

ASSIGNMENT 1

Problem 5

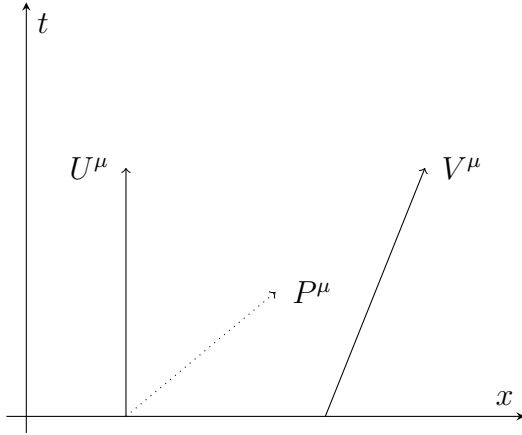


Figure 1: Setup of experiment

We have given:

$$U^\mu = (c, \mathbf{0}) \quad (1)$$

$$V^\mu = (\gamma_v c, \gamma_v \mathbf{v}) \quad (2)$$

$$P^\mu = \left(\frac{h\nu}{c}, \mathbf{p} \right) \quad (3)$$

We use following relation in this problem:

$$E = -P^\mu V_\mu \quad (4)$$

This expression is Lorentz invariant and can be calculated in non-moving frame. So we plug in Eq. 2 in this expression to obtain

$$E = -P^\mu V_\mu = \frac{h\nu}{c} \gamma_v c - \gamma_v \mathbf{v} \mathbf{p} = \gamma_v (h\nu - |\mathbf{v}| |\mathbf{p}| \cos(\theta)) = \left\{ |\mathbf{p}| = \frac{h\nu}{c} \right\} = \gamma_v h\nu \left(1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) \quad (5)$$

But it is still photon, but with different energy (for moving observer) So

$$\gamma_v h\nu \left(1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) = h\nu' \quad (6)$$

So ratio of those two frequencies is

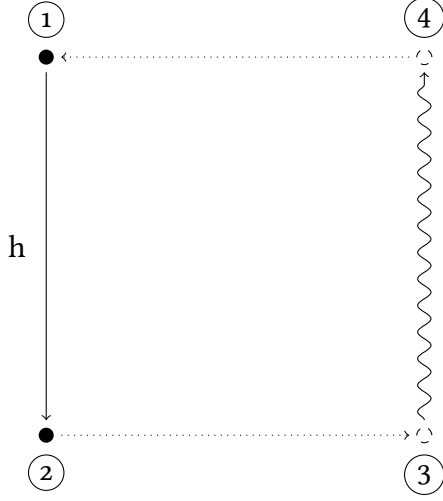
$$\frac{\nu'}{\nu} = \gamma_v \left(1 - \frac{|\mathbf{v}|}{c} \cos(\theta) \right) \quad (7)$$

If $\theta = 0$ and $\frac{v}{c} \ll 1 \Rightarrow \gamma_v \simeq 1$ then we obtain:

$$\boxed{\nu' = \nu \left(1 - \frac{v}{c} \right)} \quad (8)$$

ASSIGNMENT 2

Problem 1a



Let's take a look at energy changes in above diagram:

$$\textcircled{1} \quad E_1 = mc^2$$

$$\textcircled{2} \quad E_2 = mc^2 + mgh$$

$$\textcircled{3} \quad E_3 = h\nu = mc^2 + mgh$$

$$\textcircled{4} \quad E_4 = h\nu' = mc^2 + mgh$$

but $E_4 = E_1$ because of energy conservation. It means that photon has to have different frequency at the height h than it has at the ground. So $E_4 = h\nu' = mc^2$. From it follows

$$\frac{\nu}{\nu'} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \quad (9)$$

and it is easy to calculate redshift

$$z = \frac{\nu - \nu'}{\nu'} = \frac{gh}{c^2} \quad (10)$$

Figure 2: Mass falling in graitational field ($1 \rightarrow 2$), converting to photon ($2 \rightarrow 3$), photon traveling up ($3 \rightarrow 4$) and converting back to mass ($4 \rightarrow 1$)

