Assignment 1

Problem 5

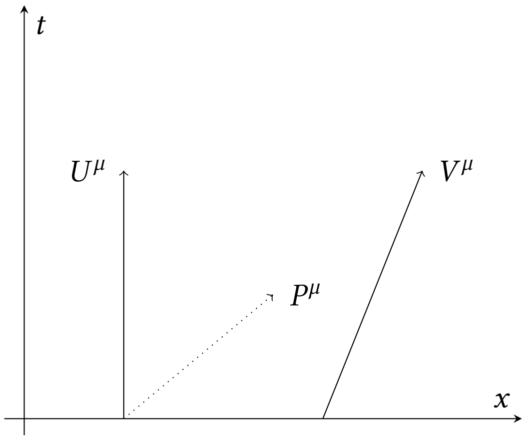

Figure 1.1: Setup of experiment

We have given:

$$U^\mu = (c, \mathbf{0}) \tag{1.1}$$

$$V^\mu = (\gamma_v c, \gamma_v \boldsymbol{v}) \tag{1.2}$$

$$P^\mu = \left(\frac{h\nu}{c}, \boldsymbol{p}\right) \tag{1.3}$$

We use following relation in this problem:

$$E = -P^\mu V_\mu \tag{1.4}$$

This expression is Lorentz invariant and can be calculated in non-moving frame. So we plug in Eq. 1.2 in this expression to obtain

$$E = -P^\mu V_\mu = \frac{h\nu}{c}\gamma_v c - \gamma_v \boldsymbol{v}\boldsymbol{p} = \gamma_v \left(h\nu - |\boldsymbol{v}||\boldsymbol{p}|\cos(\theta)\right) = \left\{|\boldsymbol{p}| = \frac{h\nu}{c}\right\} =$$

$$\gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{1.5}$$

But it is still photon, but with different energy (for moving observer) So

$$\gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) = h\nu' \tag{1.6}$$

So ratio of those two frequencies is

$$\frac{\nu'}{\nu} = \gamma_v \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{1.7}$$

If $\theta = 0$ and $\frac{v}{c} \ll 1 \Rightarrow \gamma_v \simeq 1$ then we obtain:

$$\boxed{\nu' = \nu\left(1 - \frac{v}{c}\right)} \tag{1.8}$$

Assignment 2

Problem 1

Problem 1a

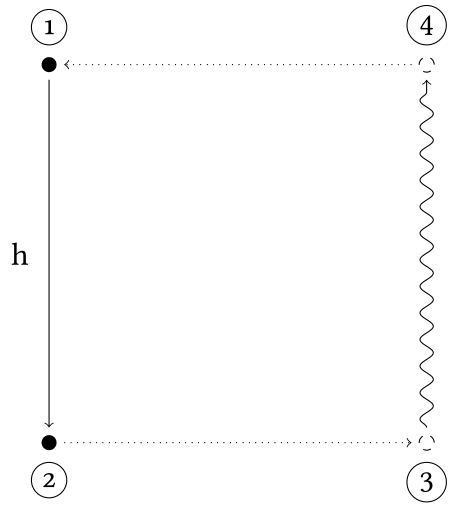

Figure 2.1: Mass falling in graitational field (1⟶2), converting to photon (2⟶3), photon traveling up (3⟶4) and converting back to mass (4⟶1)

Let's take a look at energy changes in above diagram:

1. $E_1 = mc^2$
2. $E_2 = mc^2 + mgh$
3. $E_3 = h\nu = mc^2 + mgh$
4. $E_4 = h\nu = mc^2 + mgh$

but $E_4 = E_1$ because of energy conservation. It means that photon has to have different frequency at the height h than it has at the ground. So $E_4 = h\nu' = mc^2$. From it follows

$$\frac{\nu}{\nu'} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \tag{2.1}$$

and it is easy to calculate redshift

$$\boxed{z = \frac{\nu - \nu'}{\nu'} = \frac{gh}{c^2}} \tag{2.2}$$

Problem 1b

Let's calculate time which light needs to reach observer 2

$$t = \frac{s}{c} = \frac{h - \frac{gt^2}{2}}{c} \tag{2.3}$$

From this expression we get quadratic equation

$$\frac{g}{2}t^2 + ct - h = 0 \tag{2.4}$$

for which solution is given by

$$t = \frac{-c + \sqrt{c^2 + 2gh}}{g} \tag{2.5}$$

Velocity of observer 2 after this time is equal

$$v(t) = \frac{-c + \sqrt{c^2 + 2gh}}{g} \cdot g = -c + \sqrt{c^2 + 2gh} \tag{2.6}$$

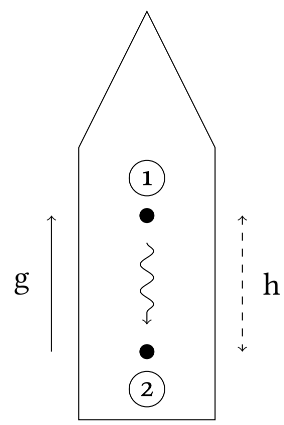

Figure 2.2: Two observers in a rocket sending photon

Then redshift formula is given in following way

$$\frac{\nu'}{\nu} = 1 - \frac{v}{c} = 1 - \frac{-c + \sqrt{c^2 + 2gh}}{c} = 2 - \sqrt{1 - \frac{2gh}{c^2}} \tag{2.7}$$

We can use Taylor expansion $\sqrt{1-x} = 1 - \frac{x}{2}$ we get

$$\boxed{\frac{\nu'}{\nu} = 2 - 1 + \frac{gh}{c^2} = 1 + \frac{gh}{c^2}} \tag{2.8}$$

It is exactly the same result as Eq. 2.2.

Problem 2

Observer O is traveling with acceleration g in direction x_1. To calculate his worldline we will use following three conditions

$$U^\mu U_\mu = -1 \qquad U^\mu A_\mu = 0 \qquad A^\mu A_\mu = g^2 \tag{2.9}$$

where U^μ is four-velocity and A^μ is four-acceleration. First of them can be obtained by straightforward calculation, second by applying derivative to first equation i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(U^\mu U_\mu\right) = 0 \quad \Rightarrow \quad \left(A^\mu U_\mu\right) = 0 \tag{2.10}$$

Third is Lorentz invariant and it can be calculated in the moment of launch namely when $A^\mu = (0, g, 0, 0)$.

Knowing those three we can write them in explicite form

$$-U_0^2 + \boldsymbol{U}^2 = -1 \qquad \boldsymbol{U}\boldsymbol{A} = U_0 A_0 \qquad -A_0^2 + \boldsymbol{A}^2 = g^2 \tag{2.11}$$

where bolded letters mean three-vectors.

We square middle equation and plug in left and right equation to obtained

$$(U_0^2 - 1)\boldsymbol{A}^2 = U_0^2(\boldsymbol{A}^2 - g^2) \tag{2.12}$$

Eventually we obtain:

$$\boldsymbol{A}^2 = g^2 U_0^2 \tag{2.13}$$

and plugin this expression to other equation we also obtain:[1]

$$A_0^2 = g^2 \boldsymbol{U}^2 \tag{2.14}$$

We can simplify those equation using the fact that this motion is one dimensional namely $x_2 = x_3 = 0$ and then

$$A_1 = gU_0 \qquad A_0 = gU_1 \tag{2.15}$$

But $U^\mu = \dot{X}^\mu$ and $A^\mu = \ddot{X}^\mu$ [2]. Substituting

$$\ddot{X}_1 = g\dot{X}_0 \qquad \ddot{X}_0 = g\dot{X}_1 \tag{2.16}$$

Taking a derivative of left equation and substituting right equation into it we get

$$\dddot{X}_1 = g^2 \dot{X}_1 \quad \overset{\text{after integration}}{\Rightarrow} \quad \ddot{X}_1 = g^2 X_1 \tag{2.17}$$

Solution is

$$X_1 = A\sinh(g\tau) + B\cosh(g\tau) \tag{2.18}$$

Let's choose initial conditions such as $X_1(0) = g^{-1}$ and $\dot{X}_1 = 0$. Then

$$X_1 = g^{-1}\cosh(g\tau) \tag{2.19}$$

And finally we have

$$X_0 = g^{-1}\sinh(g\tau) \qquad X_1 = g^{-1}\cosh(g\tau) \qquad X_2 = 0 \qquad X_3 = 0 \tag{2.20}$$

[1] plug it into right equation and then use left equation
[2] dot means derivation with respect to proper time

