Assignment 1

Problem 5

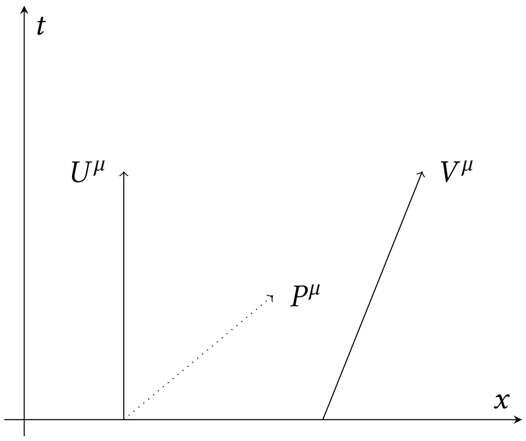

Figure 1: Setup of experiment

We have given:

$$U^\mu = (c, \mathbf{0}) \tag{1}$$

$$V^\mu = (\gamma_v c, \gamma_v \boldsymbol{v}) \tag{2}$$

$$P^\mu = \left(\frac{h\nu}{c}, \boldsymbol{p}\right) \tag{3}$$

We use following relation in this problem:

$$E = -P^\mu V_\mu \tag{4}$$

This expression is Lorentz invariant and can be calculated in non-moving frame. So we plug in Eq. 2 in this expression to obtain

$$E = -P^\mu V_\mu = \frac{h\nu}{c}\gamma_v c - \gamma_v \boldsymbol{v}\boldsymbol{p} = \gamma_v \left(h\nu - |\boldsymbol{v}||\boldsymbol{p}|\cos(\theta)\right) = \left\{|\boldsymbol{p}| = \frac{h\nu}{c}\right\} = \gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{5}$$

But it is still photon, but with different energy (for moving observer) So

$$\gamma_v h\nu \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) = h\nu' \tag{6}$$

So ratio of those two frequencies is

$$\frac{\nu'}{\nu} = \gamma_v \left(1 - \frac{|\boldsymbol{v}|}{c}\cos(\theta)\right) \tag{7}$$

If $\theta = 0$ and $\frac{v}{c} \ll 1 \Rightarrow \gamma_v \simeq 1$ then we obtain:

$$\boxed{\nu' = \nu\left(1 - \frac{v}{c}\right)} \tag{8}$$

Assignment 2

Problem 1

Problem 1a

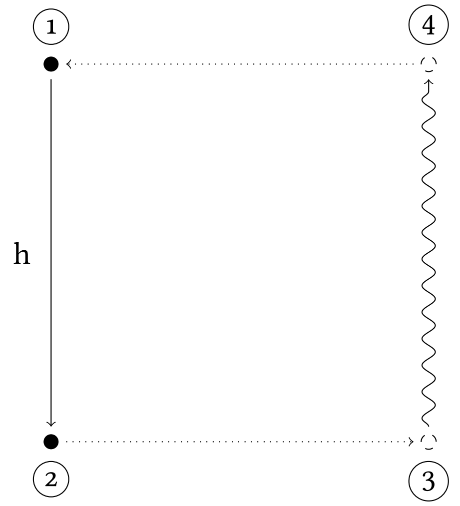

Figure 2: Mass falling in graitational field (1⟶2), converting to photon (2⟶3), photon traveling up (3⟶4) and converting back to mass (4⟶1)

Let's take a look at energy changes in above diagram:

1 $E_1 = mc^2$

2 $E_2 = mc^2 + mgh$

3 $E_3 = h\nu = mc^2 + mgh$

4 $E_4 = h\nu = mc^2 + mgh$

but $E_4 = E_1$ because of energy conservation. It means that photon has to have different frequency at the height h than it has at the ground. So $E_4 = h\nu' = mc^2$. From it follows

$$\frac{\nu}{\nu'} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \tag{9}$$

and it is easy to calculate redshift

$$\boxed{z = \frac{\nu - \nu'}{\nu'} = \frac{gh}{c^2}} \tag{10}$$

Problem 1b

Let's calculate time which light needs to reach observer 2

$$t = \frac{s}{c} = \frac{h - \frac{gt^2}{2}}{c} \tag{11}$$

From this expression we get quadratic equation

$$\frac{g}{2}t^2 + ct - h = 0 \tag{12}$$

for which solution is given by

$$t = \frac{-c + \sqrt{c^2 + 2gh}}{g} \tag{13}$$

Velocity of observer 2 after this time is equal

$$v(t) = \frac{-c + \sqrt{c^2 + 2gh}}{g} \cdot g = -c + \sqrt{c^2 + 2gh} \tag{14}$$

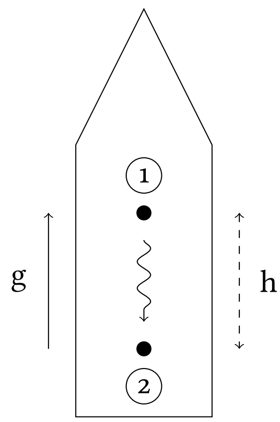

Figure 3: Two observers in a rocket sending photon

Then redshift formula is given in following way

$$\frac{\nu'}{\nu} = 1 - \frac{v}{c} = 1 - \frac{-c + \sqrt{c^2 + 2gh}}{c} = 2 - \sqrt{1 - \frac{2gh}{c^2}} \tag{15}$$

We can use Taylor expansion $\sqrt{1 - x} = 1 - \frac{x}{2}$ we get

$$\boxed{\frac{\nu'}{\nu} = 2 - 1 + \frac{gh}{c^2} = 1 + \frac{gh}{c^2}} \tag{16}$$

It is exactly the same result as Eq. 10.

Problem 2

Observer O is traveling with acceleration g in direction x_1. To calculate his worldline we will use following three conditions

$$U^\mu U_\mu = -1 \qquad U^\mu A_\mu = 0 \qquad A^\mu A_\mu = g^2 \tag{17}$$

where U^μ is four-velocity and A^μ is four-acceleration. First of them can be obtained by straightforward calculation, second by applying derivative to first equation i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(U^\mu U_\mu\right) = 0 \quad \Rightarrow \quad \left(A^\mu U_\mu\right) = 0 \tag{18}$$

Third is Lorentz invariant and it can be calculated in the moment of launch namely when $A^\mu = (0, g, 0, 0)$.

Knowing those three we can write them in explicite form

$$-U_0^2 + \boldsymbol{U}^2 = -1 \qquad \boldsymbol{U}\boldsymbol{A} = U_0 A_0 \qquad -A_0^2 + \boldsymbol{A}^2 = g^2 \tag{19}$$

where bolded letters mean three-vectors.

We square middle equation and plug in left and right equation to obtained

$$(U_0^2 - 1)\boldsymbol{A}^2 = U_0^2(\boldsymbol{A}^2 - g^2) \tag{20}$$

Eventually we obtain:

$$\boldsymbol{A}^2 = g^2 U_0^2 \tag{21}$$

and plugin this expression to other equation we also obtain:[1]

$$A_0^2 = g^2 \boldsymbol{U}^2 \tag{22}$$

We can simplify those equation using the fact that this motion is one dimensional namely $x_2 = x_3 = 0$ and then

$$A_1 = gU_0 \qquad A_0 = gU_1 \tag{23}$$

But $U^\mu = \dot{X}^\mu$ and $A^\mu = \ddot{X}^\mu$ [2]. Substituting

$$\ddot{X}_1 = g\dot{X}_0 \qquad \ddot{X}_0 = g\dot{X}_1 \tag{24}$$

Taking a derivative of left equation and substituting right equation into it we get

$$\dddot{X}_1 = g^2\dot{X}_1 \quad \overset{\text{after integration}}{\Rightarrow} \quad \ddot{X}_1 = g^2 X_1 \tag{25}$$

Solution is

$$X_1 = A\sinh(g\tau) + B\cosh(g\tau) \tag{26}$$

Let's choose initial conditions such as $X_1(0) = g^{-1}$ and $\dot{X}_1 = 0$. Then

$$X_1 = g^{-1}\cosh(g\tau) \tag{27}$$

And finally we have

$$X_0 = g^{-1}\sinh(g\tau) \qquad X_1 = g^{-1}\cosh(g\tau) \qquad X_2 = 0 \qquad X_3 = 0 \tag{28}$$

[1] plug it into right equation and then use left equation
[2] dot means derivation with respect to proper time

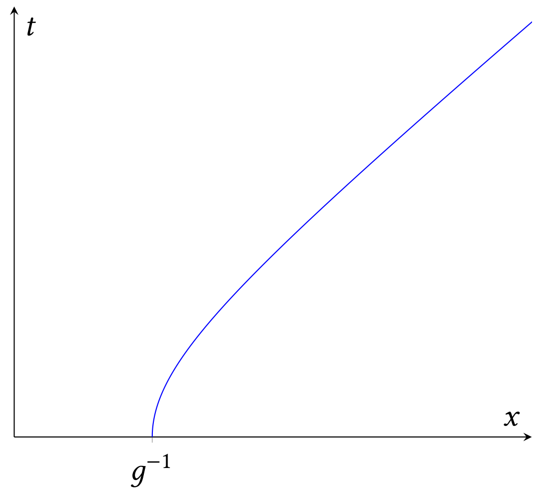

Figure 4: Trajectory of O

Problem 3

As a first basis vector we can choose four-velocity namely

$$\boldsymbol{e}_0 = \left(\dot{X}_0, \dot{X}_1, \dot{X}_2, \dot{X}_3\right) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{29}$$

As a basis vectors in directions x_2 and x_3 we simply choose

$$\boldsymbol{e}_2 = (0, 0, 1, 0) \tag{30}$$

$$\boldsymbol{e}_3 = (0, 0, 0, 1) \tag{31}$$

And finally we choose vector $\boldsymbol{e}_1$ in a form $\boldsymbol{e}_1 = \left(e_1^0, e_1^1, 0, 0\right)$ where e_1^0 and e_1^1 are chosen in order to satisfy $\boldsymbol{e}_0\boldsymbol{e}_1 = 0$ and $(\boldsymbol{e}_0)^2 = 1$ i.e.

$$-e_1^0 \cosh(g\tau) + e_1^1 \sinh(g\tau) = 0 \tag{32}$$

$$-(e_1^0)^2 + (e_1^1)^2 = 1 \tag{33}$$

We square first equation and substitute second equation

$$(e_1^0)^2 \cosh^2(g\tau) = (1 + (e_1^0)^2) \sinh^2(g\tau) \tag{34}$$

From this we obtain

$$(e_1^0)^2 = \sinh^2(g\tau) \qquad\qquad (e_1^1)^2 = \cosh^2(g\tau) \tag{35}$$

We can choose positive solution and eventually we get

$$\boldsymbol{e}_1 = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{36}$$

All vectors

$$\boldsymbol{e}_0(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{37}$$

$$\boldsymbol{e}_1(\tau) = (\sinh(g\tau), \cosh(g\tau), 0, 0) \tag{38}$$

$$\boldsymbol{e}_2(\tau) = (0, 0, 1, 0) \tag{39}$$

$$\boldsymbol{e}_3(\tau) = (0, 0, 0, 1) \tag{40}$$

Last thing to do is to check whether those are vectors which were obtain without any rotation. For this I will find a Lorentz boost which transforms initial basis into this one. Namely consider a boost of time-basis vector

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 0 \end{pmatrix} \tag{41}$$

So γ and β have to satisfy:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh(g\tau) \quad \Rightarrow \quad v = \tanh(g\tau) \tag{42}$$

Knowing that it is easy to calculate

$$\beta\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh(g\tau) \tag{43}$$

So indeed we obtain vector $\boldsymbol{e}_0(\tau)$ only via boost (at $v = \tanh(g\tau)$). The same can be done with vector $\boldsymbol{e}_1(\tau)$

Problem 4

We define new coordinate system ($\xi_0 \equiv \tau, \xi_1, \xi_2, \xi_3$) where basis vectors are those defined in problem before. We can write

$$\boldsymbol{x} = \xi^1 \boldsymbol{e}_1(\tau) + \xi^2 \boldsymbol{e}_2(\tau) + \xi^3 \boldsymbol{e}_3(\tau) + \boldsymbol{x}_O(\tau) \tag{44}$$

where $\boldsymbol{x}_O(\tau)$ is trajectory of moving frame.

