ASSIGMENT 1

Problem 5

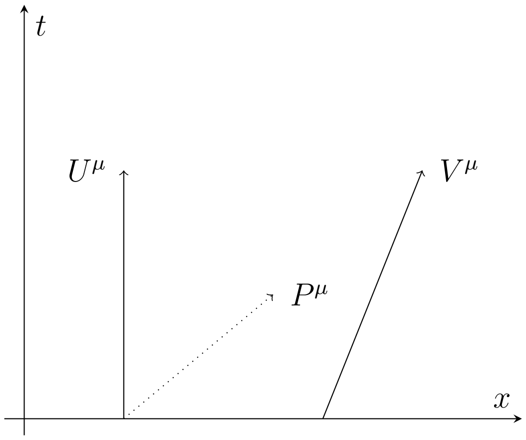

Figure 1: Setup of experiment

We have given:

$$U^\mu = (c, \mathbf{0}) \tag{1}$$

$$V^\mu = (\gamma_v c, \gamma_v \boldsymbol{v}) \tag{2}$$

$$P^\mu = \left(\frac{h\nu}{c}, \boldsymbol{p}\right) \tag{3}$$

We use following relation in this problem:

$$E = -P^\mu V_\mu \tag{4}$$

This expression is Lorentz invariant and can be calculated in non-moving frame. So we plug in Eq. 2 in this expression to obtain

$$E = -P^\mu V_\mu = \frac{h\nu}{c}\gamma_v c - \gamma_v \boldsymbol{v}\boldsymbol{p} = \gamma_v \left(h\nu - |\boldsymbol{v}||\boldsymbol{p}|\cos(\theta)\right) = \left\{|\boldsymbol{p}| = \frac{h\nu}{c}\right\} = \gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{5}$$

But it is still photon, but with different energy (for moving observer) So

$$\gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) = h\nu' \tag{6}$$

So ratio of those two frequencies is

$$\frac{\nu'}{\nu} = \gamma_v \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{7}$$

If $\theta = 0$ and $\frac{v}{c} \ll 1 \Rightarrow \gamma_v \simeq 1$ then we obtain:

$$\nu' = \nu\left(1 - \frac{v}{c}\right) \tag{8}$$

ASSIGMENT 2

Problem 1a

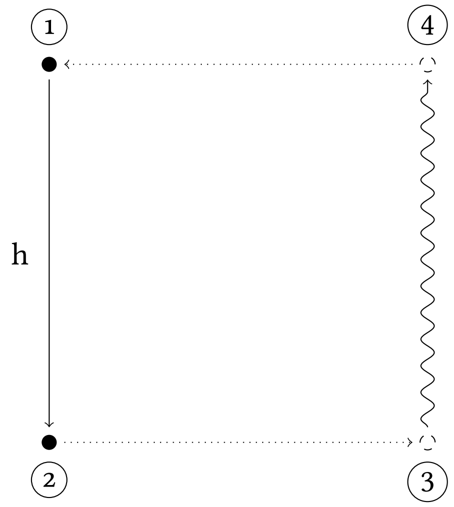

Figure 2: Mass falling in graitational field (1→2), converting to photon (2→3), photon traveling up (3→4) and converting back to mass (4→1)

Let's take a look at energy changes in above diagram:

1 $E_1 = mc^2$

2 $E_2 = mc^2 + mgh$

3 $E_3 = h\nu = mc^2 + mgh$

4 $E_4 = h\nu = mc^2 + mgh$

but $E_4 = E_1$ because of energy conservation. It means that photon has to have different frequency at the height h than it has at the ground. So $E_4 = h\nu' = mc^2$. From it follows

$$\frac{\nu}{\nu'} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \tag{9}$$

and it is easy to calculate redshift

$$z = \frac{\nu - \nu'}{\nu'} = \frac{gh}{c^2} \tag{10}$$

Problem 1b

Let's calculate time which light needs to reach observer 2

$$t = \frac{s}{c} = \frac{h - \frac{gt^2}{2}}{c} \tag{11}$$

From this expression we get quadratic equation

$$\frac{g}{2}t^2 + ct - h = 0 \tag{12}$$

for which solution is given by

$$t = \frac{-c + \sqrt{c^2 + 2gh}}{g} \tag{13}$$

Velocity of observer 2 after this time is equal

$$v(t) = \frac{-c + \sqrt{c^2 + 2gh}}{g} \cdot g = -c + \sqrt{c^2 + 2gh} \tag{14}$$

