Homework 1

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1 Homework 1

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- **Description**: Source code in the form of a `JupyterNotebook` for **Homework 1** of Summer Course `NUMAO` @ **Lund University, Sweden**.

[1]:
```python
# Imports for the module
import matplotlib.pyplot as plt
from numpy import sqrt, log, linspace, zeros
```

1.1 Task 1 - Approximating the logarithm

In this task, we create a function that **approximates a logarithm** in n steps based on the following algorithm:

- Having $x > 0$, let $a_0 = \frac{1+x}{2}, g_0 = \sqrt{x}$,
- iterate $a_{i+1} = \frac{a_i + g_i}{2}$ and $g_{i+1} = \sqrt{a_{i+1} * g_i}$,
- then $\frac{x-1}{a_i}$ is the approximation of $\ln(x)$.

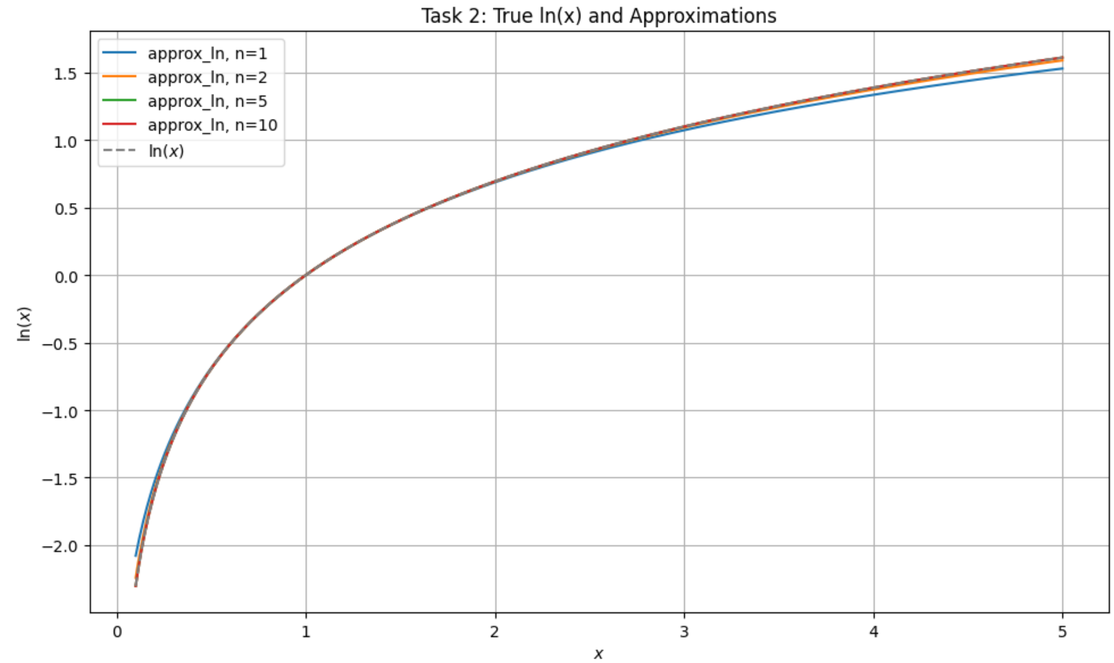
[2]:
```python
def approx_ln(x: float, n: int) -> float:
    # Base cases for x, n
    if x <= 0 or n <= 0:
        raise ValueError("x, n must be greater than 0")

    # Initial mean values
    a_curr, g_curr = (1 + x) / 2, sqrt(x)

    # Iterative calculation based on the algorithm
    for _ in range(n):
        a_next = (a_curr + g_curr) / 2
        g_next = sqrt(a_next * g_curr)
        a_curr, g_curr = a_next, g_next

    # Final approximation form
    return (x - 1) / a_curr
```

