Straightening Polygonal Arcs and Convexifying Polygonal Cycles

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Abstract

Consider a planar linkage, consisting of disjoint polygonal arcs and cycles of rigid bars joined at incident endpoints (polygonal chains), with the property that no cycle surrounds another arc or cycle. We prove that the linkage can be continuously moved so that the arcs become straight, the cycles become convex, and no bars cross while preserving the bar lengths. Furthermore, our motion is piecewise-differentiable, does not decrease the distance between any pair of vertices, and preserves any symmetry present in the initial configuration. In particular, this result settles the well-studied carpenter's rule conjecture.

1 Introduction

Consider a finite embedded polygonal arc in the plane (by an *arc* we mean a homeomorphic image of the closed interval $[0,1]$). Such a polygonal arc is often called an *open polygonal chain*. It has been an outstanding question as to whether it is possible to continuously move a polygonal arc in such a way that each edge remains a fixed length, there are no self-intersections during the motion, and at the end of the motion the arc lies on a straight line. This has come to be known as the carpenter's rule problem. A related question is whether it is possible to continuously move a polygonal simple closed curve in the plane, often called a *closed polygonal chain* or *polygon*, again without creating self-intersections or changing the length of the edges, so that it ends up a convex closed curve. We solve both problems here by showing that in both cases there is such a motion.

Physically, we think of a polygonal arc as a *linkage* or *framework* with hinges at its vertices, and rigid bars at its edges. The hinges can be folded as desired, but the bars must maintain their length and cannot cross. Motions of such linkages have been studied in

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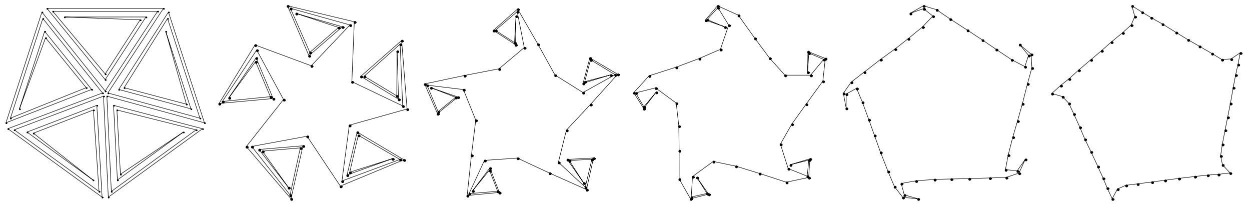

Figure 1: Convexifying a polygon that comes from doubling each edge in a locked tree. Snapshots are zoomed different amounts to improve visibility; each edge stays the same length throughout the motion. See `http://daisy.uwaterloo.ca/~eddemain/linkage` for more animations.

discrete and computational geometry [BDD+99, Erd35, Grü95, LW95, Nag39, O'R98, Sal73, Tou99, Weg93, Weg96, Whi92b], in knot theory [CJ98, Mil94], and in molecular biology and polymer physics [FK97, MOS90, McM79, Mil94, SW88, SG72, Whi83]. Applications of this field include robotics, wire bending, hydraulic tube folding, and the study of macromolecule folding [O'R98, Tou99].

We say an arc is *straightened* by a motion if at the end of the motion it lies on a straight line. We say a polygonal simple closed curve (or *cycle*) is *convexified* by a motion if at the end of the motion it is a convex closed curve. All motions must be *proper* in the sense that no self-intersections are created, and each edge length is kept fixed. It is easy to see that if any cycle can be convexified by a motion, then any arc can be straightened by a motion: simply extend each arc to a cycle and convexify it. It is then easy to straighten the portion of the cycle that is the original arc.

It seems intuitively easy to straighten an entangled chain: just grab the ends and pull them apart. Similarly, a cycle might be opened by blowing air into it until it expands. But these methods have the difficulty that they may introduce singularities, where the arc or cycle may intersect itself. Our approach is to use an *expansive* motion in which all distances between two points increase. We also show that the area of a polygon increases in such an expansive motion.

We consider the more general situation, which we call an *arc-and-cycle set* A, consisting of a finite number of polygonal arcs and polygonal simple closed curves in the plane, with none of the arcs or cycles intersecting each other or having self-intersections. We say that A is in an *outer-convex configuration* if each component of A that is not contained in any cycle of A is either straight (when it is an arc) or convex (when it is a cycle).

We say that a motion of an arc-and-cycle set A is *expansive* if for every pair of vertices of A the distance increases or stays the same during the motion. We say that the motion is *strictly expansive* if in addition, for those vertices not on a straight subarc in a component of A and not on or in a common convex cycle of A, the distance between them increases strictly. We say that A is *separated*, if there is a line L in the plane such that L is disjoint from A and at least one component of A lies on each side of L.

Our main result is the following.

Theorem 1 *Every arc-and-cycle set has a piecewise-differentiable proper motion to an outer-convex configuration. Moreover, the motion is strictly expansive until the arc-and-cycle set* *is separated.*

We can also insist that the motion be strictly expansive during the entire motion. The definition of this motion is actually even simpler than the one we use for Theorem 1, but unfortunately the proof is a great deal more complex. Taking pity on the reader, we have placed the proof of this extension in the appendix. Note that when there is just one component in the arc-and-cycle set, there is no difference between Theorem 1 and the extension in the appendix.

In contrast to this result, in dimension three there are arcs that cannot be straightened and polygons that cannot be convexified [BDD+99, CJ98]. In four and higher dimensions, no arcs, cycles, or trees can be locked, i.e., all arcs, cycles, or trees can be straightened or convexified, respectively. This is true because there are enough degrees of freedom and one can easily avoid any impending self-intersection by "moving around" it. An explicit unlocking algorithm for arcs and cycles in four dimensions was given in [CO99].

In the plane, there are examples of trees embedded in the plane that are *locked* in the sense that they cannot be properly moved so that the vertices lie nearly on a line [BDD+98]. In other words, there are two embeddings of the tree such that there is no proper motion from one configuration to the other. The important difference between trees and arc-and-cycle sets is that arc-and-cycle sets have maximum degree two. We have recently strengthened the example of [BDD+98] by constructing a locked tree with just one vertex of degree three and all other vertices of degree one or two. This shows that the restriction to arc-and-cycle sets in Theorem 1 is best possible.

	Arcs and Cycles	Trees
2-D	Not lockable (this paper)	Lockable [BDD+98]
3-D	Lockable [BDD+99, CJ98]	Lockable
4-D$^+$	Not lockable [CO99]	Not lockable

Table 1: Summary of what types of linkages can be locked.

Whether every arc in the plane can be straightened, and whether every polygon in the plane can be convexified, have been outstanding open questions until now. The problems are natural, so they have arisen independently in a variety of fields, including topology, pattern recognition, and discrete geometry. We are probably not aware of all contexts in which the problem has appeared. To our knowledge, the first person to pose the problem of convexifying cycles was Stephen Schanuel. George Bergman learned of the problem from Schanuel during Bergman's visit to the State University of New York at Buffalo in the early 1970's, and suggested the simpler question of straightening arcs. As a consequence of this line of interest, the problems are included in Kirby's *Problems in Low-Dimensional Topology* [Kir97, Problem 5.18].

During the period 1986–1989, Ulf Grenander and the members of the Pattern Theory Group at Brown University explored various problems involving the probabilistic structure when generators (e.g., points and line segments) were transformed by diffeomorphisms (e.g., Euclidean transformations) subject to global constraints (e.g., avoiding intersections). For the purposes of Bayesian image understanding, they were interested in whether the process was *ergodic*, i.e., every configuration could be reached from any other. In particular, they proved this for polygonal cycles in which the roles of angles and lengths are reversed: *angles* are fixed but *lengths* may vary [GCK91, Appendix D, pp. 108–128]. Grenander posed the problems considered here during a seminar at Indiana University in March 1987 [Gre87], and probably also at earlier talks (according to personal communication with Allan Edmonds).

In the discrete and computational community, the problems were independently posed by William Lenhart and Sue Whitesides in March 1991 and by Joseph Mitchell in December 1992 (according to personal communications with Sue Whitesides and Joseph Mitchell). Sue Whitesides first posed this problem in a talk in 1991 [LW91]. In this community the problems were first published in a technical report in 1993 [LW93] and in a journal in 1995 [LW95].

Solutions were already known for the special cases of monotone cycles [BDL$^+$99] and star-shaped cycles [ELR$^+$98], and for certain types of "externally visible" arcs [BDST99].

A fairly large group of people, mentioned in the acknowledgments, was involved in trying to construct and prove or disprove locked arcs and cycles, at various times over the past few years. Typically, someone in the group would distribute an example that s/he constructed or was given by a colleague. We would try various motions that did not work, and we would often try proving that the example was locked because it appeared so! For some examples, it took several months before we found an unlocking motion. The main difficulty was that "simple" motions that change a few vertex angles at once, while easiest to visualize, seemed to be insufficient for unlocking complex examples. Amazingly, it also seemed that nevertheless there was always a global unlocking motion, and furthermore it was felt that there was a driving principle permitting "blowing up" of the linkage. This notion was formalized by the third author with the idea that perhaps an arc could be straightened via an expansive motion.

The tools that are applied here for the first time come from the theory of mechanisms and rigid frameworks. Arcs and cycles can be regarded as frameworks. See [AR78, AR79, Con80, Con82, Con93, CW96, CW93, CW82, CW94, GSS93, RW81, Whi84a, Whi84b, Whi87, Whi88, Whi92a] for relevant information about this theory.

Our approach is to prove that for any configuration there is an infinitesimal motion that increases all distances. Because of the nature of the arc-and-cycle set, this implies that there is a motion that works at least for a small expansive perturbation. We then combine these local motions into one complete motion. These notions are described in the rest of this paper. Section 3 proves the existence of infinitesimal motions using the nonexistence of certain stresses, a notion dual to infinitesimal motions for the underlying framework. The analysis of these stresses uses a lifting theorem from the theory of rigidity that was known to James Clerk Maxwell and Luigi Cremona [CW82, CW93, Whi82] in the nineteenth century. Section 4 shows how to maneuver through the space of local motions to find a global motion with the desired properties.

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2 Basics

A *linkage* or *bar framework* $G(\mathbf{p})$ is a finite graph $G = (V, E)$ without loops or multiple edges, together with a corresponding configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of n points in the plane, where $\mathbf{p}_i$ corresponds to vertex $i \in V$. (For convenience we assume $V = \{1, \ldots, n\}$.) The edges of G constitute the set E and correspond to the bars in the framework, i.e., the links of a linkage. Arc-and-cycle sets are a particular kind of bar framework in which the graph G is a disjoint union of paths and cycles.

A *flex* or *motion* of $G(\mathbf{p})$ is a set of continuous functions $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$, defined for $0 \le t \le 1$, such that $\mathbf{p}(0) = \mathbf{p}$ and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is constant for each $\{i, j\} \in E$. We are interested in finding a motion of the arc-and-cycle set with the additional property that it is strictly expansive.

2.1 Expansiveness

We begin with some basic properties of expansive motions. Namely, we will show that if a motion expands the distance between all pairs of vertices, it also expands the distance between all pairs of points on the arc-and-cycle framework. One consequence of this is a key reason why we use expansive motions: they automatically avoid self-intersection.

Lemma 1 *In the plane, suppose that a point $\mathbf{c}$ is contained in the closed triangle formed by three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, and another point $\mathbf{q}_3$ is chosen so that it is farther from $\mathbf{p}_1$ and $\mathbf{p}_2$ than $\mathbf{p}_3$, i.e.,*

$$\|\mathbf{q}_3 - \mathbf{p}_2\| \ge \|\mathbf{p}_3 - \mathbf{p}_2\| \text{ and } \|\mathbf{q}_3 - \mathbf{p}_1\| \ge \|\mathbf{p}_3 - \mathbf{p}_1\|. \tag{1}$$

Then $\mathbf{q}_3$ is also farther from $\mathbf{c}$ than $\mathbf{p}_3$, i.e., $\|\mathbf{q}_3 - \mathbf{p}_1\| \ge \|\mathbf{p}_3 - \mathbf{p}_1\|$, with equality precisely if both inequalities of (1) are equalities.

Proof: Refer to Figure 2. The circular disk C_0 centered at $\mathbf{c}$ and of radius $\|\mathbf{p}_3 - \mathbf{c}\|$ is contained in the union of the circular disks C_1 centered at $\mathbf{p}_1$ and of radius $\|\mathbf{p}_3 - \mathbf{p}_1\|$, and C_2 centered at $\mathbf{p}_2$ and of radius $\|\mathbf{p}_3 - \mathbf{p}_2\|$. This implies the result, because $\mathbf{q}_3$ must be outside of C_1 and C_2. See also [Con82] for a proof in terms of tensegrities. □

Corollary 1 *Any expansive motion of an arc-and-cycle set only increases the distance between two points on the arc-and-cycle set (each either a vertex or on a bar). In particular, there can be no self-intersections.*

Proof: First consider the distance between a vertex $\mathbf{p}_3$ and a point $\mathbf{c}$ on a segment $\mathbf{p}_1\mathbf{p}_2$ of the arc-and-cycle set. This distance can only increase by Lemma 1. Now suppose we have another point $\mathbf{c}'$ on a segment $\mathbf{p}_3\mathbf{p}_4$ of the arc-and-cycle set. We know that $\mathbf{p}_3$ and $\mathbf{p}_4$ can only get farther from $\mathbf{c}$. Hence, $\mathbf{c}$ can only get farther from $\mathbf{c}'$. □

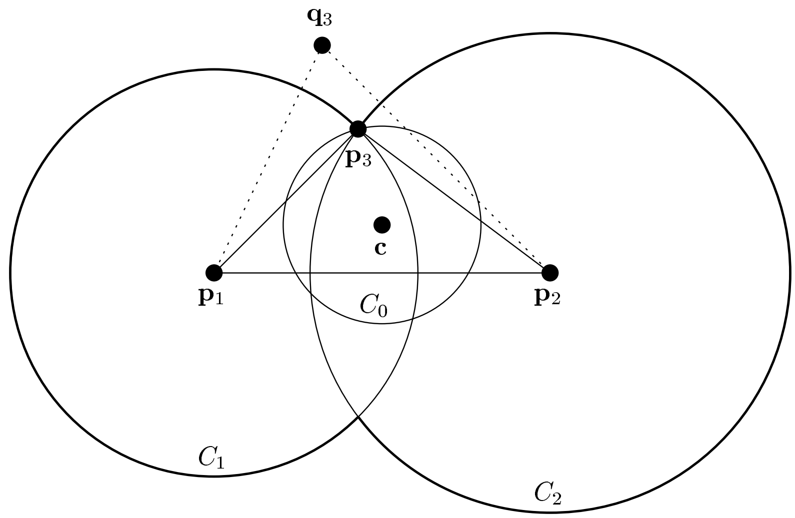

Figure 2: Illustration of Lemma 1.

2.2 The Framework $G_A(\mathbf{p})$

Given an arc-and-cycle set A that we would like to move to an outer-convex configuration, we make four modifications to A. The first three modifications simplify the problem by removing a few special cases that are easy to deal with; see Figure 3. The fourth modification will bring the problem of finding a strictly expansive motion into the area of tensegrity theory. In the end we will have defined a new framework, $G_A(\mathbf{p})$, which we will use throughout the rest of the proof.