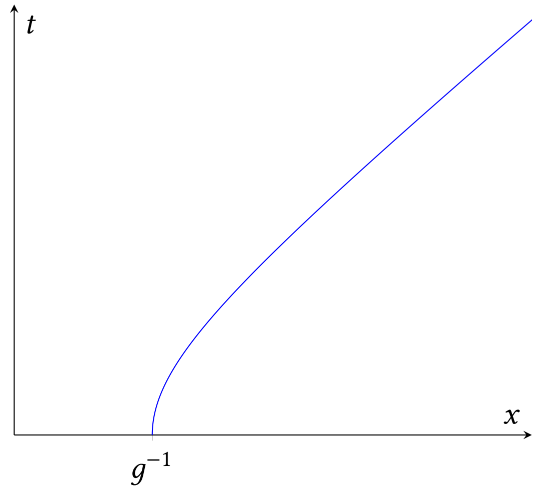

Figure 2.3: Trajectory of O

Problem 3

As a first basis vector we can choose four-velocity namely

$$\boldsymbol{e}_0 = \left(\dot{X}_0, \dot{X}_1, \dot{X}_2, \dot{X}_3\right) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{2.21}$$

As a basis vectors in directions x_2 and x_3 we simply choose

$$\boldsymbol{e}_2 = (0, 0, 1, 0) \tag{2.22}$$

$$\boldsymbol{e}_3 = (0, 0, 0, 1) \tag{2.23}$$

And finally we choose vector $\boldsymbol{e}_1$ in a form $\boldsymbol{e}_1 = \left(e_1^0, e_1^1, 0, 0\right)$ where e_1^0 and e_1^1 are chosen in order to satisfy $\boldsymbol{e}_0\boldsymbol{e}_1 = 0$ and $(\boldsymbol{e}_0)^2 = 1$ i.e.

$$-e_1^0 \cosh(g\tau) + e_1^1 \sinh(g\tau) = 0 \tag{2.24}$$

$$-(e_1^0)^2 + (e_1^1)^2 = 1 \tag{2.25}$$

We square first equation and substitute second equation

$$(e_1^0)^2 \cosh^2(g\tau) = (1 + (e_1^0)^2) \sinh^2(g\tau) \tag{2.26}$$

From this we obtain

$$(e_1^0)^2 = \sinh^2(g\tau) \qquad\qquad (e_1^1)^2 = \cosh^2(g\tau) \tag{2.27}$$

We can choose positive solution and eventually we get

$$\boldsymbol{e}_1 = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{2.28}$$

All vectors

$$\boldsymbol{e}_0(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{2.29}$$

$$\boldsymbol{e}_1(\tau) = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{2.30}$$

$$\boldsymbol{e}_2(\tau) = (0, 0, 1, 0) \tag{2.31}$$

$$\boldsymbol{e}_3(\tau) = (0, 0, 0, 1) \tag{2.32}$$

Last thing to do is to check whether those are vectors which were obtain without any rotation. For this I will find a Lorentz boost which transforms initial basis into this one. Namely consider a boost of time-basis vector

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 0 \end{pmatrix} \tag{2.33}$$

So γ and β have to satisfy:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh(g\tau) \quad \Rightarrow \quad v = \tanh(g\tau) \tag{2.34}$$

Knowing that it is easy to calculate

$$\beta\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh(g\tau) \tag{2.35}$$

So indeed we obtain vector $\boldsymbol{e}_0(\tau)$ only via boost (at $v = \tanh(g\tau)$). The same can be done with vector $\boldsymbol{e}_1(\tau)$

Problem 4

We define new coordinate system ($\xi_0 \equiv \tau, \xi_1, \xi_2, \xi_3$) where basis vectors are those defined in problem before. We can write

$$\boldsymbol{x} = \xi^1 \boldsymbol{e}_1(\tau) + \xi^2 \boldsymbol{e}_2(\tau) + \xi^3 \boldsymbol{e}_3(\tau) + \boldsymbol{x}_O(\tau) \tag{2.36}$$

where $\boldsymbol{x}_O(\tau)$ is trajectory of moving frame.

After plugging in all basis vectors explicitly we get

$$\boldsymbol{x} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \xi^1 \sinh(g\tau) \\ \xi^1 \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \xi^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^3 \end{pmatrix} + \begin{pmatrix} g^{-1} \sinh(g\tau) \\ g^{-1} \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} g^{-1} \sinh(g\tau) + \xi^1 \sinh(g\tau) \\ g^{-1} \cosh(g\tau) + \xi^1 \cosh(g\tau) \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} (g^{-1} + \xi^1) \sinh(g\xi^0) \\ (g^{-1} + \xi^1) \cosh(g\xi^0) \\ \xi^2 \\ \xi^3 \end{pmatrix} \tag{2.37}$$

Line element $\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ is then equal (we use chain rule i.e. $\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} \mathrm{d}\xi^\nu$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{2.38}$$

$$\mathrm{d}t = \frac{\partial t}{\partial \xi^\nu} \mathrm{d}\xi^\nu = (1 + g\xi_1) \cosh(g\xi_0)\, \mathrm{d}\xi_0 + \sinh(g\xi_0)\, \mathrm{d}\xi_1 \tag{2.39}$$

$$\mathrm{d}x_1 = (1 + g\xi_1) \sinh(g\xi_0)\, \mathrm{d}\xi_0 + \cosh(g\xi_0)\, \mathrm{d}\xi_1 \tag{2.40}$$

$$\mathrm{d}x_2 = \mathrm{d}\xi_2 \tag{2.41}$$

$$\mathrm{d}x_3 = \mathrm{d}\xi_3 \tag{2.42}$$

After squaring and adding them up we get

$$\begin{aligned} ds^2 = &-(1+g\xi_1)^2\cosh^2(g\xi_0)d\xi_0^2 - \sinh^2(g\xi_0)d\xi_1^2 + \\ &(1+g\xi_1)^2\sinh^2(g\xi_0)d\xi_0^2 + \cosh^2(g\xi_0)d\xi_1^2 + \\ &d\xi_2^2 + \\ &d\xi_3^2 \end{aligned} \tag{2.43}$$

After simplification

$$\boxed{ds^2 = -(1+g\xi_1)^2 d\xi_0^2 + d\xi_1^2 + d\xi_2^2 + d\xi_3^2} \tag{2.44}$$

Problem 5

For $\xi^1 \equiv \text{const}$ we can easily derive equation of motion from Eq. 2.37 namely

$$x_1^2 - t^2 = (g^{-1} + \xi^1)^2 \tag{2.45}$$

which leads to

$$x_1(t) = \sqrt{(g^{-1}+\xi^1)^2 + t^2} \tag{2.46}$$

We take derivative twice

$$\dot{x}_1(t) = \frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}} \tag{2.47}$$

$$\ddot{x}_1(t) = \frac{\sqrt{(g^{-1}+\xi^1)^2+t^2} - t\frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}}}{(g^{-1}+\xi^1)^2+t^2} = \frac{1}{\sqrt{(g^{-1}+\xi^1)^2+t^2}} - \frac{2t^2}{((g^{-1}+\xi^1)^2+t^2)^{\frac{3}{2}}} \tag{2.48}$$

So when $t = 0$

$$\boxed{\ddot{x}_1(t)\Big|_{t=0} = \frac{1}{g^{-1}+\xi^1} = \frac{g}{1+g\xi^1}} \tag{2.49}$$

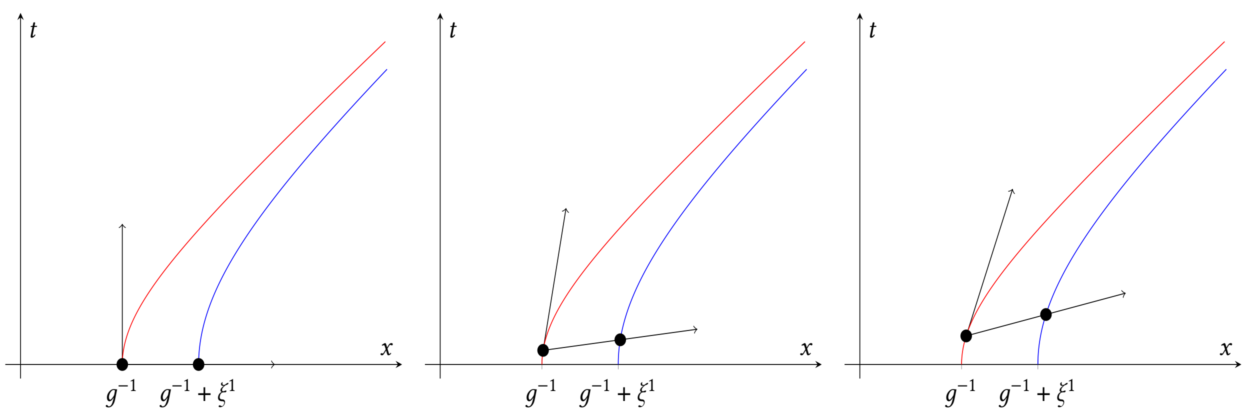

Figure 2.4: Red line is worldline of Eq. 2.20 and blue is worldline of Eq. 2.46

Problem 6

We start with equation Eq. 2.44. We can simplify it and neglect other spatial dimensions than ξ^1 namely

$$ds^2 = -(1+g\xi^1)^2(d\xi^0)^2 + (d\xi^1)^2 \tag{2.50}$$

We can change the form to

$$d\tau = ds = d\xi^0\sqrt{-(1+g\xi^1)^2 + \left(\frac{d\xi^1}{d\xi^0}\right)^2} \tag{2.51}$$