After plugging in all basis vectors explicitly we get

$$\boldsymbol{x} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \xi^1 \sinh(g\tau) \\ \xi^1 \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \xi^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^3 \end{pmatrix} + \begin{pmatrix} g^{-1} \sinh(g\tau) \\ g^{-1} \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} g^{-1} \sinh(g\tau) + \xi^1 \sinh(g\tau) \\ g^{-1} \cosh(g\tau) + \xi^1 \cosh(g\tau) \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} (g^{-1} + \xi^1) \sinh(g\xi^0) \\ (g^{-1} + \xi^1) \cosh(g\xi^0) \\ \xi^2 \\ \xi^3 \end{pmatrix} \tag{45}$$

Line element $\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ is then equal (we use chain rule i.e. $\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} \mathrm{d}\xi^\nu$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{46}$$

$$\mathrm{d}t = \frac{\partial t}{\partial \xi^\nu} \mathrm{d}\xi^\nu = (1 + g\xi_1) \cosh(g\xi_0)\, \mathrm{d}\xi_0 + \sinh(g\xi_0)\, \mathrm{d}\xi_1 \tag{47}$$

$$\mathrm{d}x_1 = (1 + g\xi_1) \sinh(g\xi_0)\, \mathrm{d}\xi_0 + \cosh(g\xi_0)\, \mathrm{d}\xi_1 \tag{48}$$

$$\mathrm{d}x_2 = \mathrm{d}\xi_2 \tag{49}$$

$$\mathrm{d}x_3 = \mathrm{d}\xi_3 \tag{50}$$

After squaring and adding them up we get

$$\begin{aligned}\mathrm{d}s^2 = &-(1+g\xi_1)^2\cosh^2(g\xi_0)\mathrm{d}\xi_0^2 - \sinh^2(g\xi_0)\mathrm{d}\xi_1^2 + \\ &(1+g\xi_1)^2\sinh^2(g\xi_0)\mathrm{d}\xi_0^2 + \cosh^2(g\xi_0)\mathrm{d}\xi_1^2 + \\ &\mathrm{d}\xi_2^2 + \\ &\mathrm{d}\xi_3^2\end{aligned} \tag{51}$$

After simplification

$$\boxed{\mathrm{d}s^2 = -(1+g\xi_1)^2\mathrm{d}\xi_0^2 + \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2} \tag{52}$$

Problem 5

For $\xi^1 \equiv \text{const}$ we can easily derive equation of motion from Eq. 45 namely

$$x_1^2 - t^2 = (g^{-1} + \xi^1)^2 \tag{53}$$

which leads to

$$x_1(t) = \sqrt{(g^{-1}+\xi^1)^2 + t^2} \tag{54}$$

We take derivative twice

$$\dot{x}_1(t) = \frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}} \tag{55}$$

$$\ddot{x}_1(t) = \frac{\sqrt{(g^{-1}+\xi^1)^2+t^2} - t\frac{2t}{2\sqrt{(g^{-1}+\xi^1)^2+t^2}}}{(g^{-1}+\xi^1)^2+t^2} = \frac{1}{\sqrt{(g^{-1}+\xi^1)^2+t^2}} - \frac{2t^2}{((g^{-1}+\xi^1)^2+t^2)^{\frac{3}{2}}} \tag{56}$$

So when $t = 0$

$$\boxed{\ddot{x}_1(t)\Big|_{t=0} = \frac{1}{g^{-1}+\xi^1} = \frac{g}{1+g\xi^1}} \tag{57}$$

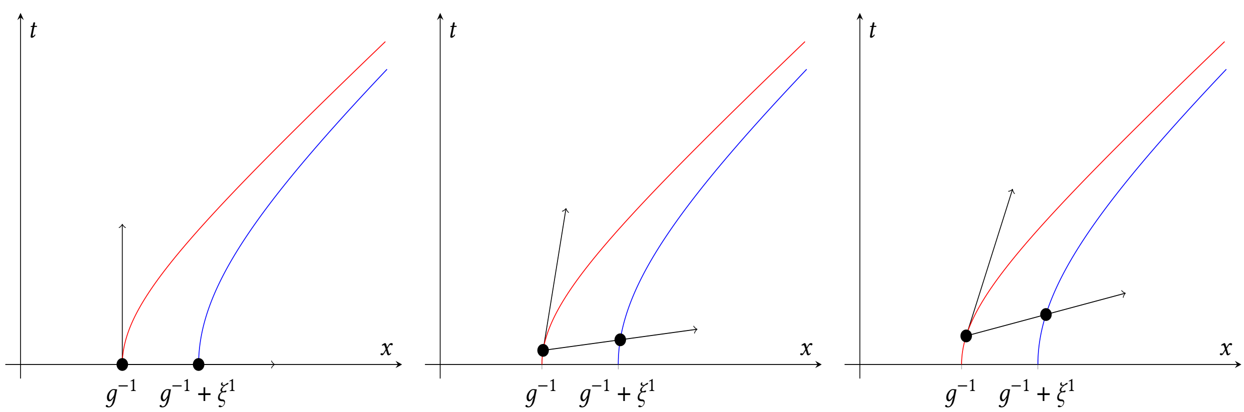

Figure 5: Red line is worldline of Eq. 28 and blue is worldline of Eq. 54

Problem 6

We start with equation Eq. 52. We can simplify it and neglect other spatial dimensions than ξ^1 namely

$$ds^2 = -(1 + g\xi^1)^2(d\xi^0)^2 + (d\xi^1)^2 \tag{58}$$

We can change the form to

$$d\tau = ds = d\xi^0 \sqrt{-(1 + g\xi^1)^2 + \left(\frac{d\xi^1}{d\xi^0}\right)^2} \tag{59}$$

We can now plug in $\xi^1 = \xi^1_{em}$ and since emiter does not move in this frame we can set $\frac{d\xi^1}{d\xi^0} = 0$:

$$d\tau_{em} = d\xi^0_{em}(1 + g\xi^1_{em}) \tag{60}$$

We can integrate both sides and obtain equation for finite differences

$$\Delta\tau_{em} = \Delta\xi^0_{em}(1 + g\xi^1_{em}) \tag{61}$$

We can do similar thing with ξ^1_{rec}:

$$\Delta\tau_{rec} = \Delta\xi^0_{rec}(1 + g\xi^1_{rec}) \tag{62}$$

But left sides of above equations are equal (since line element is invariant under changing of coordinates) and we can compare them:

$$\frac{\Delta\xi^0_{rec}}{\Delta\xi^0_{em}} = \frac{1 + g\xi^1_{em}}{1 + g\xi^1_{rec}} = 1 + \frac{g\xi^1_{em} - g\xi^1_{rec}}{1 + g\xi^1_{rec}} = 1 - \frac{gh}{1 + gh + g\xi^1_{em}} \tag{63}$$

where I put $h = \xi^1_{rec} - \xi^1_{em}$. After rearranging terms and substituting $\Delta\xi^1_{rec} = \frac{1}{\nu'}$ and $\Delta\xi^1_{em} = \frac{1}{\nu}$

$$\frac{\Delta\xi^0_{em} - \Delta\xi^0_{rec}}{\Delta\xi^0_{em}} = \frac{gh}{1 + gh + g\xi^1_{em}} \tag{64}$$

$$\frac{\frac{1}{\nu} - \frac{1}{\nu'}}{\frac{1}{\nu}} = \frac{gh}{1 + gh + g\xi^1_{em}} \quad \Rightarrow \quad \boxed{z = \frac{\nu' - \nu}{\nu'} = \frac{gh}{1 + gh + g\xi^1_{em}}} \tag{65}$$

We can now assume that g is small and using Taylor expansion $\frac{1}{1+x} \simeq 1 - x$

$$z = gh(1 - gh - g\xi^1_{em}) = gh - (gh)^2 - g^2h\xi^1_{em} \simeq gh$$

$$z = gh \tag{66}$$

so the same result as photon in gravitational field.

Assignment 3

Problem 1

Calculate EOM given the Lagrangian

$$\mathcal{L}_{\text{dyn}}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu \dot{x}^\nu, \quad \dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{67}$$

I calculate first variation of Lagrangian

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu)\mathrm{d}\lambda = \int_{\lambda 1}^{\lambda 2} \left(\frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) \mathrm{d}\lambda =$$
$$\int_{\lambda 1}^{\lambda 2} \left(\frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{1}{2} g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right) \mathrm{d}\lambda \tag{68}$$

Let's take a look at second part of the integral:

$$\frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right\} \mathrm{d}\lambda = \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta \dot{x}^\sigma \mathrm{d}\lambda =$$
$$\frac{1}{2} \int_{\lambda 1}^{\lambda 2} \frac{\partial}{\partial \lambda} \left(\left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \right) \mathrm{d}\lambda - \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \frac{\partial}{\partial \lambda} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \mathrm{d}\lambda =$$
$$\underbrace{\frac{1}{2} \left\{ g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu \right\} \delta x^\sigma \Bigg|_{\lambda 1}^{\lambda 2}}_{=0} - \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left\{ \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu + g_{\sigma\nu} \ddot{x}^\nu + \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu + g_{\sigma\mu} \ddot{x}^\mu \right\} \delta x^\sigma \mathrm{d}\lambda \tag{69}$$

Plugging result back into Eq. 68 yields

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu)\mathrm{d}\lambda = \frac{1}{2} \int_{\lambda 1}^{\lambda 2} \left(\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu \right) \delta x^\sigma \mathrm{d}\lambda \tag{70}$$

We want

$$\delta \int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu)\mathrm{d}\lambda = 0 \tag{71}$$

but since δx^σ can be arbitrary the rest has to be equal 0, namely

$$\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu = 0 \tag{72}$$

or after rearranging elements

$$\boxed{2 g_{\sigma\mu} \ddot{x}^\mu + \left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu = 0} \tag{73}$$

Problem 2

Calculate the EOM given Lagrangian:

$$\mathcal{L}_{\text{geo}}(x^\mu, \dot{x}^\mu) = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}, \quad \dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{74}$$

First we calculate variation of Lagrangian

$$\delta\int_{\lambda 1}^{\lambda 2} \mathcal{L}(x^\mu, \dot{x}^\mu)\mathrm{d}\lambda = \int_{\lambda 1}^{\lambda 2}\left(\frac{\partial\mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma}\delta x^\sigma + \frac{\partial\mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial\dot{x}^\sigma}\delta\dot{x}^\sigma\right)\mathrm{d}\lambda =$$

$$-\int_{\lambda 1}^{\lambda 2}\left(\frac{\dot{x}^\mu\dot{x}^\nu\partial_\sigma g_{\mu\nu}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\delta x^\sigma + \frac{g_{\mu\nu}\dot{x}^\mu\delta_{\nu\sigma}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\dot{x}^\sigma + \frac{g_{\mu\nu}\delta_{\mu\sigma}\dot{x}^\nu}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\dot{x}^\sigma\right)\mathrm{d}\lambda =$$

$$-\int_{\lambda 1}^{\lambda 2}\frac{1}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}}\left(\dot{x}^\mu\dot{x}^\nu\partial_\sigma g_{\mu\nu}\delta x^\sigma + g_{\mu\sigma}\dot{x}^\mu\delta\dot{x}^\sigma + g_{\sigma\nu}\dot{x}^\nu\delta\dot{x}^\sigma\right)\mathrm{d}\lambda \tag{75}$$

I change variable of differentiating and integration from $\mathrm{d}\lambda$ to $\mathrm{d}\tau = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda$. Derivatives changing as following

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{76}$$

and integral as following

$$\int\mathrm{d}\lambda = \int\frac{\mathrm{d}\tau}{\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \tag{77}$$

After plugging in those transformations into Eq. 75 we obtain

$$-\frac{1}{2}\int_{\lambda 1}^{\lambda 2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\sigma g_{\mu\nu}\delta x^\sigma + \left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{78}$$

Now let's look at the second part of this integral and transform it (using Leibniz rule)

$$\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma =$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \left\{\frac{\mathrm{d}g_{\mu\sigma}}{\mathrm{d}x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}g_{\sigma\nu}}{\mathrm{d}x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\delta x^\sigma =$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right) - \left[g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left\{\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu}\right\}\right]\delta x^\sigma \tag{79}$$