Modification 1: Remove straight vertices. First we show that our arc-and-cycle set can be assumed to have no straight vertices, i.e., vertices with angle π. Furthermore, if during an expansive motion of the arc-and-cycle set we find that a vertex becomes straight, we can proceed by induction. For once the arc-and-cycle set has a straight subarc of more than one bar, we can coalesce this subarc into a single bar, thereby preserving the straightness of the subarc throughout the motion once it becomes straight. This reduces the number of bars and the number of vertices in the framework. By induction, this reduced framework has a motion according to Theorem 1, and such a motion extends directly to the original framework. The resulting motion is also strictly expansive by Lemma 1.

Modification 2: Rigidify convex polygons. Once a cycle becomes convex, we no longer have to expand it, and indeed we can hold it rigid from that point on. Of course, we allow a convex cycle to translate or rotate in the plane, but its vertex angles are not allowed to change. This can be directly modeled in the bar framework by introducing bars in addition to the arc-and-set cycle. Specifically, we add the edges of a triangulation of a cycle once that cycle becomes convex.

Modification 3: Remove components nested within convex cycles. The previous modification did not address the fact that components can be nested within cycles. Once

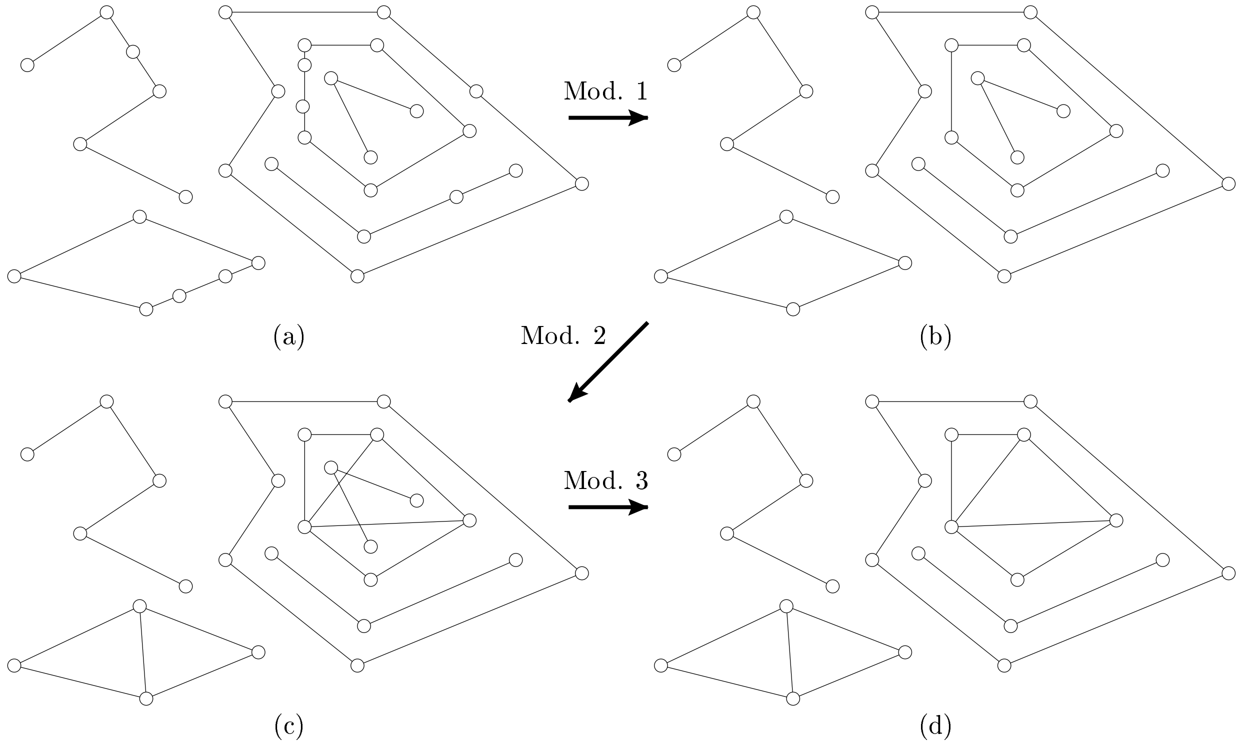

Figure 3: (a) Original arc-and-cycle framework. (b) With straight vertices removed. (c) With convex cycles rigidified. (d) With components nested within convex cycles removed.

a cycle becomes convex, not only can we rigidify it, but we can also rigidify any nested components, and treat them as moving in synchrony with the convex cycle. We do this by removing from the framework any components nested within a convex cycle. Assuming there were some nested components to deal with, this results in a framework with fewer vertices and fewer bars. By induction, this reduced framework has a motion according to Theorem 1. This motion can be extended to apply to the original framework by defining nested components to follow the rigid motion of the containing convex cycle. By the following consequence of Lemma 1, the resulting motion is also strictly expansive.

Lemma 2 *Extending a motion to apply to components nested within convex cycles preserves strict expansiveness.*

Proof: Consider some vertex $\mathbf{c}$ on a component inside some convex cycle, and a vertex $\mathbf{p}_3$ outside the cycle. We first consider the case that $\mathbf{p}_3$ does not lie inside another convex cycle. Extend the ray from $\mathbf{p}_3\mathbf{c}$ beyond $\mathbf{c}$, and let $\mathbf{p}_1\mathbf{p}_2$ the edge through which this ray exits the cycle. Thus, $\mathbf{c}$ is in the triangle $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, so Lemma 1 applies, and the distance $\mathbf{p}_3\mathbf{c}$ increases.

For two points $\mathbf{c}_1$ and $\mathbf{c}_2$ in two different cycles C_1 and C_2, we extend the ray $\mathbf{c}_1\mathbf{c}_2$ to identify the edge $\mathbf{p}_1\mathbf{p}_2$ on C_2 where the ray leaves C_2. From the first part of the proof we conclude that $\mathbf{c}_1\mathbf{p}_1$ and $\mathbf{c}_1\mathbf{p}_2$ increase, and by Lemma 1, the distance $\mathbf{c}_1\mathbf{c}_2$ increases. $\square$

Modification 4: Add struts. In order to model the expansive property we need, we apply the theory of tensegrity frameworks, in which frameworks can consist of both bars and "struts." In contrast to a bar which must stay the same length throughout a motion, a *strut* is permitted to increase in length, or stay the same length. Specifically, we add a strut between nearly every pair of vertices in the framework. The exceptions are those vertices already connected by a bar, and vertices on a common convex cycle, because in both cases we cannot hope to strictly increase the distance.

Final framework: $G_A(\mathbf{p})$. The above modifications define a tensegrity (bar-and-strut) framework $G_A(\mathbf{p})$ in terms of the arc-and-cycle set A. Specifically, assume that A has no straight vertices or components nested within convex components. We call such an arc-and-cycle set *reduced*. We define the set B of bars by starting with the set of bars from the arc-and-cycle set, and adding a triangulation of every convex cycle. The set S of struts consists of all vertex pairs which are not in B and which do not belong to the same convex cycle. See Figure 4 for an example.

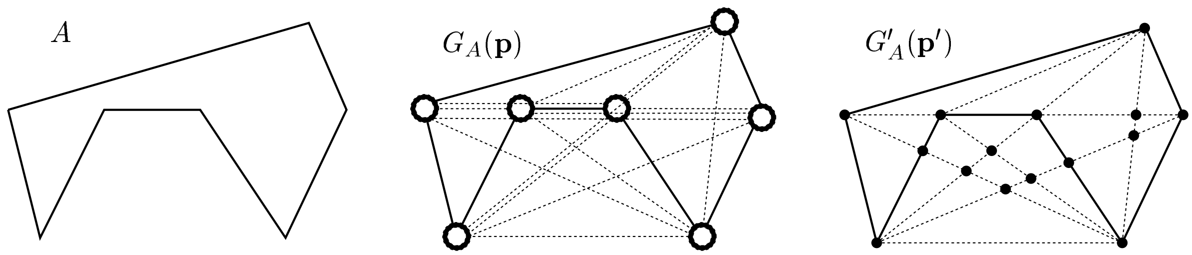

Figure 4: Construction of the frameworks $G_A(\mathbf{p})$ and $G'_A(\mathbf{p}')$. Solid lines denote bars, and dashed lines denote struts.

Our goal in the proof of Theorem 1 is to find a motion such that all bars maintain their length, while all struts strictly increase in length, in other words, a motion of $G_A(\mathbf{p})$ that is *strict* on all struts.

Thus, we want to find a motion $\mathbf{p}(t)$ for $0 \le t \le 1$ such that $\mathbf{p}(0) = \mathbf{p}$ and

$$\begin{aligned} \tfrac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &= 0 \quad \text{for } \{i,j\} \in B, \\ \tfrac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &> 0 \quad \text{for } \{i,j\} \in S. \end{aligned}$$

Differentiating the *squared* distances $\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|^2 = (\mathbf{p}_j(t) - \mathbf{p}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t))$ and denoting the velocity vectors by $\mathbf{v}_i(t) := \frac{d}{dt}\mathbf{p}_i(t)$, we obtain the following equivalent conditions.

$$\begin{aligned} (\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) &= 0 \quad \text{for } \{i,j\} \in B, \\ (\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) &> 0 \quad \text{for } \{i,j\} \in S. \end{aligned}$$

Intuitively, the first-order change in the distance between vertex i and j is modeled by projecting the velocity vectors onto the line segment between the two vertices; see Figure 5.

2.3 Infinitesimal Motions

A *strict infinitesimal motion* or *strict infinitesimal flex* $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ specifies the first derivative of a strictly expansive motion at time 0. In other words, it assigns a velocity

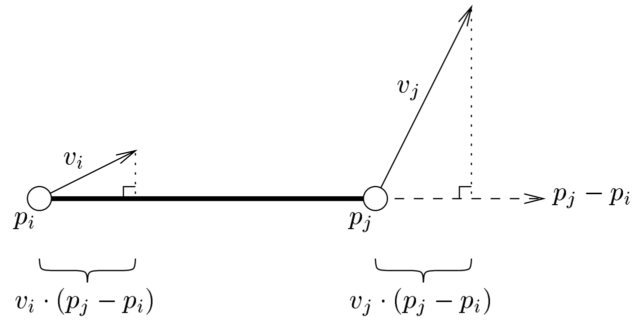

Figure 5: The dot product $(\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t))$ is zero if the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ stays the same to the first order, positive if the distance increases, and negative if the distance decreases.

vector $\mathbf{v}_i$ to each vertex i so that it preserves the length of the bars to the first order, and strictly increases the length of struts to the first order. More precisely, a strict infinitesimal motion must satisfy

$$\begin{aligned} (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i,j\} \in B, \\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) > 0 \text{ for } \{i,j\} \in S, \end{aligned} \tag{2}$$

where $\mathbf{p}_i$ denotes the initial position of vertex i.

In the next section, we prove that such a strict infinitesimal motion always exists. In Section 4 we show how this leads to motions for small amounts of time. These motions are then shown to continue globally until the configuration reaches an outer-convex configuration.

3 Local Motion

Recall that an arc-and-cycle set is called *reduced* if adjacent collinear bars have been coalesced, and components nested within cycles have been removed. In this section, we prove the following:

Theorem 2 *For any reduced arc-and-cycle set A there is an infinitesimal flex $\mathbf{v}$ of the corresponding bar-and-strut framework $G_A(\mathbf{p})$ satisfying* (2).

3.1 Equilibrium Stresses

The equations and inequalities in (2) form a linear feasibility problem that is common for tensegrity frameworks. But in order to solve this problem it is helpful to restate it in terms of the dual problem. This leads to the study of equilibrium stresses in tensegrity frameworks.

A *stress* in a framework $G(\mathbf{p})$ is an assignment of a scalar $\omega_{i,j} = \omega_{j,i}$ to each edge $\{i,j\}$ of G (a bar or strut). The whole stress is denoted by $\omega = (\ldots, \omega_{i,j}, \ldots)$. We say that the stress ω is an *equilibrium stress* if for each vertex i of G the following equilibrium equation holds:

$$\sum_{j\,:\,\{i,j\} \in B \cup S} \omega_{ij} (\mathbf{p}_j - \mathbf{p}_i) = 0 \tag{3}$$

We say that the stress ω is *proper* if furthermore for all struts $\{i,j\}$, $\omega_{i,j} \geq 0$. There is no sign condition for bars. (Terminology and sign conditions are not uniform in the literature: An equilibrium stress is also called a *self-stress* or plainly a *stress*; all stresses that we deal with are equilibrium stresses.)

Now we state the duality between equilibrium stresses and infinitesimal motions.

Theorem 3 *The framework $G_A(\mathbf{p})$ corresponding to a reduced arc-and-cycle set A has only the zero proper equilibrium stress.*

Lemma 3 *Theorem 3 implies Theorem 2.*

This equivalence is a standard technique in the theory of rigidity. See [CW96, Theorem 2.3.2] for a similar statement. For completeness, we give a brief proof here based on linear programming duality:

Proof: By rescaling the velocities in (2), we obtain the following equivalent linear feasibility problem:

$$\begin{array}{ll} \text{minimize} & 0 \cdot \mathbf{v} \\ \text{subject to} & (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i,j\} \in B, \\ & (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) \geq 1 \text{ for } \{i,j\} \in S, \end{array} \tag{4}$$

Theorem 2 claims that this linear program has a feasible solution, that is, an optimal solution of value 0. By linear-programming duality (the Farkas lemma), it suffices to show that the dual linear program

$$\begin{array}{lll} \text{maximize} & \sum_{\{i,j\}\in S} \omega_{i,j} & \\ \text{subject to} & \displaystyle\sum_{j:\{i,j\}\in B\cup S} \omega_{i,j}(\mathbf{p}_j - \mathbf{p}_i) = 0 & \text{for } i \in V, \\ & \omega_{i,j} = \omega_{j,i} & \text{for } \{i,j\} \in B \cup S, \\ & \omega_{i,j} \geq 0 & \text{for } \{i,j\} \in S, \end{array} \tag{5}$$

has an optimal solution of value 0. This linear program exactly specifies the constraints of a proper equilibrium stress. Thus, it suffices to show that every proper equilibrium stress has $\omega_{i,j} = 0$ for all $\{i,j\} \in S$. In particular, it suffices to show that every proper equilibrium stress is identically zero. □

3.2 Planarization

To prove that only the zero equilibrium stress exists (i.e., to prove Theorem 3), we use another tool in rigidity called the Maxwell-Cremona theorem. Before we can apply this tool, we need to transform the framework $G_A(\mathbf{p})$ into a planar framework $G'_A(\mathbf{p}')$. (Refer to the framework on the right of Figure 4.) We introduce new vertices at all intersection points between edges of $G_A(\mathbf{p})$, and subdivide the bars and struts accordingly. Any multiple edges resulting from this operation are merged. We define the resulting framework $G'_A(\mathbf{p}')$ to have bars precisely covering the bars of $G_A(\mathbf{p})$. All the other edges of $G'_A(\mathbf{p}')$ are struts. $G'_A(\mathbf{p}')$ is *planar* in the sense that two edges (bars or struts) intersect only at a common endpoint.

Despite the added points in this modification, the planar framework $G'_A(\mathbf{p}')$ is equivalent to the original framework $G_A(\mathbf{p})$ in the sense of equilibrium stresses. Indeed, the following stronger statement holds. We call a stress *outer-zero* if the only edges that carry a nonzero stress are edges of convex cycles and edges interior to convex cycles, and we call it *outer-nonzero* otherwise.