We can now plug in $\xi^1 = \xi^1_{\text{em}}$ and since emiter does not move in this frame we can set $\frac{d\xi^1}{d\xi^0} = 0$:

$$d\tau_{\text{em}} = d\xi^0_{\text{em}}(1+g\xi^1_{\text{em}}) \tag{2.52}$$

We can integrate both sides and obtain equation for finite differences

$$\Delta\tau_{\text{em}} = \Delta\xi^0_{\text{em}}(1+g\xi^1_{\text{em}}) \tag{2.53}$$

We can do similar thing with ξ^1_{rec}:

$$\Delta\tau_{\text{rec}} = \Delta\xi^0_{\text{rec}}(1+g\xi^1_{\text{rec}}) \tag{2.54}$$

But left sides of above equations are equal (since line element is invariant under changing of coordinates) and we can compare them:

$$\frac{\Delta\xi^0_{\text{rec}}}{\Delta\xi^0_{\text{em}}} = \frac{1+g\xi^1_{\text{em}}}{1+g\xi^1_{\text{rec}}} = 1 + \frac{g\xi^1_{\text{em}} - g\xi^1_{\text{rec}}}{1+g\xi^1_{\text{rec}}} = 1 - \frac{gh}{1+gh+g\xi^1_{\text{em}}} \tag{2.55}$$

where I put $h = \xi^1_{\text{rec}} - \xi^1_{\text{em}}$. After rearranging terms and substituting $\Delta\xi^1_{\text{rec}} = \frac{1}{\nu'}$ and $\Delta\xi^1_{\text{em}} = \frac{1}{\nu}$

$$\frac{\Delta\xi^0_{\text{em}} - \Delta\xi^0_{\text{rec}}}{\Delta\xi^0_{\text{em}}} = \frac{gh}{1+gh+g\xi^1_{\text{em}}} \tag{2.56}$$

$$\frac{\frac{1}{\nu} - \frac{1}{\nu'}}{\frac{1}{\nu}} = \frac{gh}{1+gh+g\xi^1_{\text{em}}} \quad \Rightarrow \quad \boxed{z = \frac{\nu'-\nu}{\nu'} = \frac{gh}{1+gh+g\xi^1_{\text{em}}}} \tag{2.57}$$

We can now assume that g is small and using Taylor expansion $\frac{1}{1+x} \simeq 1-x$

$$z = gh(1-gh-g\xi^1_{\text{em}}) = gh - (gh)^2 - g^2h\xi^1_{\text{em}} \simeq gh$$

$$z = gh \tag{2.58}$$

so the same result as photon in gravitational field.

Assignment 3

Problem 1

Calculate EOM given the Lagrangian

$$\mathcal{L}_{\text{dyn}}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu \dot{x}^\nu, \quad \dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{3.1}$$

I calculate first variation of Lagrangian

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu) \mathrm{d}\lambda = \int_{\lambda 1}^{\lambda 2} \left(\frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) \mathrm{d}\lambda =$$

$$\int_{\lambda 1}^{\lambda 2} \left(\frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{1}{2} g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right) \mathrm{d}\lambda \tag{3.2}$$

Let's take a look at second part of the integral:

$$\frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right\} \mathrm{d}\lambda = \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta \dot{x}^\sigma \mathrm{d}\lambda =$$

$$\frac{1}{2} \int_{\lambda 1}^{\lambda 2} \frac{\partial}{\partial \lambda} \left(\left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \right) \mathrm{d}\lambda - \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \frac{\partial}{\partial \lambda} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \mathrm{d}\lambda =$$

$$\underbrace{\frac{1}{2} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \Big|_{\lambda 1}^{\lambda 2}}_{=0} - \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu + g_{\sigma\nu} \ddot{x}^\nu + \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu + g_{\sigma\mu} \ddot{x}^\mu \right\} \delta x^\sigma \mathrm{d}\lambda \tag{3.3}$$

Plugging result back into Eq. 3.2 yields

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu) \mathrm{d}\lambda = \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left(\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu \right) \delta x^\sigma \mathrm{d}\lambda \tag{3.4}$$

We want

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu) \mathrm{d}\lambda = 0 \tag{3.5}$$

but since δx^σ can be arbitrary the rest has to be equal 0, namely

$$\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu = 0 \tag{3.6}$$

or after rearranging elements

$$\boxed{2 g_{\sigma\mu} \ddot{x}^\mu + \left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu = 0} \tag{3.7}$$

Problem 2

Calculate the EOM given Lagrangian:

$$\mathcal{L}_{\text{geo}}(x^\mu, \dot{x}^\mu) = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}, \quad \dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{3.8}$$

First we calculate variation of Lagrangian

$$\delta\int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu,\dot{x}^\mu)\mathrm{d}\lambda = \int_{\lambda 1}^{\lambda 2}\left(\frac{\partial\mathcal{L}(x^\mu,\dot{x}^\mu)}{\partial x^\sigma}\delta x^\sigma + \frac{\partial\mathcal{L}(x^\mu,\dot{x}^\mu)}{\partial\dot{x}^\sigma}\delta\dot{x}^\sigma\right)\mathrm{d}\lambda =$$

$$-\int_{\lambda 1}^{\lambda 2}\left(\frac{\dot{x}^\mu\dot{x}^\nu\partial_\sigma g_{\mu\nu}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\delta x^\sigma + \frac{g_{\mu\nu}\dot{x}^\mu\delta_{\nu\sigma}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\dot{x}^\sigma + \frac{g_{\mu\nu}\delta_{\mu\sigma}\dot{x}^\nu}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\dot{x}^\sigma\right)\mathrm{d}\lambda =$$

$$-\int_{\lambda 1}^{\lambda 2}\frac{1}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\left(\dot{x}^\mu\dot{x}^\nu\partial_\sigma g_{\mu\nu}\delta x^\sigma + g_{\mu\sigma}\dot{x}^\mu\delta\dot{x}^\sigma + g_{\sigma\nu}\dot{x}^\nu\delta\dot{x}^\sigma\right)\mathrm{d}\lambda \tag{3.9}$$

I change variable of differentiating and integration from $\mathrm{d}\lambda$ to $\mathrm{d}\tau = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda$. Derivatives changing as following

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{3.10}$$

and integral as following

$$\int\mathrm{d}\lambda = \int\frac{\mathrm{d}\tau}{\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \tag{3.11}$$

After plugging in those transformations into Eq. 3.9 we obtain

$$-\frac{1}{2}\int_{\lambda 1}^{\lambda 2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\sigma g_{\mu\nu}\delta x^\sigma + \left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{3.12}$$

Now let's look at the second part of this integral and transform it (using Leibniz rule)

$$\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma =$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \left\{\frac{\mathrm{d}g_{\mu\sigma}}{\mathrm{d}x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}g_{\sigma\nu}}{\mathrm{d}x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\delta x^\sigma =$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \left[g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left\{\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu}\right\}\right]\delta x^\sigma \tag{3.13}$$

After plugging it into Eq. 3.12 we obtain

$$
-\frac{1}{2}\int_{\lambda 1}^{\lambda 2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\sigma g_{\mu\nu}\delta x^\sigma+\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}+g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right]-\right.
$$

$$
\left.\left[g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left\{\partial_\nu g_{\mu\sigma}+\partial_\mu g_{\sigma\nu}\right\}\right]\delta x^\sigma\right)\mathrm{d}\tau=
$$

$$
-\frac{1}{2}\int_{\lambda 1}^{\lambda 2}\left[-2g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)\right]\delta x^\sigma\mathrm{d}\tau-\underbrace{\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}+g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\Bigg|_{\lambda 1}^{\lambda 2}}_{=0}=
$$

$$
-\frac{1}{2}\int_{\lambda 1}^{\lambda 2}\left[-2g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)\right]\delta x^\sigma\mathrm{d}\tau \tag{3.14}
$$

But this variation has to be equal zero no matter what the value of δx^σ is. Namely

$$
-g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{1}{2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)=0 \tag{3.15}
$$

Changing sign

$$
\boxed{g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{1}{2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\nu g_{\mu\sigma}+\partial_\mu g_{\sigma\nu}-\partial_\sigma g_{\mu\nu}\right)=0} \tag{3.16}
$$

Problem 3

Following metric is given

$$
g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]\mathrm{d}(ct)^2+(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2,\quad \boldsymbol{x}\equiv(x^1,x^2,x^3) \tag{3.17}
$$

Dividing both sides by $\mathrm{d}t^2$ yields

$$
g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]\frac{\mathrm{d}(ct)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^1)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^2)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^3)^2}{\mathrm{d}t^2}=
$$

$$
-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]c^2+\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x^3}{\mathrm{d}t}\right)^2=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]c^2+\boldsymbol{v}\cdot\boldsymbol{v}=
$$

$$
-c^2\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}\right] \tag{3.18}
$$

