

# ASSIGNMENT 1

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## Problem 5

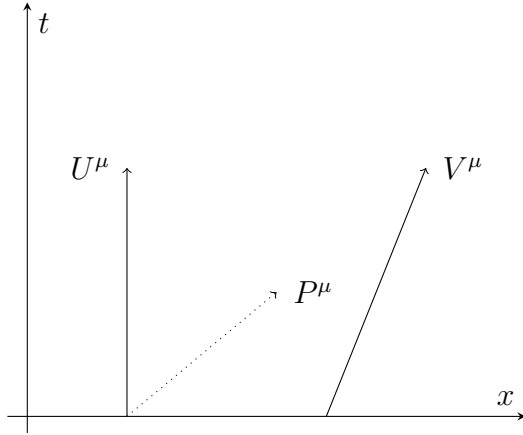


Figure 1: Setup of experiment

We have given:

$$U^\mu = (c, \mathbf{0}) \quad (1)$$

$$V^\mu = (\gamma_v c, \gamma_v \mathbf{v}) \quad (2)$$

$$P^\mu = \left( \frac{h\nu}{c}, \mathbf{p} \right) \quad (3)$$

We use following relation in this problem:

$$E = -P^\mu V_\mu \quad (4)$$

This expression is Lorentz invariant and can be calculated in non-moving frame. So we plug in Eq. 2 in this expression to obtain

$$E = -P^\mu V_\mu = \frac{h\nu}{c} \gamma_v c - \gamma_v \mathbf{v} \mathbf{p} = \gamma_v (h\nu - |\mathbf{v}| |\mathbf{p}| \cos(\theta)) = \left\{ |\mathbf{p}| = \frac{h\nu}{c} \right\} = \gamma_v h\nu \left( 1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) \quad (5)$$

But it is still photon, but with different energy (for moving observer) So

$$\gamma_v h\nu \left( 1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) = h\nu' \quad (6)$$

So ratio of those two frequencies is

$$\frac{\nu'}{\nu} = \gamma_v \left( 1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) \quad (7)$$

If  $\theta = 0$  and  $\frac{v}{c} \ll 1 \Rightarrow \gamma_v \simeq 1$  then we obtain:

$$\boxed{\nu' = \nu \left( 1 - \frac{v}{c} \right)} \quad (8)$$

## ASSIGNMENT 2

### Problem 1a

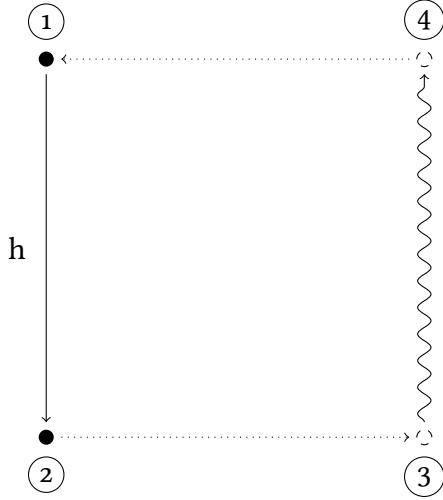


Figure 2: Mass falling in gravitational field (1→2), converting to photon (2→3), photon traveling up (3→4) and converting back to mass (4→1)

Let's take a look at energy changes in above diagram:

$$\textcircled{1} \quad E_1 = mc^2$$

$$\textcircled{2} \quad E_2 = mc^2 + mgh$$

$$\textcircled{3} \quad E_3 = h\nu = mc^2 + mgh$$

$$\textcircled{4} \quad E_4 = h\nu = mc^2 + mgh$$

but  $E_4 = E_1$  because of energy conservation. It means that photon has to have different frequency at the height  $h$  than it has at the ground. So  $E_4 = h\nu' = mc^2$ . From it follows

$$\frac{\nu}{\nu'} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \quad (9)$$

and it is easy to calculate redshift

$$z = \frac{\nu - \nu'}{\nu'} = \frac{gh}{c^2} \quad (10)$$

### Problem 1b

Let's calculate time which light needs to reach observer  $\textcircled{2}$

$$t = \frac{s}{c} = \frac{h - \frac{gt^2}{2}}{c} \quad (11)$$

From this expression we get quadratic equation

$$\frac{g}{2}t^2 + ct - h = 0 \quad (12)$$

for which solution is given by

$$t = \frac{-c + \sqrt{c^2 + 2gh}}{g} \quad (13)$$

Velocity of observer  $\textcircled{2}$  after this time is equal

$$v(t) = \frac{-c + \sqrt{c^2 + 2gh}}{g} \cdot g = -c + \sqrt{c^2 + 2gh} \quad (14)$$

Then redshift formula is given in following way

$$\frac{\nu'}{\nu} = 1 - \frac{v}{c} = 1 - \frac{-c + \sqrt{c^2 + 2gh}}{c} = 2 - \sqrt{1 - \frac{2gh}{c^2}} \quad (15)$$

We can use Taylor expansion  $\sqrt{1-x} = 1 - \frac{x}{2}$  we get

$$\frac{\nu'}{\nu} = 2 - 1 + \frac{gh}{c^2} = 1 + \frac{gh}{c^2} \quad (16)$$

It is exactly the same result as Eq. 10.

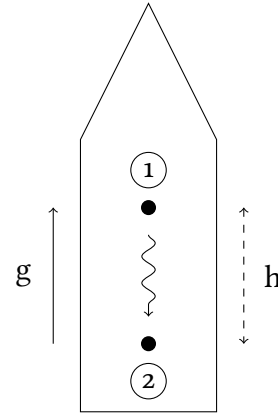


Figure 3: Two observers in a rocket sending photon

## Problem 2

Observer  $\mathcal{O}$  is traveling with acceleration  $g$  in direction  $x_1$ . To calculate his worldline we will use following three conditions

$$U^\mu U_\mu = -1 \quad U^\mu A_\mu = 0 \quad A^\mu A_\mu = g^2 \quad (17)$$

where  $U^\mu$  is four-velocity and  $A^\mu$  is four-acceleration. First of them can be obtained by straightforward calculation, second by applying derivative to first equation i.e.

$$\frac{d}{d\tau} (U^\mu U_\mu) = 0 \Rightarrow (A^\mu U_\mu) = 0 \quad (18)$$

Third is Lorentz invariant and it can be calculated in the moment of launch namely when  $A^\mu = (0, g, 0, 0)$ .

Knowing those three we can write them in explicite form

$$-U_0^2 + \mathbf{U}^2 = -1 \quad \mathbf{U} \mathbf{A} = U_0 A_0 \quad -A_0^2 + \mathbf{A}^2 = g^2 \quad (19)$$

where bolded letters mean three-vectors.