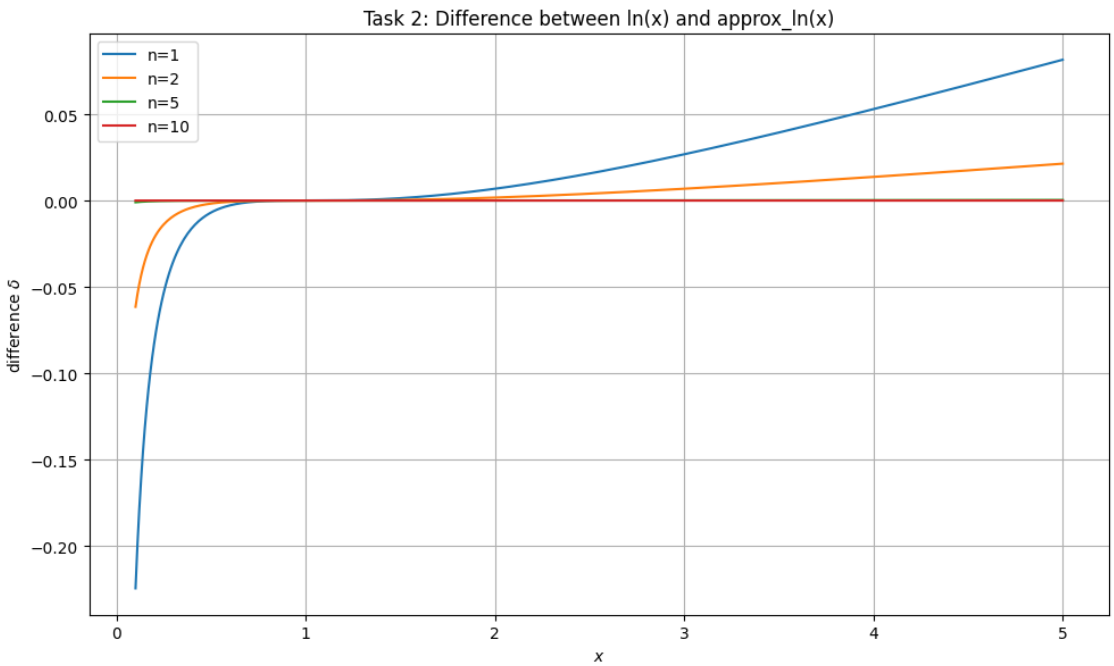
1.2 Task 2 - Plotting functions and their difference

In this task, we (i) plot the functions (`log`, `approx_log`) in a single Cartesian coordinate system (ii) and plot the **difference** of both functions given $n = \{1, 2, 5, 10\}$.

Note: an established implementation to compute $\ln(x)$ (`numpy.log`) is used here.

[7]:
```python
# Generate a range of x values
x_values = linspace(0.1, 5, 500)

# Compute the true ln(x) values
true_ln_values = log(x_values)

# Different values of n for approximation
n_values = [1, 2, 5, 10]

# Create subplots for this task
fig, axes = plt.subplots(2, 1, figsize=(10, 12))

# Subplot 1: plot ln(x) and approx_ln(x) for different n values
for n in n_values:
    approx_ln_values = [approx_ln(x, n) for x in x_values]
    axes[0].plot(x_values, approx_ln_values, label=f'approx_ln, n={n}')

# Plot actual ln(x) values
axes[0].plot(x_values, true_ln_values, label='$\\ln(x)$', color='gray',
             linestyle='--')
axes[0].set_title('Task 2: True ln(x) and Approximations')
axes[0].set_xlabel('$x$')
axes[0].set_ylabel('$\\ln(x)$')
axes[0].legend()
axes[0].grid(True)

# Subplot 2: plot the difference between ln(x) and approx_ln(x)
for n in n_values:
    approx_ln_y = [approx_ln(x, n) for x in x_values]
    delta_ln    = true_ln_values - approx_ln_y
    axes[1].plot(x_values, delta_ln, label=f'n={n}')

axes[1].set_title('Task 2: Difference between ln(x) and approx_ln(x)')
axes[1].set_xlabel('$x$')
axes[1].set_ylabel('difference $\delta$')
axes[1].legend()
axes[1].grid(True)

# Show plots
plt.tight_layout()
plt.show()
```

