Proof By Contradiction

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May 25, 2022

Prove that $\sqrt{7}$ is irrational.

$$S : \sqrt{7} \in \mathbb{Q}'$$
$$\neg S : \sqrt{7} \in \mathbb{Q}$$

We prove S by **contradiction**. That is:

Suppose S; $\neg S \to N \mid S$, where N is some **nonsense**.

We assume $\sqrt{7}$ to be rational, that implies $\sqrt{7} = \dfrac{p}{q}$, where $p \in \mathbb{Z} \wedge q \in \mathbb{Z} - \{0\}$ and that p, q are relatively prime, i.e. coprime integers $\Rightarrow \nexists d \in \mathbb{N} - \{1\} : d \mid p \wedge d \mid q$.

$\sqrt{7} = \dfrac{p}{q}$ $7 = \left(\dfrac{p}{q}\right)^2$ $7 = \dfrac{p^2}{q^2}$ $7q^2 = p^2 \Rightarrow 7 \mid p^2$

Assumption: $7 \mid p^2 \Rightarrow 7 \mid p$

$$A : 7 \mid p^2 \Rightarrow 7 \mid p$$
$$A' : 7 \nmid p \Rightarrow 7 \nmid p^2$$

Let $p = 7a + 1$, where $a \in \mathbb{Z}$: $p^2 = (7a + 1)^2 \ldots 7 \times (7a^2 + 7a) + 1$. Let $(7a^2 + 7a) = z \Rightarrow 7z + 1 \Rightarrow 7 \nmid 7z + 1 \Rightarrow 7 \nmid p^2$. The assumptions is **true**.

Thus, $7 \mid p^2 \Rightarrow 7 \mid p \Rightarrow p = 7a$, where $a \in \mathbb{Z}$.

$7q^2 = (7a)^2$ $7q^2 = 49a^2$ $q^2 = 7a^2 \Rightarrow 7 \mid q^2 \Rightarrow 7 \mid q$

Contradiction with the assumption: p, q are not relatively prime, symbolically: $\exists d \in \mathbb{N} - \{1\} : d \mid p \wedge d \mid q \Rightarrow \sqrt{7} \neq \dfrac{p}{q} \Rightarrow \sqrt{7} \notin \mathbb{Q} \Rightarrow \sqrt{7} \in \mathbb{Q}'$.

Conclusion: $\sqrt{7}$ is an **irrational** number.