Substituting this into lagrangian

$$
\mathcal{L}=-mc\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \tag{3.19}
$$

gives us

$$
\boxed{\mathcal{L}=-mc\sqrt{c^2\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}\right]}=-mc^2\sqrt{1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}}} \tag{3.20}
$$

We can now assume that both $\frac{\phi(\boldsymbol{x})}{c^2}$ and $\frac{\boldsymbol{v}^2}{c^2}$ are small. We can taylor expand square root ($\sqrt{1+x} \simeq 1 + \frac{1}{2x}$) and leave only linear terms:

$$\mathcal{L} = -mc^2 \left(1 + \frac{\phi(\boldsymbol{x})}{c^2} - \frac{\boldsymbol{v}^2}{2c^2}\right) = -mc^2 - m\phi(\boldsymbol{x}) + m\frac{\boldsymbol{v}^2}{2} \tag{3.21}$$

But adding constants (in this case mc^2) to Lagrangian doesn't change equation of motions, so effective Lagrangian can be written as

$$\boxed{\mathcal{L} = \frac{m\boldsymbol{v}^2}{2} - m\phi(\boldsymbol{x})} \tag{3.22}$$

which is exactly the Lagrangian for classical mechanics, which leads to Newton's law of motion, namely

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\nabla\phi(\boldsymbol{x}) \tag{3.23}$$

Assignment 4

Problem 1

Parallel transport of a vector $V = v^\mu \partial_\mu$ along the curve s $\gamma : \lambda \mapsto \left[x^1(\lambda), \dots, x^n(\lambda)\right]$:

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\sigma}\left[x(\lambda)\right] v^\nu \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{4.1}$$

We change coordinates, namely $V = v^\mu \partial_\mu = v^\mu \frac{\partial}{\partial x^\mu} = u^\nu \frac{\partial}{\partial y^\nu}$ and $\gamma : \lambda' \mapsto \left[y^1(\lambda'), \dots, y^n(\lambda')\right]$ First we want to obtain transormation rule for vectors namely

$$v^\nu \frac{\partial y^k}{\partial x^\nu} = V(y^k) = u^\mu \frac{\partial y^k}{\partial y^\mu} = u^k \tag{4.2}$$

and for $\Gamma^\rho_{\mu\sigma} = \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\sigma} \frac{\partial x^\rho}{\partial \xi^\mu}$

$$\Gamma'^\rho_{\nu\sigma} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial \xi^\mu} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} = \overbrace{\left[\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\kappa}{\partial y^\sigma} + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\right]}^{\frac{\partial^2 \xi^\mu}{\partial y^\nu \partial y^\sigma}\ (*)}\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} =$$

$$\underbrace{\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\Gamma^\beta_{\alpha\kappa}}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\beta}\underbrace{\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\delta^\beta_\alpha} = \Gamma^\beta_{\alpha\kappa}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\alpha} \tag{4.3}$$

Plugging those things into

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\lambda'} + \Gamma'^\mu_{\nu\sigma}\left[y(\lambda')\right] u^\nu \frac{\mathrm{d}y^\sigma}{\mathrm{d}\lambda'} = 0 \tag{4.4}$$

we obtain

$$\frac{\partial v^\beta}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} + \left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial \lambda'} = 0 \tag{4.5}$$

Let's take a look at third term of this sum

$$\left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\underbrace{\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{\frac{\partial y^\sigma}{\partial \lambda'}} =$$

$$v^\eta \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\underbrace{\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{=\frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\eta}}_{\delta^\alpha_\eta}\frac{\partial y^\mu}{\partial x^\beta} + v^\eta \underbrace{\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial x^\tau}}_{\frac{\partial^2 x^\alpha}{\partial x^\eta \partial x^\tau} - \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\ (*)}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\alpha} =$$

$$\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\alpha}}_{=\delta^\mu_\nu}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} = \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} \tag{4.6}$$

So at the end of the day we have

$$\frac{\partial v^\beta}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda'}\frac{\partial y^\mu}{\partial x^\beta}+v^\beta\frac{\partial^2 y^\mu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda'}+\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]v^\alpha\frac{\partial x^\kappa}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda'}\frac{\partial y^\mu}{\partial x^\beta}-v^\eta\frac{\partial^2 y^\mu}{\partial x^\eta\partial x^\tau}\frac{\partial x^\tau}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda'}=0 \tag{4.7}$$

Which simplifies to

$$\left(\frac{\partial v^\beta}{\partial\lambda}+\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]v^\alpha\frac{\partial x^\kappa}{\partial\lambda}\right)\frac{\partial\lambda}{\partial\lambda'}\frac{\partial y^\mu}{\partial x^\beta}=0 \tag{4.8}$$

We can divide by $\frac{\partial\lambda}{\partial\lambda'}$

$$\boxed{\left(\frac{\partial v^\beta}{\partial\lambda}+\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]v^\alpha\frac{\partial x^\kappa}{\partial\lambda}\right)\frac{\partial y^\mu}{\partial x^\beta}=0} \tag{4.9}$$

So indeed this equation is coordinate–covariant.

Problem 2

Let γ_V denote the geodesic with tangent vector $\boldsymbol{V}_p$ at point p. $\{\boldsymbol{e}_\mu\}$ is arbitrary basis chosen at the point p and normal coordinates are defined as $x(q)=(x^1,\ldots,x^n)\Leftrightarrow q=\gamma_{x^\mu\boldsymbol{e}_\mu}$ where $p=\gamma(\lambda=0)$, $q=\gamma(\lambda=1)$ and $\{x^i\}_{i=1}^n\in\mathbb{R}$.

First we let $\boldsymbol{V}_p=v^\mu\boldsymbol{e}_\mu$. But we know, since we consider geodesic, that $v^\mu=\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. On the other hand we can write normal coordinates of point q as $x(q)=(v^1,\ldots,v^n)$. So we have two conditions

$$\left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_{\lambda=0}=v^\mu \qquad \left.x^\mu\right|_{\lambda=1}=v^\mu \tag{4.10}$$

It is easy to solve this

$$x^\mu(\lambda)=v^\mu\lambda+x_\mu(0) \tag{4.11}$$

Eq. 4.11 describes straight line, because it is linear with respect to λ [1]. Substituting this expression into geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}=0 \tag{4.12}$$

gives

$$\Gamma^\mu_{\sigma\rho}v^\sigma v^\rho=0 \tag{4.13}$$

But Eq. 4.13 has to be satisfied for arbitrary v^σ and v^ρ which implies

$$\boxed{\Gamma^\mu_{\sigma\rho}=0} \tag{4.14}$$

Problem 3

A $\partial g=0$

We know that metric transforms as follow:

$$g'_{\alpha\beta}=\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}g_{\mu\nu} \tag{4.15}$$

[1] or equivalently $\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2}=0$

Now we choose transformation $x^\mu = x'^\mu - \frac{1}{2}M^\mu_{\alpha\beta}x'^\alpha x'^\beta$ with condition $x^\mu(p) = 0$. In point p we have $x'^\mu = \frac{1}{2}M^\mu_{\alpha\beta}x'^\alpha x'^\beta$. We see that $x'^\mu = 0$ is (one of the) solution to this equation. We choose this solution. This transformation will give us (M is taken in point p so it derivative is equal to zero)

$$g'_{\alpha\beta} = \frac{\partial(x'^\mu - \frac{1}{2}M^\mu_{\lambda\sigma}x'^\lambda x'^\sigma)}{\partial x'^\alpha}\frac{\partial(x'^\nu - \frac{1}{2}M^\nu_{\kappa\eta}x'^\kappa x'^\eta)}{\partial x'^\beta}g_{\mu\nu} =$$

$$\left(\delta^\mu_\alpha - \frac{1}{2}M^\mu_{\lambda\sigma}(\delta^\lambda_\alpha x'^\sigma + x'^\lambda\delta^\sigma_\alpha)\right)\left(\delta^\nu_\beta - \frac{1}{2}M^\nu_{\kappa\eta}(\delta^\kappa_\beta x'^\eta + x'^\kappa\delta^\eta_\beta)\right)g_{\mu\nu} =$$

$$\left(\delta^\mu_\alpha - \frac{1}{2}M^\mu_{\alpha\sigma}x'^\sigma - \frac{1}{2}M^\mu_{\lambda\alpha}x'^\lambda\right)\left(\delta^\nu_\beta - \frac{1}{2}M^\nu_{\beta\eta}x'^\eta - \frac{1}{2}M^\nu_{\kappa\beta}x'^\kappa\right)g_{\mu\nu} =$$

$$\left(\delta^\mu_\alpha - \frac{1}{2}x'^\sigma(M^\mu_{\alpha\sigma} + M^\mu_{\sigma\alpha})\right)\left(\delta^\nu_\beta - \frac{1}{2}x'^\eta(M^\nu_{\beta\eta} + M^\nu_{\eta\beta})\right)g_{\mu\nu} \quad (4.16)$$