Lemma 4 *If $G_A(\mathbf{p})$ has a nonzero proper equilibrium stress ω, then $G'_A(\mathbf{p}')$ has an outer-nonzero proper equilibrium stress ω'.*

Proof: During the modifications to $G_A(\mathbf{p})$ that made $G'_A(\mathbf{p}')$, we modify ω to make ω' as follows. When we subdivide an edge $\{i,j\}$ with stress $\omega_{i,j}$, each edge of the subdivision $\{k,l\}$ gets the stress $\omega_{i,j}\|p_i - p_j\|/\|p_k - p_l\|$. (The ratio of lengths is necessary because $\omega_{i,j}$ is a weight, and the actual force comes from scaling by the length of the edge $\{i,j\}$; see (3).) When merging several edges, we add the corresponding stresses. It is easy to verify that the resulting stress is proper and in equilibrium. We only have to check that positive and negative stresses do not completely cancel during the merging process, and that the stress is furthermore outer-nonzero.

First we prove that some strut $\{i,j\}$ of $G_A(\mathbf{p})$ carries a positive stress. In other words, $G_A(\mathbf{p})$ cannot be stressed only on its bars; in particular, a framework consisting exclusively of arcs, cycles, and triangulated convex cycles cannot carry a nonzero stress. This follows because, in any such bar framework, there is a degree-two vertex $\mathbf{v}$; in particular, every triangulated convex cycle has a degree-two vertex (an ear). Because the framework is reduced, the two bars incident to $\mathbf{v}$ are non-parallel, so these two bars cannot carry stress while satisfying equilibrium at $\mathbf{v}$. Removing them and proceeding inductively with the rest of the framework, we conclude that the stress is zero on the whole bar framework. Hence, the bars alone cannot carry a nonzero stress, so some strut $\{i,j\} \in G_A(\mathbf{p})$ must have a nonzero stress.

The conditions of Theorem 2 enforce that no angles at vertices of the arc-and-cycle set are π or 0: an angle of π would create a straight subarc of two bars (contradicting the assumption that framework is reduced), and an angle of 0 would violate simplicity. Thus, no strut of $G_A(\mathbf{p})$ is completely covered by bars. Therefore, for the strut $\{i,j\}$ of $G_A(\mathbf{p})$ that carries a positive stress, some portion of it in $G'_A(\mathbf{p}')$ will also have a positive stress, because a positive stress can only be canceled by a stress on a bar. In particular, ω' must be nonzero.

Furthermore, if the strut $\{i,j\}$ is exterior to all convex cycles in A, we have that ω' is outer-nonzero. Now suppose that $\{i,j\}$ is partially interior to convex cycles in A (by construction, the strut cannot be entirely within convex cycles of A). Then there is a portion of $\{i,j\}$ with the property that it is incident to a convex cycle and exterior to all convex cycles in A. This portion must be uncovered by bars, because no bar in A has this property, and the only additional bars in $G_A(\mathbf{p})$ are interior to convex cycles. Hence, the corresponding strut in $G'_A(\mathbf{p})$ carries a positive stress, so ω' is outer-nonzero in all cases. □

Thus, to prove that the original framework $G_A(\mathbf{p})$ has only the zero proper equilibrium stress, it suffices to prove that the planar framework $G'_A(\mathbf{p}')$ has only outer-zero proper equilibrium stresses.

3.3 Maxwell-Cremona Theorem

To prove that only outer-zero equilibrium stresses exist, we employ the Maxwell-Cremona correspondence between equilibrium stresses in planar frameworks and three-dimensional polyhedral graphs that project onto these frameworks. More precisely, a *polyhedral graph* or *polyhedral terrain* Γ comes from lifting a planar framework into three dimensions—that is, assigning a z coordinate (positive or negative) to each vertex in the framework—such that each face bounded by edges of the framework (including the exterior face) remains planar. The polyhedral surface Γ is then the graph of a piecewise-linear continuous function of two variables that is linear on the faces determined by $G'_A(\mathbf{p}')$.

Consider an edge $\{i, j\}$ in a planar framework, separating faces F and F'. Let $z = \mathbf{a}\cdot\mathbf{p}+b$ and $z = \mathbf{a}' \cdot \mathbf{p} + b'$ be the two linear functions specifying Γ on F and F', respectively. A straightforward calculation reveals that the vector $\mathbf{a}' - \mathbf{a}$ must be perpendicular to the edge $\{i, j\}$:

$$\mathbf{a}' - \mathbf{a} = \omega_{i,j}\mathbf{e}^{\perp}_{i,j} \tag{6}$$

where $\mathbf{e}^{\perp}_{i,j}$ is a vector of the same length as the vector $\mathbf{p}_j - \mathbf{p}_i$, perpendicular to it, and pointing from F towards F'. We call the edge $\{i, j\}$ a *valley* if $\omega_{i,j} > 0$, a *mountain* if $\omega_{i,j} < 0$, and *flat* if $\omega_{i,j} = 0$. Note that the two sides of a valley do not necessary "go up" in z (and so a valley might not carry water); however, as one crosses a valley, the slope gets steeper. A similar remark applies to mountains.

Theorem 4 (Maxwell-Cremona Theorem) *(i) For every polyhedral graph Γ that projects to a planar bar framework $G(\mathbf{p})$, the stress ω defined by* (6) *forms an equilibrium stress on $G(\mathbf{p})$.*

(ii) For every proper equilibrium stress ω in a planar framework $G(\mathbf{p})$, $G(\mathbf{p})$ can be lifted to a polyhedral graph Γ such that (6) *holds for all edges. In particular, edges with positive stress lift to valleys, edges with negative stress lift to mountains, and edges with no stress lift to flat edges. Furthermore, Γ is unique up to addition of affine-linear functions.*

A proof of this result can be found in [CW94, HK92, Whi82], which follows the idea suggested above. Another point of view [Glu74] is that the stresses are a scaling of the angular momentum vectors of the function that lifts from the plane to the graph.

3.4 Main Argument

Note in particular that the zero equilibrium stress corresponds to the trivial polyhedral graph in which all faces are coplanar (i.e., defined by the same linear function). More generally, an outer-zero equilibrium stress corresponds to a polyhedral graph that is flat on every edge exterior to all convex cycles. Therefore, to prove that only outer-zero equilibrium stresses exist, and hence prove Theorem 3, it suffices to show that only such polyhedral graphs exist.

More precisely, consider any polyhedral graph Γ that projects to the planar framework $G'_A(\mathbf{p}')$ with the property that all struts are lifted to valleys or flat edges (because they can only carry nonnegative stress), and bars are lifted to valleys, mountains, or flat edges. We need to show that nonflat edges can only appear within or on the boundary of convex cycles. Because we may add an arbitrary affine-linear function, we may conveniently assume that the exterior face of Γ is on the xy-plane. Thus the problem is to show that Γ does not lift off the xy-plane any vertex of $G'_A(\mathbf{p}')$ except possibly vertices interior to convex cycles of A.

One simple fact that we will need is the following:

Lemma 5 *Any mountain in the polyhedral graph Γ projects to a bar in the planar framework $G'_A(\mathbf{p}')$.*

Proof: A strut can only carry nonnegative stress, so by Theorem 4 it can only lift to a valley or a flat edge. □

We now come to the heart of our proof, the proof of Theorem 5. It is here that we finally show that the stress must be outer-zero, by looking at the maximum of any Maxwell-Cremona lift. Specifically, let M denote the region in the xy-plane where the z value attains its maximum in Γ. M is a nonempty union of faces, edges, and vertices of the planar framework $G'_A(\mathbf{p}')$. The following statement immediately implies Theorem 3 and hence Theorem 2:

Theorem 5 *The set M includes every face of the framework $G'_A(\mathbf{p}')$ that is exterior to all convex cycles.*

Consider the boundary ∂M, which may be empty if M fills the whole plane. Because points in M lift to maximum height, all edges of ∂M must lift to mountains. Thus by Lemma 5, all edges of ∂M must be bars in the framework. Hence, ∂M consists of disjoint vertices, paths of edges, and complete cycles of the arc-and-cycle set, together with a subset of the triangulations of the convex components. Figure 6 shows typical cases of all possibilities. We will show that the only case in Figure 6 that can actually occur is (l), in which ∂M includes a convex cycle and M includes the local exterior of that cycle.

Our main technique for arriving at a contradiction in all cases except (l) is that of *slicing* the polyhedral graph. Consider a plane Π that is parallel to the xy-plane and just below the maximum z coordinate of Γ. (By "just below" we mean that Π is above all vertices of Γ not at the maximum z coordinate.) Now take the intersection of Π with the surface Γ, and project this intersection to the xy-plane. The resulting set X is shown in Figure 7 for the various cases.

The set X captures several properties of the polyhedral graph Γ. First note that because X is the boundary of a small neighborhood of M in the plane, it is a disjoint union of cycles. It is also polygonal. Each edge of X corresponds to a face of Γ, and each vertex of X corresponds to an edge of Γ. The *angle* at a vertex of X (at the side interior to M) determines the type of edge corresponding to that vertex: the angle is π (straight) if the edge is flat, less than π (convex) if the edge is a mountain, and more than π (reflex) if the edge is a valley.

The basic idea is to show that X has "many" convex angles, and apply Lemma 5 to prove that the framework has "too many" bars. The key fact underlying the proof is that the arc-and-cycle set has maximum bar-degree two: every vertex is incident to at most two bars. In the planar framework $G'_A(\mathbf{p}')$, only vertices $\mathbf{v}$ of convex cycles can have bar-degrees greater than two, and these bars are contained in a convex wedge from $\mathbf{v}$.

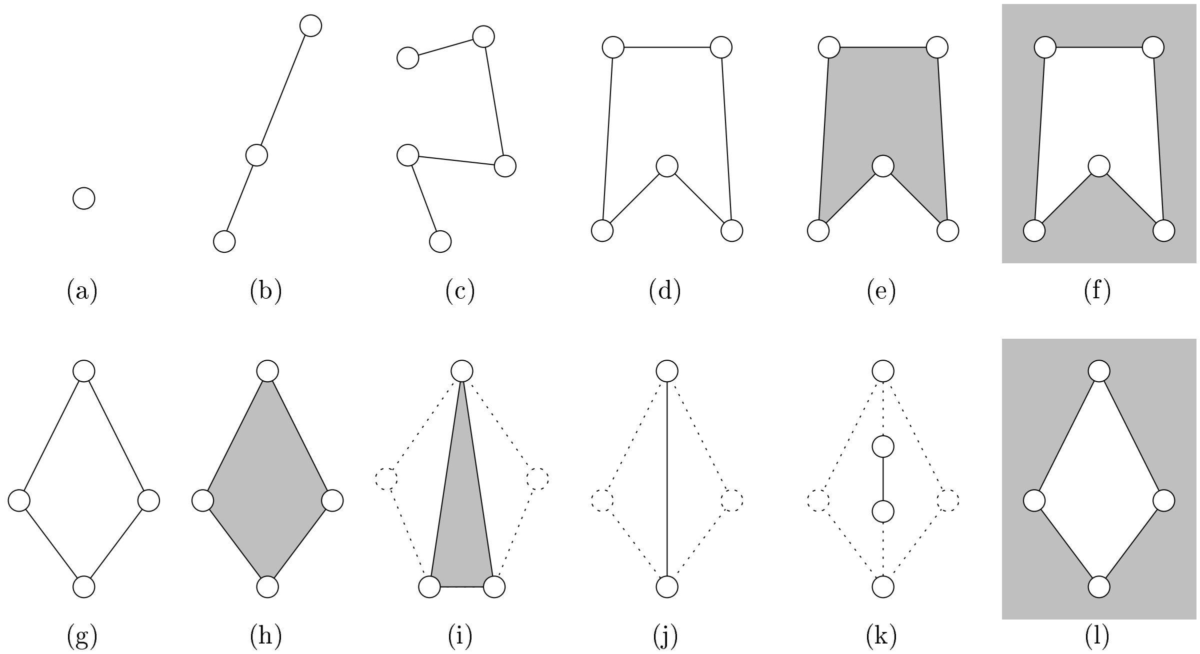

Figure 6: Hypothetical connected components of ∂M and their relation to M. Solid lines are edges of ∂M; white regions are absent from M; and shaded regions are present in M. (a) An isolated vertex. (b) A straight subarc. (c) A nonstraight subarc. (d) A nonconvex cycle. (e) A nonconvex cycle and its local interior. (f) A nonconvex cycle and its local exterior. (g–k) Various situations with a convex cycle. (l) The only possible case: A convex cycle and its local exterior.

Our proof deals with all cases at once. To illustrate the essence of the proof, we first describe it for the special case (a) in which one component of ∂M is a single vertex $\mathbf{v}$ that does not belong to a convex cycle. In this case, one component of X is a star-shaped polygonal cycle P around $\mathbf{v}$. Every polygon P has at least three convex vertices. (Because the turn angles of a polygon sum to 2π, and the maximum turn angle of a vertex is $< \pi$, every polygon has at least three vertices with positive turn angles.) These three convex vertices correspond to three mountains in Γ, all incident to a common vertex v. By Lemma 5, there are three bars incident to v, contradicting the maximum-degree-two property for vertices not on convex cycles. Therefore, case (a) cannot exist.

The general reason that cases (a–k) cannot exist is the following:

Lemma 6 *Let* $\mathbf{v}$ *be a vertex on the boundary of* M, *and let* $b_1, \ldots, b_k$ *be the bars incident to* $\mathbf{v}$ *in cyclic order. Consider a small disk* D *around* $\mathbf{v}$.

(i) *If there is an angle of at least* π *at* $\mathbf{v}$ *between two consecutive bars, say* b_i *and* b_{i+1}, *then the pie wedge* P *of* D *bounded by* b_i *and* b_{i+1} *belongs to* M (*see Figure* 8).

(ii) *If there are no bars or only one bar incident to* $\mathbf{v}$, *i.e.,* $k \leq 1$, *then the entire disk* D *belongs to* M. (*This can be viewed as a special case of* (i).)