Problem 1b

Let's calculate time which light needs to reach observer $\textcircled{2}$

$$t = \frac{s}{c} = \frac{h - \frac{gt^2}{2}}{c} \quad (11)$$

From this expression we get quadratic equation

$$\frac{g}{2}t^2 + ct - h = 0 \quad (12)$$

for which solution is given by

$$t = \frac{-c + \sqrt{c^2 + 2gh}}{g} \quad (13)$$

Velocity of observer $\textcircled{2}$ after this time is equal

$$v(t) = \frac{-c + \sqrt{c^2 + 2gh}}{g} \cdot g = -c + \sqrt{c^2 + 2gh} \quad (14)$$

Then redshift formula is given in following way

$$\frac{\nu'}{\nu} = 1 - \frac{v}{c} = 1 - \frac{-c + \sqrt{c^2 + 2gh}}{c} = 2 - \sqrt{1 - \frac{2gh}{c^2}} \quad (15)$$

We can use Taylor expansion $\sqrt{1-x} = 1 - \frac{x}{2}$ we get

$$\frac{\nu'}{\nu} = 2 - 1 + \frac{gh}{c^2} = 1 + \frac{gh}{c^2} \quad (16)$$

It is exactly the same result as Eq. 10.

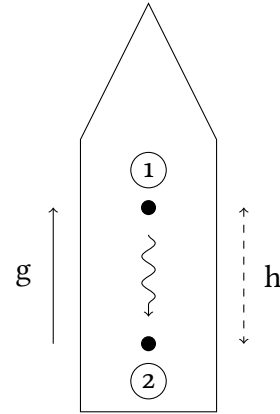


Figure 3: Two observers in a rocket sending photon

Problem 2

Observer \mathcal{O} is traveling with acceleration g in direction x_1 . To calculate his worldline we will use following three conditions

$$U^\mu U_\mu = -1 \quad U^\mu A_\mu = 0 \quad A^\mu A_\mu = g^2 \quad (17)$$

where U^μ is four-velocity and A^μ is four-acceleration. First of them can be obtained by straightforward calculation, second by applying derivative to first equation i.e.

$$\frac{d}{d\tau} (U^\mu U_\mu) = 0 \Rightarrow (A^\mu U_\mu) = 0 \quad (18)$$

Third is Lorentz invariant and it can be calculated in the moment of launch namely when $A^\mu = (0, g, 0, 0)$.

Knowing those three we can write them in explicite form

$$-U_0^2 + \mathbf{U}^2 = -1 \quad \mathbf{U} \mathbf{A} = U_0 A_0 \quad -A_0^2 + \mathbf{A}^2 = g^2 \quad (19)$$

where bolded letters mean three-vectors.

We square middle equation and plug in left and right equation to obtained

$$(U_0^2 - 1) \mathbf{A}^2 = U_0^2 (\mathbf{A}^2 - g^2) \quad (20)$$

Eventually we obtain:

$$\mathbf{A}^2 = g^2 U_0^2 \quad (21)$$

and plugin this expression to other equation we also obtain:¹

$$A_0^2 = g^2 \mathbf{U}^2 \quad (22)$$

We can simplify those equation using the fact that this motion is one dimensional namely $x_2 = x_3 = 0$ and then

$$A_1 = g U_0 \quad A_0 = g U_1 \quad (23)$$

But $U^\mu = \dot{X}^\mu$ and $A^\mu = \ddot{X}^\mu$ ². Substituting

$$\ddot{X}_1 = g \dot{X}_0 \quad \ddot{X}_0 = g \dot{X}_1 \quad (24)$$

Taking a derivative of left equation and substituting right equation into it we get

$$\ddot{X}_1 = g^2 \dot{X}_1 \xrightarrow{\text{after integration}} \ddot{X}_1 = g^2 X_1 \quad (25)$$

Solution is

$$X_1 = A \sinh(g\tau) + B \cosh(g\tau) \quad (26)$$

Let's choose initial conditions such as $X_1(0) = g^{-1}$ and $\dot{X}_1 = 0$. Then

$$X_1 = g^{-1} \cosh(g\tau) \quad (27)$$

And finally we have

$$X_0 = g^{-1} \sinh(g\tau) \quad X_1 = g^{-1} \cosh(g\tau) \quad X_2 = 0 \quad X_3 = 0 \quad (28)$$