We take $\tilde{M}^\nu_{\beta\eta} = \frac{1}{2}(M^\nu_{\beta\eta} + M^\nu_{\eta\beta})$ and write

$$g'_{\alpha\beta} = \left(\delta^\mu_\alpha - x'^\sigma\tilde{M}^\mu_{\alpha\sigma}\right)\left(\delta^\nu_\beta - x'^\eta\tilde{M}^\nu_{\beta\eta}\right)g_{\mu\nu} = g_{\alpha\beta} - g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta} - g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma} + x'^\sigma x'^\eta\tilde{M}^\mu_{\alpha\sigma}\tilde{M}^\nu_{\beta\eta} \quad (4.17)$$

Now we take derivative of both sides in point p (remember that $x(p) = x'(p) = 0$)

$$\partial'_\lambda g'_{\alpha\beta} = \partial'_\lambda g_{\alpha\beta} - \partial'_\lambda(g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta}) - \partial'_\lambda(g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma}) =$$

$$\left(\delta^\tau_\lambda - x'^\sigma\tilde{M}^\tau_{\lambda\sigma}\right)\partial_\tau g_{\alpha\beta} - \partial'_\lambda g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta} - g_{\alpha\nu}\delta^\eta_\lambda\tilde{M}^\nu_{\beta\eta} - \partial'_\lambda g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma} - g_{\mu\beta}\delta^\sigma_\lambda\tilde{M}^\mu_{\alpha\sigma} =$$

$$\partial_\lambda g_{\alpha\beta} - g_{\alpha\nu}\delta^\eta_\lambda\tilde{M}^\nu_{\beta\eta} - g_{\mu\beta}\delta^\sigma_\lambda\tilde{M}^\mu_{\alpha\sigma} = \partial_\lambda g_{\alpha\beta} - g_{\alpha\nu}\tilde{M}^\nu_{\beta\lambda} - g_{\mu\beta}\tilde{M}^\mu_{\alpha\lambda} \quad (4.18)$$

Now let's substitute Chrisroffel symbol in place of $\tilde{M}$ namely

$$\tilde{M}^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \quad (4.19)$$

We obtain (using $g_{\alpha\beta} = g_{\beta\alpha}$)

$$2\partial'_\lambda g'_{\alpha\beta} = 2\partial_\lambda g_{\alpha\beta} - \underbrace{g_{\alpha\nu}g^{\nu\sigma}}_{=\delta^\sigma_\alpha}(\partial_\beta g_{\sigma\lambda} + \partial_\lambda g_{\sigma\beta} - \partial_\sigma g_{\beta\lambda}) - \underbrace{g_{\mu\beta}g^{\mu\sigma}}_{=\delta^\sigma_\beta}(\partial_\alpha g_{\sigma\lambda} + \partial_\lambda g_{\sigma\alpha} - \partial_\sigma g_{\alpha\lambda}) =$$

$$2\partial_\lambda g_{\alpha\beta} - \underline{\partial_\beta g_{\alpha\lambda}} - \underline{\partial_\lambda g_{\alpha\beta}} + \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\lambda g_{\beta\alpha}} + \underline{\partial_\beta g_{\alpha\lambda}} = 2\partial_\lambda g_{\alpha\beta} - 2\partial_\lambda g_{\alpha\beta} = 0 \quad (4.20)$$

So eventually

$$\boxed{\partial'_\lambda g'_{\alpha\beta} = 0} \quad (4.21)$$

B $\quad g = \eta$

We try following change of coordinates

$$x'^\mu = N^\mu{}_\alpha y^\alpha \quad (4.22)$$

In those coordinates metric looks like

$$g''_{\alpha\beta} = \frac{\partial x'^\mu}{\partial y^\alpha}\frac{\partial x'^\nu}{\partial y^\beta}g'_{\mu\nu} = N^\mu{}_\alpha N^\nu{}_\beta g'_{\mu\nu} = (N^{-1})_\alpha{}^\mu g'_{\mu\nu}N^\nu{}_\beta = (N^{-1}g'N)_{\alpha\beta} \quad (4.23)$$

We can now diagonalize metric g'. We can write

$$g' = C\,\eta\,C^{-1} \tag{4.24}$$

where η is diagonal and C is a matrix which consists of eigenvectors of g'. If we will choose $N = C$ then Eq. 4.23 simplifies to

$$\boxed{g''_{\alpha\beta} = \eta_{\alpha\beta}} \tag{4.25}$$

Assignment 5

Problem 1

Problem 1a

We have given metric

$$\boldsymbol{g} = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi \tag{5.1}$$

Only non-zero elements are:

$$g_{11} = 1 \qquad\qquad g_{22} = \sin^2\theta \tag{5.2}$$

Also rmeber that $g^{\alpha\beta} = g^{-1}_{\alpha\beta}$ Let's calculate connection for that metric:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma} \right) \tag{5.3}$$

It is easy to see that

$$\partial_\nu g_{\rho\sigma} = \delta_{1\nu}\delta_{2\rho}\delta_{2\sigma} \, 2\sin\theta\cos\theta \tag{5.4}$$

where $\partial_1 \equiv \partial_\theta$ and $\partial_2 \equiv \partial_\phi$. Substituting Eq. (5.4) into connection yields

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\delta_{1\nu}\delta_{2\rho}\delta_{2\sigma} \, 2\sin\theta\cos\theta + \delta_{1\sigma}\delta_{2\rho}\delta_{2\nu} \, 2\sin\theta\cos\theta - \delta_{1\rho}\delta_{2\nu}\delta_{2\sigma} \, 2\sin\theta\cos\theta \right) =$$

$$\sin\theta\cos\theta \left(g^{2\mu}\delta_{1\nu}\delta_{2\sigma} + g^{2\mu}\delta_{1\sigma}\delta_{2\nu} - g^{1\mu}\delta_{2\nu}\delta_{2\sigma} \right) \tag{5.5}$$

The only non–zero coefficients

$$\Gamma^{1}_{22} = -\sin\theta\cos\theta \qquad\qquad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{\cos\theta}{\sin\theta} \tag{5.6}$$

Problem 1b

Geodesic equation is given by

$$\frac{\partial^2 x^\sigma}{\partial\lambda^2} + \Gamma^{\sigma}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{5.7}$$

Writing explicitly

$$\frac{\partial^2 x^1}{\partial\lambda^2} + \Gamma^{1}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{5.8a}$$

$$\frac{\partial^2 x^2}{\partial\lambda^2} + \Gamma^{2}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{5.8b}$$

After plugging in Eq. (5.6) we obtain

$$\frac{\partial^2 \theta}{\partial\lambda^2} - \sin\theta\cos\theta \frac{\partial\phi}{\partial\lambda} \frac{\partial\phi}{\partial\lambda} = 0 \tag{5.9a}$$

$$\frac{\partial^2 \phi}{\partial\lambda^2} + 2\frac{\cos\theta}{\sin\theta} \frac{\partial\theta}{\partial\lambda} \frac{\partial\phi}{\partial\lambda} = 0 \tag{5.9b}$$

Now we can set θ and ϕ as affine parameters

$\lambda \longrightarrow \theta$

$$-\sin\theta\cos\theta\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta} = 0 \tag{5.10a}$$

$$\frac{\partial^2\phi}{\partial\theta^2} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\phi}{\partial\theta} = 0 \tag{5.10b}$$

Solution to this set of equations is trivial namely

$$\frac{\partial\phi}{\partial\theta} = 0 \quad \Rightarrow \quad \phi = \text{const} \tag{5.11}$$

It means that longitudinal lines are geodesics in this metric.

$\lambda \longrightarrow \phi$

$$\frac{\partial^2\theta}{\partial\phi^2} - \sin\theta\cos\theta = 0 \tag{5.12a}$$

$$2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\phi} = 0 \tag{5.12b}$$

From second equation we have that

$$\frac{\partial\theta}{\partial\phi} = 0 \tag{5.13}$$

but it does not solve first equation for every θ. This system of equations has solutions only when

$$\sin\theta\cos\theta = 0 \quad \Rightarrow \quad \theta = 0 \ \vee \ \theta = \frac{\pi}{2} \ \vee \ \theta = \pi \tag{5.14}$$

since $\theta \in [0, \pi]$. But for $\theta = 0$ or $\theta = \pi$ geodesic line is just one point, because those are poles. Only $\theta = \frac{\pi}{2}$ gives non-trivial geodesic. This line is called equator.