After plugging it into Eq. 78 we obtain

$$-\frac{1}{2}\int\limits_{\lambda1}^{\lambda2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\sigma g_{\mu\nu}\delta x^\sigma+\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}+g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\right]-\right.$$

$$\left.\left[g_{\mu\sigma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left\{\partial_\nu g_{\mu\sigma}+\partial_\mu g_{\sigma\nu}\right\}\right]\delta x^\sigma\right)\mathrm{d}\tau=$$

$$-\frac{1}{2}\int\limits_{\lambda1}^{\lambda2}\left[-2g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)\right]\delta x^\sigma\mathrm{d}\tau-\underbrace{\left\{g_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}+g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right\}\delta x^\sigma\Bigg|_{\lambda1}^{\lambda2}}_{=0}=$$

$$-\frac{1}{2}\int\limits_{\lambda1}^{\lambda2}\left[-2g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)\right]\delta x^\sigma\mathrm{d}\tau \quad (80)$$

But this variation has to be equal zero no matter what the value of δx^σ is. Namely

$$-g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{1}{2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu}\right)=0 \quad (81)$$

Changing sign

$$\boxed{g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}+\frac{1}{2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\nu g_{\mu\sigma}+\partial_\mu g_{\sigma\nu}-\partial_\sigma g_{\mu\nu}\right)=0} \quad (82)$$

Problem 3

Following metric is given

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]\mathrm{d}(ct)^2+(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2,\quad \boldsymbol{x}\equiv(x^1,x^2,x^3) \quad (83)$$

Dividing both sides by $\mathrm{d}t^2$ yields

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]\frac{\mathrm{d}(ct)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^1)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^2)^2}{\mathrm{d}t^2}+\frac{(\mathrm{d}x^3)^2}{\mathrm{d}t^2}=$$

$$-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]c^2+\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x^3}{\mathrm{d}t}\right)^2=-\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}\right]c^2+\boldsymbol{v}\cdot\boldsymbol{v}=$$

$$-c^2\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}\right] \quad (84)$$

Substituting this into lagrangian

$$\mathcal{L}=-mc\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \quad (85)$$

gives us

$$\boxed{\mathcal{L}=-mc\sqrt{c^2\left[1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}\right]}=-mc^2\sqrt{1+2\frac{\phi(\boldsymbol{x})}{c^2}-\frac{\boldsymbol{v}^2}{c^2}}} \quad (86)$$

We can now assume that both $\frac{\phi(\boldsymbol{x})}{c^2}$ and $\frac{\boldsymbol{v}^2}{c^2}$ are small. We can taylor expand square root ($\sqrt{1+x} \simeq 1 + \frac{1}{2x}$) and leave only linear terms:

$$\mathcal{L} = -mc^2 \left(1 + \frac{\phi(\boldsymbol{x})}{c^2} - \frac{\boldsymbol{v}^2}{2c^2}\right) = -mc^2 - m\phi(\boldsymbol{x}) + m\frac{\boldsymbol{v}^2}{2} \tag{87}$$

But adding constants (in this case mc^2) to Lagrangian doesn't change equation of motions, so effective Lagrangian can be written as

$$\boxed{\mathcal{L} = \frac{m\boldsymbol{v}^2}{2} - m\phi(\boldsymbol{x})} \tag{88}$$

which is exactly the Lagrangian for classical mechanics, which leads to Newton's law of motion, namely

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\nabla\phi(\boldsymbol{x}) \tag{89}$$

Assignment 4

Problem 1

Parallel transport of a vector $V = v^\mu \partial_\mu$ along the curve s $\gamma : \lambda \mapsto \left[x^1(\lambda), \ldots, x^n(\lambda)\right]$:

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\sigma}\left[x(\lambda)\right] v^\nu \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{90}$$

We change coordinates, namely $V = v^\mu \partial_\mu = v^\mu \frac{\partial}{\partial x^\mu} = u^\nu \frac{\partial}{\partial y^\nu}$ and $\gamma : \lambda' \mapsto \left[y^1(\lambda'), \ldots, y^n(\lambda')\right]$ First we want to obtain transormation rule for vectors namely

$$v^\nu \frac{\partial y^k}{\partial x^\nu} = V(y^k) = u^\mu \frac{\partial y^k}{\partial y^\mu} = u^k \tag{91}$$

and for $\Gamma^\rho_{\mu\sigma} = \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\sigma} \frac{\partial x^\rho}{\partial \xi^\mu}$

$$\Gamma'^\rho_{\nu\sigma} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial \xi^\mu} = \frac{\partial}{\partial y^\sigma}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} = \overbrace{\left[\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\kappa}{\partial y^\sigma} + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\right]}^{\frac{\partial^2 \xi^\mu}{\partial y^\nu \partial y^\sigma}\ (*)}\frac{\partial y^\rho}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\mu} =$$

$$\underbrace{\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\kappa}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\Gamma^\beta_{\alpha\kappa}}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\beta}\underbrace{\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \xi^\mu}}_{=\delta^\beta_\alpha} = \Gamma^\beta_{\alpha\kappa}\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\rho}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\rho}{\partial x^\alpha} \tag{92}$$

Plugging those things into

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\lambda'} + \Gamma'^\mu_{\nu\sigma}\left[y(\lambda')\right] u^\nu \frac{\mathrm{d}y^\sigma}{\mathrm{d}\lambda'} = 0 \tag{93}$$

we obtain

$$\frac{\partial v^\beta}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} + v^\beta \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} + \left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial \lambda'} = 0 \tag{94}$$

Let's take a look at third term of this sum

$$\left\{\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\mu}{\partial x^\alpha}\right\} v^\eta \frac{\partial y^\nu}{\partial x^\eta}\underbrace{\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{\frac{\partial y^\sigma}{\partial \lambda'}} =$$

$$v^\eta \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right]\underbrace{\frac{\partial x^\kappa}{\partial y^\sigma}\frac{\partial y^\sigma}{\partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}_{=\frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\eta}}_{\delta^\alpha_\eta}\frac{\partial y^\mu}{\partial x^\beta} + v^\eta \underbrace{\frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\sigma}\frac{\partial y^\nu}{\partial x^\eta}\frac{\partial y^\sigma}{\partial x^\tau}}_{\frac{\partial^2 x^\alpha}{\partial x^\eta \partial x^\tau} - \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\ (*)}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\alpha} =$$

$$\Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\nu}{\partial x^\eta \partial x^\tau}\underbrace{\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\alpha}}_{=\delta^\mu_\nu}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} = \Gamma^\beta_{\alpha\kappa}\left[x(\lambda)\right] v^\alpha \frac{\partial x^\kappa}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^\mu}{\partial x^\beta} - v^\eta \frac{\partial^2 y^\mu}{\partial x^\eta \partial x^\tau}\frac{\partial x^\tau}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} \tag{95}$$

So at the end of the day we have

$$\frac{\partial v^{\beta}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^{\mu}}{\partial x^{\beta}} + v^{\beta}\frac{\partial^2 y^{\mu}}{\partial x^{\beta}\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} + \Gamma^{\beta}_{\alpha\kappa}\left[x(\lambda)\right] v^{\alpha}\frac{\partial x^{\kappa}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^{\mu}}{\partial x^{\beta}} - v^{\eta}\frac{\partial^2 y^{\mu}}{\partial x^{\eta}\partial x^{\tau}}\frac{\partial x^{\tau}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda'} = 0 \tag{96}$$

Which simplifies to

$$\left(\frac{\partial v^{\beta}}{\partial \lambda} + \Gamma^{\beta}_{\alpha\kappa}\left[x(\lambda)\right] v^{\alpha}\frac{\partial x^{\kappa}}{\partial \lambda}\right)\frac{\partial \lambda}{\partial \lambda'}\frac{\partial y^{\mu}}{\partial x^{\beta}} = 0 \tag{97}$$

We can divide by $\frac{\partial \lambda}{\partial \lambda'}$

$$\boxed{\left(\frac{\partial v^{\beta}}{\partial \lambda} + \Gamma^{\beta}_{\alpha\kappa}\left[x(\lambda)\right] v^{\alpha}\frac{\partial x^{\kappa}}{\partial \lambda}\right)\frac{\partial y^{\mu}}{\partial x^{\beta}} = 0} \tag{98}$$

So indeed this equation is coordinate–covariant.

Problem 2

Let γ_V denote the geodesic with tangent vector $\boldsymbol{V}_p$ at point p. $\{\boldsymbol{e}_\mu\}$ is arbitrary basis chosen at the point p and normal coordinates are defined as $x(q) = (x^1, \dots, x^n) \Leftrightarrow q = \gamma_{x^\mu \boldsymbol{e}_\mu}$ where $p = \gamma(\lambda = 0)$, $q = \gamma(\lambda = 1)$ and $\{x^i\}_{i=1}^n \in \mathbb{R}$.

First we let $\boldsymbol{V}_p = v^\mu \boldsymbol{e}_\mu$. But we know, since we consider geodesic, that $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. On the other hand we can write normal coordinates of point q as $x(q) = (v^1, \dots, v^n)$. So we have two conditions

$$\left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_{\lambda=0} = v^\mu \qquad \left.x^\mu\right|_{\lambda=1} = v^\mu \tag{99}$$

It is easy to solve this

$$x^\mu(\lambda) = v^\mu \lambda + x_\mu(0) \tag{100}$$

Eq. 100 describes straight line, because it is linear with respect to λ [3]. Substituting this expression into geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0 \tag{101}$$

gives

$$\Gamma^{\mu}_{\sigma\rho} v^\sigma v^\rho = 0 \tag{102}$$

But Eq. 102 has to be satisfied for arbitrary v^σ and v^ρ which implies

$$\boxed{\Gamma^{\mu}_{\sigma\rho} = 0} \tag{103}$$

Problem 3

A $\quad \partial g = 0$

We know that metric transforms as follow:

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \tag{104}$$

[3] or equivalently $\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = 0$

Now we choose transformation $x^\mu = x'^\mu - \frac{1}{2}M^\mu_{\alpha\beta}x'^\alpha x'^\beta$ with condition $x^\mu(p) = 0$. In point p we have $x'^\mu = \frac{1}{2}M^\mu_{\alpha\beta}x'^\alpha x'^\beta$. We see that $x'^\mu = 0$ is (one of the) solution to this equation. We choose this solution. This transformation will give us (M is taken in point p so it derivative is equal to zero)

$$
\begin{aligned}
g'_{\alpha\beta} = \frac{\partial(x'^\mu - \frac{1}{2}M^\mu_{\lambda\sigma}x'^\lambda x'^\sigma)}{\partial x'^\alpha}\frac{\partial(x'^\nu - \frac{1}{2}M^\nu_{\kappa\eta}x'^\kappa x'^\eta)}{\partial x'^\beta}g_{\mu\nu} = \\
\left(\delta^\mu_\alpha - \frac{1}{2}M^\mu_{\lambda\sigma}(\delta^\lambda_\alpha x'^\sigma + x'^\lambda\delta^\sigma_\alpha)\right)\left(\delta^\nu_\beta - \frac{1}{2}M^\nu_{\kappa\eta}(\delta^\kappa_\beta x'^\eta + x'^\kappa\delta^\eta_\beta)\right)g_{\mu\nu} = \\
\left(\delta^\mu_\alpha - \frac{1}{2}M^\mu_{\alpha\sigma}x'^\sigma - \frac{1}{2}M^\mu_{\lambda\alpha}x'^\lambda\right)\left(\delta^\nu_\beta - \frac{1}{2}M^\nu_{\beta\eta}x'^\eta - \frac{1}{2}M^\nu_{\kappa\beta}x'^\kappa\right)g_{\mu\nu} = \\
\left(\delta^\mu_\alpha - \frac{1}{2}x'^\sigma(M^\mu_{\alpha\sigma} + M^\mu_{\sigma\alpha})\right)\left(\delta^\nu_\beta - \frac{1}{2}x'^\eta(M^\nu_{\beta\eta} + M^\nu_{\eta\beta})\right)g_{\mu\nu}
\end{aligned}
\tag{105}
$$