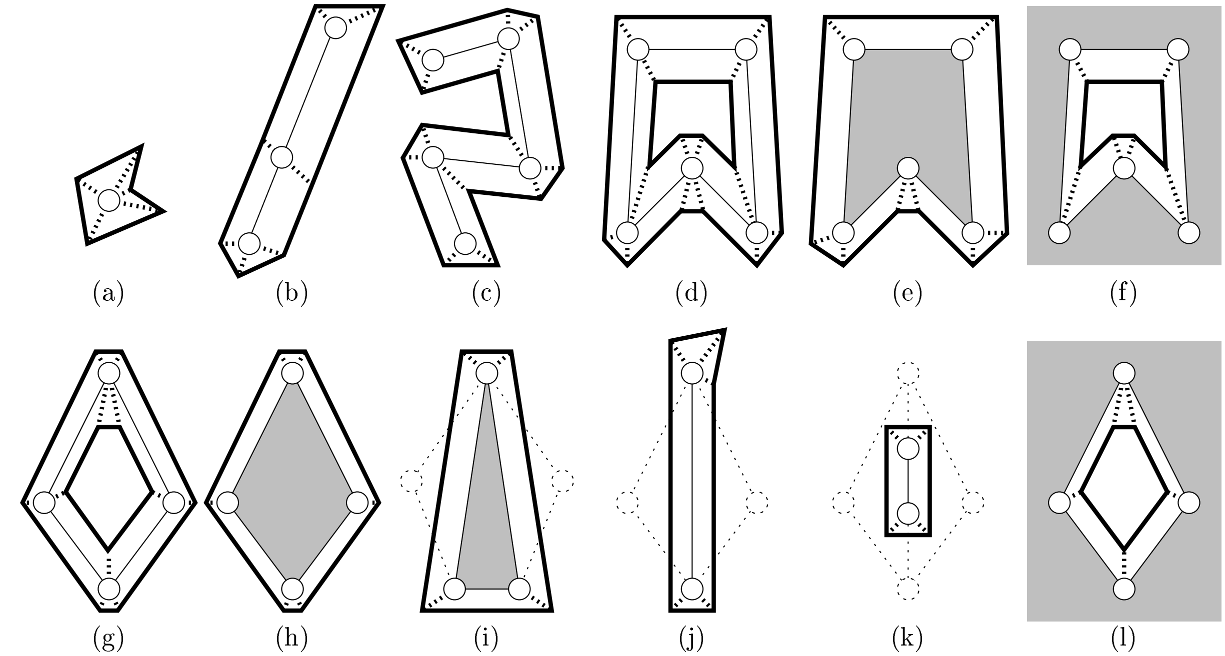

Figure 7: Slicing the polyhedral graph Γ just below the maximum z coordinate, in each case corresponding to those in Figure 6. Thick lines denote the slice intersection X, and thick dotted lines denote the corresponding edges in the polyhedral graph Γ.

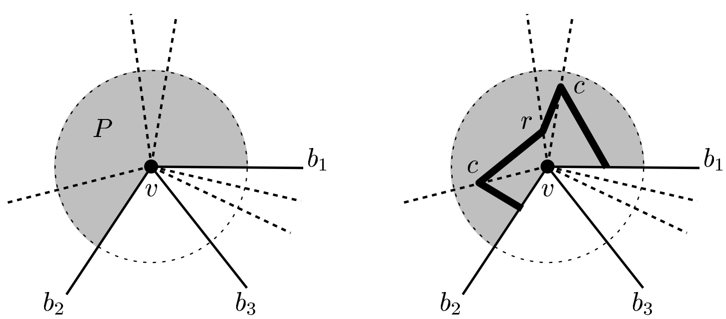

Figure 8: (Left) Illustration of Lemma 6: solid lines are bars, dotted lines are struts, and the shaded pie wedge P must be contained in M. (Right) Illustration of the proof; the thick lines form the portion of X inside P, and the symbols c and r denote convex and reflex vertices, respectively.

Proof: (i) Because there are no bars in the pie wedge P, and hence no edges of ∂M in P, P must be completely contained in or disjoint from M. Assume to the contrary that P is disjoint from M. Then the intersection of the slice X with the pie wedge P is a star-shaped polygonal arc around $\mathbf{v}$ starting from a point on b_i and ending at a point on b_{i+1}. By the properties of X, convex vertices on this arc correspond to mountains emanating from $\mathbf{v}$, and reflex vertices correspond to valleys emanating from $\mathbf{v}$. Because the angle of the pie wedge P is at least π, the arc must have at least one convex vertex in P. (The turn angles along the arc must sum to a positive number, so some vertex must have a positive turn angle.) By

Lemma 5, there must be a bar in P, a contradiction.

(ii) If $k = 1$, the bars b_i and b_{i+1} coincide, and the same proof applies. The star-shaped polygonal arc becomes a star-shaped polygonal cycle, which must have at least two convex vertices not on $b_i = b_{i+1}$. If $k = 0$, X also has a star-shaped polygonal cycle around $\mathbf{v}$, which must have at least three convex vertices, yet $\mathbf{v}$ has no incident bars. $\square$

Note that this lemma applies to every vertex in our planar framework $G'_A(\mathbf{p}')$, because every vertex either has bar-degree at most two or is a vertex of a convex cycle, and in either case there is a nonconvex angle between two consecutive bars.

One can immediately verify that the examples shown in Figure 7(a–k) contradict Lemma 6. For example, applying the lemma to any vertex of ∂M shows that M should contain a positive two-dimensional area incident to that vertex. This immediately rules out cases (a–d), (g), (j), and (k).

A general proof is also easy with Lemma 6 in hand:

Proof (Theorem 5): Consider first a degree-0 or degree-1 vertex $\mathbf{v}$ in ∂M. (Such a point would appear when M has a component that is an isolated point or an arc of bars.) Because Lemma 6 applies to every vertex of the framework, we know that some positive two-dimensional area in the vicinity of $\mathbf{v}$ belongs to M, contradicting that $\mathbf{v}$ has degree 0 or 1 in ∂M. This rules out cases (a–c) and (j–k).

It follows that ∂M is a union of cycles. A component of ∂M can be of two kinds:

(i) If it is formed from the edges of a convex cycle and its triangulation, Lemma 6 applies to any vertex in it, and we conclude that M contains the face of the framework immediately exterior to the cycle. This rules out cases (g–i).

(ii) If it consists of a complete nonconvex cycle, we can apply Lemma 6 to some convex vertex and to some reflex vertex (they must both exist), and we conclude that M contains both the face of the framework immediately interior and the face immediately exterior to the cycle. This rules out cases (d–f).

In the end, the only faces of the framework that can be missing from M are those interior to convex cycles (case (l)). This completes the proof of Theorem 5 and of Theorems 3 and 2. $\square$

4 Global Motion

In this section, we combine the infinitesimal motions into a global motion, thereby proving Theorem 1, the main theorem. In Theorem 2 we have established the existence of *some* direction of motion $\mathbf{v}$. Now we select a unique vector $\mathbf{v} := f(\mathbf{p})$ for each configuration $\mathbf{p}$ as the solution of a convex optimization problem (7–9). We then set up the differential equation

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)).$$

The solution of this differential equation moves the linkage to a configuration where an angle between two bars becomes straight. At this point we merge the two bars and continue with the reduced framework that has one vertex less. This procedure is iterated until the framework is outer-convex and no further expansive motion is possible.

It is convenient for the proof of Theorem 1 to effectively pin an edge in the configuration. Choose any edge, say $\{\mathbf{p}_1, \mathbf{p}_2\}$, that is a bar. During the motion we will arrange matters so that this bar is stationary.

We now go into the details of the proof. We use the following nonlinear minimization problem to define a unique direction $\mathbf{v}$ for every configuration $\mathbf{p}$ of a reduced arc-and-cycle set.

$$\begin{aligned}
\text{minimize} \quad & \sum_{i\in V} \|\mathbf{v}_i\|^2 + \sum_{\{i,j\}\in S} \left[(\mathbf{v}_i - \mathbf{v}_j)\cdot(\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|\right]^{-1} && (7)\\
\text{subject to} \quad & (\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{p}_j - \mathbf{p}_i) \;>\; \|\mathbf{p}_j - \mathbf{p}_i\|, \quad \text{for } \{i,j\}\in S && (8)\\
& (\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{p}_j - \mathbf{p}_i) \;=\; 0, \quad \text{for } \{i,j\}\in B && (9)\\
& \mathbf{v}_1 = \mathbf{v}_2 \;=\; 0 && (10)
\end{aligned}$$

The restrictions (8) place a uniform constraint on the growth of the struts S: $\ell'_{ij} > 1$. Since the system (2) is homogeneous, the system (8–9) is feasible for any choice of right-hand sides in (8). This particular right-hand side has been chosen for convenience in the proof.

The objective function (7) includes the norm of $\mathbf{v}$ as a quadratic term, plus a barrier-type penalty term that keeps the solution away from the boundary (8) of the feasible region. This penalty term is necessary to achieve a smooth dependence of the solution on the data. Now, because the objective function is strictly convex, and it goes to infinity if $\mathbf{v}$ increases to infinity or approaches the boundary, there is a unique solution $\mathbf{v} =: f(\mathbf{p})$ for every $\mathbf{p}$.

The function $f(\mathbf{p})$ is defined on an open set $U \subset \mathbf{R}^{2n}$ that is characterized by the conditions of Theorem 2: no angles are $0°$ or $180°$, no vertex touches a bar, and at least one cycle is nonconvex or at least one open arc is not straight.

4.1 Smoothness

We will show that f is differentiable in the domain U. This follows from the stability theory of convex programming under equality constraints, as applied to parametric optimization problems of the type

$$\min\{\, g(p,x) : x \in \Omega(p) \subseteq \mathbb{R}^n,\ A(p)x = b(p)\,\} \tag{11}$$

where $A(p)$ is an $m\times n$ matrix and $b(p)$ is an m-vector. The objective function g, the domain $\Omega(p)$, and the linear constraints (A,b) depend on a parameter p that ranges over an open region $U \subseteq \mathbb{R}^k$.

For such an optimization problem, the following lemma establishes the smooth dependence of the solution vector on the data.

Lemma 7 *Suppose that the following conditions are satisfied in the optimization problem* (11).

(a) *The objective function $g(p,x)$ is twice continuously differentiable and strictly convex as a function of $x \in \Omega(p)$, with a positive definite Hessian H_g, for every $p \in U$.*

(b) *The domain* $\Omega(p)$ *is an open set, for every* $p \in U$.

(c) *The rows of the constraint matrix* $A(p)$ *are linearly independent, for every* $p \in U$.

(d) *The gradient* ∇g *of* g *with respect to* x *and the data* $A(p)$ *and* $b(p)$ *are continuously differentiable in* p, *for* $p \in U$.

(e) *The optimum point* $x^*(p)$ *of the problem* (11) *exists for every* $p \in U$ (*and is unique, by* (a)).

Then $x^*(p)$ *is continuously differentiable in* U.

Proof: The proof is based on the implicit function theorem and follows the standard lines of the proof in this area, cf. [BS74, in particular Section 4] or [Fia76, Theorem 2.1] for more general theorems where inequalities are also allowed. For the benefit of the reader, we sketch the proof here. From (a) and (e) it follows that x^* can be found as part of the unique solution (x^*, λ) of the system of equations $h(p, x, \lambda) = 0$ that represents the first-order necessary optimality conditions. Here, λ is a k-vector of Lagrange multipliers, and $h \colon U \times \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is given by

$$h = \begin{pmatrix} \nabla g - \lambda^{\mathrm{T}} A^{\mathrm{T}} \\ Ax - b \end{pmatrix}.$$

The implicit function theorem guarantees the local existence of $x(p)$ (and $\lambda(p)$) as a solution of $h(p, x(p), \lambda(p)) = 0$ in a neighborhood of x if the Jacobian $J = \partial h / \partial(x, \lambda)$ is an invertible matrix for every $p \in U$. Moreover, differentiability is ensured if h is continuously differentiable. The Jacobi matrix is given by

$$J = \frac{\partial h(p, x, \lambda)}{\partial(x, \lambda)} = \begin{pmatrix} H_g & A^{\mathrm{T}} \\ A & 0 \end{pmatrix}.$$

Differentiability of h follows from the assumptions; we only have to check that A is invertible. Since H_g is invertible by (a), we have

$$\det J = \det H_g \cdot \det(-A H_g^{-1} A^{\mathrm{T}}).$$

By assumption (c), A has full row rank, and the matrix $A H_g^{-1} A^{\mathrm{T}}$ is positive definite. Therefore $\det J \neq 0$. $\square$

Lemma 8 *f is differentiable on* U.

Proof: The objective function is the sum of the quadratic function $\sum \|\mathbf{v}_i\|^2$, which has a positive definite (constant) Hessian, and additional convex terms, and therefore assumption (a) of Lemma 7 holds. We use the inequalities (8) to define the feasible domain Ω. So we only need to consider the linear constraints (9). All other conditions of Lemma 7 are trivial to check except for the linear independence of the equations (9).

This linear independence is easy to see: we have to check whether the system (9) has a solution for any choice of right-hand sides instead of 0. Even when we consider the set of bars B that are augmented by the triangulation edges that were added in the modification step 2 of section 2.2, There is always a vertex which is incident to at most two bars, and moreover, these two bars cannot be parallel. The corresponding unknown vector $\mathbf{v}_i$ appears in two equations in which the scalar products with two vectors $\mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{p}_i - \mathbf{p}_k$ are taken; as these vectors are not parallel, there is always a solution for $\mathbf{v}_i$ regardless of the values of the other variables. The case when node i is incident to one bar or to no bar is easy. $\square$

4.2 Solving the Differential Equation and Proving Theorem 1

Differentiability of f on U is sufficient to ensure that the initial-value problem

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)), \quad \mathbf{p}(0) = \mathbf{p}_0 \tag{12}$$

has a (unique) *maximal solution* $\mathbf{p}(t)$, $0 \le t < T$, that cannot be extended beyond some positive bound $T \le \infty$; see for example [Wal96, Section II.XXI]. This means that one of three cases occurs:

(a) $\mathbf{p}(t)$ exists for all t, i.e., $T = \infty$.

(b) T is finite, and $\mathbf{p}(t)$ becomes unbounded as $t \to T$.

(c) T is finite, and $\mathbf{p}(t)$ approaches the boundary of U as $t \to T$.

The last case (c) is the case we want: at the boundary of U, some angle becomes straight, and we can reduce the linkage.

Case (a) can be excluded very easily. By assumption, the bar-and-strut framework $G_A(\mathbf{p})$ has some strut $\{i, j\}$ between two points in the same component of the bar framework; their distance increases at least with rate 1, by (8), but it is bounded from above because i and j are linked by a sequence of bars. It follows that the solution cannot exist indefinitely and T must be finite.

If there is a line L that separates the components of the arc-and-cycle set A, this partitions A into two nonempty sets, and each of these can be treated separately and recursively. Unfortunately, the guarantee for the expansive property between different members of the partition is lost. But for the purposes of proving Theorem 1 we may assume that there is no such separation. Then the sum of the maximum diameters of each of the components of A is a uniform a-priori bound on the diameter of A for all time. This eliminates case (b).

Thus we are left with case (c) only. We show that $\mathbf{p}(t)$ converves as $t \to T$. Observe that all pairwise distances of vertices $\mathbf{p}(t)$ are monotonically increasing, and by condition (10) $\mathbf{p}_1$ and $\mathbf{p}_2$ are fixed during the motion. Thus, all other vertices are determined up to reflection, and the whole configuration is determined up to reflection. Thus $\mathbf{p}(t) \to \mathbf{p}$ for some configuration $\mathbf{p}$ as $t \to T$. The configuration $\mathbf{p}$ is on the boundary of U and thus must have some vertex with a straight angle. Then we inductively continue with a simpler linkage. This completes the proof of Theorem 1. $\square$

In this proof, the easy exclusion of possibility (b), that the motion becomes unbounded, depends crucially on the fact that the diameter of A is bounded, and the motion is stopped as soon as there is a separating line. Boundedness is valid even without this precaution, as is stated in the following lemma.