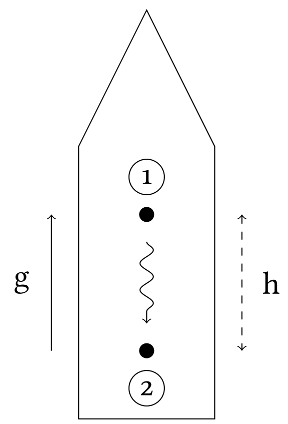

Figure 3: Two observers in a rocket sending photon

Then redshift formula is given in following way

$$\frac{\nu'}{\nu} = 1 - \frac{v}{c} = 1 - \frac{-c + \sqrt{c^2 + 2gh}}{c} = 2 - \sqrt{1 - \frac{2gh}{c^2}} \tag{15}$$

We can use Taylor expansion $\sqrt{1-x} = 1 - \frac{x}{2}$ we get

$$\frac{\nu'}{\nu} = 2 - 1 + \frac{gh}{c^2} = 1 + \frac{gh}{c^2} \tag{16}$$

It is exactly the same result as Eq. 10.

Problem 2

Observer $\mathcal{O}$ is traveling with acceleration g in direction x_1. To calculate his worldline we will use following three conditions

$$U^\mu U_\mu = -1 \qquad U^\mu A_\mu = 0 \qquad A^\mu A_\mu = g^2 \tag{17}$$

where U^μ is four-velocity and A^μ is four-acceleration. First of them can be obtained by straightforward calculation, second by applying derivative to first equation i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(U^\mu U_\mu) = 0 \quad \Rightarrow \quad (A^\mu U_\mu) = 0 \tag{18}$$

Third is Lorentz invariant and it can be calculated in the moment of launch namely when $A^\mu = (0, g, 0, 0)$.

Knowing those three we can write them in explicite form

$$-U_0^2 + \boldsymbol{U}^2 = -1 \qquad \boldsymbol{U}\boldsymbol{A} = U_0 A_0 \qquad -A_0^2 + \boldsymbol{A}^2 = g^2 \tag{19}$$

where bolded letters mean three-vectors.

We square middle equation and plug in left and right equation to obtained

$$(U_0^2 - 1)\boldsymbol{A}^2 = U_0^2(\boldsymbol{A}^2 - g^2) \tag{20}$$

Eventually we obtain:

$$\boldsymbol{A}^2 = g^2 U_0^2 \tag{21}$$

and plugin this expression to other equation we also obtain:[1]

$$A_0^2 = g^2 \boldsymbol{U}^2 \tag{22}$$

We can simplify those equation using the fact that this motion is one dimensional namely $x_2 = x_3 = 0$ and then

$$A_1 = gU_0 \qquad A_0 = gU_1 \tag{23}$$

But $U^\mu = \dot{X}^\mu$ and $A^\mu = \ddot{X}^\mu$ [2]. Substituting

$$\ddot{X}_1 = g\dot{X}_0 \qquad \ddot{X}_0 = g\dot{X}_1 \tag{24}$$

Taking a derivative of left equation and substituting right equation into it we get

$$\dddot{X}_1 = g^2 \dot{X}_1 \quad \overset{\text{after integration}}{\Rightarrow} \quad \ddot{X}_1 = g^2 X_1 \tag{25}$$

Solution is

$$X_1 = A\sinh(g\tau) + B\cosh(g\tau) \tag{26}$$

Let's choose initial conditions such as $X_1(0) = g^{-1}$ and $\dot{X}_1 = 0$. Then

$$X_1 = g^{-1}\cosh(g\tau) \tag{27}$$

And finally we have

$$X_0 = g^{-1}\sinh(g\tau) \qquad X_1 = g^{-1}\cosh(g\tau) \qquad X_2 = 0 \qquad X_3 = 0 \tag{28}$$