We take $\tilde{M}^\nu_{\beta\eta} = \frac{1}{2}(M^\nu_{\beta\eta} + M^\nu_{\eta\beta})$ and write

$$
g'_{\alpha\beta} = \left(\delta^\mu_\alpha - x'^\sigma\tilde{M}^\mu_{\alpha\sigma}\right)\left(\delta^\nu_\beta - x'^\eta\tilde{M}^\nu_{\beta\eta}\right)g_{\mu\nu} = g_{\alpha\beta} - g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta} - g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma} + x'^\sigma x'^\eta\tilde{M}^\mu_{\alpha\sigma}\tilde{M}^\nu_{\beta\eta}
\tag{106}
$$

Now we take derivative of both sides in point p (remember that $x(p) = x'(p) = 0$)

$$
\begin{aligned}
\partial'_\lambda g'_{\alpha\beta} = \partial'_\lambda g_{\alpha\beta} - \partial'_\lambda(g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta}) - \partial'_\lambda(g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma}) = \\
\left(\delta^\tau_\lambda - x'^\sigma\tilde{M}^\tau_{\lambda\sigma}\right)\partial_\tau g_{\alpha\beta} - \partial'_\lambda g_{\alpha\nu}x'^\eta\tilde{M}^\nu_{\beta\eta} - g_{\alpha\nu}\delta^\eta_\lambda\tilde{M}^\nu_{\beta\eta} - \partial'_\lambda g_{\mu\beta}x'^\sigma\tilde{M}^\mu_{\alpha\sigma} - g_{\mu\beta}\delta^\sigma_\lambda\tilde{M}^\mu_{\alpha\sigma} = \\
\partial_\lambda g_{\alpha\beta} - g_{\alpha\nu}\delta^\eta_\lambda\tilde{M}^\nu_{\beta\eta} - g_{\mu\beta}\delta^\sigma_\lambda\tilde{M}^\mu_{\alpha\sigma} = \partial_\lambda g_{\alpha\beta} - g_{\alpha\nu}\tilde{M}^\nu_{\beta\lambda} - g_{\mu\beta}\tilde{M}^\mu_{\alpha\lambda}
\end{aligned}
\tag{107}
$$

Now let's substitute Chrisroffel symbol in place of $\tilde{M}$ namely

$$
\tilde{M}^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta})
\tag{108}
$$

We obtain (using $g_{\alpha\beta} = g_{\beta\alpha}$)

$$
\begin{aligned}
2\partial'_\lambda g'_{\alpha\beta} = 2\partial_\lambda g_{\alpha\beta} - \underbrace{g_{\alpha\nu}g^{\nu\sigma}}_{=\delta^\sigma_\alpha}(\partial_\beta g_{\sigma\lambda} + \partial_\lambda g_{\sigma\beta} - \partial_\sigma g_{\beta\lambda}) - \underbrace{g_{\mu\beta}g^{\mu\sigma}}_{=\delta^\sigma_\beta}(\partial_\alpha g_{\sigma\lambda} + \partial_\lambda g_{\sigma\alpha} - \partial_\sigma g_{\alpha\lambda}) = \\
2\partial_\lambda g_{\alpha\beta} - \underline{\partial_\beta g_{\alpha\lambda}} - \underline{\partial_\lambda g_{\alpha\beta}} + \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\alpha g_{\beta\lambda}} - \underline{\partial_\lambda g_{\beta\alpha}} + \underline{\partial_\beta g_{\alpha\lambda}} = 2\partial_\lambda g_{\alpha\beta} - 2\partial_\lambda g_{\alpha\beta} = 0
\end{aligned}
\tag{109}
$$

So eventually

$$
\boxed{\partial'_\lambda g'_{\alpha\beta} = 0}
\tag{110}
$$

B $g = \eta$

We try following change of coordinates

$$
x'^\mu = N^\mu{}_\alpha y^\alpha
\tag{111}
$$

In those coordinates metric looks like

$$
g''_{\alpha\beta} = \frac{\partial x'^\mu}{\partial y^\alpha}\frac{\partial x'^\nu}{\partial y^\beta}g'_{\mu\nu} = N^\mu{}_\alpha N^\nu{}_\beta g'_{\mu\nu} = (N^{-1})_\alpha{}^\mu g'_{\mu\nu}N^\nu{}_\beta = (N^{-1}g'N)_{\alpha\beta}
\tag{112}
$$

We can now diagonalize metric g'. We can write

$$g' = C\,\eta\,C^{-1} \tag{113}$$

where η is diagonal and C is a matrix which consists of eigenvectors of g'. If we will choose $N = C$ then Eq. 112 simplifies to

$$\boxed{g''_{\alpha\beta} = \eta_{\alpha\beta}} \tag{114}$$

Assignment 5

Problem 1

Problem 1a

We have given metric

$$\boldsymbol{g} = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \, \mathrm{d}\phi \otimes \mathrm{d}\phi \tag{115}$$

Only non-zero elements are:

$$g_{11} = 1 \qquad\qquad g_{22} = \sin^2\theta \tag{116}$$

Also rmeber that $g^{\alpha\beta} = g^{-1}_{\alpha\beta}$ Let's calculate connection for that metric:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma}\right) \tag{117}$$

It is easy to see that

$$\partial_\nu g_{\rho\sigma} = \delta_{1\nu}\delta_{2\rho}\delta_{2\sigma} \, 2\sin\theta\cos\theta \tag{118}$$

where $\partial_1 \equiv \partial_\theta$ and $\partial_2 \equiv \partial_\phi$. Substituting Eq. (118) into connection yields

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\delta_{1\nu}\delta_{2\rho}\delta_{2\sigma} \, 2\sin\theta\cos\theta + \delta_{1\sigma}\delta_{2\rho}\delta_{2\nu} \, 2\sin\theta\cos\theta - \delta_{1\rho}\delta_{2\nu}\delta_{2\sigma} \, 2\sin\theta\cos\theta\right) =$$

$$\sin\theta\cos\theta \left(g^{2\mu}\delta_{1\nu}\delta_{2\sigma} + g^{2\mu}\delta_{1\sigma}\delta_{2\nu} - g^{1\mu}\delta_{2\nu}\delta_{2\sigma}\right) \tag{119}$$

The only non–zero coefficients

$$\Gamma^{1}_{22} = -\sin\theta\cos\theta \qquad\qquad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{\cos\theta}{\sin\theta} \tag{120}$$

Problem 1b

Geodesic equation is given by

$$\frac{\partial^2 x^\sigma}{\partial\lambda^2} + \Gamma^{\sigma}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{121}$$

Writing explicitly

$$\frac{\partial^2 x^1}{\partial\lambda^2} + \Gamma^{1}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{122a}$$

$$\frac{\partial^2 x^2}{\partial\lambda^2} + \Gamma^{2}_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda} \frac{\partial x^\nu}{\partial\lambda} = 0 \tag{122b}$$

After plugging in Eq. (120) we obtain

$$\frac{\partial^2 \theta}{\partial\lambda^2} - \sin\theta\cos\theta \frac{\partial\phi}{\partial\lambda} \frac{\partial\phi}{\partial\lambda} = 0 \tag{123a}$$

$$\frac{\partial^2 \phi}{\partial\lambda^2} + 2\frac{\cos\theta}{\sin\theta} \frac{\partial\theta}{\partial\lambda} \frac{\partial\phi}{\partial\lambda} = 0 \tag{123b}$$

Now we can set θ and ϕ as affine parameters

$\lambda \longrightarrow \theta$

$$-\sin\theta\cos\theta\frac{\partial\phi}{\partial\theta}\frac{\partial\phi}{\partial\theta} = 0 \tag{124a}$$

$$\frac{\partial^2\phi}{\partial\theta^2} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\phi}{\partial\theta} = 0 \tag{124b}$$

Solution to this set of equations is trivial namely

$$\frac{\partial\phi}{\partial\theta} = 0 \quad \Rightarrow \quad \phi = \text{const} \tag{125}$$

It means that longitudinal lines are geodesics in this metric.

$\lambda \longrightarrow \phi$

$$\frac{\partial^2\theta}{\partial\phi^2} - \sin\theta\cos\theta = 0 \tag{126a}$$

$$2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\phi} = 0 \tag{126b}$$

From second equation we have that

$$\frac{\partial\theta}{\partial\phi} = 0 \tag{127}$$

but it does not solve first equation for every θ. This system of equations has solutions only when

$$\sin\theta\cos\theta = 0 \quad \Rightarrow \quad \theta = 0 \ \lor \ \theta = \frac{\pi}{2} \ \lor \ \theta = \pi \tag{128}$$

since $\theta \in [0, \pi]$. But for $\theta = 0$ or $\theta = \pi$ geodesic line is just one point, because those are poles. Only $\theta = \frac{\pi}{2}$ gives non-trivial geodesic. This line is called equator.

Problem 1c

$\theta = \frac{\pi}{2},\ \phi = 0 \longrightarrow \theta = 0,\ \phi = 0$

We use equation of parallel transport of vector $\boldsymbol{v}$ along curve γ with $\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} = w^\mu$

$$\nabla_{\boldsymbol{w}}\boldsymbol{v} = 0 \tag{129}$$

We can write it explicitly

$$w^\nu\partial_\nu v^\mu + w^\nu\Gamma^\mu_{\nu\sigma}v^\sigma = 0 \tag{130}$$

or substituting connections I've calculated before

$$w^1\partial_1 v^1 + w^2\partial_2 v^1 + w^2\Gamma^1_{22}v^2 = 0 \tag{131a}$$

$$w^1\partial_1 v^2 + w^2\partial_2 v^2 + w^1\Gamma^2_{12}v^2 + w^2\Gamma^2_{21}v^1 = 0 \tag{131b}$$

If we move along meridian then we can take $\lambda \longrightarrow \theta$. From this we get coefficients w namely $w^1 = \frac{\mathrm{d}\theta}{\mathrm{d}\theta} = 1$ and $w^2 = \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = 0$. Putting this and connection coefficients into equations we obtain

$$\partial_1 v^1 = 0 \tag{132a}$$

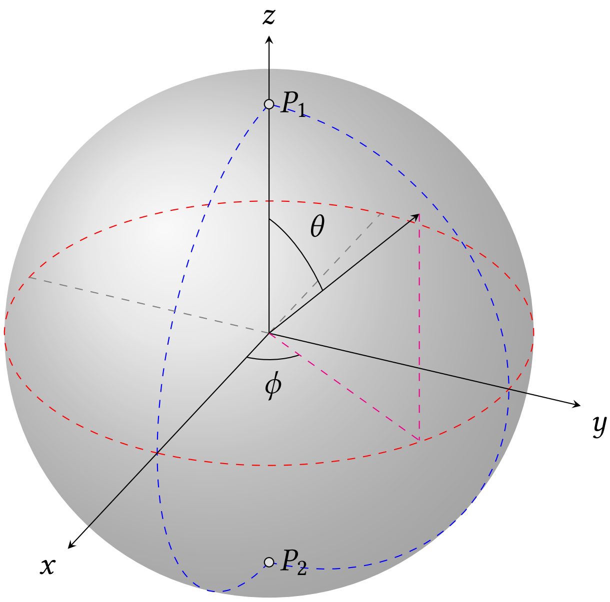

Figure 6: Visualization of geodesics – red line is equator, blue lines are two possible meridians, P_1 and P_2 are poles.

$$\partial_1 v^2 + \frac{\cos\theta}{\sin\theta} v^2 = 0 \tag{132b}$$

Second equation we multiply by $\sin\theta$ and simplify

$$\partial_1 v^1 = 0 \tag{133a}$$

$$\partial_1 \left(v^2 \sin\theta\right) = 0 \tag{133b}$$

This gives use

$$v^1 = C_1 \tag{134a}$$

$$v^2 = \frac{C_2}{\sin\theta} \tag{134b}$$

Constants C_1 and C_2 depends on the vector we transport:

- for $\frac{\partial}{\partial\theta}$ we have at the beginning ($\theta = \frac{\pi}{2}$) $v = (1, 0)$ so $C_1 = 1$ and $C_2 = 0$ so the transport

  $$\boldsymbol{v} = (1, 0) \longrightarrow (1, 0) = \boldsymbol{v}' \tag{135}$$

  does not change this vector.

