ASSIGMENT 4

Problem 1

Parallel transport of a vector $V = v^\mu \partial_\mu$ along the curve s $\gamma : \lambda \mapsto [x^1(\lambda), \ldots, x^n(\lambda)]$:

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\sigma}\left[x(\lambda)\right] v^\nu \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{1}$$

We change coordinates, namely $V = v^\mu \partial_\mu = v^\mu \frac{\partial}{\partial x^\mu} = u^\nu \frac{\partial}{\partial y^\nu}$ and $\gamma : \lambda' \mapsto [y^1(\lambda'), \ldots, y^n(\lambda')]$ First we want to obtain transormation rule for vectors namely

$$v^\nu \frac{\partial y^k}{\partial x^\nu} = V(y^k) = u^\mu \frac{\partial y^k}{\partial y^\mu} = u^k \tag{2}$$

and for $\Gamma^\rho_{\mu\sigma} = \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\sigma} \frac{\partial x^\rho}{\partial \xi^\mu}$

$$\Gamma'^\rho_{\nu\sigma} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial \xi^\mu} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} = \overbrace{\left[\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\kappa}{\partial y^\sigma} + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\right]}^{\frac{\partial^2 \xi^\mu}{\partial y^\nu \partial y^\sigma}\ (*)}\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} =$$

$$\underbrace{\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\Gamma^\beta_{\alpha\kappa}}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\beta}\underbrace{\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\delta^\beta_\alpha} = \Gamma^\beta_{\alpha\kappa}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\alpha} \tag{3}$$

Plugging those things into

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\lambda'} + \Gamma'^\mu_{\nu\sigma}\left[y(\lambda')\right] u^\nu \frac{\mathrm{d}y^\sigma}{\mathrm{d}\lambda'} = 0 \tag{4}$$

we obtain

$$\frac{\partial v^\beta}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} + \left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial \lambda'} = 0 \tag{5}$$

Let's take a look at third term of this sum

$$\left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\underbrace{\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{\frac{\partial y^\sigma}{\partial \lambda'}} =$$

$$v^\eta \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\underbrace{\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{=\frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\eta}}_{\delta^\alpha_\eta}\frac{\partial y^\mu}{\partial x^\beta} + v^\eta \underbrace{\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial x^\tau}}_{\frac{\partial^2 x^\alpha}{\partial x^\eta \partial x^\tau} - \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\ (*)}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\alpha} =$$

$$\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\alpha}}_{=\delta^\mu_\nu}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} = \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} \tag{6}$$

So at the end of the day we have

$$\frac{\partial v^\beta}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} + \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} = 0 \tag{7}$$

Which simplifies to

$$\left(\frac{\partial v^{\beta}}{\partial \lambda}+\Gamma^{\beta}_{\alpha\kappa}\left[x(\lambda)\right]v^{\alpha}\frac{\partial x^{\kappa}}{\partial \lambda}\right)\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^{\mu}}{\partial x^{\beta}}=0 \tag{8}$$

We can divide by $\frac{\partial \lambda}{\partial \lambda'}$

$$\boxed{\left(\frac{\partial v^{\beta}}{\partial \lambda}+\Gamma^{\beta}_{\alpha\kappa}\left[x(\lambda)\right]v^{\alpha}\frac{\partial x^{\kappa}}{\partial \lambda}\right)\frac{\partial y^{\mu}}{\partial x^{\beta}}=0} \tag{9}$$

So indeed this equation is coordinate–covariant.

Problem 2

Let $\gamma_{\boldsymbol{V}}$ denote the geodesic with tangent vector $\boldsymbol{V}_p$ at point p. $\{\boldsymbol{e}_\mu\}$ is arbitrary basis chosen at the point p and normal coordinates are defined as $x(q)=(x^1,\dots,x^n)\Leftrightarrow q=\gamma_{x^\mu \boldsymbol{e}_\mu}$ where $p=\gamma(\lambda=0)$, $q=\gamma(\lambda=1)$ and $\{x^i\}_{i=0}^{n}\in\mathbb{R}$.