[1] plug it into right equation and then use left equation

[2] dot means derivation with respect to proper time

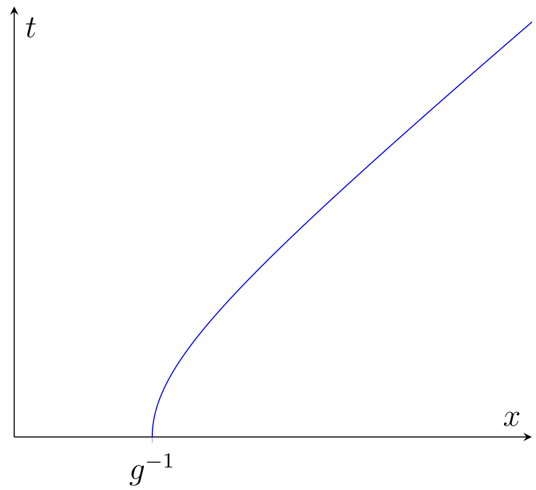

Figure 4: Trajectory of $\mathcal{O}$

Problem 3

As a first basis vector we can choose four-velocity namely

$$\boldsymbol{e}_0 = \left(\dot{X}_0, \dot{X}_1, \dot{X}_2, \dot{X}_3\right) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{29}$$

As a basis vectors in directions x_2 and x_3 we simply choose

$$\boldsymbol{e}_2 = (0, 0, 1, 0) \tag{30}$$

$$\boldsymbol{e}_3 = (0, 0, 0, 1) \tag{31}$$

And finally we choose vector $\boldsymbol{e}_1$ in a form $\boldsymbol{e}_1 = (e_1^0, e_1^1, 0, 0)$ where e_1^0 and e_1^1 are chosen in order to satisfy $\boldsymbol{e}_0\boldsymbol{e}_1 = 0$ and $(\boldsymbol{e}_0)^2 = 1$ i.e.

$$-e_1^0 \cosh(g\tau) + e_1^1 \sinh(g\tau) = 0 \tag{32}$$

$$-(e_1^0)^2 + (e_1^1)^2 = 1 \tag{33}$$

We square first equation and substitute second equation

$$(e_1^0)^2 \cosh^2(g\tau) = (1 + (e_1^0)^2) \sinh^2(g\tau) \tag{34}$$

From this we obtain

$$(e_1^0)^2 = \sinh^2(g\tau) \qquad\qquad (e_1^1)^2 = \cosh^2(g\tau) \tag{35}$$

We can choose positive solution and eventually we get

$$\boldsymbol{e}_1 = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{36}$$

All vectors

$$\boldsymbol{e}_0(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{37}$$

$$\boldsymbol{e}_1(\tau) = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{38}$$

$$\boldsymbol{e}_2(\tau) = (0, 0, 1, 0) \tag{39}$$

$$\boldsymbol{e}_3(\tau) = (0, 0, 0, 1) \tag{40}$$

Last thing to do is to check whether those are vectors which were obtain without any rotation. For this I will find a Lorentz boost which transforms initial basis into this one. Namely consider a boost of time-basis vector

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 0 \end{pmatrix} \tag{41}$$

So γ and β have to satisfy:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh(g\tau) \quad \Rightarrow \quad v = \tanh(g\tau) \tag{42}$$

Knowing that it is easy to calculate

$$\beta\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh(g\tau) \tag{43}$$

So indeed we obtain vector $\boldsymbol{e}_0(\tau)$ only via boost (at $v = \tanh(g\tau)$). The same can be done with vector $\boldsymbol{e}_1(\tau)$

Problem 4

We define new coordinate system $(\xi_0 \equiv \tau, \xi_1, \xi_2, \xi_3)$ where basis vectors are those defined in problem before. We can write

$$\boldsymbol{x} = \xi^1 \boldsymbol{e}_1(\tau) + \xi^2 \boldsymbol{e}_2(\tau) + \xi^3 \boldsymbol{e}_3(\tau) + \boldsymbol{x}_{\mathcal{O}}(\tau) \tag{44}$$

where $\boldsymbol{x}_{\mathcal{O}}(\tau)$ is trajectory of moving frame.