Lemma 9 (Boundedness Lemma) *Let* $\mathbf{p}(t)$ *be the motion given by the differential equation* (12)*, where* $\mathbf{v} = f(\mathbf{p})$ *is given as the solution of the optimization problem* (7–9)*. Then the motion of every vertex* i *is bounded:*

$$\|\mathbf{p}_i(t) - \mathbf{p}_i(0)\| \le \int_0^t \|\mathbf{v}_i(t)\|\, dt \le K_{B,S,\mathbf{p}_0}(t),$$

where $K_{B,S,\mathbf{p}_0}(t)$ *is an explicit function of* t *that depends only on the combinatorial structure of the arc-and-cycle set* (B *and* S) *and on the initial configuration* $\mathbf{p}_0$*.*

Note that the definition of $\mathbf{v}$ does not involves the pinning constraints (10). The lemma implies that it is not necessary to treat separated components separately. The proof of the lemma is complicated, and it is given in the appendix.

4.3 Alternative Approaches

There are many ways to select a local motion $\mathbf{v}$ among the feasible local motions whose existence is guaranteed by Theorem 2. We have chosen one possibility that is most convenient for the proof.

As a possible alternative approach, we might consider a *linear* programming problem, with some arbitrary artificial linear objective function $\mathbf{c}$, and some linear normalization condition to ensure boundedness, pinning down some bar $(i_1, i_2) \in B$:

$$\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{i \in V} \mathbf{c}_i \cdot \mathbf{v}_i \\
\text{subject to} \quad & (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i,j\} \in B, && (13)\\
& (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) \ge 0 \text{ for } \{i,j\} \in S, && (14)\\
& \textstyle\sum_{i \in V} \mathbf{d}_i \cdot \mathbf{v}_i = 1, && (15)\\
& \mathbf{v}_{i_1} = \mathbf{v}_{i_2} = 0, && (16)
\end{aligned}$$

We have given up *strict* expansiveness in (14), The set of vectors given by (13), (14), and (16) forms a polyhedral cone C. Theorem 2 guarantees that there are nonzero solutions. One can check that the pinning constraints (16) ensure that the cone is pointed. The idea is now to use an extreme ray of the cone C for the motion. A vector $\mathbf{d}$ can be found which ensures that the feasible set (13–16) is a bounded set. Any basic feasible solution of the linear program will correspond to an extreme ray of the cone C. It will have a few inequalities of (14) fulfilled with equality. The resulting framework obtained by inserting "artificial" bars corresponding to the nonbasic inequalities of (14), will have a unique vector of velocities $\mathbf{v}$ subject to the normalization constraint (15). This means that the framework is a *mechanism*, allowing one degree of freedom; as the mechanism follows this forced motion, all nonfixed distances will increase, at least for some time.

So one follows the paradigm of parametric linear programming: The optimal basic feasible solution will continue to remain feasible as the coefficients $\mathbf{p}_i$ in the constraints (14) change smoothly. At some point, one of these constraints will threaten to become violated: this is the time to make a *pivot*, exchanging one of the artificial bars for a new one which allows the motion to be continued.

The above discussion has ignored several issues, such as possible degeneracy of the linear program. However, this approach might be more attractive from a conceptual, as well as a practical point of view.

Recently, Streinu [Str00] has found a class of such mechanisms, so-called *pseudo-triangulations*. These structures have some nice properties; for example, they form a planar framework of bars. Streinu [Str00] has proved that a polygonal arc can be opened by a sequence of at most $O(n^2)$ motions, where each motion is given by the mechanism of a single pseudo-triangulation.

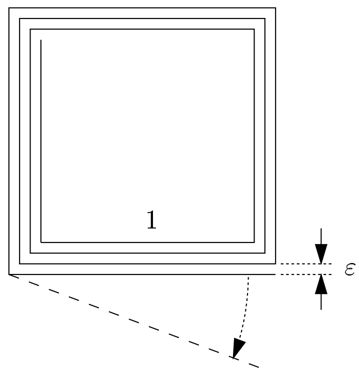

Figure 9: An arc that is numerically difficult to unfold.

4.4 Comparison of Approaches

The approach based on mechanisms might avoid some of the numerical difficulties associated with solving the optimization problem (7–9). For example, consider a spiral n-bar arc winding around a unit square in layers of thickness ε (Figure 9). In the solution of (8–9), a rough estimate shows that the outermost vertex must move with a speed of at least ε^{4-n}, as $\varepsilon \to 0$. On the other hand, the "natural" solution of unwinding the spiral one bar at a time fits nicely into the setup of mechanisms and the parametric linear program approach.

Our proof has certain nonconstructive aspects: the direction $\mathbf{v}$ of movement is specified implicitly as the solution of an optimization problem, and the global motion arises as the solution of a differential equation. Both of these items are numerically well-understood, and our approach lends itself to a practical implementation. Indeed, we implemented our approach to produce animations such as Figure 1. However, this does not necessarily lead to a finite algorithm in the strict sense. The optimization problem (7–9), having an objective function which is rational, can in principle be solved exactly by solving the system $h(p, x, \lambda) = 0$ of algebraic equations as in Lemma 7. The differential equation cannot be solved explicitly, but it may be possible to bound the convergence and solve the differential equation up to a given error bound.

Since the motions of a mechanism are described by algebraic equations, Streinu's algorithm leads to a finite algorithm for a digital computer, at least in principle. It remains to be seen how a practical implementation competes with our approach; in any case, as an algorithm for a direct realization of the motion by a mechanical device, Streinu's algorithm appears attractive.

On the other hand, the nonlinear programming approach might be preferable because it produces a "canonical" movement. As a consequence of this, any symmetry of the starting configuration is preserved (see Corollary 6).

5 Additional Properties and Related Problems

5.1 Symmetry

We show that the deformation that we have defined in Theorem 1 preserves any symmetries that the original configuration might have. We say that the arc-and-cycle set A has some group H of congruences of the plane as *symmetry group* if each element of H permutes the vertices and edges of A.

Theorem 6 *If an arc-and-cycle set has a symmetry group H, then there is a piecewise-differentiable proper motion to an outer-convex configuration, which is expansive until it is separated, and the symmetry group H is preserved during the deformation.*

Proof: Since A is bounded, there must be a point which is fixed by all elements of H. Let this point be the origin, and let $\mathbf{p}_1$ be any vertex of the configuration distinct from the origin. Consider the infinitesimal flex defined by the conditions (7), (8), and (9) but not (10). There is a unique solution $\mathbf{v}$ to this minimization problem. This solution must be symmetric with respect to the symmetry group H. If not, then the action of some element of H takes $\mathbf{v}$ to another distinct solution say $\mathbf{v}'$, contradicting the uniqueness of the solution. There is now a unique infinitesimal rotation that we can add to $\mathbf{v}$ so that $\mathbf{p}_1$ and $\mathbf{v}_1$ are parallel. This still maintains the symmetry of the infinitesimal flex $\mathbf{v}$. Now it is clear that the limit exists as before in the proof of Theorem 1, and the symmetry of H is preserved. $\square$

5.2 Increasing Area

A natural question is whether the expansive property of the motion implies that the area bounded by each polygonal cycle increases. Here we show that indeed it does. First we show how to extend any given expansive infinitesimal motion to any point in the plane.

Lemma 10 *Given a set $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of points in $\mathbf{E}^d$, an infinitesimal flex $\mathbf{v}$ of $\mathbf{p}$ that is infinitesimally expansive (i.e., $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{v}_i - \mathbf{v}_j) \geq 0$ for all i, j), and another point $\mathbf{p}_0$ in $\mathbf{E}^d$, the infinitesimal flex can be extended to $\mathbf{p}_0$ in such a way that $(\mathbf{v}_0, \mathbf{v})$ is still expansive and all the new inequalities are strict unless $\mathbf{p}_0$ is in the convex hull of some subset of points for which the infinitesimal flex is trivial.*

Proof: We have two proofs of this statement: one proof is based on a direct calculation; the second proof is not so straightforward, but it provides some geometric insight.

We first consider the case where we want to prove strict expansiveness. By the Farkas lemma, the desired inequalities $(\mathbf{v}_0 - \mathbf{v}_i) \cdot (\mathbf{p}_0 - \mathbf{p}_i) > 0$, can be fulfilled by some unknown vector $\mathbf{v}_0$ if and only if the dual system

$$\sum_{i=1}^{n} \lambda_i (\mathbf{p}_0 - \mathbf{p}_i) = 0 \tag{17}$$

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i) \geq 0 \tag{18}$$

$$\lambda_i \geq 0$$

has no solution except the trivial solution $\lambda \equiv 0$. Suppose for contradiction that a non-trivial solution λ exists. Without loss of generality, we may assume

$$\sum_{i=1}^{n} \lambda_i = 1.$$

Then we get from (17) a representation of $\mathbf{p}_0$ as a convex combination

$$\mathbf{p}_0 = \sum_{i=1}^{n} \lambda_i \mathbf{p}_i. \tag{19}$$

Substituting this into (18) yields

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (\mathbf{v}_i \cdot \mathbf{p}_j) - \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i \cdot \mathbf{p}_i) \geq 0. \tag{20}$$

On the other hand, multiplying the given inequalities

$$\mathbf{v}_i \cdot \mathbf{p}_i - \mathbf{v}_i \cdot \mathbf{p}_j - \mathbf{v}_j \cdot \mathbf{p}_i + \mathbf{v}_j \cdot \mathbf{p}_j \geq 0 \tag{21}$$

by $-\lambda_j \lambda_j / 2$ and summing them over $i, j = 1, \ldots, n$ (including the trivial cases for $i = j$) yields

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (\mathbf{v}_i \cdot \mathbf{p}_j) - \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i \cdot \mathbf{p}_i) \leq 0. \tag{22}$$

By the assumption of the lemma, we have $\lambda_i > 0$ in (19) for at least two points $\mathbf{p}_i$ and $\mathbf{p}_j$ whose distance expands strictly. This means that the corresponding strict inequality in (21) will hold in (22) too, a contradiction to (20). This finishes the case when $\mathbf{p}_0$ does not lie in the convex hull of some points which move rigidly.

In the other case, non-strict expansiveness can be shown by a variation of the above argument. Alternatively, we can appeal to Lemma 2 (or its higher-dimensional extension) and let the point $\mathbf{p}_0$ move rigidly with the rigid point set in whose convex hull it lies. The resulting motion is expansive; the distance from $\mathbf{p}_0$ to the other points will expand strictly, with the obvious exception of the points with whom it moves rigidly. This concludes the first proof of the lemma.

The other proof establishes a connection to tensegrity frameworks and is more intuitive. However, we have to go through a sequence of steps to reduce the statement of the lemma to the basic case that the points $\mathbf{p}_1, \ldots, \mathbf{p}_n$ form the vertices of a simplex that contains the point $\mathbf{p}_0$ in its interior.

We proceed by induction on the dimension d. There is nothing to prove in case $d = 0$. So we assume the statement for $0, \ldots, d-1$ and $d \geq 1$. The desired inequalities $(\mathbf{v}_0 - \mathbf{v}_i) \cdot (\mathbf{p}_0 - \mathbf{p}_i) > 0$ define open half spaces

$$H_i = \{\mathbf{v}_0 \mid \mathbf{v}_0 \cdot (\mathbf{p}_0 - \mathbf{p}_i) > \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i)\}.$$

This finite collection of half spaces is nonempty precisely if every set of $d+1$ of them is nonempty by Helly's theorem. So we first consider any subset of $d+1$ points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1}$ of $\mathbf{p}$. We consider the case when $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1}$ are affine-independent, i.e., they are the vertices of a d-dimensional simplex σ in $\mathbf{E}^d$.

If $\mathbf{p}_0$ is outside σ, then choose $\mathbf{v}_0$ in a direction along a normal to a hyperplane separating $\mathbf{p}_0$ and σ, pointing away from σ. If the magnitude of $\mathbf{v}_0$ is large enough, then the desired inequalities will be satisfied.

If $\mathbf{p}_0$ is interior to σ, suppose that the inequalities defined above do not have a solution. Then if we look at the complementary half-spaces defined by

$$H_i^- = \{\mathbf{v}_0 \mid \mathbf{v}_0 \cdot (\mathbf{p}_0 - \mathbf{p}_i) \leq \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i)\},$$

they do have a solution. Let $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1})$ be a solution to those new inequalities. Now $\mathbf{v}$ is an infinitesimal flex of the tensegrity that is obtained by having cables from $\mathbf{p}_0$ and the rest all struts as before. But this tensegrity is easily seen to be infinitesimally rigid in E^d. (For example, apply Theorem 5.2(c) of [RW81] observing that the underlying bar framework is infinitesimally rigid, and there must be a proper stress that is non-zero on all struts and cables. An explicit calculation of the proper stress is given in [BC99]. The calculation is similar to the calculations (19)–(22) in the first part of the proof, with $\lambda_i \lambda_j$ being interpreted as stress.) So $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ must be a trivial infinitesimal flex, which can only happen if the flex is trivial on all of σ.

If $\mathbf{p}_0$ lies in a proper face of σ or the vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1}$ are not affine independent, they lie in an affine $(d-1)$-dimensional space S with all the points of $\mathbf{p}$ either in S or in one of the open half-spaces defined S. Orthogonally project into S all of the $\mathbf{v}_i$ that correspond to $\mathbf{p}_i$ that are in S. By the induction hypothesis, there is a vector $\mathbf{v}_S$ in S such that it together with the vertices in S is infinitesimally expansive with respect to the projected $\mathbf{v}_i$ and hence the $\mathbf{v}_i$ themselves. It is strict unless $\mathbf{p}_0$ is in the convex hull of a simplex, where the infinitesimal motion is trivial. Let $\mathbf{v}_N$ be a large vector perpendicular to S pointing away from any other points in $\mathbf{p}$. If $\mathbf{v}_N$ is large enough, $\mathbf{v}_0 = \mathbf{v}_S + \mathbf{v}_N$ together with the other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ will be expansive.

Thus in any case we see that there is a strict solution to the inequalities for $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1}$ or $\mathbf{p}_0$ is in the convex hull of points where the infinitesimal motion acts as bars for all surrounding points. This finishes the proof. $\square$

Using this result inductively, we can extend the expansive motion to any finite set of points. We apply Lemma 10 to the vertices of an appropriately chosen triangulation of the region bounded by any cyclic component. The following result can be found in [BGR88] and later faster algorithms in [BMR95, Epp97].

Lemma 11 *Any simple closed polygonal curve in the plane can be triangulated such that all the triangles are nonobtuse.*

There has been some interest in providing acute triangulations and subdivisions (in contrast to nonobtuse triangulations) of various planar polygonal objects. For example, the column of Martin Gardner [Gar60] (see also [Gar95] and [Man60]) asks for a dissection of an obtuse triangle into acute trianges. But we do not know of a result guaranteeing an acute triangulation for a general polygon. Fortunately, the following is sufficient for the area-expanding property that we need:

Lemma 12 *Let* $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *be an infinitesimal flex of a nonobtuse triangle* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ *such that for* $i \neq j$,

$$(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq 0. \tag{23}$$

Then the infinitesimal change in the area of $\mathbf{p}$ *is always nonnegative. If any of the inequalities in* (23) *are strict and the triangle* $\mathbf{p}$ *is acute, then the infinitesimal change in the area of* $\mathbf{p}$ *is positive. If* $\mathbf{p}$ *is a right triangle and* (23) *is strict for either of the legs of* $\mathbf{p}$*, then the infinitesimal change in the area of the triangle is positive.*

Proof: Let the lengths of the sides of the triangle be denoted by a, b, c, and let the area of the triangle be denoted by A. If we differentiate Heron's formula

$$16A^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

and rearrange terms, denoting derivatives by a', b', c', A', we get

$$8AA' = (b^2 + c^2 - a^2)aa' + (a^2 + c^2 - b^2)bb' + (a^2 + b^2 - c^2)cc'. \tag{24}$$

We can regard aa', bb', cc' as the left hand side of (23). Each of the terms in parenthesis in (24) is nonnegative since $\mathbf{p}$ is nonobtuse. Thus $A' \geq 0$. Strict positivity ($A' > 0$) in the cases indicated in the lemma can be checked easily. □

Note that with a right triangle it is possible for the first derivative of the length of the hypotenuse to be positive while the first derivative of the length of the two legs is 0, and in this case the first derivative of the area will still be 0. This is the reason for the condition on the legs of the triangle.