First we know that tangent vector when parallel transport along a geodesic stays tangent. So let $\boldsymbol{V}_p=v^\mu \boldsymbol{e}_\mu$ but since it is parrarel transport $\boldsymbol{V}_p=\boldsymbol{V}_q$. If so we can write normal coordinates of point q as $x(q)=(v^1,\dots,v^n)$. On the other hand v^i is defined as $v^i(\lambda)=\frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda}$. We can solve this equation (near point $\lambda=1$) and get expression for

$$x^i(\lambda)=v^i\lambda+x^i(0) \tag{10}$$

Eq. 10 describes straight line, because it is linear with respect to λ [1]. Substituting this expression into geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^{\mu}_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}=0 \tag{11}$$

gives

$$\Gamma^{\mu}_{\sigma\rho}v^\sigma v^\rho=0 \tag{12}$$

But Eq. 12 has to be satisfied for arbitrary v^σ and v^ρ which implies

$$\boxed{\Gamma^{\mu}_{\sigma\rho}=0} \tag{13}$$

Problem 3

A $\partial g = 0$

We know that metric transforms as follow:

$$g'_{\alpha\beta}=\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}g_{\mu\nu} \tag{14}$$

Now choosing $x^\mu=x'^\mu-\frac{1}{2}M^{\mu}_{\alpha\beta}x'^\alpha x'^\beta$ will give us

$$\begin{aligned}
g'_{\alpha\beta}=&\frac{\partial(x'^\mu-\frac{1}{2}M^{\mu}_{\lambda\sigma}x'^\lambda x'^\sigma)}{\partial x'^\alpha}\frac{\partial(x'^\nu-\frac{1}{2}M^{\nu}_{\kappa\eta}x'^\kappa x'^\eta)}{\partial x'^\beta}g_{\mu\nu}=\\
&\left(\delta^\mu_\alpha-\frac{1}{2}M^{\mu}_{\lambda\sigma}(\delta^\lambda_\alpha x'^\sigma+x'^\lambda\delta^\sigma_\alpha)\right)\left(\delta^\nu_\beta-\frac{1}{2}M^{\nu}_{\kappa\eta}(\delta^\kappa_\beta x'^\eta+x'^\kappa\delta^\eta_\beta)\right)g_{\mu\nu}=\\
&\left(\delta^\mu_\alpha-\frac{1}{2}M^{\mu}_{\alpha\sigma}x'^\sigma-\frac{1}{2}M^{\mu}_{\lambda\alpha}x'^\lambda\right)\left(\delta^\nu_\beta-\frac{1}{2}M^{\nu}_{\beta\eta}x'^\eta-\frac{1}{2}M^{\nu}_{\kappa\beta}x'^\kappa\right)g_{\mu\nu}=\\
&\left(\delta^\mu_\alpha-\frac{1}{2}x'^\sigma(M^{\mu}_{\alpha\sigma}+M^{\mu}_{\sigma\alpha})\right)\left(\delta^\nu_\beta-\frac{1}{2}x'^\eta(M^{\nu}_{\beta\eta}+M^{\nu}_{\eta\beta})\right)g_{\mu\nu}
\end{aligned} \tag{15}$$

[1] or equivalently $\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2}=0$

We take $\tilde{M}^{\nu}_{\beta\eta} = \frac{1}{2}(M^{\nu}_{\beta\eta} + M^{\nu}_{\eta\beta})$ and write (keeping only linear terms in x)

$$g'_{\alpha\beta} = \left(\delta^{\mu}_{\alpha} - x'^{\sigma}\tilde{M}^{\mu}_{\alpha\sigma}\right)\left(\delta^{\nu}_{\beta} - x'^{\eta}\tilde{M}^{\nu}_{\beta\eta}\right) g_{\mu\nu} = g_{\alpha\beta} - g_{\alpha\nu}x'^{\eta}\tilde{M}^{\nu}_{\beta\eta} - g_{\mu\beta}x'^{\sigma}\tilde{M}^{\mu}_{\alpha\sigma} = \tag{16}$$

Now we differentiate both sides

$$\begin{aligned}\partial'_{\lambda}g'_{\alpha\beta} &= \partial'_{\lambda}g_{\alpha\beta} - \partial'_{\lambda}(g_{\alpha\nu}x'^{\eta}\tilde{M}^{\nu}_{\beta\eta}) - \partial'_{\lambda}(g_{\mu\beta}x'^{\sigma}\tilde{M}^{\mu}_{\alpha\sigma}) = \\ &\partial'_{\lambda}g_{\alpha\beta} - \partial'_{\lambda}g_{\alpha\nu}x'^{\eta}\tilde{M}^{\nu}_{\beta\eta} - g_{\alpha\nu}\delta^{\eta}_{\lambda}\tilde{M}^{\nu}_{\beta\eta} - \partial'_{\lambda}g_{\mu\beta}x'^{\sigma}\tilde{M}^{\mu}_{\alpha\sigma} - g_{\mu\beta}\delta^{\sigma}_{\lambda}\tilde{M}^{\mu}_{\alpha\sigma}\end{aligned} \tag{17}$$