¹plug it into right equation and then use left equation

²dot means derivation with respect to proper time

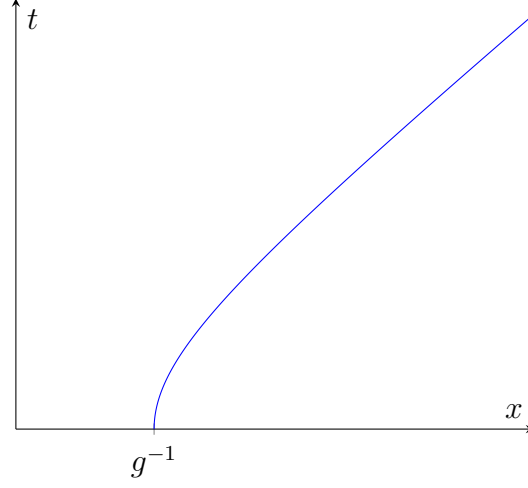


Figure 4: Trajectory of \mathcal{O}

Problem 3

As a first basis vector we can choose four-velocity namely

$$\mathbf{e}_0 = (\dot{X}_0, \dot{X}_1, \dot{X}_2, \dot{X}_3) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \quad (29)$$

As a basis vectors in directions x_2 and x_3 we simply choose

$$\mathbf{e}_2 = (0, 0, 1, 0) \quad (30)$$

$$\mathbf{e}_3 = (0, 0, 0, 1) \quad (31)$$

And finally we choose vector \mathbf{e}_1 in a form $\mathbf{e}_1 = (e_1^0, e_1^1, 0, 0)$ where e_1^0 and e_1^1 are chosen in order to satisfy $\mathbf{e}_0 \mathbf{e}_1 = 0$ and $(\mathbf{e}_0)^2 = 1$ i.e.

$$-e_1^0 \cosh(g\tau) + e_1^1 \sinh(g\tau) = 0 \quad (32)$$

$$-(e_1^0)^2 + (e_1^1)^2 = 1 \quad (33)$$

We square first equation and substitute second equation

$$(e_1^0)^2 \cosh^2(g\tau) = (1 + (e_1^0)^2) \sinh^2(g\tau) \quad (34)$$

From this we obtain

$$(e_1^0)^2 = \sinh^2(g\tau) \quad (e_1^1)^2 = \cosh^2(g\tau) \quad (35)$$

We can choose positive solution and eventually we get

$$\mathbf{e}_1 = (\sinh(g\tau), \cosh(g\tau), 0, 0) \quad (36)$$

All vectors

$$\mathbf{e}_0(\tau) = (\cosh(g\tau), \sinh(g\tau), 0, 0) \quad (37)$$

$$\mathbf{e}_1(\tau) = (\sinh(g\tau), \cosh(g\tau), 0, 0) \quad (38)$$

$$\mathbf{e}_2(\tau) = (0, 0, 1, 0) \quad (39)$$

$$\mathbf{e}_3(\tau) = (0, 0, 0, 1) \quad (40)$$

Last thing to do is to check whether those are vectors which were obtain without any rotation. For this I will find a Lorentz boost which transforms initial basis into this one. Namely consider a boost of time-basis vector

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 0 \end{pmatrix} \quad (41)$$

So γ and β have to satisfy:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh(g\tau) \quad \Rightarrow \quad v = \tanh(g\tau) \quad (42)$$

Knowing that it is easy to calculate

$$\beta\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh(g\tau) \quad (43)$$

So indeed we obtain vector $e_0(\tau)$ only via boost (at $v = \tanh(g\tau)$). The same can be done with vector $e_1(\tau)$

Problem 4

We define new coordinate system ($\xi_0 \equiv \tau, \xi_1, \xi_2, \xi_3$) where basis vectors are those defined in problem before. We can write

$$\mathbf{x} = \xi^1 \mathbf{e}_1(\tau) + \xi^2 \mathbf{e}_2(\tau) + \xi^3 \mathbf{e}_3(\tau) + \mathbf{x}_O(\tau) \quad (44)$$

where $\mathbf{x}_O(\tau)$ is trajectory of moving frame.