Theorem 7 *Any smooth expansive noncongruent motion of a simple closed polygonal curve* C *in the plane, fixing the lengths of its edges, must increase the area of the interior of* C *during the motion.*

Proof: Consider the vector field $\mathbf{v}_t$, $0 \leq t \leq 1$ defined as the derivative at each vertex of C at time t. Apply Lemma 11 to find a triangulation T of the area bounded by C with all triangles nonobtuse. Apply Lemma 10 to extend the vector field to the vertices of T.

To get a strictly increasing area, we have to show that the triangulation T has an acute triangle with an edge interior to C, or a right triangle with a leg interior to C. Otherwise, T would be a single triangle, or it would exclusively consist of right triangles with both legs on C, hence it would be a convex quadrilateral with two opposite corners having right angles. These cases are excluded because a triangle or a convex quadrilateral (or any convex polygon) does not have an expansive noncongruent motion.

So we have established that T must have an acute triangle with an edge interior to C or a right triangle with a leg interior to C. Because the motion is expansive and the derivative of at least one of those lengths must be positive for all but a finite number of times, the derivative of the area of at least one of those triangles must by strictly positive, and they all are nonnegative by Lemma 12. So the derivative of the area bounded by C must be positive for all but a finite number of times $0 \leq t \leq 1$. Thus the area must strictly increase. □

Notice in the proof above that we use the property that the edges of C have fixed lengths when we extend the vector field to points in the interior of edges of C. These preserve the motion on C. If the lengths of edges of C are not fixed, then, for example, an obtuse triangle can have an expansive motion that decreases its area.

5.3 Topology of Configuration Spaces

It is natural to ask more about the structure of the *configuration space* of an arc-and-cycle set. Let $X(G, L)$ denote the space of all configurations of embeddings in the plane of of a bar graph G consisting of a finite number arcs and cycles, without self-intersections, where the edge lengths are determined by $L = (\dots, \ell_{i,j}, \dots)$. This inherits a natural topology from considering all the coordinates of all the vertices as part of a large dimensional Euclidean space. Let $X_0(G, L) \subset X(G, L)$ denote the subspace of outer-convex configurations. We assume that L is chosen so that there is at least one realization in the plane. We mention some results without proof.

Theorem 8 *The space of outer-convex realizations $X_0(G)$ is a strong deformation retract of $X(G)$.*

The main point to remember is that the limit in Theorem 1 depends continuously on the initial starting configuration. The following is a natural consequence of Theorem 8.

Corollary 2 *If the underlying graph G is a single arc or a single cycle, then $X(G, L)$ modulo congruences (including orientation reversing ones) is contractible.*

Here the main task is to show that the space of convex realizations is contractible.

It is interesting to compare $X(G, L)$, as we have defined it, to the space of realizations of an arc or cycle in the plane with fixed edge lengths, but where crossings are allowed. See e.g. [LW95, KM99, KM95a, KM95b, KM96] for results about this space.

5.4 Open Problems

Another direction is to explore what happens when the arc-and-cycle set is allowed to touch but not cross:

Conjecture 1 *If G is a single arc or a single cycle, then the closure of $X(G, L)$ modulo congruences is contractible.*

We conjecture that motions can be realized by a sequence of relatively simple motions:

Conjecture 2 *If A is an arc-and-cycle set in the plane, then there is a flex that takes it to an outer-convex configuration, by a finite sequence of motions, where each motion changes at most four vertex angles.*

It also remains open precisely how many such moves are needed.

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A Appendix: Expansive Motion for Several Components

We show the following stronger version of Theorem 1:

Theorem 9 *Every arc-and-cycle set has a piecewise-differentiable proper motion to an outer-convex configuration that is strictly expansive for all time.*

Proof: As mentioned in Section 4.2, the motion is defined in the same way as for Theorem 1, except that the explicit pinning constraint (10) is removed from the definition of $\mathbf{v} = f(\mathbf{p})$.

As in the case of Theorem 1, we know that the initial value problem (12) has a unique maximal solution $\mathbf{p}(t)$, $0 \le t < T$, that cannot be extended beyond T, and we have $T < \infty$. We have to show that, for $t \to T$, the configuration $\mathbf{p}(t)$ converges to a configuration on the boundary of U. From the Boundedness Lemma (Lemma 9 in Section 4.2), whose proof if given below, we conclude that the monotone limit

$$K_i := \lim_{t \to T} \int_0^t \|\mathbf{v}_i\| \, dt$$

exists. Therefore, for all $t_2 \ge t_1 \ge t_0$, we have

$$\|\mathbf{p}_i(t_2) - \mathbf{p}_i(t_1)\| \le \int_{t_1}^{t_2} \|\mathbf{v}_i(t)\| \, dt \le K_i - \int_0^{t_0} \|\mathbf{v}_i(t)\| \, dt.$$

The last bound goes to zero as $t_0 \to T$, and therefore $\mathbf{p}_i(t)$ converges. The limit configuration must lie on the boundary of the domain U because otherwise the solution of (12) could be extended beyond T. $\square$

A.1 Proof of the Boundedness Lemma

We would like to show that under the expansive motion given by the optimization problem program (7–9), the motion of individual points is limited (Lemma 9).

Before embarking on the proof, let us see consider a possibly difficult situation, which the proof has to handle. This might aid the reader in appreciating the arguments of the proof.

We know that the distances change monotonically, and therefore their total change is bounded if the distances remain bounded. However, it is conceivable that the points move back and forth without increasing the distances too much. Consider three collinear points $\mathbf{p}_1$, $\mathbf{p}$ and $\mathbf{p}_2$ in a horizontal row. If the point $\mathbf{p}$ wiggles vertically with a small amplitude, one can maintain an expansive motion where all distances remain bounded although $\mathbf{p}$ moves on a path of infinite length. This conceivable situation must be avoided.

We may get arbitrarily close to this situation described if all points are very nearly collinear. We can however get a quantitative estimate of the motion by placing bounds on the minimum width and on the diameter of the point set. If the angles in the triangle $\mathbf{p}_1\mathbf{p}\mathbf{p}_2$ are not arbitrarily small, the motion of $\mathbf{p}$ relative to $\mathbf{p}_1$ and $\mathbf{p}_2$ can be bounded in terms of the increase of pairwise distances.

A.2 Bounding the motion in terms of angles and distance changes

We give two geometric lemmas that allow us to bound the motions in terms of quantities that can only change monotonically.

Lemma 13 *Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be the sequence of vertices of a component with the edges of fixed length $\|\mathbf{p}_j - \mathbf{p}_{j+1}\| = \ell_{j,j+1}$ for $j = 1, \ldots, n-1$. (We are ignoring a possible edge from $\mathbf{p}_n$ to $\mathbf{p}_1$.) Say $\mathbf{p}_1$ and $\mathbf{p}_2$ are pinned, i.e., $\mathbf{v}_1 = \mathbf{v}_2 = 0$. Let θ_j be the internal angle at vertex j ($0 < \theta_j < \pi$; we disregard the orientation of the angles). Then, for $k = 2, \ldots, n$,*

$$\|\mathbf{v}_k\| \le (\ell_{1,2} + \cdots + \ell_{k-1,k}) \cdot (\theta'_2 + \cdots + \theta'_{k-1})$$

Proof: Using complex notation,

$$\frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{\ell_{j,j+1}} = \frac{\mathbf{p}_{j-1} - \mathbf{p}_j}{\ell_{j-1,j}} \cdot \exp(\pm\theta_j i).$$

Taking derivatives gives

$$\frac{\mathbf{v}_{j+1} - \mathbf{v}_j}{\ell_{j,j+1}} = \frac{\mathbf{v}_{j-1} - \mathbf{v}_j}{\ell_{j-1,j}} \cdot \underbrace{\exp(\pm\theta_j i)}_{|\ |=1} + \underbrace{\frac{\mathbf{p}_{j-1} - \mathbf{p}_j}{\ell_{j-1,j}}}_{\|\ \|=1} \cdot \underbrace{\exp(\pm\theta_j i)}_{|\ |=1} \cdot (\pm\theta'_j i).$$

Taking the absolute value of both sides and using the triangle inequality,

$$\frac{\|\mathbf{v}_{j+1} - \mathbf{v}_j\|}{\ell_{j,j+1}} \le \frac{\|\mathbf{v}_{j-1} - \mathbf{v}_j\|}{\ell_{j-1,j}} + |\theta'_j| \le |\theta'_2| + \cdots + |\theta'_j|,$$

and thus

$$\|\mathbf{v}_{j+1} - \mathbf{v}_j\| \le \ell_{j,j+1} \cdot (|\theta'_2| + \cdots + |\theta'_j|) \le \ell_{j,j+1} \cdot (|\theta'_2| + \cdots + |\theta'_{k-1}|),$$

for $j \le k$. Applying induction we obtain the desired result. □

The above lemma works only for single components. For disconnected components, we want to bound the speed of motion $\mathbf{v}$ in terms of distance changes. However, it is always possible to move the point set as a whole without changing any distances at all. Therefore we will consider a modified motion in which one vertex $\mathbf{p}_1$ is fixed ("pinned"). Another vertex $\mathbf{p}_2$ remains on a fixed line through $\mathbf{p}_1$, preventing the complete set from spinning. This modification corresponds to adding a rigid motion of the point set $\mathbf{p}$ to the given velocities $\mathbf{v}$. It does not change the lengths ℓ_{ij} or their derivatives, $(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j)$.

The *width* of a set of points is the minimum width of a strip (infinitely long rectangle) containing the set.

Lemma 14 *Consider a set $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of points with distances $\ell_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$, moving with velocities $\mathbf{v}_i = \mathbf{p}'_i$ under an expansive motion: $\ell'_{ij} \ge 0$. Denote the width by w. Without loss of generality, we assume that the diameter is $\ell_{12} =: D$. We normalize the motion by pinning $\mathbf{p}_1$ and the direction of the axis from $\mathbf{p}_1$ to $\mathbf{p}_2$. In other words, we assume that $\mathbf{v}_1 = 0$ and $\mathbf{v}_2$ is parallel to $\mathbf{p}_2 - \mathbf{p}_1$. Then, for every $k \in \{1, \ldots, n\}$,*

$$\|\mathbf{v}_k\| \le C_0 \cdot \sum_{i<j} \ell'_{ij},$$

where $C_0 = 666D^3/w^3$.

The proof is given in Section A.6.

A.3 Constant Quantities

We define the following constants. We denote by D_0 the maximum sum of the edge lengths in a single component. This is an upper bound on the diameter of a component. As we shall see, $D_{\max} := D_0 6^{n-1}$ is an upper bound on the distance between two points in the same cluster (to be defined). Let m_0 ($0 < m_0 \leq 1$) be a lower bound on the distance between two different components. (This may be the distance between two vertices or the distance from a vertex to an edge.) Let m_1 ($0 < m_1 \leq 1$) be a lower bound on the distance between between any two vertices (in the same component or in difference components). Since these distances can only increase during the expansive motion, we just need to choose m_0 and m_1 smaller than the minimum initial distance between two different components or between two vertices.

The constant $C = 5328 D_{\max}^3 / m_0^3$ will play an important role in Lemma 18 and subsequently.

A.4 Combining components into clusters

In the proof we distinguish between components that are "close" to each other or even entangled within each other and components that are separated. Namely, we call two nonempty sets of points A and B *well-separated* if they can be separated by a line, and moreover, the two inner common tangents separating A and B form an angle less than 60°. See Figure 10(a).

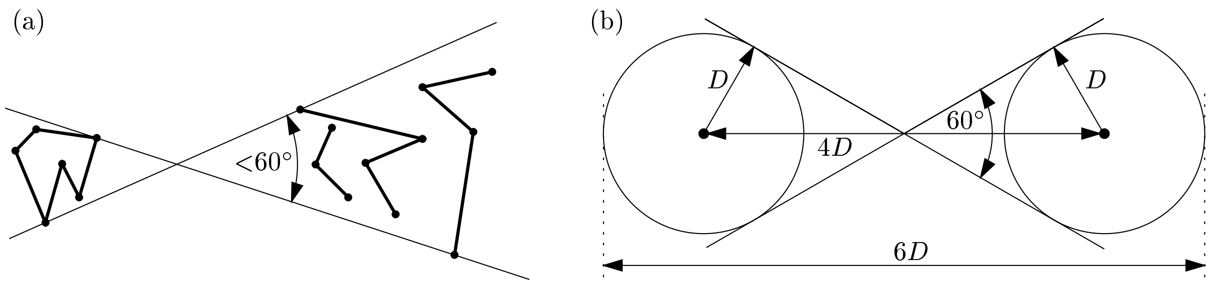

Figure 10: (a) Two well-separated clusters. (b) Inductive proof of Lemma 15. $D = D_0 6^{k-2}$ denotes the bound on the diameter of the two subclusters.

We now define a clustering procedure that will combine the components into clusters. (This procedure is similar to bottom-up hierarchical clustering [Eve93, HJ97], except that we pay no attention to the order in which we merge clusters.) We start with each component as a separate cluster. Whenever two clusters are not well-separated, we merge them into one cluster. We repeat this until we end up with a single cluster or with a set of clusters that are (pairwise) well-separated. The order in which we merge clusters has no effect on the outcome. However, the clustering may change over time as the framework moves.

Lemma 15 *A cluster with k components has diameter at most $D_0 6^{k-1}$.*

Proof: By induction on k. Because a single component is connected, the case $k = 1$ follows. A cluster with $k \geq 2$ components results from merging two clusters of at most $k-1$

components that are not well-separated. By induction, these two clusters are contained in two disks of radius at most $D_0 6^{k-2}$. If the distance between the disk centers were bigger than $4 \cdot D_0 6^{k-2}$, the two sets would be well-separated; see Figure 10(b). Hence the union of the two circles, which contains the cluster, has diameter at most $(2+4) \cdot D_0 6^{k-2} = D_0 6^{k-1}$. $\square$

So the quantity $D_{\max} := D_0 6^{n-1}$, which has already been introduced, is indeed an upper bound on the maximum possible distance between two points in the same cluster.

Lemma 16 *A cluster with at least two components has width at least* $m_0/2$.