We square middle equation and plug in left and right equation to obtained

$$(U_0^2 - 1) \mathbf{A}^2 = U_0^2 (\mathbf{A}^2 - g^2) \quad (20)$$

Eventually we obtain:

$$\mathbf{A}^2 = g^2 U_0^2 \quad (21)$$

and plugin this expression to other equation we also obtain:<sup>1</sup>

$$A_0^2 = g^2 \mathbf{U}^2 \quad (22)$$

We can simplify those equation using the fact that this motion is one dimensional namely  $x_2 = x_3 = 0$  and then

$$A_1 = g U_0 \quad A_0 = g U_1 \quad (23)$$

But  $U^\mu = \dot{X}^\mu$  and  $A^\mu = \ddot{X}^\mu$ <sup>2</sup>. Substituting

$$\ddot{X}_1 = g \dot{X}_0 \quad \ddot{X}_0 = g \dot{X}_1 \quad (24)$$

Taking a derivative of left equation and substituting right equation into it we get

$$\ddot{X}_1 = g^2 \dot{X}_1 \xrightarrow{\text{after integration}} \ddot{X}_1 = g^2 X_1 \quad (25)$$

Solution is

$$X_1 = A \sinh(g\tau) + B \cosh(g\tau) \quad (26)$$

Let's choose initial conditions such as  $X_1(0) = g^{-1}$  and  $\dot{X}_1 = 0$ . Then

$$X_1 = g^{-1} \cosh(g\tau) \quad (27)$$

And finally we have

$$X_0 = g^{-1} \sinh(g\tau) \quad X_1 = g^{-1} \cosh(g\tau) \quad X_2 = 0 \quad X_3 = 0 \quad (28)$$

<sup>1</sup>plug it into right equation and then use left equation

<sup>2</sup>dot means derivation with respect to proper time

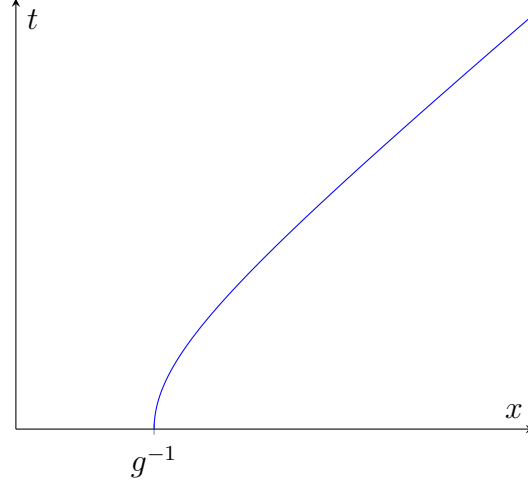


Figure 4: Trajectory of  $\mathcal{O}$

### Problem 3

As a first basis vector we can choose four-velocity namely

$$\mathbf{e}_0 = (\dot{X}_0, \dot{X}_1, \dot{X}_2, \dot{X}_3) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \quad (29)$$

As a basis vectors in directions  $x_2$  and  $x_3$  we simply choose

$$\mathbf{e}_2 = (0, 0, 1, 0) \quad (30)$$

$$\mathbf{e}_3 = (0, 0, 0, 1) \quad (31)$$

And finally we choose vector  $\mathbf{e}_1$  in a form  $\mathbf{e}_1 = (e_1^0, e_1^1, 0, 0)$  where  $e_1^0$  and  $e_1^1$  are chosen in order to satisfy  $\mathbf{e}_0 \mathbf{e}_1 = 0$  and  $(\mathbf{e}_0)^2 = 1$  i.e.

$$-e_1^0 \cosh(g\tau) + e_1^1 \sinh(g\tau) = 0 \quad (32)$$

$$-(e_1^0)^2 + (e_1^1)^2 = 1 \quad (33)$$

We square first equation and substitute second equation

$$(e_1^0)^2 \cosh^2(g\tau) = (1 + (e_1^0)^2) \sinh^2(g\tau) \quad (34)$$

From this we obtain

$$(e_1^0)^2 = \sinh^2(g\tau) \quad (e_1^1)^2 = \cosh^2(g\tau) \quad (35)$$

We can choose positive solution and eventually we get

$$\mathbf{e}_1 = (\sinh(g\tau), \cosh(g\tau), 0, 0) \quad (36)$$

All vectors

$$\mathbf{e}_0(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \quad (37)$$

$$\mathbf{e}_1(\tau) = (\sinh(g\tau), \cosh(g\tau), 0, 0) \quad (38)$$

$$\mathbf{e}_2(\tau) = (0, 0, 1, 0) \quad (39)$$

$$\mathbf{e}_3(\tau) = (0, 0, 0, 1) \quad (40)$$

Last thing to do is to check whether those are vectors which were obtain without any rotation. For this I will find a Lorentz boost which transforms initial basis into this one. Namely consider a boost of time-basis vector

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 0 \end{pmatrix} \quad (41)$$

So  $\gamma$  and  $\beta$  have to satisfy:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh(g\tau) \quad \Rightarrow \quad v = \tanh(g\tau) \quad (42)$$

Knowing that it is easy to calculate

$$\beta\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh(g\tau) \quad (43)$$

So indeed we obtain vector  $e_0(\tau)$  only via boost (at  $v = \tanh(g\tau)$ ). The same can be done with vector  $e_1(\tau)$

## Problem 4

We define new coordinate system ( $\xi_0 \equiv \tau, \xi_1, \xi_2, \xi_3$ ) where basis vectors are those defined in problem before. We can write

$$\mathbf{x} = \xi^1 \mathbf{e}_1(\tau) + \xi^2 \mathbf{e}_2(\tau) + \xi^3 \mathbf{e}_3(\tau) + \mathbf{x}_O(\tau) \quad (44)$$

where  $\mathbf{x}_O(\tau)$  is trajectory of moving frame.

After plugging in all basis vectors explicitly we get

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \xi^1 \sinh(g\tau) \\ \xi^1 \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \xi^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^3 \end{pmatrix} + \begin{pmatrix} g^{-1} \sinh(g\tau) \\ g^{-1} \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} g^{-1} \sinh(g\tau) + \xi^1 \sinh(g\tau) \\ g^{-1} \cosh(g\tau) + \xi^1 \cosh(g\tau) \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} (g^{-1} + \xi^1) \sinh(g\xi_0) \\ (g^{-1} + \xi^1) \cosh(g\xi_0) \\ \xi^2 \\ \xi^3 \end{pmatrix} \end{aligned} \quad (45)$$

Line element  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$  is then equal (we use chain rule i.e.  $dx^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} d\xi^\nu$ )

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \quad (46)$$

$$dt = \frac{\partial t}{\partial \xi^\nu} d\xi^\nu = (1 + g\xi_1) \cosh(g\xi_0) d\xi_0 + \sinh(g\xi_0) d\xi_1 \quad (47)$$

$$dx_1 = (1 + g\xi_1) \sinh(g\xi_0) d\xi_0 + \cosh(g\xi_0) d\xi_1 \quad (48)$$

$$dx_2 = d\xi_2 \quad (49)$$

$$dx_3 = d\xi_3 \quad (50)$$

After squaring and adding them up we get

$$\begin{aligned} ds^2 = & -(1 + g\xi_1)^2 \cosh^2(g\xi_0) d\xi_0^2 - \sinh^2(g\xi_0) d\xi_1^2 + \\ & (1 + g\xi_1)^2 \sinh^2(g\xi_0) d\xi_0^2 + \cosh^2(g\xi_0) d\xi_1^2 + \\ & d\xi_2^2 + \\ & d\xi_3^2 \end{aligned} \quad (51)$$

After simplification

$$\boxed{ds^2 = -(1 + g\xi_1)^2 d\xi_0^2 + d\xi_1^2 + d\xi_2^2 + d\xi_3^2} \quad (52)$$

## Problem 5

For  $\xi^1 \equiv \text{const}$  we can easily derive equation of motion from Eq. 45 namely

$$x_1^2 - t^2 = (g^{-1} + \xi^1)^2 \quad (53)$$

which leads to

$$x_1(t) = \sqrt{(g^{-1} + \xi^1)^2 + t^2} \quad (54)$$

We take derivative twice

$$\dot{x}_1(t) = \frac{2t}{2\sqrt{(g^{-1} + \xi^1)^2 + t^2}} \quad (55)$$

$$\ddot{x}_1(t) = \frac{\sqrt{(g^{-1} + \xi^1)^2 + t^2} - t \frac{2t}{2\sqrt{(g^{-1} + \xi^1)^2 + t^2}}}{(g^{-1} + \xi^1)^2 + t^2} = \frac{1}{\sqrt{(g^{-1} + \xi^1)^2 + t^2}} - \frac{2t^2}{((g^{-1} + \xi^1)^2 + t^2)^{\frac{3}{2}}} \quad (56)$$