After plugging in all basis vectors explicitly we get

$$\boldsymbol{x} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \xi^1 \sinh(g\tau) \\ \xi^1 \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \xi^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^3 \end{pmatrix} + \begin{pmatrix} g^{-1} \sinh(g\tau) \\ g^{-1} \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} g^{-1} \sinh(g\tau) + \xi^1 \sinh(g\tau) \\ g^{-1} \cosh(g\tau) + \xi^1 \cosh(g\tau) \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} (g^{-1} + \xi^1) \sinh(g\xi^0) \\ (g^{-1} + \xi^1) \cosh(g\xi^0) \\ \xi^2 \\ \xi^3 \end{pmatrix} \tag{45}$$

Line element $\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ is then equal (we use chain rule i.e. $\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} \mathrm{d}\xi^\nu$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{46}$$

$$\mathrm{d}t = \frac{\partial t}{\partial \xi^\nu} \mathrm{d}\xi^\nu = (1 + g\xi_1) \cosh(g\xi_0) \mathrm{d}\xi_0 + \sinh(g\xi_0) \mathrm{d}\xi_1 \tag{47}$$

$$\mathrm{d}x_1 = (1 + g\xi_1) \sinh(g\xi_0) \mathrm{d}\xi_0 + \cosh(g\xi_0) \mathrm{d}\xi_1 \tag{48}$$

$$\mathrm{d}x_2 = \mathrm{d}\xi_2 \tag{49}$$

$$\mathrm{d}x_3 = \mathrm{d}\xi_3 \tag{50}$$

After squaring and adding them up we get

$$\begin{aligned}\mathrm{d}s^2 = &-(1+g\xi_1)^2\cosh^2(g\xi_0)\mathrm{d}\xi_0^2 - \sinh^2(g\xi_0)\mathrm{d}\xi_1^2 + \\ &(1+g\xi_1)^2\sinh^2(g\xi_0)\mathrm{d}\xi_0^2 + \cosh^2(g\xi_0)\mathrm{d}\xi_1^2 + \\ &\mathrm{d}\xi_2^2 + \\ &\mathrm{d}\xi_3^2\end{aligned} \tag{51}$$

After simplification

$$\mathrm{d}s^2 = -(1+g\xi_1)^2\mathrm{d}\xi_0^2 + \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2 \tag{52}$$

Problem 5

For $\xi^1 \equiv$ const we can easily derive equation of motion from Eq. 45 namely

$$x_1^2 - t^2 = (g^{-1} + \xi^1)^2 \tag{53}$$

which leads to

$$x_1(t) = \sqrt{(g^{-1}+\xi^1)^2 + t^2} \tag{54}$$

We take derivative twice

$$\dot{x_1}(t) = \frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}} \tag{55}$$

$$\ddot{x_1}(t) = \frac{\sqrt{(g^{-1}+\xi^1)^2+t^2} - t\frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}}}{(g^{-1}+\xi^1)^2+t^2} = \frac{1}{\sqrt{(g^{-1}+\xi^1)^2+t^2}} - \frac{2t^2}{((g^{-1}+\xi^1)^2+t^2)^{\frac{3}{2}}} \tag{56}$$

So when $t = 0$

$$\ddot{x_1}(t)\Big|_{t=0} = \frac{1}{g^{-1}+\xi^1} = \frac{g}{1+g\xi^1} \tag{57}$$