After plugging in all basis vectors explicitly we get

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \xi^1 \sinh(g\tau) \\ \xi^1 \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \xi^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^3 \end{pmatrix} + \begin{pmatrix} g^{-1} \sinh(g\tau) \\ g^{-1} \cosh(g\tau) \\ 0 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} g^{-1} \sinh(g\tau) + \xi^1 \sinh(g\tau) \\ g^{-1} \cosh(g\tau) + \xi^1 \cosh(g\tau) \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} (g^{-1} + \xi^1) \sinh(g\xi_0) \\ (g^{-1} + \xi^1) \cosh(g\xi_0) \\ \xi^2 \\ \xi^3 \end{pmatrix} \end{aligned} \quad (45)$$

Line element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ is then equal (we use chain rule i.e. $dx^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} d\xi^\nu$)

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \quad (46)$$

$$dt = \frac{\partial t}{\partial \xi^\nu} d\xi^\nu = (1 + g\xi_1) \cosh(g\xi_0) d\xi_0 + \sinh(g\xi_0) d\xi_1 \quad (47)$$

$$dx_1 = (1 + g\xi_1) \sinh(g\xi_0) d\xi_0 + \cosh(g\xi_0) d\xi_1 \quad (48)$$

$$dx_2 = d\xi_2 \quad (49)$$

$$dx_3 = d\xi_3 \quad (50)$$

After squaring and adding them up we get

$$\begin{aligned} ds^2 = & -(1 + g\xi_1)^2 \cosh^2(g\xi_0) d\xi_0^2 - \sinh^2(g\xi_0) d\xi_1^2 + \\ & (1 + g\xi_1)^2 \sinh^2(g\xi_0) d\xi_0^2 + \cosh^2(g\xi_0) d\xi_1^2 + \\ & d\xi_2^2 + \\ & d\xi_3^2 \end{aligned} \quad (51)$$

After simplification

$$\boxed{ds^2 = -(1 + g\xi_1)^2 d\xi_0^2 + d\xi_1^2 + d\xi_2^2 + d\xi_3^2} \quad (52)$$

Problem 5

For $\xi^1 \equiv \text{const}$ we can easily derive equation of motion from Eq. 45 namely

$$x_1^2 - t^2 = (g^{-1} + \xi^1)^2 \quad (53)$$

which leads to

$$x_1(t) = \sqrt{(g^{-1} + \xi^1)^2 + t^2} \quad (54)$$

We take derivative twice

$$\dot{x}_1(t) = \frac{2t}{2\sqrt{(g^{-1} + \xi^1)^2 + t^2}} \quad (55)$$

$$\ddot{x}_1(t) = \frac{\sqrt{(g^{-1} + \xi^1)^2 + t^2} - t \frac{2t}{2\sqrt{(g^{-1} + \xi^1)^2 + t^2}}}{(g^{-1} + \xi^1)^2 + t^2} = \frac{1}{\sqrt{(g^{-1} + \xi^1)^2 + t^2}} - \frac{2t^2}{((g^{-1} + \xi^1)^2 + t^2)^{\frac{3}{2}}} \quad (56)$$

So when $t = 0$

$$\boxed{\ddot{x}_1(t)|_{t=0} = \frac{1}{g^{-1} + \xi^1} = \frac{g}{1 + g\xi^1}} \quad (57)$$