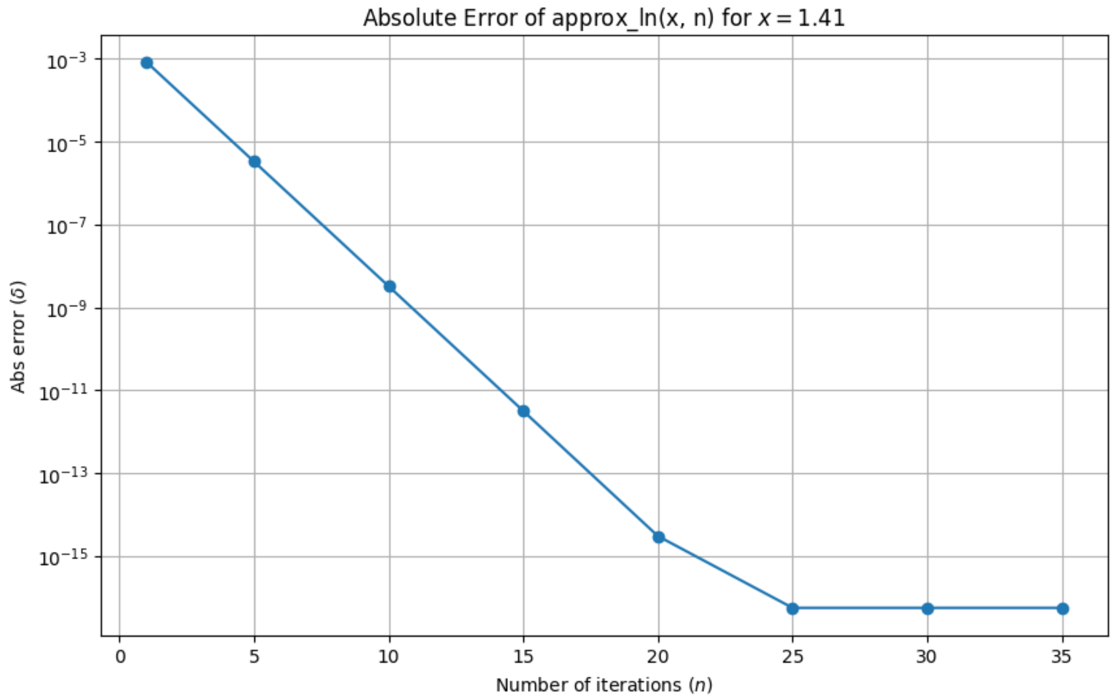




2 Task 3 - Error given some x

In this example, we let $x = 1.41$. We then proceed to plot the absolute value of the error δ against n. We let n be $1, 5, 10, \ldots, 35$. This graph is represented with a **dotted line** where the x values represent the individual steps (n) and y represent the absolute value of the error.

[4]:
```python
x = 1.41
ns = [1] + [i for i in range(5, 40, 5)] # 1, 5, 10, ..., 35 (n step values)

approx_ln_ys = [approx_ln(x, n) for n in ns]
delta_y      = [abs(y - log(x)) for y in approx_ln_ys]

plt.figure(figsize=(10, 6))
plt.plot(ns, delta_y, "o-") # use a dotted line
plt.yscale("log")
plt.ylabel("Abs error ($\\delta$)")
plt.title(f"Absolute Error of approx_ln(x, n) for $x = {x}$")
plt.xlabel("Number of iterations ($n$)")
plt.grid(True)
plt.show()
```



3 Task 4 - Improving the algorithm

Based on the provided article by *B. C. Carlsson*, the following method is suggested to accelerate the convergence of the previous implementation. In short, it says that

- we iterate $i = 0$ through n,
- let $d_{0,i} = a_i$ and continue with
- $d_{k,i} = \frac{d_{k-1,i} - 4^{-k} d_{k-1,i-1}}{1-4^{-k}}$, such that $k = 1, \dots, i$ whenever $i > 0$.

Lastly, an approximation of $\ln(x)$ is taken as $\frac{x-1}{d_{n,n}}$. This is an extension of the previously implemented `approx_ln()` method.

```
[5]: def fast_approx_ln(x: float, n: int) -> float:
         # Base cases for x, n
         if x <= 0 or n <= 0:
             raise ValueError("x, n must be greater than 0")

         # Initial mean values
         a, g = (1 + x) / 2, sqrt(x)

         # Initialize '(n+1) x (n+1)' matrix called `d` with zeros
         d = zeros((n + 1, n + 1))

         # Iterate in 0..n steps, instantiate d_{0,i}
         for i in range(n + 1):
             d[0, i] = a
             a = (a + g) / 2
             g = sqrt(a * g)

         # Compute remaining d_{k,i} s.t. k = 1..i, whenever i > 0.
         for i in range(n + 1):
             for k in range(1, i + 1):
                 # Apply the said formula
                 d[k, i] = (d[k - 1, i] - 4**(-k) * d[k - 1, i-1]) / (1 - 4**(-k))

         # An approximation to ln(x) is taken as the following
         return (x - 1) / d[n, n]
```

4 Task 5 - Plotting the improved algorithm

Herein, we replicate the plot provided in the homework using the improved `fast_approx_ln` method.

```
[6]: xs = linspace(0.1, 20, 1500) # x = (0,20]

     for i in range(2, 7): # iterations 2..6
         fast_approx_ln_ys = [fast_approx_ln(x, i) for x in xs]
         # Compare each iteration against actual log(x)
         delta_y = [abs(y - log(x)) for x, y in zip(xs, fast_approx_ln_ys)]
         plt.plot(xs, delta_y, marker="o", markersize=3, linestyle="None",
                  label=f"iteration {i}")

     plt.title("Error behavior of the accelerated Carlsson method for the $\\log$")
     plt.yscale("log") # logarithmic scale for y-axis
     plt.xlabel("$x$")
     plt.ylabel("error")
```

```
plt.legend()
plt.show()
```