Proof: Assume for contradiction that a cluster with at least two components lies in a strip of width less than $m_0/2$, which we can without loss of generality assume to be horizontal, as in Figure 11. Consider any two components in the cluster. By the definition of m_0, the distance between these two components is at least $m_0 > m_0/2$. Hence, there must be a vertical line separating the two components. Consider one of the two inner common tangents. The segment between the tangent points has length at least m_0 and lies in the strip. Therefore the angle between the tangent and the horizontal axis is less than 30°, and the same is true for the other tangent. It follows that any two components in the strip are well-separated, and therefore they do not belong to the same cluster. $\square$

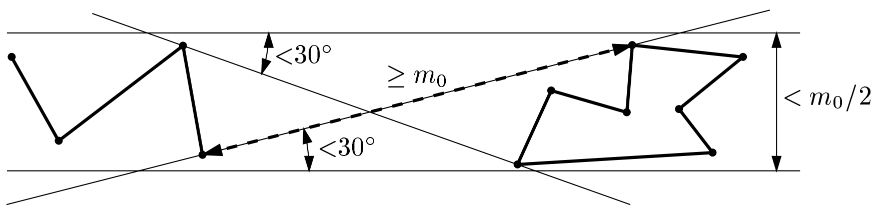

Figure 11: Proof of Lemma 16: two components in a thin strip.

Lemma 17 *Let* $\mathbf{p}_1 \in V_1$ *and* $\mathbf{p}_2 \in V_2$ *be points in two well-separated clusters* V_1 *and* V_2, *and let* $\bar{\mathbf{c}}^{(1)}$ *and* $\bar{\mathbf{c}}^{(2)}$ *denote the center of gravity of the vertices in* V_1 *and* V_2, *respectively. Then the angle between the vectors* $\mathbf{p}_2 - \mathbf{p}_1$ *and* $\bar{\mathbf{c}}^{(2)} - \bar{\mathbf{c}}^{(1)}$ *is at most* 60°. $\square$

A.5 Rounding up the proof

Using Lemmas 13 and 14, we can bound all velocities $\mathbf{v}_i$ in terms of the sum of ℓ'_{ij} over all pairs $(\mathbf{p}_i, \mathbf{p}_j)$ in the *same* cluster (plus some θ'-terms to accommodate single-component clusters). These quantities cannot grow indefinitely: we have an upper bound on the diameter of a cluster.

Lemma 18 *Let* V_1 *denote the set of vertices of a cluster and* $T_1 \subseteq V_1$ *the set of degree-2 vertices, with angles* θ_j *for* $j \in T_1, 0 < \theta_j < \pi$. *If the motion is pinned as in Lemma 13 (for* *the case of a single-component cluster) or Lemma 14, (for a multi-component cluster), then, for every $k \in V_1$,*

$$\|\mathbf{v}_k\| \le C \cdot \sum_{i,j \in V_1} \ell'_{ij} + D_0 \cdot \sum_{i \in T_1} \theta'_i$$

where $C = 5328 D_{\max}^3 / m_0^3$.

Proof: The first term handles a cluster with at least two components by Lemma 14. We have width $w \ge m_0/2$ by Lemma 16 and diameter $D \le D_{\max}$ by Lemma 15. The second term accounts for a single-component cluster and follows from Lemma 13. □

Everything so far has been generic to expansive motions. Now we will use the precise form of (7–9).

Lemma 19 *Let $D(t) = \max\{1, \max_{i,j \in V} \ell_{ij}(t)\}$ denote the diameter at time t raised to be at least one. By $\bar{\ell}_{ij}(t) := \min\{\ell_{ij}(t), D_{\max}\}$ we denote the "truncated" distances. $T \subseteq V$ denotes the set of degree-two vertices. Then the solution $\mathbf{v}_i$ of the minimization problem* (7–9) *is bounded by*

$$\|\mathbf{v}_k\| \le \frac{6n}{m_0 m_1} \left(C \sum_{i,j \in V} \bar{\ell}'_{ij} + D_0 \sum_{i \in T} \theta'_i + 2 \right) \cdot D(t)$$

Proof: The strategy of the proof is as follows:

1. Start with $\mathbf{v}_i$ given by the optimization problem (7–9).

2. In each cluster separately, "normalize" ("pin") $\mathbf{v}_i$ as indicated in Lemma 18, yielding "well-bounded" $\mathbf{v}_i$'s for each cluster.

3. Combine these pinned motions into an overall expansive motion by adding a vector to each cluster that ensures that the clusters will fly away from each other. We have control over those vectors, and we guarantee that the overall motion is still well-bounded (bounded in terms of the sum of ℓ'_{ij} over all pairs i, j in the same cluster.)

4. Thus we have shown the existence of a well-bounded solution. Using the minimizing property of the optimization problem we conclude that its solution is also well-bounded.

Here are the details. Assume that there are K clusters with vertex sets $V_1, \ldots, V_K$. We first consider the motion $\hat{\mathbf{v}}^{(r)}$ that is obtained from $\mathbf{v}$ by restricting it to a cluster V_r and pinning it as in Lemma 13 or 14. We know that

$$\|\hat{\mathbf{v}}_k^{(r)}\| \le C \cdot \sum_{i,j \in V_r} \bar{\ell}'_{ij} + D_0 \cdot \sum_{i \in T_r} \theta'_i =: H^{(r)} \tag{25}$$

for all $k \in V_r$.

By construction, all distance increases $\|\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(r)}\|$ satisfy the conditions (8–9) for i, j in the same cluster V_r. We will now add a vector to the motion of each cluster in order to ensure that the expansion (8) also holds for points in different clusters. Intuitively, we want the clusters to "fly apart," but in a controlled manner.

Let $\bar{\mathbf{c}}^{(r)}$ denote the centroid of cluster r. We set

$$\bar{\mathbf{v}}_k := 2 \cdot \hat{\mathbf{v}}_k^{(r)} + E \cdot (\bar{\mathbf{c}}^{(r)} - \mathbf{p}_1) \tag{26}$$

for $k \in V_r$, with an arbitrarily chosen fixed vertex $\mathbf{p}_1$ and with the abbreviation

$$E := 2(H^{(1)} + H^{(2)} + \cdots + H^{(K)} + 2)/m_0 > 4.$$

Note that $m_0 E/2$ is the value in parentheses in the bound of Lemma 19 that we want to prove. For vertices in the same cluster, we get $\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j = 2(\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(r)})$, and hence the constraint

$$(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq \|\mathbf{p}_i - \mathbf{p}_j\| \tag{8}$$

is "twice over-fulfilled:"

$$(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq 2 \cdot \|\mathbf{p}_i - \mathbf{p}_j\|$$

So we get the penalty term

$$\left[(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_i - \mathbf{p}_j\|\right]^{-1} \leq \left[\|\mathbf{p}_i - \mathbf{p}_j\|\right]^{-1} \leq 1/m_1. \tag{27}$$

We claim that this inequality also holds for points i and j in two different clusters V_r and V_s. We have

$$\begin{aligned}
(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) &= (\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(s)}) \cdot (\mathbf{p}_i - \mathbf{p}_j) + E\left(\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}\right) \cdot (\mathbf{p}_i - \mathbf{p}_j) \\
&\geq \left|E\left(\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}\right) \cdot (\mathbf{p}_i - \mathbf{p}_j)\right| - \left|\left(\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(s)}\right) \cdot (\mathbf{p}_i - \mathbf{p}_j)\right| \\
&\geq E \cdot \underbrace{\frac{1}{2}\|\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}\| \cdot \|\mathbf{p}_i - \mathbf{p}_j\|}_{\text{by Lemma 17}} - (H^{(r)} + H^{(s)}) \cdot \|\mathbf{p}_i - \mathbf{p}_j\| \\
&\geq \|\mathbf{p}_i - \mathbf{p}_j\| \left[E m_0/2 - (H^{(r)} + H^{(s)})\right] \\
&\geq 2 \cdot \|\mathbf{p}_i - \mathbf{p}_j\|
\end{aligned}$$

by the definition of E. So $\bar{\mathbf{v}}$ is indeed a feasible solution to (8–9), and (27) holds for all struts $\{i,j\} \in S$. From (25) and $\|\bar{\mathbf{c}}^{(r)} - \mathbf{p}_1\| \leq D(t)$ we obtain the bound

$$\|\bar{\mathbf{v}}_k\| \leq 2H^{(r)} + E \cdot D(t) \leq E m_0 + E \cdot D(t) \leq 2E \cdot D(t)$$

because we defined $D(t)$ to be at least 1.

From this and from (27), the objective function (7) for the solution $\bar{\mathbf{v}}$ that we have constructed satisfies

$$\sum_{i \in V} \|\bar{\mathbf{v}}_i\|^2 + \sum_{\{i,j\} \in S} \left[(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|\right]^{-1} \leq n \cdot (2E \cdot D(t))^2 + n^2/m_1$$

$$\leq n^2 \cdot (2E \cdot D(t))^2 \cdot \frac{1}{m_1} + \frac{n^2}{m_1} \cdot E^2 \cdot D(t)^2 = \frac{5n^2}{m_1} \cdot E^2 \cdot D(t)^2 \leq \left(\frac{3n}{m_1} \cdot E \cdot D(t)\right)^2. \tag{28}$$

It follows that the optimal solution $\mathbf{v}$, from which we started, must also satisfy this inequality. Therefore we obtain,

$$\sum_{i\in V}\|\mathbf{v}_i\|^2 \le \sum_{i\in V}\|\mathbf{v}_i\|^2 + \sum_{\{i,j\}\in S}\left[(\mathbf{v}_i-\mathbf{v}_j)\cdot(\mathbf{p}_i-\mathbf{p}_j)-\|\mathbf{p}_j-\mathbf{p}_i\|\right]^{-1} \le \left(\frac{3n}{m_1}\cdot E\cdot D(t)\right)^2.$$

and hence, for every $k\in V$,

$$\|\mathbf{v}_k\| \le \sqrt{\sum\|\mathbf{v}_i\|^2} \le \frac{3n}{m_1}\cdot E\cdot D(t) \qquad \square$$

Corollary 3 *$D(t)\le D(0)e^{F(t)}$ where $F(t)=\frac{12n}{m_0m_1}(Cn^2D_{\max}+D_0n\pi+2t)$.*

Proof: By noting the relation $D'(t)\le 2\max_{k\in V}\|\mathbf{v}_k\|$, we can apply the previous lemma to obtain an upper bound on $D'(t)/D(t)=\frac{d}{dt}\log D(t)$:

$$\frac{D'(t)}{D(t)} \le \frac{6nE}{m_1} \le \frac{12n}{m_0m_1}\left(C\sum_{i,j\in V}\bar\ell'_{ij}+D_0\sum_{i\in T}\theta'_i+2\right).$$

Integrating and noting the bounds $\bar\ell_{ij}\le D_{\max}$ and $\theta_i\le\pi$, we obtain

$$\log D(t)-\log D(0)=\int_0^t\frac{D'(t)}{D(t)}dt \le \frac{12n}{m_0m_1}\left(Cn^2D_{\max}+D_0n\pi+2t\right). \qquad \square$$

We finally establish the result of the Boundedness Lemma (Lemma 9).

Corollary 4 *The path length of a vertex $\mathbf{p}_k$ until time T is bounded by*

$$\int_{t=0}^{T}\|\mathbf{v}_k\|\,dt \le \frac{D(0)F(T)}{2}e^{F(T)}$$

Proof: This follows from Lemma 19 and the previous corollary:

$$\int_0^T\|\mathbf{v}_k(t)\|\,dt \le \int_0^T\frac{3nE}{m_1}D(t)\,dt \le \int_0^T\frac{3nE}{m_1}dt\cdot D(T) \le \frac{F(T)}{2}\cdot D(0)\cdot e^{F(T)} \qquad \square$$

Remark: By a different choice of shifting vectors in (26) instead of $E\cdot\bar{\mathbf{c}}^{(r)}$ for each cluster it is possible to avoid the exponential "blow-up" in the above bounds.

A.6 Proof of Lemma 14

Unfortunately, the proof of Lemma 14 is quite elaborate. It is based on the special case of three points, which is elementary.

Lemma 20 *Consider a triangle with sides a,b,c bounded by $m\le a,b,c\le D$ and angles α,β,γ that are bounded by $\sin\alpha,\sin\beta,\sin\gamma\ge\mu>0$. If the sides of the triangle are expanded with velocities $a',b',c'\ge 0$, then the change of the angles is bounded by*

$$|\alpha'|,|\beta'|,|\gamma'| \le \frac{2D}{m^2\mu}\cdot(a'+b'+c')$$

Proof: The cosine law gives $-\cos\alpha = -(b^2+c^2-a^2)/(2bc) = \frac{1}{2}[a^2/(bc) - b/c - c/b]$. Taking the derivative and omitting negative terms on the right-hand side leads to

$$\alpha' \sin\alpha \le \frac{1}{2}\left(\frac{2aa'}{bc} + \frac{bc'}{c^2} + \frac{cb'}{b^2}\right) \le \frac{1}{2}\left(2a' + c' + b'\right)\frac{D}{m^2} \le \left(a' + c' + b'\right)\frac{D}{m^2}$$

and thus, $\alpha' \le (a'+b'+c')\cdot D/(m^2\mu)$. We get the same upper bound for β' and γ'. To obtain a lower bound, note that $\alpha+\beta+\gamma = \pi$ and hence $\alpha' = -\beta' - \gamma' \ge -(a'+b'+c')\cdot 2D/(m^2\mu)$. By symmetry, the same bound holds for the derivatives of the other two angles. $\square$

Proof (Lemma 14): Let $\mathbf{p}_3$ be the point whose distance h from the segment $\mathbf{p}_1\mathbf{p}_2$ is largest. By assumption, $w/2 \le h \le \sqrt{3/4}\cdot D$. So the points are contained in a rectangle K of length D and width $2h$, see Figure 12. Without loss of generality, we draw $\mathbf{p}_1\mathbf{p}_2$ horizontal from left to right and $\mathbf{p}_3$ above $\mathbf{p}_1\mathbf{p}_2$.

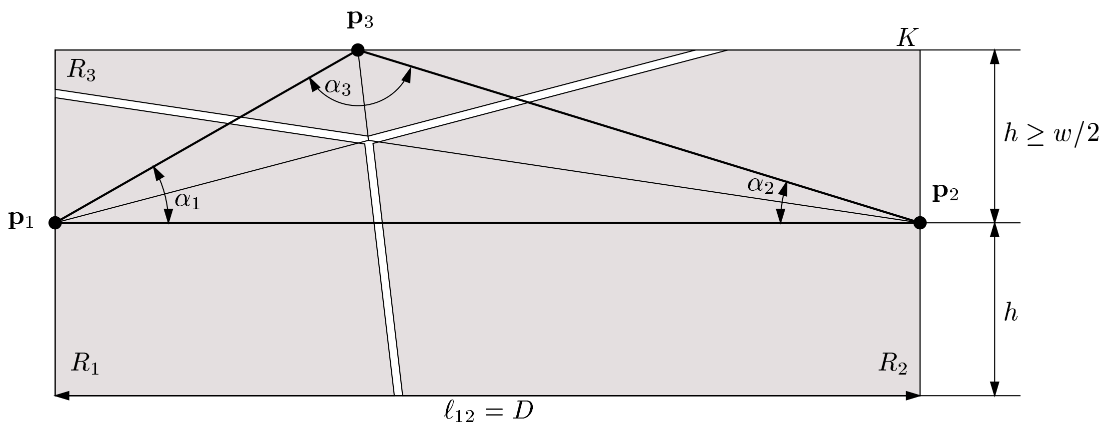

Figure 12: Proof of Lemma 14.