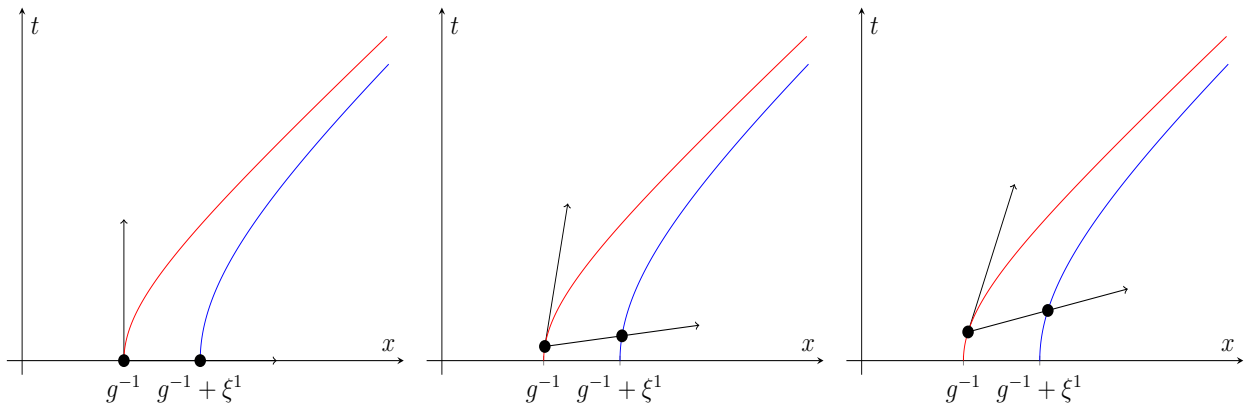


Figure 5: Red line is worldline of Eq. 28 and blue is worldline of Eq. 54

Problem 6

We start with equation Eq. 52. We can simplify it and neglect other spatial dimensions than ξ^1 namely

$$ds^2 = -(1 + g\xi^1)^2 (d\xi^0)^2 + (d\xi^1)^2 \quad (58)$$

We can change the form to

$$d\tau = ds = d\xi^0 \sqrt{-(1 + g\xi^1)^2 + \left(\frac{d\xi^1}{d\xi^0}\right)^2} \quad (59)$$

We can now plug in $\xi^1 = \xi_{\text{em}}^1$ and since emitter does not move in this frame we can set $\frac{d\xi^1}{d\xi^0} = 0$:

$$d\tau_{\text{em}} = d\xi_{\text{em}}^0 (1 + g\xi_{\text{em}}^1) \quad (60)$$

We can integrate both sides and obtain equation for finite differences

$$\Delta\tau_{\text{em}} = \Delta\xi_{\text{em}}^0 (1 + g\xi_{\text{em}}^1) \quad (61)$$

We can do similar thing with ξ_{rec}^1 :

$$\Delta\tau_{\text{rec}} = \Delta\xi_{\text{rec}}^0 (1 + g\xi_{\text{rec}}^1) \quad (62)$$

But left sides of above equations are equal (since line element is invariant under changing of coordinates) and we can compare them:

$$\frac{\Delta\xi_{\text{rec}}^0}{\Delta\xi_{\text{em}}^0} = \frac{1 + g\xi_{\text{em}}^1}{1 + g\xi_{\text{rec}}^1} = 1 + \frac{g\xi_{\text{em}}^1 - g\xi_{\text{rec}}^1}{1 + g\xi_{\text{rec}}^1} = 1 - \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \quad (63)$$

where I put $h = \xi_{\text{rec}}^1 - \xi_{\text{em}}^1$. After rearranging terms and substituting $\Delta\xi_{\text{rec}}^1 = \frac{1}{\nu'}$ and $\Delta\xi_{\text{em}}^1 = \frac{1}{\nu}$

$$\frac{\Delta\xi_{\text{em}}^0 - \Delta\xi_{\text{rec}}^0}{\Delta\xi_{\text{em}}^0} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \quad (64)$$

$$\frac{\frac{1}{\nu} - \frac{1}{\nu'}}{\frac{1}{\nu}} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1} \Rightarrow \boxed{z = \frac{\nu' - \nu}{\nu'} = \frac{gh}{1 + gh + g\xi_{\text{em}}^1}} \quad (65)$$

We can now assume that g is small and using Taylor expansion $\frac{1}{1+x} \simeq 1 - x$

$$z = gh(1 - gh - g\xi_{\text{em}}^1) = gh - (gh)^2 - g^2 h \xi_{\text{em}}^1 \simeq gh$$

$$z = gh \quad (66)$$

so the same result as photon in gravitational field.