So when  $t = 0$

$$\boxed{\ddot{x}_1(t)|_{t=0} = \frac{1}{g^{-1} + \xi^1} = \frac{g}{1 + g\xi^1}} \quad (57)$$

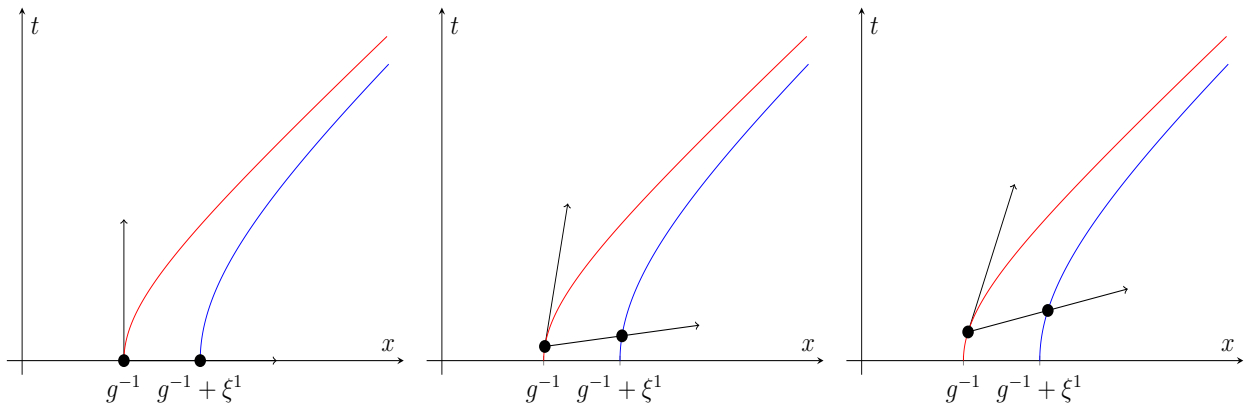


Figure 5: Red line is worldline of Eq. 28 and blue is worldline of Eq. 54

## Problem 6

We start with equation Eq. 52. We can simplify it and neglect other spatial dimensions than  $\xi^1$  namely

$$ds^2 = -(1 + g\xi^1)^2 (d\xi^0)^2 + (d\xi^1)^2 \quad (58)$$

We can change the form to

$$d\tau = ds = d\xi^0 \sqrt{-(1 + g\xi^1)^2 + \left(\frac{d\xi^1}{d\xi^0}\right)^2} \quad (59)$$

We can now plug in  $\xi^1 = \xi_{\text{em}}^1$  and since emitter does not move in this frame we can set  $\frac{d\xi^1}{d\xi^0} = 0$ :

$$d\tau_{\text{em}} = d\xi_{\text{em}}^0 (1 + g\xi_{\text{em}}^1) \quad (60)$$

We can integrate both sides and obtain equation for finite differences

$$\Delta\tau_{\text{em}} = \Delta\xi_{\text{em}}^0 (1 + g\xi_{\text{em}}^1) \quad (61)$$

We can do similar thing with  $\xi_{\text{rec}}^1$ :

$$\Delta\tau_{\text{rec}} = \Delta\xi_{\text{rec}}^0 (1 + g\xi_{\text{rec}}^1) \quad (62)$$

But left sides of above equations are equal (since line element is invariant under changing of coordinates) and we can compare them:

$$\frac{\Delta\xi_{\text{rec}}^0}{\Delta\xi_{\text{em}}^0} = \frac{1 + g\xi_{\text{em}}^1}{1 + g\xi_{\text{rec}}^1} = 1 + \frac{g\xi_{\text{em}}^1 - g\xi_{\text{rec}}^1}{1 + g\xi_{\text{rec}}^1} = 1 - \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \quad (63)$$

where I put  $h = \xi_{\text{rec}}^1 - \xi_{\text{em}}^1$ . After rearranging terms and substituting  $\Delta\xi_{\text{rec}}^1 = \frac{1}{\nu'}$  and  $\Delta\xi_{\text{em}}^1 = \frac{1}{\nu}$

$$\frac{\Delta\xi_{\text{em}}^0 - \Delta\xi_{\text{rec}}^0}{\Delta\xi_{\text{em}}^0} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \quad (64)$$

$$\frac{\frac{1}{\nu} - \frac{1}{\nu'}}{\frac{1}{\nu}} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \Rightarrow \boxed{z = \frac{\nu' - \nu}{\nu'} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1}} \quad (65)$$

We can now assume that  $g$  is small and using Taylor expansion  $\frac{1}{1+x} \simeq 1 - x$

$$z = gh(1 - gh - g\xi_{\text{em}}^1) = gh - (gh)^2 - g^2 h \xi_{\text{em}}^1 \simeq gh$$

$$z = gh \quad (66)$$

so the same result as photon in gravitational field.

# ASSIGNMENT 3

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## Problem 1

Calculate EOM given the Lagrangian

$$\mathcal{L}_{\text{dyn}}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu}[x^\mu(\lambda)] \dot{x}^\mu \dot{x}^\nu, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \quad (67)$$

I calculate first variation of Lagrangian

$$\begin{aligned} \delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda &= \int_{\lambda_1}^{\lambda_2} \left( \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) d\lambda = \\ &= \int_{\lambda_1}^{\lambda_2} \left( \frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{1}{2} g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right) d\lambda \quad (68) \end{aligned}$$

Let's take a look at second part of the integral:

$$\begin{aligned} \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma\} d\lambda &= \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta \dot{x}^\sigma d\lambda = \\ &= \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \frac{\partial}{\partial \lambda} (\{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma) d\lambda - \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \frac{\partial}{\partial \lambda} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma d\lambda = \\ &= \underbrace{\frac{1}{2} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma \Big|_{\lambda_1}^{\lambda_2}}_{=0} - \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{\partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu + g_{\sigma\nu} \ddot{x}^\nu + \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu + g_{\sigma\mu} \ddot{x}^\mu\} \delta x^\sigma d\lambda \quad (69) \end{aligned}$$

Plugging result back into Eq. 68 yields

$$\delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda = \frac{1}{2} \int_{\lambda_1}^{\lambda_2} (\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu) \delta x^\sigma d\lambda \quad (70)$$

We want

$$\delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda = 0 \quad (71)$$

but since  $\delta x^\sigma$  can be arbitrary the rest has to be equal 0, namely

$$\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu = 0 \quad (72)$$

or after rearranging elements

$$\boxed{2g_{\sigma\mu} \ddot{x}^\mu + (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu = 0} \quad (73)$$



## Problem 2

Calculate the EOM given Lagrangian:

$$\mathcal{L}_{\text{geo}}(x^\mu, \dot{x}^\mu) = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \quad (74)$$

First we calculate variation of Lagrangian

$$\begin{aligned} \delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda &= \int_{\lambda_1}^{\lambda_2} \left( \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) d\lambda = \\ &- \int_{\lambda_1}^{\lambda_2} \left( \frac{\dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \delta x^\sigma + \frac{g_{\mu\nu} \dot{x}^\mu \delta_{\nu\sigma}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \dot{x}^\sigma + \frac{g_{\mu\nu} \delta_{\mu\sigma} \dot{x}^\nu}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \dot{x}^\sigma \right) d\lambda = \\ &- \int_{\lambda_1}^{\lambda_2} \frac{1}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} (\dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + g_{\mu\sigma} \dot{x}^\mu \delta \dot{x}^\sigma + g_{\sigma\nu} \dot{x}^\nu \delta \dot{x}^\sigma) d\lambda \quad (75) \end{aligned}$$