- for $\frac{\partial}{\partial\phi}$ we have at the beginning $v = (0, 1)$ so $C_1 = 0$ and $C_2 = 1$ so the transport

  $$\boldsymbol{v} = (0, 1) \longrightarrow (0, \frac{1}{\sin\theta}) = \boldsymbol{v}' \tag{136}$$

  is undefined at point $\theta = 0$

$\theta = \frac{\pi}{2},\ \phi = 0 \longrightarrow \theta = \frac{\pi}{2},\ \phi = \frac{\pi}{4}$

If we move along equator then we can take $\lambda \rightarrow \phi$. From this we get coefficients w namely $w^1 = \frac{d\theta}{d\phi} = 0$ and $w^2 = \frac{d\phi}{d\phi} = 1$. Putting this and connection coefficients into equations we obtain

$$\partial_2 v^1 - \sin\theta\cos\theta v^2 = 0 \tag{137a}$$

$$\partial_2 v^2 + \frac{\cos\theta}{\sin\theta} v^1 = 0 \tag{137b}$$

which simplifies to

$$\partial_2 v^1 = 0 \tag{138a}$$

$$\partial_2 v^2 = 0 \tag{138b}$$

because $\cos\theta\Big|_{\theta=\frac{\pi}{2}} = 0$

$$v^1 = C_1 \tag{139a}$$

$$v^2 = C_2 \tag{139b}$$

Constants C_1 and C_2 depends on the vector we transport:

- for $\frac{\partial}{\partial\theta}$ we have at the beginning ($\phi = 0$) $v = (1, 0)$ so $C_1 = 1$ and $C_2 = 0$ so the transport

  $$\boldsymbol{v} = (1, 0) \longrightarrow (1, 0) = \boldsymbol{v}' \tag{140}$$

  does not change this vector.
- for $\frac{\partial}{\partial\phi}$ we have at the beginning $v = (0, 1)$ so $C_1 = 0$ and $C_2 = 1$ so the transport

  $$\boldsymbol{v} = (0, 1) \longrightarrow (0, 1) = \boldsymbol{v}' \tag{141}$$

  does not change this vector.

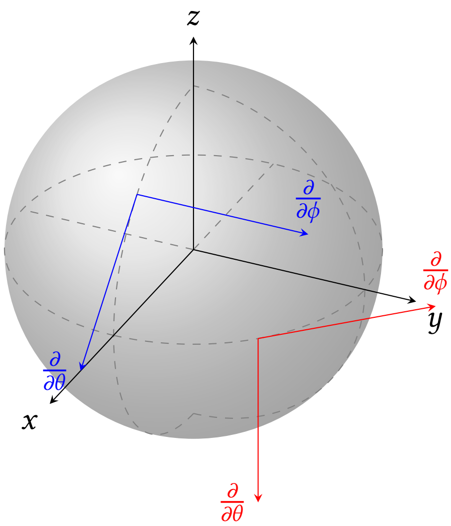

Figure 7: Visualization of parallel transport – red is transport along equator, blue is transport along meridians.

Problem 2

Materic is given by

$$g_{\nu\mu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{142}$$

We can calculate connection coefficients using relation

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\partial_{\nu} g_{\rho\sigma} + \partial_{\sigma} g_{\rho\nu} - \partial_{\rho} g_{\nu\sigma} \right) \tag{143}$$

Becasue metric is diagonal we can simplify this expression:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\mu} \left(\partial_{\nu} g_{\mu\sigma} + \partial_{\sigma} g_{\mu\nu} - \partial_{\mu} g_{\nu\sigma} \right) \tag{144}$$

$$\Gamma^{0} = -\frac{1}{2} \frac{1}{B(r)} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{145a}$$

$$\Gamma^{1} = \frac{1}{2} \frac{1}{A(r)} \begin{pmatrix} \dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r \sin^2\theta \end{pmatrix} \tag{145b}$$

$$\Gamma^{2} = \frac{1}{2} \frac{1}{r^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & -r^2 2 \sin\theta \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{145c}$$

$$\Gamma^{3} = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r \sin^2\theta \\ 0 & 0 & 0 & r^2 2 \sin\theta\cos\theta \\ 0 & 2r\sin^2\theta & r^2 2 \sin\theta\cos\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos\theta}{\sin\theta} \\ 0 & \frac{1}{r} & \frac{\cos\theta}{\sin\theta} & 0 \end{pmatrix} \tag{145d}$$

Geodesic equation

$$\frac{\partial^2 x^{\sigma}}{\partial\lambda^2} + \Gamma^{\sigma}_{\mu\nu} \frac{\partial x^{\mu}}{\partial\lambda} \frac{\partial x^{\nu}}{\partial\lambda} = 0 \tag{146}$$

$\sigma = 0$

$$\frac{\partial^2 t}{\partial\lambda^2} - \frac{1}{2}\frac{1}{B(t)} \begin{pmatrix} \frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda} \end{pmatrix} \begin{pmatrix} 0 & \dot{B} & 0 & 0 \\ \dot{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial\lambda} \\ \frac{\partial r}{\partial\lambda} \\ \frac{\partial\theta}{\partial\lambda} \\ \frac{\partial\phi}{\partial\lambda} \end{pmatrix} = 0 \tag{147}$$

$$\frac{\partial^2 t}{\partial\lambda^2} - \frac{1}{2}\frac{1}{B(t)} \left(-2\dot{B} \frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda} \right) = 0 \tag{148}$$

$$\frac{\partial^2 t}{\partial\lambda^2} + \frac{\dot{B}}{B}\frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda} = 0 \tag{149}$$

$\sigma = 1$

$$\frac{\partial^2 r}{\partial\lambda^2} + \frac{1}{2}\frac{1}{A(r)} \begin{pmatrix} \frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda} \end{pmatrix} \begin{pmatrix} \dot{B} & 0 & 0 & 0 \\ 0 & \dot{A} & 0 & 0 \\ 0 & 0 & -2r & 0 \\ 0 & 0 & 0 & -2r\sin^2\theta \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial\lambda} \\ \frac{\partial r}{\partial\lambda} \\ \frac{\partial\theta}{\partial\lambda} \\ \frac{\partial\phi}{\partial\lambda} \end{pmatrix} = 0 \tag{150}$$

$$\frac{\partial^2 r}{\partial\lambda^2} + \frac{\dot{B}}{2A}\left(\frac{\partial t}{\partial\lambda}\right)^2 + \frac{\dot{A}}{2A}\left(\frac{\partial r}{\partial\lambda}\right)^2 - \frac{r}{A}\left(\frac{\partial\theta}{\partial\lambda}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{\partial\phi}{\partial\lambda}\right)^2 = 0 \tag{151}$$

$\sigma = 2$

$$\frac{\partial^2\theta}{\partial\lambda^2} + \begin{pmatrix}\frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial t}{\partial\lambda}\\ \frac{\partial r}{\partial\lambda}\\ \frac{\partial\theta}{\partial\lambda}\\ \frac{\partial\phi}{\partial\lambda}\end{pmatrix} = 0 \tag{152}$$

$$\frac{\partial^2\theta}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\theta}{\partial\lambda} - \sin\theta\cos\theta\left(\frac{\partial\phi}{\partial\lambda}\right)^2 = 0 \tag{153}$$

$\sigma = 2$

$$\frac{\partial^2\phi}{\partial\lambda^2} + \begin{pmatrix}\frac{\partial t}{\partial\lambda} & \frac{\partial r}{\partial\lambda} & \frac{\partial\theta}{\partial\lambda} & \frac{\partial\phi}{\partial\lambda}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \frac{\cos\theta}{\sin\theta}\\ 0 & \frac{1}{r} & \frac{\cos\theta}{\sin\theta} & 0\end{pmatrix}\begin{pmatrix}\frac{\partial t}{\partial\lambda}\\ \frac{\partial r}{\partial\lambda}\\ \frac{\partial\theta}{\partial\lambda}\\ \frac{\partial\phi}{\partial\lambda}\end{pmatrix} = 0 \tag{154}$$

$$\frac{\partial^2\phi}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} = 0 \tag{155}$$

Gathering all equations

$$0 = \frac{\partial^2 t}{\partial\lambda^2} + \frac{\dot{B}}{B}\frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda} \tag{156a}$$

$$0 = \frac{\partial^2 r}{\partial\lambda^2} + \frac{\dot{B}}{2A}\left(\frac{\partial t}{\partial\lambda}\right)^2 + \frac{\dot{A}}{2A}\left(\frac{\partial r}{\partial\lambda}\right)^2 - \frac{r}{A}\left(\frac{\partial\theta}{\partial\lambda}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{\partial\phi}{\partial\lambda}\right)^2 \tag{156b}$$

$$0 = \frac{\partial^2\theta}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\theta}{\partial\lambda} - \sin\theta\cos\theta\left(\frac{\partial\phi}{\partial\lambda}\right)^2 \tag{156c}$$

$$0 = \frac{\partial^2\phi}{\partial\lambda^2} + \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial\theta}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} \tag{156d}$$

Problem 3

Metric is given by

$$\boldsymbol{g} = -c^2\left(1 - 2\frac{GM}{rc^2}\right)\mathrm{d}t\otimes\mathrm{d}t + \mathrm{d}\boldsymbol{x}\otimes\mathrm{d}\boldsymbol{x} \tag{157}$$

We can calculate infinitesimal interval of two events in this metric

$$\mathrm{d}s^2 = -c^2\left(1 - 2\frac{GM}{rc^2}\right)\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 \tag{158}$$

Simplifying this and assuming that the person has not been moving for whole year we can write

$$\mathrm{d}s^2 = \mathrm{d}t^2 \left(-c^2 \left(1 - 2\frac{GM}{rc^2} \right) + \frac{\mathrm{d}\boldsymbol{x}^2}{\mathrm{d}t^2} \right) = -\mathrm{d}t^2 c^2 \left(1 - 2\frac{GM}{rc^2} \right) \tag{159}$$

We can now calculate the proper time

$$c^2 \mathrm{d}\tau^2 = \mathrm{d}t^2 c^2 \left(1 - 2\frac{GM}{rc^2} \right) \quad \Rightarrow \quad \mathrm{d}\tau = \mathrm{d}t \sqrt{1 - 2\frac{GM}{rc^2}} \tag{160}$$

and finite version

$$\Delta\tau = \Delta t \sqrt{1 - 2\frac{GM}{rc^2}} \tag{161}$$

Now I take $\Delta\tau_0$ to be proper time of person's feet and $\Delta\tau_H$ for head (in distance H from feet). Now I can calculate the difference

$$\Delta\tau_H - \Delta\tau_0 = \Delta t \left(\sqrt{1 - 2\frac{GM}{(r+H)c^2}} - \sqrt{1 - 2\frac{GM}{rc^2}} \right) \tag{162}$$

Because expressions under square are small I can Taylor expand them (using $\sqrt{1-2x} \simeq 1 - x$)

$$\Delta\tau_H - \Delta\tau_0 = \Delta t \left(-\frac{GM}{(r+H)c^2} + \frac{GM}{rc^2} \right) = \Delta t \frac{GM}{c^2} \left(\frac{1}{r} - \frac{1}{r+H} \right) = \Delta t \frac{GM}{c^2 r} \frac{H}{r+H} \tag{163}$$

Plugging all the constants and assuming that person has height 1.8 m after 1 year time difference is

$$\Delta\tau_H - \Delta\tau_0 = 6.3 \times 10^{-9} \text{ s} \tag{164}$$

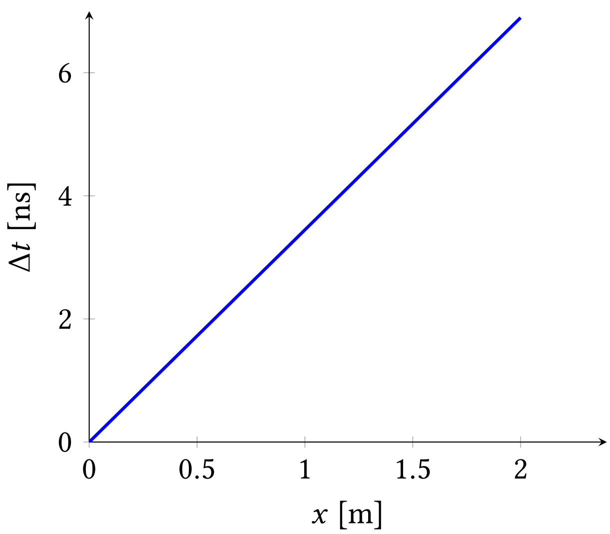

Figure 8: Time difference between head and feet after one year with respect to height of a person