ASSIGNMENT 3

Problem 1

Calculate EOM given the Lagrangian

$$\mathcal{L}_{\text{dyn}}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu}[x^\mu(\lambda)] \dot{x}^\mu \dot{x}^\nu, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \quad (67)$$

I calculate first variation of Lagrangian

$$\begin{aligned} \delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda &= \int_{\lambda_1}^{\lambda_2} \left(\frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) d\lambda = \\ &= \int_{\lambda_1}^{\lambda_2} \left(\frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{1}{2} g_{\mu\nu} (\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma \right) d\lambda \quad (68) \end{aligned}$$

Let's take a look at second part of the integral:

$$\begin{aligned} \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{g_{\mu\nu}(\delta_{\mu\sigma} \dot{x}^\nu + \delta_{\nu\sigma} \dot{x}^\mu) \delta \dot{x}^\sigma\} d\lambda &= \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta \dot{x}^\sigma d\lambda = \\ &= \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \frac{\partial}{\partial \lambda} (\{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma) d\lambda - \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \frac{\partial}{\partial \lambda} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma d\lambda = \\ &= \underbrace{\frac{1}{2} \{g_{\sigma\nu} \dot{x}^\nu + g_{\mu\sigma} \dot{x}^\mu\} \delta x^\sigma \Big|_{\lambda_1}^{\lambda_2}}_{=0} - \frac{1}{2} \int_{\lambda_1}^{\lambda_2} \{\partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu + g_{\sigma\nu} \ddot{x}^\nu + \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu + g_{\sigma\mu} \ddot{x}^\mu\} \delta x^\sigma d\lambda \quad (69) \end{aligned}$$

Plugging result back into Eq. 68 yields

$$\delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda = \frac{1}{2} \int_{\lambda_1}^{\lambda_2} (\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu) \delta x^\sigma d\lambda \quad (70)$$

We want

$$\delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda = 0 \quad (71)$$

but since δx^σ can be arbitrary the rest has to be equal 0, namely

$$\partial_\sigma g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \partial_\mu g_{\sigma\nu} \dot{x}^\mu \dot{x}^\nu - g_{\sigma\nu} \ddot{x}^\nu - \partial_\nu g_{\sigma\mu} \dot{x}^\nu \dot{x}^\mu - g_{\sigma\mu} \ddot{x}^\mu = 0 \quad (72)$$

or after rearranging elements

$$\boxed{2g_{\sigma\mu} \ddot{x}^\mu + (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu = 0} \quad (73)$$

Problem 2

Calculate the EOM given Lagrangian:

$$\mathcal{L}_{\text{geo}}(x^\mu, \dot{x}^\mu) = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \quad (74)$$

First we calculate variation of Lagrangian

$$\begin{aligned} \delta \int_{\lambda_1}^{\lambda_2} \mathcal{L}(x^\mu, \dot{x}^\mu) d\lambda &= \int_{\lambda_1}^{\lambda_2} \left(\frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial x^\sigma} \delta x^\sigma + \frac{\partial \mathcal{L}(x^\mu, \dot{x}^\mu)}{\partial \dot{x}^\sigma} \delta \dot{x}^\sigma \right) d\lambda = \\ &= \int_{\lambda_1}^{\lambda_2} \left(\frac{\dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \delta x^\sigma + \frac{g_{\mu\nu} \dot{x}^\mu \delta_{\nu\sigma}}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \dot{x}^\sigma + \frac{g_{\mu\nu} \delta_{\mu\sigma} \dot{x}^\nu}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \dot{x}^\sigma \right) d\lambda = \\ &= \int_{\lambda_1}^{\lambda_2} \frac{1}{2\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} (\dot{x}^\mu \dot{x}^\nu \partial_\sigma g_{\mu\nu} \delta x^\sigma + g_{\mu\sigma} \dot{x}^\mu \delta \dot{x}^\sigma + g_{\sigma\nu} \dot{x}^\nu \delta \dot{x}^\sigma) d\lambda \quad (75) \end{aligned}$$

I change variable of differentiating and integration from $d\lambda$ to $d\tau = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu} d\lambda$. Derivatives changing as following

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu} \frac{dx^\mu}{d\tau} \quad (76)$$

and integral as following

$$\int d\lambda = \int \frac{d\tau}{\sqrt{-g_{\mu\nu}[x^\mu(\lambda)]\dot{x}^\mu\dot{x}^\nu}} \quad (77)$$