I change variable of differentiating and integration from  $d\lambda$  to  $d\tau = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu} d\lambda$ . Derivatives changing as following

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu} \frac{dx^\mu}{d\tau} \quad (76)$$

and integral as following

$$\int d\lambda = \int \frac{d\tau}{\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \quad (77)$$

After plugging in those transformations into Eq. 75 we obtain

$$- \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left( \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \partial_\sigma g_{\mu\nu} \delta x^\sigma + \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta \frac{dx^\sigma}{d\tau} \right) d\tau \quad (78)$$

Now let's look at the second part of this integral and transform it (using Leibniz rule)

$$\begin{aligned} \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta \frac{dx^\sigma}{d\tau} &= \frac{d}{d\tau} \left( \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) - \frac{d}{d\tau} \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma = \\ \frac{d}{d\tau} \left( \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) &- \left\{ \frac{dg_{\mu\sigma}}{dx^\nu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} + g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{dg_{\sigma\nu}}{dx^\mu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} \right\} \delta x^\sigma = \\ \frac{d}{d\tau} \left( \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) &- \left[ g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \{ \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} \} \right] \delta x^\sigma \quad (79) \end{aligned}$$

After plugging it into Eq. 78 we obtain

$$\begin{aligned}
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left( \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{d}{d\tau} \left[ \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right] - \right. \\
& \quad \left. \left[ g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \{ \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} \} \right] \delta x^\sigma \right) d\tau = \\
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left[ -2g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) \right] \delta x^\sigma d\tau - \underbrace{\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma}_{=0} \Big|_{\lambda_1}^{\lambda_2} = \\
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left[ -2g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) \right] \delta x^\sigma d\tau \quad (80)
\end{aligned}$$

But this variation has to be equal zero no matter what the value of  $\delta x^\sigma$  is. Namely

$$-g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{1}{2} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) = 0 \quad (81)$$

Changing sign

$$\boxed{g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{1}{2} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) = 0} \quad (82)$$

### Problem 3

Following metric is given

$$g_{\mu\nu} dx^\mu dx^\nu = - \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] d(ct)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad \mathbf{x} \equiv (x^1, x^2, x^3) \quad (83)$$

Dividing both sides by  $dt^2$  yields

$$\begin{aligned}
g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} &= - \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] \frac{d(ct)^2}{dt^2} + \frac{(dx^1)^2}{dt^2} + \frac{(dx^2)^2}{dt^2} + \frac{(dx^3)^2}{dt^2} = \\
&= - \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] c^2 + \left( \frac{dx^1}{dt} \right)^2 + \left( \frac{dx^2}{dt} \right)^2 + \left( \frac{dx^3}{dt} \right)^2 = - \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] c^2 + \mathbf{v} \cdot \mathbf{v} = \\
&= -c^2 \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2} \right] \quad (84)
\end{aligned}$$

Substituting this into lagrangian

$$\mathcal{L} = -mc \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \quad (85)$$

gives us

$$\boxed{\mathcal{L} = -mc \sqrt{c^2 \left[ 1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2} \right]} = -mc^2 \sqrt{1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2}}} \quad (86)$$

We can now assume that both  $\frac{\phi(\mathbf{x})}{c^2}$  and  $\frac{v^2}{c^2}$  are small. We can Taylor expand square root ( $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ ) and leave only linear terms:

$$\mathcal{L} = -mc^2 \left( 1 + \frac{\phi(\mathbf{x})}{c^2} - \frac{v^2}{2c^2} \right) = -mc^2 - m\phi(\mathbf{x}) + m\frac{v^2}{2} \quad (87)$$

But adding constants (in this case  $mc^2$ ) to Lagrangian doesn't change equation of motions, so effective Lagrangian can be written as

$$\boxed{\mathcal{L} = \frac{mv^2}{2} - m\phi(\mathbf{x})} \quad (88)$$

which is exactly the Lagrangian for classical mechanics, which leads to Newton's law of motion, namely

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi(\mathbf{x}) \quad (89)$$

# ASSIGNMENT 4

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## Problem 1

Parallel transport of a vector  $V = v^\mu \partial_\mu$  along the curve  $s \gamma : \lambda \mapsto [x^1(\lambda), \dots, x^n(\lambda)]$ :

$$\frac{dv^\mu}{d\lambda} + \Gamma_{\nu\sigma}^\mu [x(\lambda)] v^\nu \frac{dx^\sigma}{d\lambda} = 0 \quad (90)$$

We change coordinates, namely  $V = v^\mu \partial_\mu = v^\mu \frac{\partial}{\partial x^\mu} = u^\nu \frac{\partial}{\partial y^\nu}$  and  $\gamma : \lambda' \mapsto [y^1(\lambda'), \dots, y^n(\lambda')]$  First we want to obtain transformation rule for vectors namely

$$v^\nu \frac{\partial y^k}{\partial x^\nu} = V(y^k) = u^\mu \frac{\partial y^k}{\partial y^\mu} = u^k \quad (91)$$

and for  $\Gamma_{\mu\sigma}^\rho = \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\sigma} \frac{\partial x^\rho}{\partial \xi^\mu}$

$$\begin{aligned} \Gamma_{\nu\sigma}^{\rho'} &= \frac{\partial}{\partial y^\sigma} \left( \frac{\partial \xi^\mu}{\partial y^\nu} \right) \frac{\partial y^\rho}{\partial \xi^\mu} = \frac{\partial}{\partial y^\sigma} \left( \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\nu} \right) \frac{\partial y^\rho}{\partial x^\beta} \frac{\partial x^\beta}{\partial \xi^\mu} = \overbrace{\left[ \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial x^\kappa}{\partial y^\sigma} + \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \right]}^{\frac{\partial^2 \xi^\mu}{\partial y^\nu \partial y^\sigma} (*)} \frac{\partial y^\rho}{\partial x^\beta} \frac{\partial x^\beta}{\partial \xi^\mu} = \\ &= \underbrace{\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa} \frac{\partial x^\beta}{\partial \xi^\mu}}_{=\Gamma_{\alpha\kappa}^\beta} \frac{\partial x^\beta}{\partial y^\sigma} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\rho}{\partial x^\beta} + \underbrace{\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \frac{\partial y^\rho}{\partial x^\beta} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \xi^\mu}}_{=\delta_\alpha^\beta} = \Gamma_{\alpha\kappa}^\beta \frac{\partial x^\kappa}{\partial y^\sigma} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \frac{\partial y^\rho}{\partial x^\alpha} \quad (92) \end{aligned}$$

Plugging those things into

$$\frac{du^\mu}{d\lambda'} + \Gamma_{\nu\sigma}^{\mu'} [y(\lambda')] u^\nu \frac{dy^\sigma}{d\lambda'} = 0 \quad (93)$$

we obtain

$$\frac{\partial v^\beta}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} + \left\{ \Gamma_{\alpha\kappa}^\beta [x(\lambda)] \frac{\partial x^\kappa}{\partial y^\sigma} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \frac{\partial y^\mu}{\partial x^\alpha} \right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta} \frac{\partial y^\sigma}{\partial \lambda'} = 0 \quad (94)$$