Assignment 6

Problem 1

Show that

$$R(\boldsymbol{u},\boldsymbol{v})z = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z \tag{165}$$

satisfies

$$R(\boldsymbol{u},\boldsymbol{v})(fz) = fR(\boldsymbol{u},\boldsymbol{v})z \tag{166}$$

$$\begin{aligned}
R(\boldsymbol{u},\boldsymbol{v})(fz) = \nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}(fz)\right) - \nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}(fz)\right) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}(fz) = \\
\nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}f\ z + f\nabla_{\boldsymbol{v}}z\right) - \nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}f\ z + f\nabla_{\boldsymbol{u}}z\right) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}f\ z - f\ \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\nabla_{\boldsymbol{u}}\left(\boldsymbol{v}(f)\ z + f\nabla_{\boldsymbol{v}}z\right) - \nabla_{\boldsymbol{v}}\left(\boldsymbol{u}(f)\ z + f\nabla_{\boldsymbol{u}}z\right) - [\boldsymbol{u},\boldsymbol{v}](f)\ z - f\ \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\nabla_{\boldsymbol{u}}\boldsymbol{v}(f)\ z + \boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z + \nabla_{\boldsymbol{u}}f\nabla_{\boldsymbol{v}}z + f\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \\
\nabla_{\boldsymbol{v}}\boldsymbol{u}(f)\ z - \boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}f\nabla_{\boldsymbol{u}}z - f\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \\
[\boldsymbol{u},\boldsymbol{v}](f)\ z - f\ \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
\underline{\boldsymbol{u}(\boldsymbol{v}(f))\ z} + \underline{\boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z} + \underline{\boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z} + f\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \\
\underline{\boldsymbol{v}(\boldsymbol{u}(f))\ z} - \underline{\boldsymbol{u}(f)\nabla_{\boldsymbol{v}}z} - \underline{\boldsymbol{v}(f)\nabla_{\boldsymbol{u}}z} - f\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \\
\underline{[\boldsymbol{u},\boldsymbol{v}](f)\ z} - f\ \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z = \\
f\left(\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}z - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}z - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}z\right)
\end{aligned} \tag{167}$$

Problem 2

Show that

$$\delta\left(\ln\det\boldsymbol{M}\right) = \operatorname{Tr}\left(\boldsymbol{M}^{-1}\delta\boldsymbol{M}\right) \tag{168}$$

using relation

$$\det(\boldsymbol{1} + \epsilon\boldsymbol{A}) = 1 + \epsilon\operatorname{Tr}\boldsymbol{A} + O(\epsilon^2) \tag{169}$$

Doing straight forward calculations

$$\begin{aligned}
\delta(\ln\det\boldsymbol{M}) = \lim_{\epsilon\to 0}\frac{\ln\det(\boldsymbol{M}+\epsilon\delta\boldsymbol{M}) - \ln\det\boldsymbol{M}}{\epsilon} = \lim_{\epsilon\to 0}\frac{\ln\frac{\det(\boldsymbol{M}+\epsilon\delta\boldsymbol{M})}{\det\boldsymbol{M}}}{\epsilon} = \lim_{\epsilon\to 0}\frac{\ln\det\left(\boldsymbol{1}+\epsilon\delta\boldsymbol{M}\boldsymbol{M}^{-1}\right)}{\epsilon} = \\
\lim_{\epsilon\to 0}\frac{\ln\left\{1+\operatorname{Tr}(\epsilon\delta\boldsymbol{M}\boldsymbol{M}^{-1})\right\}}{\epsilon} = /\text{I use Taylor expansion for } \ln(1+x) \simeq x/\rightarrow \\
\lim_{\epsilon\to 0}\frac{\operatorname{Tr}(\epsilon\delta\boldsymbol{M}\boldsymbol{M}^{-1})}{\epsilon} = \operatorname{Tr}(\delta\boldsymbol{M}\boldsymbol{M}^{-1}) = \operatorname{Tr}(\boldsymbol{M}^{-1}\delta\boldsymbol{M})
\end{aligned} \tag{170}$$

Problem 3

Show that

$$\Gamma^{\nu}_{\nu\mu} = \partial_\mu \ln \sqrt{|\boldsymbol{g}|} \tag{171}$$

We first calculate

$$\Gamma^{\nu}_{\nu\mu} = \frac{1}{2} g^{\nu\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\nu\mu} \right) = \frac{1}{2} g^{\nu\alpha} \partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\nu\alpha} \partial_\nu g_{\mu\alpha} - \frac{1}{2} g^{\nu\alpha} \partial_\alpha g_{\nu\mu} = \quad (\nu \leftrightarrow \alpha)$$

$$\frac{1}{2} g^{\nu\alpha} \partial_\mu g_{\nu\alpha} + \frac{1}{2} g^{\alpha\nu} \partial_\alpha g_{\mu\nu} - \frac{1}{2} g^{\nu\alpha} \partial_\alpha g_{\nu\mu} = \frac{1}{2} g^{\nu\alpha} \partial_\mu g_{\nu\alpha} = \frac{1}{2} \left(\boldsymbol{g}^{-1} \right)_{\alpha\nu} \left(\partial_\mu \boldsymbol{g} \right)_{\nu\alpha} = \frac{1}{2} \mathrm{Tr} \left(\boldsymbol{g}^{-1} \partial_\mu \boldsymbol{g} \right) \tag{172}$$

And now we calculate

$$\partial_\mu \ln \sqrt{|\boldsymbol{g}|} = \partial_\mu \ln |\boldsymbol{g}|^{\frac{1}{2}} = \frac{1}{2} \partial_\mu \ln |\boldsymbol{g}| \overset{Eq.\ (168)}{=} \frac{1}{2} \mathrm{Tr} \left(\boldsymbol{g}^{-1} \partial_\mu \boldsymbol{g} \right) \tag{173}$$

So both sides of equation are equal

Problem 4

Show that

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\mu \left(\sqrt{|\boldsymbol{g}|} A^\mu \right) \tag{174}$$

We start with

$$\frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\mu \left(\sqrt{|\boldsymbol{g}|} A^\mu \right) = \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\mu \sqrt{|\boldsymbol{g}|} A^\mu + \frac{1}{\sqrt{|\boldsymbol{g}|}} \sqrt{|\boldsymbol{g}|} \partial_\mu A^\mu = \partial_\mu \left(\ln \sqrt{|\boldsymbol{g}|} \right) A^\mu + \frac{1}{\sqrt{|\boldsymbol{g}|}} \sqrt{|\boldsymbol{g}|} \partial_\mu A^\mu =$$

$$\Gamma^{\nu}_{\nu\mu} A^\mu + \partial_\mu A^\mu \tag{175}$$

So

$$\nabla_\mu A^\mu = \Gamma^{\nu}_{\nu\mu} A^\mu + \partial_\mu A^\mu \tag{176}$$

which is true by definition of covariant derivative

Problem 5

We start with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left\{ \boldsymbol{g} \left(\partial_{\sigma^*}, \boldsymbol{v} \right) \right\} = 0 \tag{177}$$

Since we stay on geodesic we can write

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left\{ \boldsymbol{g} \left(\partial_{\sigma^*}, \boldsymbol{v} \right) \right\} = \nabla_{\boldsymbol{v}} \left\{ \boldsymbol{g} \left(\partial_{\sigma^*}, \boldsymbol{v} \right) \right\} \tag{178}$$

We can now expand

$$\boldsymbol{g} \left(\partial_{\sigma^*}, \boldsymbol{v} \right) = \delta^{\nu}_{\sigma^*} g_{\nu\mu} v^\mu \tag{179}$$

So eventually

$$\nabla_{\boldsymbol{v}} \left\{ \delta^{\nu}_{\sigma^*} g_{\nu\mu} v^\mu \right\} = g_{\nu\mu} \nabla_{\boldsymbol{v}} \left\{ \delta^{\nu}_{\sigma^*} v^\mu \right\} = g_{\nu\mu} \nabla_{\boldsymbol{v}} \delta^{\nu}_{\sigma^*} v^\mu + g_{\nu\mu} \delta^{\nu}_{\sigma^*} \nabla_{\boldsymbol{v}} v^\mu =$$

$$g_{\nu\mu}\Gamma^{\nu}_{\rho\kappa}v^{\kappa}\delta^{\rho}_{\sigma^*}v^{\mu} + g_{\nu\mu}\delta^{\nu}_{\sigma^*}v^{\rho}\partial_{\rho}v^{\mu} + g_{\nu\mu}\delta^{\nu}_{\sigma^*}\Gamma^{\mu}_{\rho\kappa}v^{\rho}v^{\kappa} =$$

$$g_{\nu\mu}\Gamma^{\nu}_{\sigma^*\kappa}v^{\kappa}v^{\mu} + g_{\sigma^*\mu}v^{\rho}\partial_{\rho}v^{\mu} + g_{\sigma^*\mu}\Gamma^{\mu}_{\rho\kappa}v^{\rho}v^{\kappa} = \quad (180)$$

Now I calculate

$$g_{\nu\mu}\Gamma^{\nu}_{\sigma^*\kappa} = \frac{1}{2}\underbrace{g_{\nu\mu}g^{\nu\alpha}}_{=\delta^{\alpha}_{\mu}}(\partial_{\kappa}g_{\sigma^*\alpha} + \underbrace{\partial_{\sigma^*}g_{\kappa\alpha}}_{=0} - \partial_{\alpha}g_{\kappa\sigma^*}) = \frac{1}{2}(\partial_{\kappa}g_{\sigma^*\mu} - \partial_{\mu}g_{\kappa\sigma^*}) \quad (181)$$

and

$$g_{\sigma^*\mu}\Gamma^{\mu}_{\rho\kappa} = \frac{1}{2}\underbrace{g_{\sigma^*\mu}g^{\mu\alpha}}_{=\delta^{\alpha}_{\sigma^*}}(\partial_{\kappa}g_{\rho\alpha} + \partial_{\rho}g_{\kappa\alpha} - \partial_{\alpha}g_{\kappa\rho}) = \frac{1}{2}(\partial_{\kappa}g_{\rho\sigma^*} + \partial_{\rho}g_{\kappa\sigma^*} - \underbrace{\partial_{\sigma^*}g_{\kappa\rho}}_{=0}) = \frac{1}{2}(\partial_{\kappa}g_{\rho\sigma^*} + \partial_{\rho}g_{\kappa\sigma^*}) \quad (182)$$

Plugging down everything we have

$$g_{\nu\mu}\Gamma^{\nu}_{\sigma^*\kappa}v^{\kappa}v^{\mu} + g_{\sigma^*\mu}v^{\rho}\partial_{\rho}v^{\mu} + g_{\sigma^*\mu}\Gamma^{\mu}_{\rho\kappa}v^{\rho}v^{\kappa} =$$

$$\frac{1}{2}\underbrace{(\partial_{\kappa}g_{\sigma^*\mu} - \partial_{\mu}g_{\kappa\sigma^*})v^{\kappa}v^{\mu}}_{=0} + g_{\sigma^*\mu}v^{\rho}\partial_{\rho}v^{\mu} + \frac{1}{2}(\partial_{\kappa}g_{\rho\sigma^*} + \partial_{\rho}g_{\kappa\sigma^*})v^{\rho}v^{\kappa} =$$

$$g_{\sigma^*\mu}v^{\rho}\partial_{\rho}v^{\mu} + \partial_{\kappa}g_{\rho\sigma^*}v^{\rho}v^{\kappa} = v^{\rho}\partial_{\rho}\left(g_{\sigma^{\star}\mu}v^{\mu}\right) \quad (183)$$