Problem 1c

$\theta = \frac{\pi}{2},\ \phi = 0 \longrightarrow \theta = 0,\ \phi = 0$

We use equation of parallel transport of vector $\boldsymbol{v}$ along curve γ with $\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} = w^\mu$

$$\nabla_{\boldsymbol{w}}\boldsymbol{v} = 0 \tag{5.15}$$

We can write it explicitly

$$w^\nu\partial_\nu v^\mu + w^\nu\Gamma^\mu_{\nu\sigma}v^\sigma = 0 \tag{5.16}$$

or substituting connections I've calculated before

$$w^1\partial_1 v^1 + w^2\partial_2 v^1 + w^2\Gamma^1_{22}v^2 = 0 \tag{5.17a}$$

$$w^1\partial_1 v^2 + w^2\partial_2 v^2 + w^1\Gamma^2_{12}v^2 + w^2\Gamma^2_{21}v^1 = 0 \tag{5.17b}$$

If we move along meridian then we can take $\lambda \longrightarrow \theta$. From this we get coefficients w namely $w^1 = \frac{\mathrm{d}\theta}{\mathrm{d}\theta} = 1$ and $w^2 = \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = 0$. Putting this and connection coefficients into equations we obtain

$$\partial_1 v^1 = 0 \tag{5.18a}$$

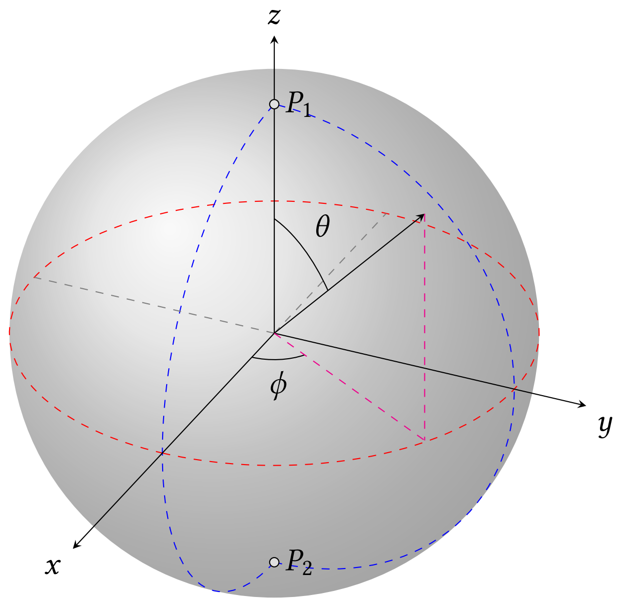

Figure 5.1: Visualization of geodesics – red line is equator, blue lines are two possible meridians, P_1 and P_2 are poles.

$$\partial_1 v^2 + \frac{\cos\theta}{\sin\theta} v^2 = 0 \tag{5.18b}$$

Second equation we multiply by $\sin\theta$ and simplify

$$\partial_1 v^1 = 0 \tag{5.19a}$$

$$\partial_1 \left(v^2 \sin\theta\right) = 0 \tag{5.19b}$$

This gives use

$$v^1 = C_1 \tag{5.20a}$$

$$v^2 = \frac{C_2}{\sin\theta} \tag{5.20b}$$

Constants C_1 and C_2 depends on the vector we transport:

- for $\frac{\partial}{\partial\theta}$ we have at the beginning ($\theta = \frac{\pi}{2}$) $v = (1, 0)$ so $C_1 = 1$ and $C_2 = 0$ so the transport

$$\boldsymbol{v} = (1, 0) \longrightarrow (1, 0) = \boldsymbol{v}' \tag{5.21}$$

  does not change this vector.

- for $\frac{\partial}{\partial\phi}$ we have at the beginning $v = (0, 1)$ so $C_1 = 0$ and $C_2 = 1$ so the transport

$$\boldsymbol{v} = (0, 1) \longrightarrow (0, \frac{1}{\sin\theta}) = \boldsymbol{v}' \tag{5.22}$$

  is undefined at point $\theta = 0$

$\theta = \frac{\pi}{2},\ \phi = 0 \longrightarrow \theta = \frac{\pi}{2},\ \phi = \frac{\pi}{4}$

If we move along equator then we can take $\lambda \longrightarrow \phi$. From this we get coefficients w namely $w^1 = \frac{d\theta}{d\phi} = 0$ and $w^2 = \frac{d\phi}{d\phi} = 1$. Putting this and connection coefficients into equations we obtain

$$\partial_2 v^1 - \sin\theta\cos\theta v^2 = 0 \tag{5.23a}$$

$$\partial_2 v^2 + \frac{\cos\theta}{\sin\theta} v^1 = 0 \tag{5.23b}$$

which simplifies to

$$\partial_2 v^1 = 0 \tag{5.24a}$$

$$\partial_2 v^2 = 0 \tag{5.24b}$$

because $\cos\theta\Big|_{\theta=\frac{\pi}{2}} = 0$

$$v^1 = C_1 \tag{5.25a}$$

$$v^2 = C_2 \tag{5.25b}$$

Constants C_1 and C_2 depends on the vector we transport:

- for $\frac{\partial}{\partial\theta}$ we have at the beginning ($\phi = 0$) $v = (1, 0)$ so $C_1 = 1$ and $C_2 = 0$ so the transport

  $$\boldsymbol{v} = (1, 0) \longrightarrow (1, 0) = \boldsymbol{v}' \tag{5.26}$$

  does not change this vector.
- for $\frac{\partial}{\partial\phi}$ we have at the beginning $v = (0, 1)$ so $C_1 = 0$ and $C_2 = 1$ so the transport

  $$\boldsymbol{v} = (0, 1) \longrightarrow (0, 1) = \boldsymbol{v}' \tag{5.27}$$

  does not change this vector.

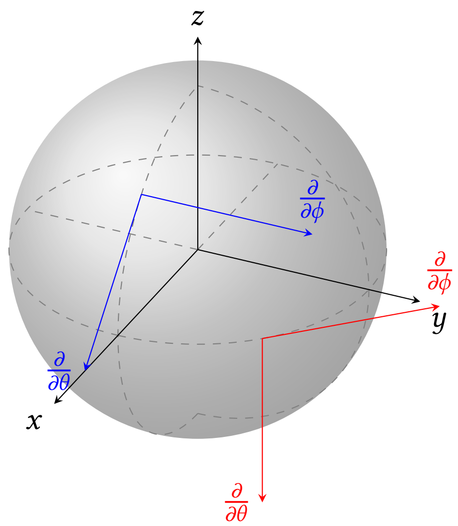

Figure 5.2: Visualization of parallel transport – red is transport along equator, blue is transport along meridians.

Problem 2

Materic is given by

$$g_{\nu\mu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{5.28}$$

We can calculate connection coefficients using relation

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma} \right) \tag{5.29}$$

Becasue metric is diagonal we can simplify this expression:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\mu} \left(\partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} \right) \tag{5.30}$$

$$\Gamma^0 = -\frac{1}{2}\frac{1}{B(r)} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.31a}$$

$$\Gamma^1 = \frac{1}{2}\frac{1}{A(r)} \begin{pmatrix} \dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r\sin^2\theta \end{pmatrix} \tag{5.31b}$$

$$\Gamma^2 = \frac{1}{2}\frac{1}{r^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & -r^2 2\sin\theta\cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{5.31c}$$

$$\Gamma^3 = \frac{1}{2}\frac{1}{r^2\sin^2\theta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r\sin^2\theta \\ 0 & 0 & 0 & r^2 2\sin\theta\cos\theta \\ 0 & 2r\sin^2\theta & r^2 2\sin\theta\cos\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos\theta}{\sin\theta} \\ 0 & \frac{1}{r} & \frac{\cos\theta}{\sin\theta} & 0 \end{pmatrix} \tag{5.31d}$$

Geodesic equation

$$\frac{\partial^2 x^\sigma}{\partial\lambda^2} + \Gamma^{\sigma}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{5.32}$$

$\sigma = 0$

$$\frac{\partial^2 t}{\partial\lambda^2} - \frac{1}{2}\frac{1}{B(t)} \begin{pmatrix} \frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda} \end{pmatrix} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial\lambda} \\ \frac{\partial r}{\partial\lambda} \\ \frac{\partial\theta}{\partial\lambda} \\ \frac{\partial\phi}{\partial\lambda} \end{pmatrix} = 0 \tag{5.33}$$

$$\frac{\partial^2 t}{\partial\lambda^2} - \frac{1}{2}\frac{1}{B(t)} \left(-2\dot{B} \frac{\partial t}{\partial\lambda} \frac{\partial r}{\partial\lambda} \right) = 0 \tag{5.34}$$

$$\frac{\partial^2 t}{\partial\lambda^2} + \frac{\dot{B}}{B} \frac{\partial t}{\partial\lambda} \frac{\partial r}{\partial\lambda} = 0 \tag{5.35}$$

$\sigma = 1$

$$\frac{\partial^2 r}{\partial\lambda^2} + \frac{1}{2}\frac{1}{A(r)} \begin{pmatrix} \frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda} \end{pmatrix} \begin{pmatrix} \dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r\sin^2\theta \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial\lambda} \\ \frac{\partial r}{\partial\lambda} \\ \frac{\partial\theta}{\partial\lambda} \\ \frac{\partial\phi}{\partial\lambda} \end{pmatrix} = 0 \tag{5.36}$$

$$\frac{\partial^2 r}{\partial\lambda^2} + \frac{\dot{B}}{2A}\left(\frac{\partial t}{\partial\lambda}\right)^2 + \frac{\dot{A}}{2A}\left(\frac{\partial r}{\partial\lambda}\right)^2 - \frac{r}{A}\left(\frac{\partial\theta}{\partial\lambda}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{\partial\phi}{\partial\lambda}\right)^2 = 0 \tag{5.37}$$