Let's take a look at third term of this sum

$$\begin{aligned} &\left\{ \Gamma_{\alpha\kappa}^\beta [x(\lambda)] \frac{\partial x^\kappa}{\partial y^\sigma} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \frac{\partial y^\mu}{\partial x^\alpha} \right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta} \underbrace{\frac{\partial y^\sigma}{\partial \lambda'}}_{\frac{\partial y^\sigma}{\partial \lambda'}} = \\ &= v^\eta \Gamma_{\alpha\kappa}^\beta [x(\lambda)] \underbrace{\frac{\partial x^\kappa}{\partial y^\sigma} \frac{\partial y^\sigma}{\partial x^\tau} \frac{\partial x^\tau}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'}}_{=\frac{\partial x^\kappa}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'}} \underbrace{\frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\nu}{\partial x^\eta} \frac{\partial y^\mu}{\partial x^\beta}}_{\delta_\eta^\alpha} + v^\eta \underbrace{\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma} \frac{\partial y^\nu}{\partial x^\eta} \frac{\partial y^\sigma}{\partial x^\tau} \frac{\partial x^\tau}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\alpha}}_{\frac{\partial^2 x^\alpha}{\partial x^\eta \partial x^\tau} - \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau} (*)} = \\ &\Gamma_{\alpha\kappa}^\beta [x(\lambda)] v^\alpha \frac{\partial x^\kappa}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} - v^\eta \underbrace{\frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau} \frac{\partial x^\alpha}{\partial y^\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial x^\tau}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'}}_{=\delta_\nu^\mu} = \Gamma_{\alpha\kappa}^\beta [x(\lambda)] v^\alpha \frac{\partial x^\kappa}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau} \frac{\partial x^\tau}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \quad (95) \end{aligned}$$

So at the end of the day we have

$$\frac{\partial v^\beta}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} + \Gamma_{\alpha\kappa}^\beta [x(\lambda)] v^\alpha \frac{\partial x^\kappa}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau} \frac{\partial x^\tau}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda'} = 0 \quad (96)$$

Which simplifies to

$$\left( \frac{\partial v^\beta}{\partial \lambda} + \Gamma_{\alpha\kappa}^\beta [x(\lambda)] v^\alpha \frac{\partial x^\kappa}{\partial \lambda} \right) \frac{\partial \lambda}{\partial \lambda'} \frac{\partial y^\mu}{\partial x^\beta} = 0 \quad (97)$$

We can divide by  $\frac{\partial \lambda}{\partial \lambda'}$

$$\boxed{\left( \frac{\partial v^\beta}{\partial \lambda} + \Gamma_{\alpha\kappa}^\beta [x(\lambda)] v^\alpha \frac{\partial x^\kappa}{\partial \lambda} \right) \frac{\partial y^\mu}{\partial x^\beta} = 0} \quad (98)$$

So indeed this equation is coordinate-covariant.

## Problem 2

Let  $\gamma_V$  denote the geodesic with tangent vector  $V_p$  at point  $p$ .  $\{e_\mu\}$  is arbitrary basis chosen at the point  $p$  and normal coordinates are defined as  $x(q) = (x^1, \dots, x^n) \Leftrightarrow q = \gamma_{x^\mu e_\mu}$  where  $p = \gamma(\lambda = 0)$ ,  $q = \gamma(\lambda = 1)$  and  $\{x^i\}_{i=1}^n \in \mathbb{R}$ .

First we let  $V_p = v^\mu e_\mu$ . But we know, since we consider geodesic, that  $v^\mu = \frac{dx^\mu}{d\lambda}$ . On the other hand we can write normal coordinates of point  $q$  as  $x(q) = (v^1, \dots, v^n)$ . So we have two conditions

$$\left. \frac{dx^\mu}{d\lambda} \right|_{\lambda=0} = v^\mu \quad \left. x^\mu \right|_{\lambda=1} = v^\mu \quad (99)$$

It is easy to solve this

$$x^\mu(\lambda) = v^\mu \lambda + x_\mu(0) \quad (100)$$

Eq. 100 describes straight line, because it is linear with respect to  $\lambda$ <sup>3</sup>. Substituting this expression into geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = 0 \quad (101)$$

gives

$$\Gamma_{\sigma\rho}^\mu v^\sigma v^\rho = 0 \quad (102)$$

But Eq. 102 has to be satisfied for arbitrary  $v^\sigma$  and  $v^\rho$  which implies

$$\boxed{\Gamma_{\sigma\rho}^\mu = 0} \quad (103)$$

## Problem 3

**A**  $\partial g = 0$

We know that metric transforms as follow:

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \quad (104)$$

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<sup>3</sup>or equivalently  $\frac{d^2 x^i}{d\lambda^2} = 0$

Now choosing  $x^\mu = x'^\mu - \frac{1}{2}M_{\alpha\beta}^\mu x'^\alpha x'^\beta$  will give us

$$\begin{aligned}
g'_{\alpha\beta} &= \frac{\partial(x'^\mu - \frac{1}{2}M_{\lambda\sigma}^\mu x'^\lambda x'^\sigma)}{\partial x'^\alpha} \frac{\partial(x'^\nu - \frac{1}{2}M_{\kappa\eta}^\nu x'^\kappa x'^\eta)}{\partial x'^\beta} g_{\mu\nu} = \\
&\left( \delta_\alpha^\mu - \frac{1}{2}M_{\lambda\sigma}^\mu (\delta_\alpha^\lambda x'^\sigma + x'^\lambda \delta_\alpha^\sigma) \right) \left( \delta_\beta^\nu - \frac{1}{2}M_{\kappa\eta}^\nu (\delta_\beta^\kappa x'^\eta + x'^\kappa \delta_\beta^\eta) \right) g_{\mu\nu} = \\
&\left( \delta_\alpha^\mu - \frac{1}{2}M_{\alpha\sigma}^\mu x'^\sigma - \frac{1}{2}M_{\lambda\alpha}^\mu x'^\lambda \right) \left( \delta_\beta^\nu - \frac{1}{2}M_{\beta\eta}^\nu x'^\eta - \frac{1}{2}M_{\kappa\beta}^\nu x'^\kappa \right) g_{\mu\nu} = \\
&\left( \delta_\alpha^\mu - \frac{1}{2}x'^\sigma (M_{\alpha\sigma}^\mu + M_{\sigma\alpha}^\mu) \right) \left( \delta_\beta^\nu - \frac{1}{2}x'^\eta (M_{\beta\eta}^\nu + M_{\eta\beta}^\nu) \right) g_{\mu\nu} \quad (105)
\end{aligned}$$

We take  $\tilde{M}_{\beta\eta}^\nu = \frac{1}{2}(M_{\beta\eta}^\nu + M_{\eta\beta}^\nu)$  and write (keeping only linear terms in  $x$ )

$$g'_{\alpha\beta} = \left( \delta_\alpha^\mu - x'^\sigma \tilde{M}_{\alpha\sigma}^\mu \right) \left( \delta_\beta^\nu - x'^\eta \tilde{M}_{\beta\eta}^\nu \right) g_{\mu\nu} = g_{\alpha\beta} - g_{\alpha\nu} x'^\eta \tilde{M}_{\beta\eta}^\nu - g_{\mu\beta} x'^\sigma \tilde{M}_{\alpha\sigma}^\mu = \quad (106)$$

