is negated} \tag{2}$$

$$\exists x \forall y \neg\big([P(x,y) \wedge Q(x)]\big) \text{ - the inner quantifier is negated} \tag{3}$$

$$\exists x \forall y [\neg P(x,y) \vee \neg Q(x)] \text{ - De Morgan's law}^* \text{ is applied} \tag{4}$$

$$\text{The resulting expression.} \ ∎ \tag{5}$$

*In this case, it represents the negation of the conjunction, namely: $\neg\big(A(x) \wedge B(x)\big) \equiv \neg A(x) \vee \neg B(x)$ for some predicates A and B given x.