$\sigma = 2$

$$\frac{\partial^2\theta}{\partial\lambda^2} + \begin{pmatrix}\frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial t}{\partial\lambda}\\ \frac{\partial r}{\partial\lambda}\\ \frac{\partial\theta}{\partial\lambda}\\ \frac{\partial\phi}{\partial\lambda}\end{pmatrix} = 0 \tag{5.38}$$

$$\frac{\partial^2\theta}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\theta}{\partial\lambda} - \sin\theta\cos\theta\left(\frac{\partial\phi}{\partial\lambda}\right)^2 = 0 \tag{5.39}$$

$\sigma = 2$

$$\frac{\partial^2\phi}{\partial\lambda^2} + \begin{pmatrix}\frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \frac{\cos\theta}{\sin\theta}\\ 0 & \frac{1}{r} & \frac{\cos\theta}{\sin\theta} & 0\end{pmatrix}\begin{pmatrix}\frac{\partial t}{\partial\lambda}\\ \frac{\partial r}{\partial\lambda}\\ \frac{\partial\theta}{\partial\lambda}\\ \frac{\partial\phi}{\partial\lambda}\end{pmatrix} = 0 \tag{5.40}$$

$$\frac{\partial^2\phi}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} = 0 \tag{5.41}$$

Gathering all equations

$$0 = \frac{\partial^2 t}{\partial\lambda^2} + \frac{\dot{B}}{B}\frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda} \tag{5.42a}$$

$$0 = \frac{\partial^2 r}{\partial\lambda^2} + \frac{\dot{B}}{2A}\left(\frac{\partial t}{\partial\lambda}\right)^2 + \frac{\dot{A}}{2A}\left(\frac{\partial r}{\partial\lambda}\right)^2 - \frac{r}{A}\left(\frac{\partial\theta}{\partial\lambda}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{\partial\phi}{\partial\lambda}\right)^2 \tag{5.42b}$$

$$0 = \frac{\partial^2\theta}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\theta}{\partial\lambda} - \sin\theta\cos\theta\left(\frac{\partial\phi}{\partial\lambda}\right)^2 \tag{5.42c}$$

$$0 = \frac{\partial^2\phi}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} \tag{5.42d}$$

Problem 3

Metric is given by

$$\boldsymbol{g} = -c^2\left(12\frac{GM}{rc^2}\right)\mathrm{d}t\otimes\mathrm{d}t + \mathrm{d}\boldsymbol{x}\otimes\mathrm{d}\boldsymbol{x} \tag{5.43}$$

We can calculate infinitesimal interval of two events in this metric

$$\mathrm{d}s^2 = -c^2\left(12\frac{GM}{rc^2}\right)\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 \tag{5.44}$$

Simplifying this and assuming that the person has not been moving for whole year we can write

$$ds^2 = dt^2 \left(-c^2 \left(12\frac{GM}{rc^2} \right) + \frac{d\boldsymbol{x}^2}{dt^2} \right) = -dt^2 c^2 \left(12\frac{GM}{rc^2} \right) \tag{5.45}$$

We can now calculate the proper time

$$c^2 d\tau^2 = dt^2 c^2 \left(12\frac{GM}{rc^2} \right) \quad \Rightarrow \quad d\tau = dt\sqrt{12\frac{GM}{rc^2}} \tag{5.46}$$

and finite version

$$\Delta\tau = \Delta t\sqrt{12\frac{GM}{rc^2}} \tag{5.47}$$

Now I take $\Delta\tau_0$ to be proper time of person's feet and $\Delta\tau_H$ for head (in distance H from feet). Now I can calculate the difference

$$\Delta\tau_H - \Delta\tau_0 = \Delta t \left(\sqrt{12\frac{GM}{(r+H)c^2}} - \sqrt{12\frac{GM}{rc^2}} \right) \tag{5.48}$$

Because expressions under square are small I can Taylor expand them (using $\sqrt{1-2x} \simeq 1-x$)

$$\Delta\tau_H - \Delta\tau_0 = \Delta t \left(-\frac{GM}{(r+H)c^2} + \frac{GM}{rc^2} \right) = \Delta t \frac{GM}{c^2} \left(\frac{1}{r} - \frac{1}{r+H} \right) = \Delta t \frac{GM}{c^2 r} \frac{H}{r+H} \tag{5.49}$$

Plugging all the constants and assuming that person has height 1.8 m after 1 year time difference is

$$\Delta\tau_H - \Delta\tau_0 = 6.3 \times 10^{-9}\ \text{s} \tag{5.50}$$

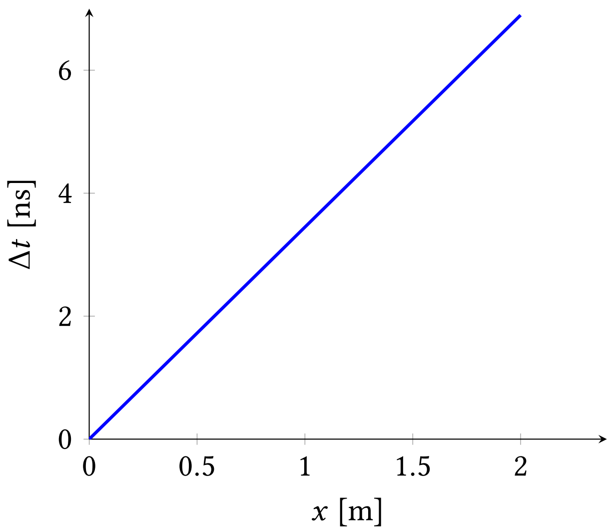

Figure 5.3: Time difference between head and feet after one year with respect to height of a person

Assignment 6

Problem 1

Show that

$$R(\boldsymbol{u},\boldsymbol{v})z = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z \tag{6.1}$$

satisfies

$$R(\boldsymbol{u},\boldsymbol{v})(fz) = fR(\boldsymbol{u},\boldsymbol{v})z \tag{6.2}$$

$$\begin{gathered}
R(\boldsymbol{u},\boldsymbol{v})(fz) = \nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}(fz)\right) - \nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}(fz)\right) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}(fz) = \\
\nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}f\; z + f\nabla_{\boldsymbol{v}}z\right) - \nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}f\; z + f\nabla_{\boldsymbol{u}}z\right) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}f\; z - f\;\nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\nabla_{\boldsymbol{u}}\left(\boldsymbol{v}(f)\; z + f\nabla_{\boldsymbol{v}}z\right) - \nabla_{\boldsymbol{v}}\left(\boldsymbol{u}(f)\; z + f\nabla_{\boldsymbol{u}}z\right) - [\boldsymbol{u},\boldsymbol{v}](f)\; z - f\;\nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\nabla_{\boldsymbol{u}}\boldsymbol{v}(f)\; z + \boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z + \nabla_{\boldsymbol{u}}f\nabla_{\boldsymbol{v}}z + f\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \\
\nabla_{\boldsymbol{v}}\boldsymbol{u}(f)\; z - \boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}f\nabla_{\boldsymbol{u}}z - f\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \\
[\boldsymbol{u},\boldsymbol{v}](f)\; z - f\;\nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\underline{\boldsymbol{u}(\boldsymbol{v}(f))\; z} + \underline{\boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z} + \underline{\boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z} + f\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \\
\underline{\boldsymbol{v}(\boldsymbol{u}(f))\; z} - \underline{\boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z} - \underline{\boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z} - f\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \\
\underline{[\boldsymbol{u},\boldsymbol{v}](f)\; z} - f\;\nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
f\left(\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z\right)
\end{gathered} \tag{6.3}$$

Problem 2

Show that

$$\delta\left(\ln\det\boldsymbol{M}\right) = \operatorname{Tr}\left(\boldsymbol{M}^{-1}\delta\boldsymbol{M}\right) \tag{6.4}$$

First we notice that

$$\delta\left(\ln\det\boldsymbol{M}\right) = \frac{\delta\det\boldsymbol{M}}{\det\boldsymbol{M}} \tag{6.5}$$

Now using following relation

$$\det(\mathbf{1} + \epsilon\boldsymbol{A}) = 1 + \epsilon\operatorname{Tr}\boldsymbol{A} + \mathcal{O}(\epsilon^2) \tag{6.6}$$

We can calculate the expression we will use later

$$\det(\boldsymbol{M} + \epsilon\delta\boldsymbol{M}) = \det\left(\mathbf{1} + \epsilon\delta\boldsymbol{M}\boldsymbol{M}^{-1}\right)\det(\boldsymbol{M}) = \det\boldsymbol{M} + \epsilon\operatorname{Tr}(\delta\boldsymbol{M}\boldsymbol{M}^{-1})\det\boldsymbol{M} \tag{6.7}$$

We can calculate derivative using its definition

$$\delta(\det\boldsymbol{M}) = \lim_{\epsilon\to 0}\frac{\det(\boldsymbol{M} + \epsilon\delta\boldsymbol{M}) - \det\boldsymbol{M}}{\epsilon} =$$

$$\lim_{\epsilon\to 0}\frac{\det \boldsymbol{M}+\epsilon\mathrm{Tr}(\delta\boldsymbol{M}\boldsymbol{M}^{-1})\det\boldsymbol{M}-\det\boldsymbol{M}}{\epsilon}=\mathrm{Tr}(\delta\boldsymbol{M}\boldsymbol{M}^{-1})\det\boldsymbol{M} \quad (6.8)$$

If we plug this into Eq. (6.5) we obtain final result (because trace is cyclic).