We use the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ with angles $\alpha_1, \alpha_2, \alpha_3$ as a "reference frame," and we measure the motion of the other points with respect to this frame. In particular, every point will be assigned to one of the sides of the reference triangle: We partition the rectangle K into three regions R_1, R_2, R_3 by extending the three angular bisectors from the incenter into the direction away from the vertices from which they emanate. R_i is the region containing the point $\mathbf{p}_i$, for $i = 1, 2, 3$. Now, every point $\mathbf{p}_j$ is assigned to the triangle side which lies *opposite* to the region in which it lies. For example, a point $\mathbf{p}_j$ in region R_1 is assigned to $\mathbf{p}_2\mathbf{p}_3$. We use this side as a *basis* of the triangle $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_j$ that is associated to $\mathbf{p}_j$, and we select one of the angles at the basis as the *reference angle* β_j for $\mathbf{p}_j$. If $\mathbf{p}_j$ lies in R_1, we select the angle at $\mathbf{p}_2$ in the triangle $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_j$. If $\mathbf{p}_j$ lies in R_2, we take the angle at $\mathbf{p}_1$ in the triangle $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ as the reference angle β_j for $\mathbf{p}_j$, see Figure 13; and for $\mathbf{p}_j$ in R_3, β_j is the angle at $\mathbf{p}_1$ in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_j$. Note that these assignments are not completely symmetric. The rationale is to select a reference angle incident to $\mathbf{p}_1$ or $\mathbf{p}_2$.

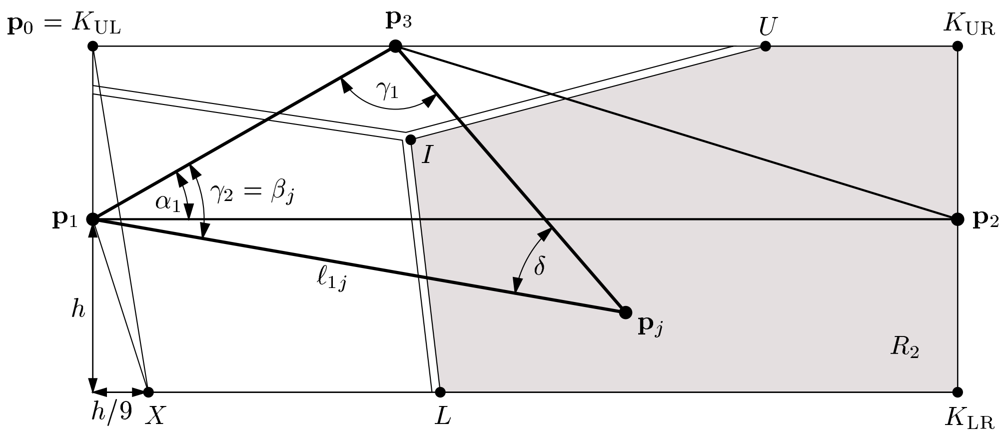

Figure 13: The triangle $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ associated to a point $\mathbf{p}_j \in R_2$ in the proof of Lemma 21.

The idea is that the triangle associated to each point cannot have arbitrarily small angles, and hence is produces a "firm grip," tying the point to its base: The point cannot move an unlimited distance without increasing the pairwise distances ℓ_{ij} by a noticeable amount.

Lemma 21 *The reference triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ satisfies the assumptions of Lemma 20: The sines of the angles are at least $\mu := w/(3D)$ and the sides are at least $m := w/2$.*

The triangles associated to each point satisfies the assumptions of Lemma 20: The sines of the angles are at least $\mu := w/(20D)$ and the sides are at least $m := w/4$.

This lemma is not difficult to see but the precise proof is a bit technical and it is given at the end. Lemmas 20 and 21 allow us to conclude:

$$|\alpha_1'|, |\alpha_3'|, |\alpha_3'| \le \frac{2D}{(w/2)^2 \cdot w/(3D)} \cdot (\ell_{12}' + \ell_{23}' + \ell_{13}') \le \frac{24D^2}{w^3} \cdot L,$$

with $L := \sum_{i<j} \ell_{ij}'$, and for every point $\mathbf{p}_j$:

$$|\beta_j'| \le \frac{2D}{(w/4)^2 \cdot w/(20D)} \cdot L \le \frac{640D^2}{w^3} \cdot L.$$

Now we can bound the motion of $\mathbf{p}_j$ as follows. For $\mathbf{p}_j \in R_3$, the angle β_j between the fixed ray $\mathbf{p}_1\mathbf{p}_2$ and the ray $\mathbf{p}_1\mathbf{p}_j$, and the length ℓ_{1j} determine the position of $\mathbf{p}_j$ like polar coordinates. We therefore have

$$\|\mathbf{p}_j'\| = \sqrt{(\ell_{1j}\beta_j')^2 + (\ell_{1j}')^2} \le \ell_{1j}|\beta_j'| + \ell_{1j}'$$

For $\mathbf{p}_j \in R_2$, the situation is similar, except that the angle between the "x-axis" $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_1\mathbf{p}_j$ is given by $\alpha_1 - \beta_j$, see Figure 13:

$$\|\mathbf{p}_j'\| \le \ell_{1j}(|\alpha_1'| + |\beta_j'|) + \ell_{1j}'$$

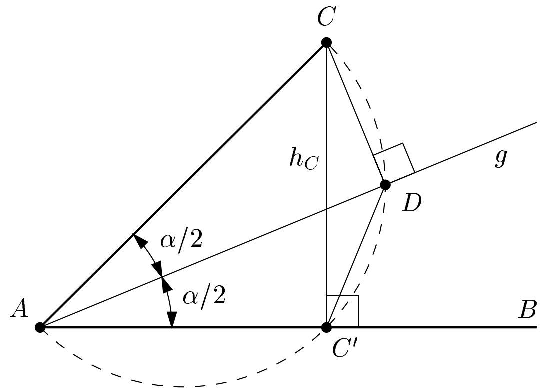

Figure 14: Proof of Lemma 22

For $\mathbf{p}_j \in R_1$, the situation still more complicated, because the "origin" $\mathbf{p}_2$ of the coordinate system moves at a speed of ℓ'_{12}. The angle between the "negative x-axis" $\mathbf{p}_2\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_j$ is given by $\alpha_2 - \beta_j$.

$$\|\mathbf{p}'_j\| \le \|\mathbf{p}'_2\| + \|\mathbf{p}'_l - \mathbf{p}'_2\| \le \ell'_{12} + \ell_{2j}(|\alpha'_2| + |\beta'_j|) + \ell'_{2j}$$

Putting everything together, we get for every j,

$$\|\mathbf{v}_j\| = \|\mathbf{p}'_j\| \le L + D \cdot \left(\frac{24D^2}{w^3} \cdot L + \frac{640D^2}{w^3} \cdot L\right) + L = \left(2 + \frac{664D^3}{w^3}\right) \cdot L \le \frac{666D^3}{w^3} \cdot L,$$

using $D \ge w$. This concludes the proof of Lemma 14. $\square$

Finally, we have to prove Lemma 21. An elementary geometric fact is needed.

Lemma 22 *In a triangle ABC with height h_C through C, the distance of C from the angular bisector g of the angle α at A is at least $h_C/2$.*

Proof: Let C' be the foot of h_C, and let D be the point on g closest to C, see Figure 14. Then C' and D lie on the circle over the diameter AC. The chords CD and $C'D$ are equal since their peripheral angles at A are both equal to $\alpha/2$. (If $\alpha > \pi/2$, the peripheral angle over $C'D$ is not $\alpha/2$ but $\pi - \alpha/2$, and for $\alpha = \pi/2$ the equality of the chords follows directly.) From the isosceles triangle $CC'D$ we get $CD \ge h_C/2$. $\square$

Proof (Lemma 21): We first prove the lower bound on the lengths of the edges. The basis edges $\mathbf{p}_1\mathbf{p}_2$, $\mathbf{p}_2\mathbf{p}_3$, and $\mathbf{p}_3\mathbf{p}_1$ all have lengths at least $h \ge w/2$. By the construction of the three regions, the point $\mathbf{p}_j$ is separated from the point $\mathbf{p}_i$ on its base segment ($i = 1, 2,$ or 3) by the angular bisector of the angle at $\mathbf{p}_i$ in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. From Lemma 22 it follows that the distance $\mathbf{p}_j\mathbf{p}_i$ is at least $1/2$ times one of the three heights of the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. The smallest height of this triangle is opposite to the longest edge, which is $\mathbf{p}_1\mathbf{p}_2$. This height is h, and we obtain $\ell_{ij} \ge h/2 \ge w/4$.

Now let us consider the angles. Note that it suffices to check the condition of Lemma 20 for the smallest angle: if α is the smallest angle in a triangle with angles α, β, γ, then $\alpha \le \beta, \gamma \le \pi - \alpha$, and hence $\sin \beta, \sin \gamma \ge \sin \alpha$.

The angle α_1 in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is bigger than the angle between $\mathbf{p}_1\mathbf{p}_2$ and the ray from $\mathbf{p}_1$ to the upper right corner K_{UR} of the rectangle K:

$$\sin\alpha_1 \geq \frac{h}{\sqrt{D^2+h^2}} \geq \frac{h}{\sqrt{2D^2}} \geq \frac{w/2}{\sqrt{2}\cdot D} \geq \frac{w}{3D}$$

and similarly for α_2. The angle α_3 lies opposite to the longest edge of $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$, and hence it is the largest angle. So we have shown the length and angle bounds for the reference triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$.

Let us now consider the triangle associated with a point $\mathbf{p}_j$. We denote the basis angles of the triangle by γ_1 and γ_2 and the angle at the apex $\mathbf{p}_j$ by δ, see Figure 13. By construction of the angular bisectors, an angle γ_j at the base is at least $1/2$ times the respective angle α_i in the reference triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. So we have $\sin\gamma_1, \sin\gamma_2 \geq \sin\frac{\alpha_i}{2} \geq (\sin\alpha_i)/2 \geq w/(6D)$.

So the only angle which remains to be checked is the third angle δ at $\mathbf{p}_j$. If $\mathbf{p}_j$ lies in R_3, δ lies opposite to the longest side of the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_j$, and we are done.

In the remainder of the proof we show rather tediously that $\sin\delta \geq w/(20D)$ when $\mathbf{p}_j \in R_1$ or $\mathbf{p}_j \in R_2$. Let us consider a triangle $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ with $\mathbf{p}_j \in R_2$, see Figure 13. (The case $\mathbf{p}_j \in R_1$ is symmetric.) Keeping $\mathbf{p}_j$ and $\mathbf{p}_1$ fixed, δ is smallest when $\mathbf{p}_3$ moves on the upper side of the rectangle K to the upper left corner $\mathbf{p}_0 = K_{\mathrm{UL}}$. (This position of $\mathbf{p}_3$ is hypothetical because ℓ_{23} would be larger than D.) Keeping $\mathbf{p}_1$ and $\mathbf{p}_0$ fixed, the angle $\delta = \angle\mathbf{p}_1\mathbf{p}\mathbf{p}_0$ is a quasi-concave function of $\mathbf{p} = \mathbf{p}_j$ as $\mathbf{p}$ moves inside K. (This means that the region in K where δ is bigger than some given threshold is alway convex.) Hence the minimum value of δ on any line segment occurs at one of its endpoints, and it suffices to check the lower bound for δ for the corners of the region R_2. More specifically, it is sufficient to check it for three points: (i) $\mathbf{p} = K_{\mathrm{UR}}$, the upper right corner of the rectangle K; (ii) $\mathbf{p} = K_{\mathrm{LR}}$, the lower right corner of K; and (iii) $\mathbf{p} = X$, a point on the lower edge of K at distance $h/9$ from the lower left corner.

The other corners of R_2 can be treated as follows: If $\mathbf{p}$ is the intersection U of the bisector of α_1 with the upper edge of K, then δ decreases as $\mathbf{p}$ moves right towards K_{UR}. If the bisector of α_1 intersects the boundary of K in the right edge, then the intersection point lies on the segment between K_{UR} and K_{LR}, and hence it is dominated by one of these points. (This situation occurs for region R_1 in Figure 13.) The incenter I can be treated in the same way as U, by moving $\mathbf{p}$ horizontally to the right until it hits the right edge.

Another corner is the intersection L of the bisector of α_3 with the lower edge of K. We cover this case by showing that the point X lies in region R_1. Therefore L lies on the segment between X and K_{LR}, and hence it is dominated by these two points. If the bisector of α_3 intersects the boundary of K in the right edge, it is dominated by K_{UR} and K_{LR}. The bisector cannot intersect the boundary of K in the left edge because of the point X.

Let us now consider the points $\mathbf{p} = K_{\mathrm{UR}}$ and $\mathbf{p} = K_{\mathrm{LR}}$. By symmetry, $\mathbf{p} = K_{\mathrm{UR}}$ yields the same angle $\delta = \angle\mathbf{p}_0\mathbf{p}\mathbf{p}_1$ as $\mathbf{p} = \mathbf{p}_2$ and that cannot be the minimum because it lies between K_{UR} and K_{LR}. For $\mathbf{p} = K_{\mathrm{LR}}$, the sine law for the triangle $\mathbf{p}_0\mathbf{p}\mathbf{p}_1$ with the angle $\delta = \angle\mathbf{p}_0\mathbf{p}\mathbf{p}_1$ gives

$$\frac{\sin\delta}{h} = \frac{\sin\angle\mathbf{p}_0\mathbf{p}_1\mathbf{p}}{\mathbf{p}_0\mathbf{p}} = \frac{D/\sqrt{D^2+h^2}}{\sqrt{D^2+(2h)^2}} \geq \frac{D}{\sqrt{2D^2}\cdot\sqrt{5D^2}},$$

using $h \le D$. So we get

$$\sin\delta \ge \frac{1}{\sqrt{10}} \cdot \frac{h}{D} \ge \frac{1}{2\sqrt{10}} \cdot \frac{w}{D},$$

and the lemma holds in this case.

Let us finally consider the point X. Since $h \le \sqrt{3/4} \cdot D$, the point X lies on the lower edge of K. We show that the distance of X from the ray $\mathbf{p}_3\mathbf{p}_1$ is smaller than the distance from the ray $\mathbf{p}_3\mathbf{p}_2$, and therefore X lies in R_1. The distance of X from the ray $\mathbf{p}_3\mathbf{p}_1$ is at most the distance $X\mathbf{p}_1 = \sqrt{1 + 1/81} \cdot h < 1.0062\, h$. The point on the ray $\mathbf{p}_3\mathbf{p}_2$ which is closest to X lies either on the segment $\mathbf{p}_3\mathbf{p}_2$ or in the region to the right of the rectangle. In the latter case, the distance is at least $D - h/9 \ge (\sqrt{4/3} - 1/9)h > 1.043\, h$, which is larger than $X\mathbf{p}_1$. In the former case, the distance between X and the segment $\mathbf{p}_3\mathbf{p}_2$ decreases if $\mathbf{p}_3$ or $\mathbf{p}_2$ are moved horizontally to the left. In the extreme cases, when $\mathbf{p}_3 = K_{\mathrm{UL}} = \mathbf{p}_0$ and $\ell_{12} = D = \sqrt{4/3} \cdot h$, this distance can be calculated as $> 1.439\, h$, which is also larger than $X\mathbf{p}_1$. A final calculation show that the angle $\delta = \angle \mathbf{p}_0 X \mathbf{p}_1$ satisfies $\sin\delta > 0.055 > 1/20 \ge \frac{w}{20D}$, since $w \le D$, and the proof is complete. $\square$