Now we drop linear terms in x (of the form $x\partial g$)

$$\begin{aligned}\partial'_{\lambda}g'_{\alpha\beta} &= \partial'_{\lambda}g_{\alpha\beta} - g_{\alpha\nu}\tilde{M}^{\nu}_{\beta\lambda} - g_{\mu\beta}\tilde{M}^{\mu}_{\alpha\lambda} = \partial_{\tau}g_{\alpha\beta}\partial'_{\lambda}x^{\tau} - g_{\alpha\nu}\tilde{M}^{\nu}_{\beta\lambda} - g_{\mu\beta}\tilde{M}^{\mu}_{\alpha\lambda} = \\ &\partial_{\tau}g_{\alpha\beta}\left(\delta^{\tau}_{\lambda} - x'^{\sigma}\tilde{M}^{\tau}_{\lambda\sigma}\right) - g_{\alpha\nu}\tilde{M}^{\nu}_{\beta\lambda} - g_{\mu\beta}\tilde{M}^{\mu}_{\alpha\lambda} \simeq \partial_{\lambda}g_{\alpha\beta} - g_{\alpha\nu}\tilde{M}^{\nu}_{\beta\lambda} - g_{\mu\beta}\tilde{M}^{\mu}_{\alpha\lambda}\end{aligned} \tag{18}$$

Now let's substitute Chrisroffel symbol in place of $\tilde{M}$ namely

$$\tilde{M}^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\sigma}(\partial_{\alpha}g_{\sigma\beta} + \partial_{\beta}g_{\sigma\alpha} - \partial_{\sigma}g_{\alpha\beta}) \tag{19}$$

We obtain (using $g_{\alpha\beta} = g_{\beta\alpha}$)

$$\begin{aligned}2\partial'_{\lambda}g'_{\alpha\beta} &= 2\partial_{\lambda}g_{\alpha\beta} - \underbrace{g_{\alpha\nu}g^{\nu\sigma}}_{=\delta^{\sigma}_{\alpha}}(\partial_{\beta}g_{\sigma\lambda} + \partial_{\lambda}g_{\sigma\beta} - \partial_{\sigma}g_{\beta\lambda}) - \underbrace{g_{\mu\beta}g^{\mu\sigma}}_{=\delta^{\sigma}_{\beta}}(\partial_{\alpha}g_{\sigma\lambda} + \partial_{\lambda}g_{\sigma\alpha} - \partial_{\sigma}g_{\alpha\lambda}) = \\ &2\partial_{\lambda}g_{\alpha\beta} - \underline{\partial_{\beta}g_{\alpha\lambda}} - \underline{\partial_{\lambda}g_{\alpha\beta}} + \underline{\partial_{\alpha}g_{\beta\lambda}} - \underline{\partial_{\alpha}g_{\beta\lambda}} - \underline{\partial_{\lambda}g_{\beta\alpha}} + \underline{\partial_{\beta}g_{\alpha\lambda}} = 2\partial_{\lambda}g_{\alpha\beta} - 2\partial_{\lambda}g_{\alpha\beta} = 0\end{aligned} \tag{20}$$

So eventually

$$\boxed{\partial'_{\lambda}g'_{\alpha\beta} = 0} \tag{21}$$

B $g = \eta$

We try following change of coordinates

$$x'^{\mu} = N^{\mu}{}_{\alpha}y^{\alpha} \tag{22}$$

In those coordinates metric looks like

$$g''_{\alpha\beta} = \frac{\partial x'^{\mu}}{\partial y^{\alpha}}\frac{\partial x'^{\nu}}{\partial y^{\beta}}g'_{\mu\nu} = N^{\mu}{}_{\alpha}N^{\nu}{}_{\beta}g'_{\mu\nu} = (N^{-1})_{\alpha}{}^{\mu}g'_{\mu\nu}N^{\nu}{}_{\beta} = (N^{-1}g'N)_{\alpha\beta} \tag{23}$$

We can now diagonalize metric g'. We can write

$$g' = C\,\eta\,C^{-1} \tag{24}$$

where η is diagonal and C is a matrix which consists of eigenvectors of g'. If we will choose $N = C$ then Eq. 23 simplifies to

$$\boxed{g''_{\alpha\beta} = \eta_{\alpha\beta}} \tag{25}$$