Problem 3

Show that

$$\Gamma^{\nu}_{\nu\mu}=\partial_\mu \ln\sqrt{|\boldsymbol{g}|} \quad (6.9)$$

We first calculate

$$\Gamma^{\nu}_{\nu\mu}=\frac{1}{2}g^{\nu\alpha}\left(\partial_\mu g_{\nu\alpha}+\partial_\nu g_{\mu\alpha}-\partial_\alpha g_{\nu\mu}\right)=\frac{1}{2}g^{\nu\alpha}\partial_\mu g_{\nu\alpha}+\frac{1}{2}g^{\nu\alpha}\partial_\nu g_{\mu\alpha}-\frac{1}{2}g^{\nu\alpha}\partial_\alpha g_{\nu\mu}= \quad (\nu\leftrightarrow\alpha)$$
$$\frac{1}{2}g^{\nu\alpha}\partial_\mu g_{\nu\alpha}+\frac{1}{2}g^{\alpha\nu}\partial_\alpha g_{\mu\nu}-\frac{1}{2}g^{\nu\alpha}\partial_\alpha g_{\nu\mu}=\frac{1}{2}g^{\nu\alpha}\partial_\mu g_{\nu\alpha}=\frac{1}{2}\left(\boldsymbol{g}^{-1}\right)_{\alpha\nu}\left(\partial_\mu\boldsymbol{g}\right)_{\nu\alpha}=\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{g}^{-1}\partial_\mu\boldsymbol{g}\right) \quad (6.10)$$

And now we calculate

$$\partial_\mu\ln\sqrt{|\boldsymbol{g}|}=\partial_\mu\ln|\sqrt{\boldsymbol{g}}|\overset{Eq.\ (6.4)}{=}\mathrm{Tr}\left(\sqrt{\boldsymbol{g}^{-1}}\partial_\mu\sqrt{\boldsymbol{g}}\right)=\mathrm{Tr}\left(\boldsymbol{g}^{-\frac{1}{2}}\boldsymbol{g}^{-\frac{1}{2}}\frac{1}{2}\partial_\mu\boldsymbol{g}\right)=\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{g}^{-1}\partial_\mu\boldsymbol{g}\right) \quad (6.11)$$

So both sides of equation are equal

Problem 4

Show that

$$\nabla_\mu A^\mu=\frac{1}{\sqrt{|\boldsymbol{g}|}}\partial_\mu\left(\sqrt{|\boldsymbol{g}|}A^\mu\right) \quad (6.12)$$

We start with

$$\frac{1}{\sqrt{|\boldsymbol{g}|}}\partial_\mu\left(\sqrt{|\boldsymbol{g}|}A^\mu\right)=\frac{1}{\sqrt{|\boldsymbol{g}|}}\partial_\mu\sqrt{|\boldsymbol{g}|}A^\mu+\frac{1}{\sqrt{|\boldsymbol{g}|}}\sqrt{|\boldsymbol{g}|}\partial_\mu A^\mu=\partial_\mu\left(\ln\sqrt{|\boldsymbol{g}|}\right)A^\mu+\frac{1}{\sqrt{|\boldsymbol{g}|}}\sqrt{|\boldsymbol{g}|}\partial_\mu A^\mu=$$
$$\Gamma^{\nu}_{\nu\mu}A^\mu+\partial_\mu A^\mu \quad (6.13)$$

So

$$\nabla_\mu A^\mu=\Gamma^{\nu}_{\nu\mu}A^\mu+\partial_\mu A^\mu \quad (6.14)$$

which is true by definition of covariant derivative

Problem 5

We start with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\boldsymbol{g}\left(\partial_{\sigma^*},\boldsymbol{v}\right)\right\}=0 \quad (6.15)$$

Since we stay on geodesic we can write

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\boldsymbol{g}\left(\partial_{\sigma^*},\boldsymbol{v}\right)\right\}=\nabla_{\boldsymbol{v}}\left\{\boldsymbol{g}\left(\partial_{\sigma^*},\boldsymbol{v}\right)\right\} \quad (6.16)$$

We can now expand

$$\boldsymbol{g}\left(\partial_{\sigma^*}, \boldsymbol{v}\right) = \delta^{\nu}_{\sigma^*} g_{\nu\mu} v^{\mu} \tag{6.17}$$

So eventually

$$\begin{aligned}
\nabla_{\boldsymbol{v}} \left\{\delta^{\nu}_{\sigma^*} g_{\nu\mu} v^{\mu}\right\} = g_{\nu\mu} \nabla_{\boldsymbol{v}} \left\{\delta^{\nu}_{\sigma^*} v^{\mu}\right\} = g_{\nu\mu} \nabla_{\boldsymbol{v}} \delta^{\nu}_{\sigma^*} v^{\mu} + g_{\nu\mu} \delta^{\nu}_{\sigma^*} \nabla_{\boldsymbol{v}} v^{\mu} = \\
g_{\nu\mu} \Gamma^{\nu}_{\rho\kappa} v^{\kappa} \delta^{\rho}_{\sigma^*} v^{\mu} + g_{\nu\mu} \delta^{\nu}_{\sigma^*} v^{\rho} \partial_{\rho} v^{\mu} + g_{\nu\mu} \delta^{\nu}_{\sigma^*} \Gamma^{\mu}_{\rho\kappa} v^{\rho} v^{\kappa} = \\
g_{\nu\mu} \Gamma^{\nu}_{\sigma^*\kappa} v^{\kappa} v^{\mu} + g_{\sigma^*\mu} v^{\rho} \partial_{\rho} v^{\mu} + g_{\sigma^*\mu} \Gamma^{\mu}_{\rho\kappa} v^{\rho} v^{\kappa} =
\end{aligned} \tag{6.18}$$

Now I calculate

$$g_{\nu\mu} \Gamma^{\nu}_{\sigma^*\kappa} = \frac{1}{2} \underbrace{g_{\nu\mu} g^{\nu\alpha}}_{=\delta^{\alpha}_{\mu}} (\partial_{\kappa} g_{\sigma^*\alpha} + \underbrace{\partial_{\sigma^*} g_{\kappa\alpha}}_{=0} - \partial_{\alpha} g_{\kappa\sigma^*}) = \frac{1}{2} (\partial_{\kappa} g_{\sigma^*\mu} - \partial_{\mu} g_{\kappa\sigma^*}) \tag{6.19}$$

and

$$g_{\sigma^*\mu} \Gamma^{\mu}_{\rho\kappa} = \frac{1}{2} \underbrace{g_{\sigma^*\mu} g^{\mu\alpha}}_{=\delta^{\alpha}_{\sigma^*}} (\partial_{\kappa} g_{\rho\alpha} + \partial_{\rho} g_{\kappa\alpha} - \partial_{\alpha} g_{\kappa\rho}) = \frac{1}{2} (\partial_{\kappa} g_{\rho\sigma^*} + \partial_{\rho} g_{\kappa\sigma^*} - \underbrace{\partial_{\sigma^*} g_{\kappa\rho}}_{=0}) = \frac{1}{2} (\partial_{\kappa} g_{\rho\sigma^*} + \partial_{\rho} g_{\kappa\sigma^*}) \tag{6.20}$$

Plugging down everything we have

$$\begin{aligned}
g_{\nu\mu} \Gamma^{\nu}_{\sigma^*\kappa} v^{\kappa} v^{\mu} + g_{\sigma^*\mu} v^{\rho} \partial_{\rho} v^{\mu} + g_{\sigma^*\mu} \Gamma^{\mu}_{\rho\kappa} v^{\rho} v^{\kappa} = \\
\frac{1}{2} \underbrace{(\partial_{\kappa} g_{\sigma^*\mu} - \partial_{\mu} g_{\kappa\sigma^*}) v^{\kappa} v^{\mu}}_{=0} + g_{\sigma^*\mu} v^{\rho} \partial_{\rho} v^{\mu} + \frac{1}{2} (\partial_{\kappa} g_{\rho\sigma^*} + \partial_{\rho} g_{\kappa\sigma^*}) v^{\rho} v^{\kappa} = \\
g_{\sigma^*\mu} v^{\rho} \partial_{\rho} v^{\mu} + \partial_{\kappa} g_{\rho\sigma^*} v^{\rho} v^{\kappa} =
\end{aligned} \tag{6.21}$$