Assignment 7

Problem 1

Energy – momentum tensor of perfect fluid

$$T^{\mu\nu} = (\rho + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu}. \tag{184}$$

First thing I do is base change

$$\boldsymbol{T} = T^{\mu\nu}\partial_\mu\partial_\nu = T^{ab}\boldsymbol{e}_a\boldsymbol{e}_b = T^{ab}\boldsymbol{e}_a\boldsymbol{e}_b = T^{ab}e_a^\mu e_b^\nu\partial_\mu\partial_\nu. \tag{185}$$

From this it can be seen that

$$T^{\mu\nu} = T^{ab}e_a^\mu e_b^\nu. \tag{186}$$

I multiply both sides by corresponding one-froms $\bar{e}_\mu^c\bar{e}_\nu^d$ which satisfy $\bar{e}_\mu^c e_d^\mu = \delta_d^c$

$$T^{\mu\nu}\bar{e}_\mu^a\bar{e}_\nu^b = T^{ab}. \tag{187}$$

From it I obtain

$$T^{ab} = (\rho + p)\frac{u^\mu\bar{e}_\mu^a\, u^\nu\bar{e}_\nu^b}{c^2} + pg^{\mu\nu}\bar{e}_\mu^a\bar{e}_\nu^b. \tag{188}$$

Now I choose base $\{\boldsymbol{e}\}$ to be orthogonal (so $g_{\mu\nu}e_a^\mu e_b^\nu = \eta_{ab}$ and $g^{\mu\nu}\bar{e}_\mu^a\bar{e}_\nu^b = \eta^{ab}$) and $u^\mu = ce_0^\mu$. Substituting those things yields

$$T^{ab} = (\rho + p)\frac{\cancel{c}e_0^\mu\bar{e}_\mu^a\ \cancel{c}e_0^\nu\bar{e}_\nu^b}{\cancel{c^2}} + p\eta^{ab} = (\rho + p)\delta_0^a\ \delta_0^b + p\eta^{ab}. \tag{189}$$

So final result is

$$T^{ab} \overset{*}{=} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{190}$$

Problem 2

Calculate

$$\partial_\mu T^{\mu\nu} = \partial_\mu\left\{(\rho + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu}\right\} \tag{191}$$

$$\begin{aligned}\partial_\mu\left\{(\rho + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu}\right\} &= \partial_\mu(\rho + p)\frac{u^\mu u^\nu}{c^2} + (\rho + p)\partial_\mu\frac{u^\mu u^\nu}{c^2} + \partial_\mu p\ g^{\mu\nu} = \\ \partial_\mu\rho\frac{u^\mu u^\nu}{c^2} + \partial_\mu p\frac{u^\mu u^\nu}{c^2} &+ \frac{1}{c^2}(\rho + p)\partial_\mu u^\mu u^\nu + \frac{1}{c^2}(\rho + p)u^\mu\partial_\mu u^\nu + \partial_\mu p\ g^{\mu\nu} = \\ \frac{1}{c^2}\left(u^\nu u^\mu\partial_\mu\rho + u^\nu u^\mu\partial_\mu p\right. &\left.+ (\rho + p)\partial_\mu u^\mu u^\nu + (\rho + p)u^\mu\partial_\mu u^\nu + c^2\partial_\mu p\ g^{\mu\nu}\right)\end{aligned} \tag{192}$$

Now we define $\frac{\mathrm{d}}{\mathrm{d}\tau} \equiv u^\mu\partial_\mu$ and plug it into equation

$$\frac{1}{c^2}\left(u^\nu\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+u^\nu u^\mu\partial_\mu p+u^\nu(\rho+p)\partial_\mu u^\mu+(\rho+p)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}+c^2\partial_\mu p\, g^{\mu\nu}\right)=$$

$$\frac{1}{c^2}u^\nu\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+(\rho+p)\partial_\mu u^\mu\right)+\frac{1}{c^2}(\rho+p)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}+\left(\frac{u^\nu u^\mu}{c^2}+g^{\mu\nu}\right)\partial_\mu p \tag{193}$$

We also define $P^{\mu\nu}\equiv\frac{u^\nu u^\mu}{c^2}+g^{\mu\nu}$. Then condition $\partial_\mu T^{\mu\nu}=0$ is equivalent to

$$u^\nu\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+(\rho+p)\partial_\mu u^\mu\right)+(\rho+p)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}+c^2P^{\mu\nu}\partial_\mu p=0 \tag{194}$$

Now we will show that ${P^\mu}_\nu$ is projector on the plane orthogonal to $\boldsymbol{u}$.

$${P^\mu}_\nu=P^{\mu\sigma}g_{\sigma\nu}=\frac{u^\sigma u^\mu}{c^2}g_{\sigma\nu}+g^{\mu\sigma}g_{\sigma\nu}=\frac{u_\nu u^\mu}{c^2}+\delta^\mu_\nu \tag{195}$$

1. Idempotency

$$P^2=P\implies {P^\mu}_\rho {P^\rho}_\nu={P^\mu}_\nu \tag{196}$$

$$\left(\frac{u_\rho u^\mu}{c^2}+\delta^\mu_\rho\right)\left(\frac{u_\nu u^\rho}{c^2}+\delta^\rho_\nu\right)=\frac{u_\rho u^\rho u^\mu u_\nu}{c^2}+\frac{u_\rho u^\mu}{c^2}\delta^\rho_\nu+\delta^\mu_\rho\frac{u_\nu u^\rho}{c^2}+\delta^\mu_\rho\delta^\rho_\nu=$$

$$-\frac{u^\mu u_\nu}{c^2}+\frac{u_\nu u^\mu}{c^2}+\frac{u_\nu u^\mu}{c^2}+\delta^\mu_\nu=\frac{u_\nu u^\mu}{c^2}+\delta^\mu_\nu={P^\mu}_\nu \tag{197}$$

2. Gives 0 when act on $\boldsymbol{u}$

$$P\boldsymbol{u}=0\implies {P^\mu}_\nu u^\nu=0 \tag{198}$$

$$\left(\frac{u_\nu u^\mu}{c^2}+\delta^\mu_\nu\right)u^\nu=\frac{u^\nu u_\nu u^\mu}{c^2}+\delta^\mu_\nu u^\nu=-u^\mu+u^\mu=0 \tag{199}$$

3. Gives $\boldsymbol{v}$ when act on $\boldsymbol{v}\perp\boldsymbol{u}$

$$P\boldsymbol{v}=\boldsymbol{v}\implies {P^\mu}_\nu v^\nu=v^\mu \tag{200}$$

$$\left(\frac{u_\nu u^\mu}{c^2}+\delta^\mu_\nu\right)v^\nu=\frac{\overbrace{v^\nu u_\nu}^{=0}\,u^\mu}{c^2}+\delta^\mu_\nu v^\nu=v^\mu \tag{201}$$

So now we can see that Eq. (194) consist of two parts which are orthogonal (so independent) [4].

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+(\rho+p)\partial_\mu u^\mu=0 \tag{202a}$$

$$(\rho+p)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}+c^2P^{\mu\nu}\partial_\mu p=0 \tag{202b}$$

[4] 4-acceleration is also orthogonal to 4-velocity

Problem 3

Now we can substitute $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\mu \partial_\mu$ and $u^\mu = \gamma(c, v^i)$ we obtain

$$\frac{1}{c}u^0\partial_t\rho + u^i\partial_i\rho + \frac{1}{c}(\rho+p)\partial_t u^0 + (\rho+p)\partial_i u^i = 0 \tag{203a}$$

$$(\rho+p)u^\mu\partial_\mu u^\nu + u^\nu u^\mu \partial_\mu p + c^2 g^{\mu\nu}\partial_\mu p = 0 \tag{203b}$$

Let's first investigate first equation

$$\begin{aligned}
&\frac{1}{c}u^0\partial_t\rho + u^i\partial_i\rho + \frac{1}{c}(\rho+p)\partial_t u^0 + (\rho+p)\partial_i u^i = \\
&\quad \gamma\left(\partial_t\rho + v^i\nabla_i\rho + (\rho+p)\nabla_i v^i\right) = \gamma\left(\partial_t\rho + \nabla_i\left(v^i\rho\right) + p\nabla_i v^i\right) = \gamma\left(\partial_t\rho + \boldsymbol{\nabla}\left(\boldsymbol{v}\rho\right) + p\boldsymbol{\nabla}\boldsymbol{v}\right) = \\
&\quad\quad /\text{neglecting terms with } p/ \qquad \gamma\left(\partial_t\rho + \boldsymbol{\nabla}\left(\boldsymbol{v}\rho\right)\right) = 0
\end{aligned} \tag{204}$$

which gives

$$\boxed{\partial_t\rho + \boldsymbol{\nabla}\left(\boldsymbol{v}\rho\right) = 0} \tag{205}$$

Now second equation

$$\begin{aligned}
&(\rho+p)u^\mu\partial_\mu u^\nu + u^\nu u^\mu\partial_\mu p + c^2 g^{\mu\nu}\partial_\mu p = \\
&\quad \frac{1}{c}(\rho+p)u^0\partial_t u^\nu + (\rho+p)u^i\partial_i u^\nu + \frac{1}{c}u^\nu u^0\partial_t p + u^\nu u^i\partial_i p + c g^{0\nu}\partial_t p + c^2 g^{i\nu}\partial_i p = \\
&\quad\quad \gamma(\rho+p)\partial_t u^\nu + \gamma(\rho+p)v^i\nabla_i u^\nu + \gamma u^\nu\partial_t p + \gamma u^\nu v^i\nabla_i p - \delta^\nu_0 c\partial_t p + \delta^\nu_i c^2\nabla_i p
\end{aligned} \tag{206}$$

I split this equation into two cases: $\nu = 0$ and $\nu = i$

$\nu = 0$

$$\gamma(\rho+p)\partial_t u^0 + \gamma(\rho+p)v^i\nabla_i u^0 + \gamma u^0\partial_t p + \gamma u^0 v^i\nabla_i p - \delta^0_0 c\partial_t p = \gamma^2 c\partial_t p + \gamma^2 c v^i\nabla_i p - c\partial_t p \tag{207}$$

Plugging in $\gamma = 1$ gives

$$\boxed{\boldsymbol{v}\boldsymbol{\nabla}p = 0} \tag{208}$$

$\nu = i$

$$\begin{aligned}
&\gamma(\rho+p)\partial_t u^i + \gamma(\rho+p)v^i\nabla_i u^i + \gamma u^i\partial_t p + \gamma u^i v^i\nabla_i p + c^2\nabla_i p = \\
&\quad \gamma(\rho+p)\partial_t v^j + \gamma(\rho+p)v^i\nabla_i v^j + \gamma v^j\partial_t p + \gamma v^j v^i\nabla_i p + c^2\nabla_j p = \\
&\quad \gamma\left(p\partial_t v^j + v^j\partial_t p\right) + \gamma\left(p v^i\nabla_i v^j + v^j v^i\nabla_i p\right) + \gamma\rho v^i\nabla_i v^j + \gamma\rho\partial_t v^j + c^2\nabla_j p = \\
&\quad \gamma\partial_t\left(p v^j\right) + \gamma v^i\nabla_i\left(p v^j\right) + \gamma\rho\left(\partial_t + v^i\nabla_i\right)v^j + c^2\nabla_j p = \\
&\quad\quad \gamma\left(\partial_t + \boldsymbol{v}\boldsymbol{\nabla}\right)\left(p\boldsymbol{v}\right) + \gamma\rho\left(\partial_t + \boldsymbol{v}\boldsymbol{\nabla}\right)\boldsymbol{v} + c^2\boldsymbol{\nabla}p = 0
\end{aligned} \tag{209}$$

Dividing both sides by c^2, neglecting terms $\frac{p}{c^2}$ and plugging $\gamma = 1$ gives us

$$\boxed{\frac{\rho}{c^2}\left(\partial_t + \boldsymbol{v}\boldsymbol{\nabla}\right)\boldsymbol{v} + \boldsymbol{\nabla}p = 0} \tag{210}$$

Final result is

$$\partial_t \rho + \boldsymbol{\nabla} \left(\rho \boldsymbol{v} \right) = 0 \tag{211a}$$

$$\frac{\rho}{c^2} \left(\partial_t + \boldsymbol{v} \boldsymbol{\nabla} \right) \boldsymbol{v} = -\boldsymbol{\nabla} p \tag{211b}$$

The first equation is **continuity equations**. It states that change in energy density in some volume is precisely the flow of the energy desity out (or in) this volume. So overall energy is conserved.