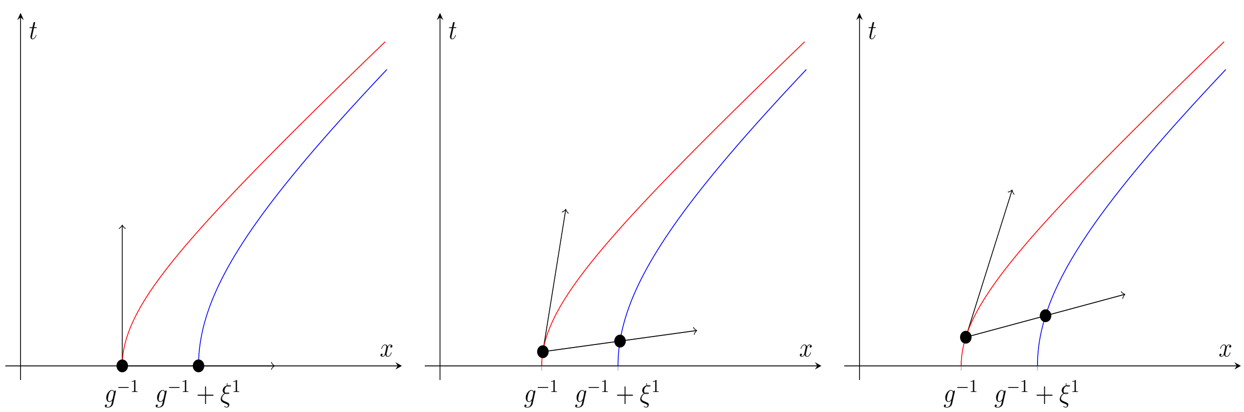

Figure 5: Red line is worldline of Eq. 28 and blue is worldline of Eq. 54

Problem 6

We start with equation Eq. 52. We can simplify it and neglect other spatial dimensions than ξ^1 namely

$$\mathrm{d}s^2 = -(1+g\xi^1)^2(\mathrm{d}\xi^0)^2 + (\mathrm{d}\xi^1)^2 \tag{58}$$

We can change the form to

$$d\tau = ds = d\xi^0 \sqrt{-(1+g\xi^1)^2 + \left(\frac{d\xi^1}{d\xi^0}\right)^2} \tag{59}$$

We can now plug in $\xi^1 = \xi^1_{\text{em}}$ and since emiter does not move in this frame we can set $\frac{d\xi^1}{d\xi^0} = 0$:

$$d\tau_{\text{em}} = d\xi^0_{\text{em}}(1 + g\xi^1_{\text{em}}) \tag{60}$$

We can integrate both sides and obtain equation for finite differences

$$\Delta\tau_{\text{em}} = \Delta\xi^0_{\text{em}}(1 + g\xi^1_{\text{em}}) \tag{61}$$

We can do similar thing with ξ^1_{rec}:

$$\Delta\tau_{\text{rec}} = \Delta\xi^0_{\text{rec}}(1 + g\xi^1_{\text{rec}}) \tag{62}$$

But left sides of above equations are equal (since line element is invariant under changing of coordinates) and we can compare them:

$$\frac{\Delta\xi^0_{\text{rec}}}{\Delta\xi^0_{\text{em}}} = \frac{1 + g\xi^1_{\text{em}}}{1 + g\xi^1_{\text{rec}}} = 1 + \frac{g\xi^1_{\text{em}} - g\xi^1_{\text{rec}}}{1 + g\xi^1_{\text{rec}}} = 1 - \frac{gh}{1 + gh + g\xi^1_{\text{em}}} \tag{63}$$

where I put $h = \xi^1_{\text{rec}} - \xi^1_{\text{em}}$. After rearranging terms and substituting $\Delta\xi^1_{\text{rec}} = \frac{1}{\nu'}$ and $\Delta\xi^1_{\text{em}} = \frac{1}{\nu}$

$$\frac{\Delta\xi^0_{\text{em}} - \Delta\xi^0_{\text{rec}}}{\Delta\xi^0_{\text{em}}} = \frac{gh}{1 + gh + g\xi^1_{\text{em}}} \tag{64}$$

$$\frac{\frac{1}{\nu} - \frac{1}{\nu'}}{\frac{1}{\nu}} = \frac{gh}{1 + gh + g\xi^1_{\text{em}}} \quad \Rightarrow \quad z = \frac{\nu' - \nu}{\nu'} = \frac{gh}{1 + gh + g\xi^1_{\text{em}}} \tag{65}$$

We can now assume that g is small and using Taylor expansion $\frac{1}{1+x} \simeq 1 - x$

$$z = gh(1 - gh - g\xi^1_{\text{em}}) = gh - (gh)^2 - g^2 h\xi^1_{\text{em}} \simeq gh$$

$$z = gh \tag{66}$$

so the same result as photon in gravitational field.