Now we differentiate both sides

$$\begin{aligned}
\partial'_\lambda g'_{\alpha\beta} &= \partial'_\lambda g_{\alpha\beta} - \partial'_\lambda (g_{\alpha\nu} x'^\eta \tilde{M}_{\beta\eta}^\nu) - \partial'_\lambda (g_{\mu\beta} x'^\sigma \tilde{M}_{\alpha\sigma}^\mu) = \\
&\partial'_\lambda g_{\alpha\beta} - \partial'_\lambda g_{\alpha\nu} x'^\eta \tilde{M}_{\beta\eta}^\nu - g_{\alpha\nu} \delta_\lambda^\eta \tilde{M}_{\beta\eta}^\nu - \partial'_\lambda g_{\mu\beta} x'^\sigma \tilde{M}_{\alpha\sigma}^\mu - g_{\mu\beta} \delta_\lambda^\sigma \tilde{M}_{\alpha\sigma}^\mu \quad (107)
\end{aligned}$$

Now we drop linear terms in  $x$  (of the form  $x\partial g$ )

$$\begin{aligned}
\partial'_\lambda g'_{\alpha\beta} &= \partial'_\lambda g_{\alpha\beta} - g_{\alpha\nu} \tilde{M}_{\beta\lambda}^\nu - g_{\mu\beta} \tilde{M}_{\alpha\lambda}^\mu = \partial_\tau g_{\alpha\beta} \partial'_\lambda x^\tau - g_{\alpha\nu} \tilde{M}_{\beta\lambda}^\nu - g_{\mu\beta} \tilde{M}_{\alpha\lambda}^\mu = \\
&\partial_\tau g_{\alpha\beta} \left( \delta_\lambda^\tau - x'^\sigma \tilde{M}_{\lambda\sigma}^\tau \right) - g_{\alpha\nu} \tilde{M}_{\beta\lambda}^\nu - g_{\mu\beta} \tilde{M}_{\alpha\lambda}^\mu \simeq \partial_\lambda g_{\alpha\beta} - g_{\alpha\nu} \tilde{M}_{\beta\lambda}^\nu - g_{\mu\beta} \tilde{M}_{\alpha\lambda}^\mu \quad (108)
\end{aligned}$$

Now let's substitute Chrisroffel symbol in place of  $\tilde{M}$  namely

$$\tilde{M}_{\alpha\beta}^\gamma = \frac{1}{2}g^{\gamma\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \quad (109)$$

We obtain (using  $g_{\alpha\beta} = g_{\beta\alpha}$ )

$$\begin{aligned}
2\partial'_\lambda g'_{\alpha\beta} &= 2\partial_\lambda g_{\alpha\beta} - \underbrace{g_{\alpha\nu} g^{\nu\sigma}}_{=\delta_\alpha^\sigma} (\partial_\beta g_{\sigma\lambda} + \partial_\lambda g_{\sigma\beta} - \partial_\sigma g_{\beta\lambda}) - \underbrace{g_{\mu\beta} g^{\mu\sigma}}_{=\delta_\beta^\sigma} (\partial_\alpha g_{\sigma\lambda} + \partial_\lambda g_{\sigma\alpha} - \partial_\sigma g_{\alpha\lambda}) = \\
&2\partial_\lambda g_{\alpha\beta} - \underline{\partial_\beta g_{\alpha\lambda}} - \underline{\partial_\lambda g_{\alpha\beta}} + \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\lambda g_{\beta\alpha}} + \underline{\partial_\beta g_{\alpha\lambda}} = 2\partial_\lambda g_{\alpha\beta} - 2\partial_\lambda g_{\alpha\beta} = 0 \quad (110)
\end{aligned}$$

So eventually

$$\boxed{\partial'_\lambda g'_{\alpha\beta} = 0} \quad (111)$$

**B**  $g = \eta$

We try following change of coordinates

$$x'^\mu = N^\mu_\alpha y^\alpha \quad (112)$$

In those coordinates metric looks like

$$g''_{\alpha\beta} = \frac{\partial x'^\mu}{\partial y^\alpha} \frac{\partial x'^\nu}{\partial y^\beta} g'_{\mu\nu} = N^\mu_\alpha N^\nu_\beta g'_{\mu\nu} = (N^{-1})_\alpha^\mu g'_{\mu\nu} N^\nu_\beta = (N^{-1} g' N)_{\alpha\beta} \quad (113)$$

We can now diagonalize metric  $g'$ . We can write

$$g' = C \eta C^{-1} \quad (114)$$

where  $\eta$  is diagonal and  $C$  is a matrix which consists of eigenvectors of  $g'$ . If we will choose  $N = C$  then Eq. 113 simplifies to

$$\boxed{g''_{\alpha\beta} = \eta_{\alpha\beta}} \quad (115)$$

## ASSIGNMENT 5

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### Problem 1a

We have given metric

$$\mathbf{g} = d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi \quad (116)$$

Only non-zero elements are:

$$g_{11} = 1 \quad g_{22} = \sin^2 \theta \quad (117)$$

Also remember that  $g^{\alpha\beta} = g_{\alpha\beta}^{-1}$ . Let's calculate connection for that metric:

$$\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} g^{\mu\rho} (\partial_{\nu} g_{\rho\sigma} + \partial_{\sigma} g_{\rho\nu} - \partial_{\rho} g_{\nu\sigma}) \quad (118)$$

It is easy to see that

$$\partial_{\nu} g_{\rho\sigma} = \delta_{1\nu} \delta_{2\rho} \delta_{2\sigma} 2 \sin \theta \cos \theta \quad (119)$$

where  $\partial_1 \equiv \partial_{\theta}$  and  $\partial_2 \equiv \partial_{\phi}$ . Substituting Eq. (119) into connection yields

$$\begin{aligned} \Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} g^{\mu\rho} (\delta_{1\nu} \delta_{2\rho} \delta_{2\sigma} 2 \sin \theta \cos \theta + \delta_{1\sigma} \delta_{2\rho} \delta_{2\nu} 2 \sin \theta \cos \theta - \delta_{1\rho} \delta_{2\nu} \delta_{2\sigma} 2 \sin \theta \cos \theta) = \\ \sin \theta \cos \theta (g^{2\mu} \delta_{1\nu} \delta_{2\sigma} + g^{2\mu} \delta_{1\sigma} \delta_{2\nu} - g^{1\mu} \delta_{2\nu} \delta_{2\sigma}) \end{aligned} \quad (120)$$

The only non-zero coefficients

$$\Gamma_{22}^1 = -\sin \theta \cos \theta \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\cos \theta}{\sin \theta} \quad (121)$$

### Problem 1b

Geodesic equation is given by

$$\frac{\partial^2 x^{\sigma}}{\partial \lambda^2} + \Gamma_{\mu\nu}^{\sigma} \frac{\partial x^{\mu}}{\partial \lambda} \frac{\partial x^{\nu}}{\partial \lambda} = 0 \quad (122)$$

Writing explicitly

$$\frac{\partial^2 x^1}{\partial \lambda^2} + \Gamma_{\mu\nu}^1 \frac{\partial x^{\mu}}{\partial \lambda} \frac{\partial x^{\nu}}{\partial \lambda} = 0 \quad (123a)$$

$$\frac{\partial^2 x^2}{\partial \lambda^2} + \Gamma_{\mu\nu}^2 \frac{\partial x^{\mu}}{\partial \lambda} \frac{\partial x^{\nu}}{\partial \lambda} = 0 \quad (123b)$$

After plugging in Eq. (121) we obtain

$$\frac{\partial^2 \theta}{\partial \lambda^2} - \sin \theta \cos \theta \frac{\partial \phi}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} = 0 \quad (124a)$$

$$\frac{\partial^2 \phi}{\partial \lambda^2} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial \theta}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} = 0 \quad (124b)$$

