Problem 1

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$SL_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

<u>Proof</u>

$$T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},} \quad n \in \mathbb{Z}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

WTS. A is a word in S and T.

Consider first the case $a=0$ or $c=0$.

We can just assume $c=0$, as otherwise we can premultiply by S and swap a and c:

$$SA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

So let $c=0$.

$$\det A = ad = 1 \Rightarrow \begin{array}{c} a=1 \;\&\; d=1 \\ \text{OR} \\ a=-1 \;\&\; d=-1 \end{array}$$

So $\underbrace{A = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = S^2 T^b}$

Now assume neither a nor c are zero. We can assume $|a| \geq |c|$, as otherwise we can premultiply by S and swap them.

We will describe a process of premultiplying by T, T^{-1}, S or S^{-1} that reduces $|a|+|c|$ by at least one provided neither a nor c are 0.

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If we keep repeating this process, it must end as $|a|+|c| \geq 0$. So eventually either $a=0$ or $c=0$ and the problem reduces to the case we considered earlier.

We have $|a| \geq |c|$. By the division algorithm, $\exists q, r \in \mathbb{Z}$ with $|r|<|c|$ s.t.

$$a = qc + r$$

$$T^{-q}A = \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a-cq & b-qd \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} r & b-qd \\ c & d \end{pmatrix}$$

$$|r|+|c| < |c|+|c| \leq |a|+|c|$$

so we strictly reduced the sum of the absolute values of the upper left and lower left entries.

$\square$