The second equation is **Navier–Stockes equation** which describes motion of viscous fluid.

Assignment 8

Problem 1

a)

Schwartzschild metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{212}$$

Proper time is defined as $\mathrm{d}\tau^2 = -\mathrm{d}s^2$ so

$$\mathrm{d}\tau^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{213}$$

After rearrangement

$$\left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \mathrm{d}\tau^2 - r^2\mathrm{d}\Omega^2 \tag{214}$$

and after division by $\left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}\tau^2$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \left(1 - \frac{2M}{r}\right)^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2M}{r}\right) - r^2\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2. \tag{215}$$

Taking square root

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| = \sqrt{\left(1 - \frac{2M}{r}\right)^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2M}{r}\right) - r^2\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2}. \tag{216}$$

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| = \sqrt{-\left(1 - \frac{2M}{r}\right)}\sqrt{1 - \left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2}. \tag{217}$$

We consider case $r < 2M \implies \frac{2M}{r} > 1$ so

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| = \sqrt{\frac{2M}{r} - 1}\left[1 + \underbrace{\left(\frac{2M}{r} - 1\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2}_{\geq 0}\right]^{\frac{1}{2}}. \tag{218}$$

Bracket is larger than one so we can write

$$\boxed{\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| \geq \sqrt{\frac{2M}{r} - 1}} \tag{219}$$

b)

We now assume that motion is towards the center so $\frac{\mathrm{d}r}{\mathrm{d}\tau} < 0$ it means that

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} \leq -\sqrt{\frac{2M}{r} - 1}. \tag{220}$$

Rearranging terms

$$-\frac{\mathrm{d}r}{\sqrt{\frac{2M}{r} - 1}} \geq \mathrm{d}\tau. \tag{221}$$

Integrating both sides

$$-\int \frac{\mathrm{d}r}{\sqrt{\frac{2M}{r} - 1}} \geq \int \mathrm{d}\tau. \tag{222}$$

Consider LHS

$$-\int \frac{\mathrm{d}r}{\sqrt{\frac{2M}{r} - 1}} = \quad /x = \frac{r}{2M}/ \quad = -2M \int \frac{\mathrm{d}x}{\sqrt{\frac{1}{x} - 1}} = -2M \int \frac{\sqrt{x}}{\sqrt{1-x}} \mathrm{d}x =$$

$$/y = \sqrt{x}/ \quad = -2M \int \frac{y}{\sqrt{1-y^2}}\, 2y\, \mathrm{d}y = \quad /z = \arcsin(y)/ \quad = -4M \int \frac{\sin^2(z)}{\sqrt{1-\sin^2(z)}} \cos(z)\,\mathrm{d}z =$$

$$-4M \int \frac{\sin^2(z)}{\cos(z)} \cos(z)\,\mathrm{d}z = -4M \int \sin^2(z)\mathrm{d}z = \quad /\cos(2z) = \cos^2(z) - \sin^2(z) = 1 - 2\sin^2(z)/ \quad =$$

$$-2M \int (1 - \cos(2z))\, \mathrm{d}z = -2M \left(z - \frac{1}{2}\sin(2z) \right) \mathrm{d}z = -2M\,[z - \sin(z)\cos(z)] + C =$$

$$-2M \left[\arcsin\left(\sqrt{x}\right) - \sqrt{x}\sqrt{1-x} \right] + C \tag{223}$$

So

$$\tau \leq -2M \left[\arcsin\left(\sqrt{x}\right) - \sqrt{x}\sqrt{1-x} \right] \Bigg|_1^0 = 2M \arcsin(1) = \pi M. \tag{224}$$

Final result

$$\boxed{\tau \leq \pi M}. \tag{225}$$

c)

Maximal time $\tau = \pi M$ is achieved when in Eq. (219) is equal sign. But this we get only when part

$$\left(\frac{2M}{r} - 1 \right) \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 + r^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} \right)^2 = 0. \tag{226}$$

Since every term is non-negative and coefficients can not equal 0 we deduce that

$$\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = 0} \tag{227}$$

which means that angular part does not change over time. So indeed trajectory is radial geodesic. Also time does not longer play role of a time coordinate inside horizon of BH, instead radial term does it. So we can put time to be constant.

Problem 2

Schwartzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{228}$$

It does not depend on variable t and ϕ which means that they are cyclic namely

$$g(\partial_t, U) = E = \text{const} \qquad \text{and} \qquad g(\partial_\phi, U) = L = \text{const} \tag{229}$$

which gives

$$-\left(1 - \frac{2M}{r}\right) \dot{t} = E \qquad \text{and} \qquad r^2 \sin^2\theta \dot{\phi} = L. \tag{230}$$

I consider circular ($r = \text{const}$) motion on the plane $\theta = \frac{\pi}{2} \implies \sin^2\theta = 1$. I also know that

$$-1 = g(U, U) = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + r^2 \dot{\phi}^2. \tag{231}$$

Substituting constants of motion

$$-1 = -\frac{E^2}{1 - \frac{2M}{r}} + \frac{L^2}{r^2} \implies E^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + 1\right). \tag{232}$$

Taking derivative ∂_r of both sides gives

$$0 = \frac{2M}{r^2}\left(\frac{L^2}{r^2} + 1\right) + \left(1 - \frac{2M}{r}\right)\left(-2\frac{L^2}{r^3}\right) = \frac{2ML^2}{r^4} + \frac{2M}{r^2} - \frac{2L^2}{r^3} + \frac{4ML^2}{r^4} = \frac{2L^2}{r^4}(3M - r) + \frac{2M}{r^2} \tag{233}$$

and rearranging

$$L^2(r - 3M) = Mr^2 \tag{234}$$

Substituting this into Eq. (232) gives

$$E^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{M}{r - 3M} + 1\right) \implies E^2(r - 3M) = r\left(1 - \frac{2M}{r}\right)^2 \tag{235}$$

Now we want to calculate $\frac{d\phi}{dt}$. We can use now chain rule to evaluate this derivative:

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} \implies \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} \tag{236}$$

Substituting constants of motion

$$\boxed{\left(\frac{d\phi}{dt}\right)^2 = \frac{L^2}{r^4}\frac{\left(1 - \frac{2M}{r}\right)^2}{E^2} = \frac{M}{r^3}} \tag{237}$$

Problem 3

We start with proper time in Schwartzschild metric

$$\mathrm{d}\tau^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{238}$$

For stationary object at $r = r_1$ this time is equal

$$\mathrm{d}\tau_1^2 = \left(1 - \frac{2M}{r_1}\right)\mathrm{d}t^2. \tag{239}$$

For object freely orbiting around equator ($\theta = \frac{\pi}{2}$) proper time is equal

$$\mathrm{d}\tau_2^2 = \left(1 - \frac{2M}{r_1}\right)\mathrm{d}t^2 - r_1^2\,\mathrm{d}\phi = \mathrm{d}t^2\left[1 - \frac{2M}{r_1} - r_1^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right] \tag{240}$$

Substituting result from previous problem it yields

$$\mathrm{d}\tau_2^2 = \mathrm{d}t^2\left[1 - \frac{2M}{r_1} - \frac{M}{r_1}\right] = \mathrm{d}t^2\left[1 - \frac{3M}{r_1}\right] \tag{241}$$

Finally

$$\boxed{\frac{\tau_1}{\tau_2} = \sqrt{\frac{1 - \frac{2M}{r_1}}{1 - \frac{3M}{r_1}}} = \sqrt{\frac{r_1 - 2M}{r_1 - 3M}}} \tag{242}$$

Assignment 9

Problem 1

We start with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2 \tag{243}$$

and we want to change coordinates $\boldsymbol{x} \longrightarrow \boldsymbol{x}_p = a(t)\boldsymbol{x}$. For that we first need to calculate

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}\left(\frac{\boldsymbol{x}_p}{a}\right) = \frac{\mathrm{d}\boldsymbol{x}_p a - \frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}t\ \boldsymbol{x}_p}{a^2} = \frac{a\mathrm{d}\boldsymbol{x}_p - \dot{a}\boldsymbol{x}_p\mathrm{d}t}{a^2} = \frac{\mathrm{d}\boldsymbol{x}_p}{a} - \frac{\dot{a}}{a^2}\boldsymbol{x}_p\mathrm{d}t. \tag{244}$$

Using that result I can now calculate term which is in the equation for line segment

$$a^2\mathrm{d}x^2 = (a\mathrm{d}x)^2 = \left(\mathrm{d}\boldsymbol{x}_p - \frac{\dot{a}}{a}\boldsymbol{x}_p\mathrm{d}t\right)^2 = \mathrm{d}\boldsymbol{x}_p^2 - 2\frac{\dot{a}}{a}\boldsymbol{x}_p\mathrm{d}t\mathrm{d}\boldsymbol{x}_p + \left(\frac{\dot{a}}{a}\right)^2\boldsymbol{x}_p^2\mathrm{d}t^2. \tag{245}$$

Substituting this into metric gives

$$\begin{aligned}\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}_p^2 - 2\frac{\dot{a}}{a}\boldsymbol{x}_p\mathrm{d}t\mathrm{d}\boldsymbol{x}_p + \left(\frac{\dot{a}}{a}\right)^2\boldsymbol{x}_p^2\mathrm{d}t^2 = \\ \boxed{-\left(1 - \left(\frac{\dot{a}}{a}\right)^2\boldsymbol{x}_p^2\right)\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}_p^2 - 2\frac{\dot{a}}{a}\boldsymbol{x}_p\ \mathrm{d}t\mathrm{d}\boldsymbol{x}_p}\end{aligned} \tag{246}$$

Problem 2

Line segment in 4D Euclidian space is given by

$$\mathrm{d}s^2 = \mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{247}$$

and 4D sphere of radius a can be parametrize by

$$w = a\cos\chi \tag{248a}$$

$$x = a\sin\chi\sin\theta\cos\phi \tag{248b}$$

$$y = a\sin\chi\sin\theta\sin\phi \tag{248c}$$

$$z = a\sin\chi\cos\theta. \tag{248d}$$

We can calculate

$$\mathrm{d}w = -a\sin\chi\mathrm{d}\chi \tag{249a}$$

$$\mathrm{d}x = a\cos\chi\sin\theta\cos\phi\mathrm{d}\chi + a\sin\chi\cos\theta\cos\phi\mathrm{d}\theta - a\sin\chi\sin\theta\sin\phi\mathrm{d}\phi \tag{249b}$$

$$\mathrm{d}y = a\cos\chi\sin\theta\sin\phi\mathrm{d}\chi + a\sin\chi\cos\theta\sin\phi\mathrm{d}\theta + a\sin\chi\sin\theta\cos\phi\mathrm{d}\phi \tag{249c}$$

$$\mathrm{d}z = a\cos\chi\cos\theta\mathrm{d}\chi - a\sin\chi\sin\theta\mathrm{d}\theta. \tag{249d}$$

and furthermore

$$
\begin{aligned}
\mathrm{d}x^2 + \mathrm{d}y^2 = a^2(& \cos^2\chi \sin^2\theta \mathrm{d}\chi^2 + \\
& \sin^2\chi \cos^2\theta \mathrm{d}\theta^2 + \\
& \sin^2\chi \sin^2\theta \mathrm{d}\phi^2 + \\
& 2\cos\chi \sin\chi \cos\theta \sin\theta \mathrm{d}\chi \mathrm{d}\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2(& \cos^2\chi \mathrm{d}\chi^2 + \\
& \sin^2\chi \mathrm{d}\theta^2 + \\
& \sin^2\chi \sin^2\theta \mathrm{d}\phi^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2(& \mathrm{d}\chi^2 + \\
& \sin^2\chi \mathrm{d}\theta^2 + \\
& \sin^2\chi \sin^2\theta \mathrm{d}\phi^2).
\end{aligned}
$$

So eventually

$$
\boxed{\mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = a^2 \left(\mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)} \tag{250}
$$