Now we can set  $\theta$  and  $\phi$  as affine parameters



$$\lambda \rightarrow \theta$$

$$-\sin \theta \cos \theta \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} = 0 \quad (125a)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial \phi}{\partial \theta} = 0 \quad (125b)$$

Solution to this set of equations is trivial namely

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \Rightarrow \quad \phi = \text{const} \quad (126)$$

It means that longitudinal lines are geodesics in this metric.

$$\lambda \rightarrow \phi$$

$$\frac{\partial^2 \theta}{\partial \phi^2} - \sin \theta \cos \theta = 0 \quad (127a)$$

$$2 \frac{\cos \theta}{\sin \theta} \frac{\partial \theta}{\partial \phi} = 0 \quad (127b)$$

From second equation we have that

$$\frac{\partial \theta}{\partial \phi} = 0 \quad (128)$$

but it does not solve first equation for every  $\theta$ . This system of equations has solutions only when

$$\sin \theta \cos \theta = 0 \quad \Rightarrow \quad \theta = 0 \vee \theta = \frac{\pi}{2} \vee \theta = \pi \quad (129)$$

since  $\theta \in [0, \pi]$ . But for  $\theta = 0$  or  $\theta = \pi$  geodesic line is just one point, because those are poles. Only  $\theta = \frac{\pi}{2}$  gives non-trivial geodesic. This line is called equator.

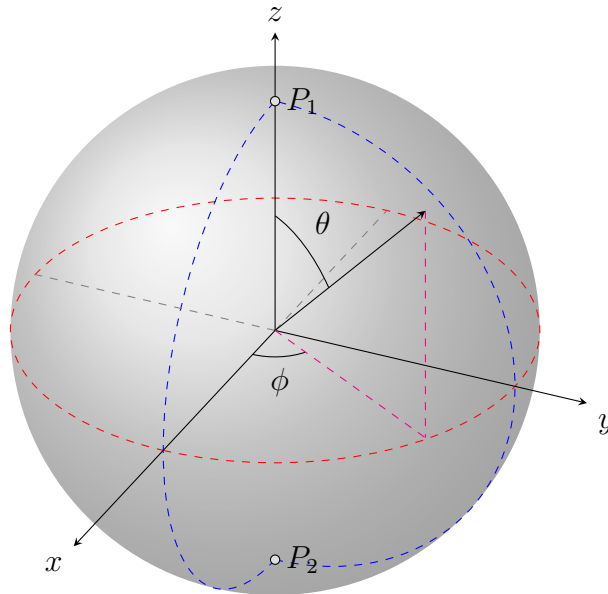


Figure 6: Visualization of geodesics – red line is equator, blue lines are two possible meridians,  $P_1$  and  $P_2$  are poles.

## Problem 1c

$$\theta = \frac{\pi}{2}, \phi = 0 \rightarrow \theta = 0, \phi = 0$$

We use equation of parallel transport of vector  $\mathbf{v}$  along curve  $\gamma$  with  $\frac{d\gamma^\mu}{d\lambda} = w^\mu$

$$\nabla_w \mathbf{v} = 0 \quad (130)$$

We can write it explicitly

$$w^\nu \partial_\nu v^\mu + w^\nu \Gamma_{\nu\sigma}^\mu v^\sigma = 0 \quad (131)$$

or substituting connections I've calculated before

$$w^1 \partial_1 v^1 + w^2 \partial_2 v^1 + w^2 \Gamma_{22}^1 v^2 = 0 \quad (132a)$$

$$w^1 \partial_1 v^2 + w^2 \partial_2 v^2 + w^1 \Gamma_{12}^2 v^2 + w^2 \Gamma_{21}^2 v^1 = 0 \quad (132b)$$

If we move along meridian then we can take  $\lambda \rightarrow \theta$ . From this we get coefficients  $w$  namely  $w^1 = \frac{d\theta}{d\theta} = 1$  and  $w^2 = \frac{d\phi}{d\theta} = 0$ . Putting this and connection coefficients into equations we obtain

$$\partial_1 v^1 = 0 \quad (133a)$$

$$\partial_1 v^2 + \frac{\cos \theta}{\sin \theta} v^2 = 0 \quad (133b)$$

Second equation we multiply by  $\sin \theta$  and simplify

$$\partial_1 v^1 = 0 \quad (134a)$$

$$\partial_1 (v^2 \sin \theta) = 0 \quad (134b)$$

This gives use

$$v^1 = C_1 \quad (135a)$$

$$v^2 = \frac{C_2}{\sin \theta} \quad (135b)$$

Constants  $C_1$  and  $C_2$  depends on the vector we transport:

- for  $\frac{\partial}{\partial \theta}$  we have at the beginning ( $\theta = \frac{\pi}{2}$ )  $v = (1, 0)$  so  $C_1 = 1$  and  $C_2 = 0$  so the transport

$$\mathbf{v} = (1, 0) \rightarrow (1, 0) = \mathbf{v}' \quad (136)$$

does not change this vector.

- for  $\frac{\partial}{\partial \phi}$  we have at the beginning  $v = (0, 1)$  so  $C_1 = 0$  and  $C_2 = 1$  so the transport

$$\mathbf{v} = (0, 1) \rightarrow (0, \frac{1}{\sin \theta}) = \mathbf{v}' \quad (137)$$

is undefined at point  $\theta = 0$

$$\theta = \frac{\pi}{2}, \phi = 0 \rightarrow \theta = \frac{\pi}{2}, \phi = \frac{\pi}{4}$$

If we move along equator then we can take  $\lambda \rightarrow \phi$ . From this we get coefficients  $w$  namely  $w^1 = \frac{d\theta}{d\phi} = 0$  and  $w^2 = \frac{d\phi}{d\phi} = 1$ . Putting this and connection coefficients into equations we obtain

$$\partial_2 v^1 - \sin \theta \cos \theta v^2 = 0 \quad (138a)$$

$$\partial_2 v^2 + \frac{\cos \theta}{\sin \theta} v^1 = 0 \quad (138b)$$

which simplifies to

$$\partial_2 v^1 = 0 \quad (139a)$$

$$\partial_2 v^2 = 0 \quad (139b)$$

because  $\cos \theta \Big|_{\theta=\frac{\pi}{2}} = 0$

$$v^1 = C_1 \quad (140a)$$

$$v^2 = C_2 \quad (140b)$$

Constants  $C_1$  and  $C_2$  depends on the vector we transport:

- for  $\frac{\partial}{\partial \theta}$  we have at the beginning ( $\phi = 0$ )  $v = (1, 0)$  so  $C_1 = 1$  and  $C_2 = 0$  so the transport

$$\mathbf{v} = (1, 0) \rightarrow (1, 0) = \mathbf{v}' \quad (141)$$

does not change this vector.

- for  $\frac{\partial}{\partial \phi}$  we have at the beginning  $v = (0, 1)$  so  $C_1 = 0$  and  $C_2 = 1$  so the transport

$$\mathbf{v} = (0, 1) \rightarrow (0, 1) = \mathbf{v}' \quad (142)$$

does not change this vector.

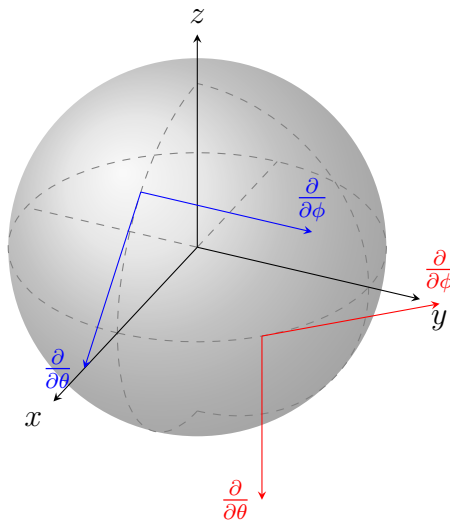


Figure 7: Visualization of parallel transport – red is transport along equator, blue is transport along meridians.