After plugging in those transformations into Eq. 75 we obtain

$$-\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \partial_\sigma g_{\mu\nu} \delta x^\sigma + \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta \frac{dx^\sigma}{d\tau} \right) d\tau \quad (78)$$

Now let's look at the second part of this integral and transform it (using Leibniz rule)

$$\begin{aligned} \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta \frac{dx^\sigma}{d\tau} &= \frac{d}{d\tau} \left(\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) - \frac{d}{d\tau} \left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma = \\ \frac{d}{d\tau} \left(\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) &- \left\{ \frac{dg_{\mu\sigma}}{dx^\nu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} + g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{dg_{\sigma\nu}}{dx^\mu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} \right\} \delta x^\sigma = \\ \frac{d}{d\tau} \left(\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right) &- \left[g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \{ \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} \} \right] \delta x^\sigma \quad (79) \end{aligned}$$

After plugging it into Eq. 78 we obtain

$$\begin{aligned}
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \partial_\sigma g_{\mu\nu} \delta x^\sigma + \frac{d}{d\tau} \left[\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma \right] - \right. \\
& \quad \left. \left[g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \{ \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} \} \right] \delta x^\sigma \right) d\tau = \\
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left[-2g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) \right] \delta x^\sigma d\tau - \underbrace{\left\{ g_{\mu\sigma} \frac{dx^\mu}{d\tau} + g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right\} \delta x^\sigma}_{=0} \Big|_{\lambda_1}^{\lambda_2} = \\
& -\frac{1}{2} \int_{\lambda_1}^{\lambda_2} \left[-2g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) \right] \delta x^\sigma d\tau \quad (80)
\end{aligned}$$

But this variation has to be equal zero no matter what the value of δx^σ is. Namely

$$-g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{1}{2} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}) = 0 \quad (81)$$

Changing sign

$$\boxed{g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{1}{2} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) = 0} \quad (82)$$

Problem 3

Following metric is given

$$g_{\mu\nu} dx^\mu dx^\nu = - \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] d(ct)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad \mathbf{x} \equiv (x^1, x^2, x^3) \quad (83)$$

Dividing both sides by dt^2 yields

$$\begin{aligned}
g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} &= - \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] \frac{d(ct)^2}{dt^2} + \frac{(dx^1)^2}{dt^2} + \frac{(dx^2)^2}{dt^2} + \frac{(dx^3)^2}{dt^2} = \\
&= - \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] c^2 + \left(\frac{dx^1}{dt} \right)^2 + \left(\frac{dx^2}{dt} \right)^2 + \left(\frac{dx^3}{dt} \right)^2 = - \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} \right] c^2 + \mathbf{v} \cdot \mathbf{v} = \\
&= -c^2 \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2} \right] \quad (84)
\end{aligned}$$

Substituting this into lagrangian

$$\mathcal{L} = -mc \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \quad (85)$$

gives us

$$\boxed{\mathcal{L} = -mc \sqrt{c^2 \left[1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2} \right]} = -mc^2 \sqrt{1 + 2 \frac{\phi(\mathbf{x})}{c^2} - \frac{\mathbf{v}^2}{c^2}}} \quad (86)$$

We can now assume that both $\frac{\phi(\mathbf{x})}{c^2}$ and $\frac{v^2}{c^2}$ are small. We can Taylor expand square root ($\sqrt{1+x} \simeq 1 + \frac{1}{2}x$) and leave only linear terms:

$$\mathcal{L} = -mc^2 \left(1 + \frac{\phi(\mathbf{x})}{c^2} - \frac{v^2}{2c^2} \right) = -mc^2 - m\phi(\mathbf{x}) + m\frac{v^2}{2} \quad (87)$$

But adding constants (in this case mc^2) to Lagrangian doesn't change equation of motions, so effective Lagrangian can be written as

$$\boxed{\mathcal{L} = \frac{mv^2}{2} - m\phi(\mathbf{x})} \quad (88)$$

which is exactly the Lagrangian for classical mechanics, which leads to Newton's law of motion, namely

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi(\mathbf{x}) \quad (89)$$