## Problem 2

Metric is given by

$$g_{\nu\mu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (143)$$

We can calculate connection coefficients using relation

$$\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} g^{\mu\rho} (\partial_{\nu} g_{\rho\sigma} + \partial_{\sigma} g_{\rho\nu} - \partial_{\rho} g_{\nu\sigma}) \quad (144)$$

Because metric is diagonal we can simplify this expression:

$$\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} g^{\mu\mu} (\partial_{\nu} g_{\mu\sigma} + \partial_{\sigma} g_{\mu\nu} - \partial_{\mu} g_{\nu\sigma}) \quad (145)$$

$$\Gamma^0 = -\frac{1}{2} \frac{1}{B(r)} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (146a)$$

$$\Gamma^1 = \frac{1}{2} \frac{1}{A(r)} \begin{pmatrix} -\dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r \sin^2 \theta \end{pmatrix} \quad (146b)$$

$$\Gamma^2 = \frac{1}{2} \frac{1}{r^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & -r^2 2 \sin \theta \cos \theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{pmatrix} \quad (146c)$$

$$\Gamma^3 = \frac{1}{2} \frac{1}{r^2 \sin^2 \theta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r \sin^2 \theta \\ 0 & 0 & 0 & r^2 2 \sin \theta \cos \theta \\ 0 & 2r \sin^2 \theta & r^2 2 \sin \theta \cos \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos \theta}{\sin \theta} \\ 0 & \frac{1}{r} & \frac{\cos \theta}{\sin \theta} & 0 \end{pmatrix} \quad (146d)$$

Geodesic equation

$$\frac{\partial^2 x^{\sigma}}{\partial \lambda^2} + \Gamma_{\mu\nu}^{\sigma} \frac{\partial x^{\mu}}{\partial \lambda} \frac{\partial x^{\nu}}{\partial \lambda} = 0 \quad (147)$$

$\sigma = 0$

$$\frac{\partial^2 t}{\partial \lambda^2} - \frac{1}{2} \frac{1}{B(t)} \begin{pmatrix} \frac{\partial t}{\partial \lambda} & \frac{\partial r}{\partial \lambda} & \frac{\partial \theta}{\partial \lambda} & \frac{\partial \phi}{\partial \lambda} \end{pmatrix} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial \lambda} \\ \frac{\partial r}{\partial \lambda} \\ \frac{\partial \theta}{\partial \lambda} \\ \frac{\partial \phi}{\partial \lambda} \end{pmatrix} = 0 \quad (148)$$

$$\frac{\partial^2 t}{\partial \lambda^2} - \frac{1}{2} \frac{1}{B(t)} \left( -2\dot{B} \frac{\partial t}{\partial \lambda} \frac{\partial r}{\partial \lambda} \right) = 0 \quad (149)$$

$$\frac{\partial^2 t}{\partial \lambda^2} + \frac{\dot{B}}{B} \frac{\partial t}{\partial \lambda} \frac{\partial r}{\partial \lambda} = 0 \quad (150)$$

$$\sigma = 1$$

$$\frac{\partial^2 r}{\partial \lambda^2} + \frac{1}{2} \frac{1}{A(r)} \begin{pmatrix} \frac{\partial t}{\partial \lambda} & \frac{\partial r}{\partial \lambda} & \frac{\partial \theta}{\partial \lambda} & \frac{\partial \phi}{\partial \lambda} \end{pmatrix} \begin{pmatrix} -\dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r \sin^2 \theta \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial \lambda} \\ \frac{\partial r}{\partial \lambda} \\ \frac{\partial \theta}{\partial \lambda} \\ \frac{\partial \phi}{\partial \lambda} \end{pmatrix} = 0 \quad (151)$$

$$\frac{\partial^2 r}{\partial \lambda^2} - \frac{\dot{B}}{2A} \left( \frac{\partial t}{\partial \lambda} \right)^2 + \frac{\dot{A}}{2A} \left( \frac{\partial r}{\partial \lambda} \right)^2 - \frac{r}{A} \left( \frac{\partial \theta}{\partial \lambda} \right)^2 - \frac{r \sin^2 \theta}{A} \left( \frac{\partial \phi}{\partial \lambda} \right)^2 = 0 \quad (152)$$

$$\sigma = 2$$

$$\frac{\partial^2 \theta}{\partial \lambda^2} + \begin{pmatrix} \frac{\partial t}{\partial \lambda} & \frac{\partial r}{\partial \lambda} & \frac{\partial \theta}{\partial \lambda} & \frac{\partial \phi}{\partial \lambda} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial \lambda} \\ \frac{\partial r}{\partial \lambda} \\ \frac{\partial \theta}{\partial \lambda} \\ \frac{\partial \phi}{\partial \lambda} \end{pmatrix} = 0 \quad (153)$$

$$\frac{\partial^2 \theta}{\partial \lambda^2} + \frac{2}{r} \frac{\partial r}{\partial \lambda} \frac{\partial \theta}{\partial \lambda} - \sin \theta \cos \theta \left( \frac{\partial \phi}{\partial \lambda} \right)^2 = 0 \quad (154)$$

$$\sigma = 2$$

$$\frac{\partial^2 \phi}{\partial \lambda^2} + \begin{pmatrix} \frac{\partial t}{\partial \lambda} & \frac{\partial r}{\partial \lambda} & \frac{\partial \theta}{\partial \lambda} & \frac{\partial \phi}{\partial \lambda} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos \theta}{\sin \theta} \\ 0 & \frac{1}{r} & \frac{\cos \theta}{\sin \theta} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial \lambda} \\ \frac{\partial r}{\partial \lambda} \\ \frac{\partial \theta}{\partial \lambda} \\ \frac{\partial \phi}{\partial \lambda} \end{pmatrix} = 0 \quad (155)$$

$$\frac{\partial^2 \phi}{\partial \lambda^2} + \frac{2}{r} \frac{\partial r}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial \theta}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} = 0 \quad (156)$$

Gathering all equations

$$0 = \frac{\partial^2 t}{\partial \lambda^2} + \frac{\dot{B}}{B} \frac{\partial t}{\partial \lambda} \frac{\partial r}{\partial \lambda} \quad (157a)$$

$$0 = \frac{\partial^2 r}{\partial \lambda^2} - \frac{\dot{B}}{2A} \left( \frac{\partial t}{\partial \lambda} \right)^2 + \frac{\dot{A}}{2A} \left( \frac{\partial r}{\partial \lambda} \right)^2 - \frac{r}{A} \left( \frac{\partial \theta}{\partial \lambda} \right)^2 - \frac{r \sin^2 \theta}{A} \left( \frac{\partial \phi}{\partial \lambda} \right)^2 \quad (157b)$$

$$0 = \frac{\partial^2 \theta}{\partial \lambda^2} + \frac{2}{r} \frac{\partial r}{\partial \lambda} \frac{\partial \theta}{\partial \lambda} - \sin \theta \cos \theta \left( \frac{\partial \phi}{\partial \lambda} \right)^2 \quad (157c)$$

$$0 = \frac{\partial^2 \phi}{\partial \lambda^2} + \frac{2}{r} \frac{\partial r}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial \theta}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} \quad (157d)$$