A mirror symmetry conjecture: *The fundamental group of the Stringy Kähler Moduli Space acts on $D^b(X)$ via spherical twists*

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The B-side: Toric Geometric Invariant Theory

We start with algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $\mathbb{C}^n$. These actions the actions are simple to write out:

$$(\lambda_1, \ldots, \lambda_r) \cdot (z_1, \ldots, z_n) = (\lambda_1{}^{q_{11}} \lambda_2{}^{q_{12}} \ldots \lambda_r{}^{q_{1r}} z_1, \ldots, \lambda_1{}^{q_{n1}} \lambda_2{}^{q_{n2}} \ldots \lambda_r{}^{q_{nr}} z_1)$$

We get an $n \times r$ integer matrix $Q = (q_{ij})$, called the weight matrix.
We construct an algebraic variety that parametrises the actions quotients. **Geometric Invariant Theory** (GIT) is the theory that tells **the unstable locus** to throw away before we quotient so that we get a good/geometric quotient [7]. GIT quotients are often denoted $X//G$.

Example 1 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ linearly, i.e. $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. If we take the quotient space, we see that the orbit of the origin cannot be separated from any other orbit. So we have to remove the origin and as expected, we get GIT quotient*

$$\mathbb{C}^2 // \mathbb{C}^* = \mathbb{C}^2 \; (0, 0)/\mathbb{C}^* = \mathbb{P}^1$$

In general GIT quotients are not unique and depend on a choice of **stability condition** $\phi \in \mathbb{Z}^r$, where for a given stability condition we denote the GIT quotient $X//_\phi G$. Different choices of ϕ have us remove different unstable loci and give us non-isomorphic (but birational) quotients.

Example 2 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^3$ via $\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$. The stability conditions space is $\mathbb{Z}$. For $\phi > 0$, we have unstable locus $Z_+ = \{x = y = 0\}$. We have GIT quotient*

$$\mathbb{C}^3 //_\phi \mathbb{C}^* = \mathbb{C}^3 \backslash \{x = y = 0\}/\mathbb{C}^* = \mathcal{O}(-1)_{\mathbb{P}^1_{xy}}.$$

For $\phi < 0$, we have unstable locus $Z_- = \{z = 0\}$.

$$\mathbb{C}^3 //_\phi \mathbb{C}^* = \mathbb{C}^3 \backslash \{z = 0\}/\mathbb{C}^* = \mathbb{A}^1_{x,y}.$$

$\mathbb{A}^1_{x,y}$ $\quad$ $\mathcal{O}(-1)_{\mathbb{P}^1_{xy}}$

-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8

Figure 1: Secondary fan of GIT problem $\mathbb{C}^3_{(1,1,-1)}$

Mirror Symmetry Conjecture Heuristics

Mirror symmetry is a series of mysterious relationships between complex and symplectic geometry. Its most basic formulation is that given a Kähler manifold X, there exists a mirror Kähler manifold $\hat{X}$ such that we have equivalence of categories

$$D^b(X) \cong \mathsf{Fuk}(\hat{X})$$

Actually, we have a whole family of mirrors, each of whom is symplectomorphic to $\hat{X}$, but has a different complex structure. We call the **Stringy Kähler moduli space** the complex structure moduli space $\mathcal{M}_{CS}$ of the symplectic manifold $\hat{X}$.

Proposition 1 *There is a monodromy action of $\pi_1(\mathcal{M}_{CS})$ on $\hat{X}$ via symplectomorphism, and hence there is a monodromy action of $\pi_1(\mathcal{M}_{CS})$ on Fuk$(\hat{X})$ via autoequivalence.*

By mirror symmetry, the action of $\pi_1(\mathcal{M}_{CS})$ carries to an action of $D^b(X)$ via autoequilvance. In the context of Calabi Yau toric GIT, the conjecture is that there is a particular way that $\pi_1(\mathcal{M}_{CS})$ acts, via spherical functors.

The program above has been done in certain cases e.g. in [6].

Calabi-Yau Toric GIT

Definition 1 *A GIT problem $(\mathbb{C}^*)^r$ acting on $\mathbb{C}^n$ is* **Calabi-Yau** *if the rows of the weight matrix Q add up to 0.*

If a GIT problem is Calabi-Yau, then all the GIT quotients are Calabi-Yau i.e. have trivial canonical bundle. In this context it also means that any two GIT quotients have equivalent derived categories [5], so that for any two stability conditions $\phi_1, \phi_2 \in \mathbb{Z}^r$:

$$D^b(\mathbb{C}^n //_{\phi_1} (\mathbb{C}^*)^r) \cong D^b(\mathbb{C}^n //_{\phi_2} (\mathbb{C}^*)^r)$$

Let $A = (a_{ij})$ be the cokernel of Q, so that we have short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n-r} \longrightarrow 0$$

A is $(n-r) \times n$ integer matrix, where we can choose the first row is 1's by the Calabi-Yau condition. The column vectors, called **rays**, which live in $\mathbb{Z}^{n-r}$ are therefore all at height 1, and span a $n - r - 1$ dimensional polytope $\mathcal{P}(A)$, called the **Primary Polytope**. Triangulations of $\mathcal{P}(A)$ give us the toric fans for our GIT quotients [2, Section 4].

The mirror family in this context is well understood [3, text]. The mirror family is a family of Landau Ginsburg Models $\left((C^*)^{n-r}, W_{\underline{\mathbf{a}}}\right)$ where $W_{\underline{\mathbf{a}}} : (\mathbb{C}^*)^{n-r} \to \mathbb{C}$:

$$W_{\underline{\mathbf{a}}}(X_1, \ldots, X_{n-r}) = \text{polynomial with coefficients } \underline{\mathbf{a}} \text{ in variables } X_1, \text{ where the powers are given by the entries of } A = (a_{ij}).$$

The mirror symmetry statement here says for any GIT quotient X:

$$D^b(X) \cong FS\left((C^*)^{n-r}, W_{\underline{\mathbf{a}}}\right)$$

where FS is for Fukaya-Seidel category [8]. Our coefficients parametrise our mirror family and live in $\underline{\mathbf{a}} \in ((\mathbb{C}^*)^n \backslash \Delta_A)/A$, where the hypersurface $\Delta_A \subset (\mathbb{C}^*)^n$ is called the A-**discriminant** which was constructed in [4] for any integer matrix A. In short, Δ_A generalises the discriminant of the quadratic to any polynomial with any number of variables. By our mirror symmetry heuristics, this means the Stringy Kähler Moduli space in this context is the parameter space

$$\mathcal{M}_{CS} = ((\mathbb{C}^*)^n \backslash \Delta_A)/A \cong (\mathbb{C}^*)^r \backslash \nabla_A$$

Example 3 *Suppose $A = (2\,1\,0)$. Then $W_{a,b,c}(X) = aX^2 + bX + c$. Then $\Delta_A = \{b^2 = 4ac\} \subset (\mathbb{C}^*)^3$, and using A to scale out b, we get $\left((\mathbb{C}^*)^3 \backslash \Delta_A\right)/A \cong \left((\mathbb{C}^*)^2\right) \backslash \{ac = 1/4\}$.*

The main observation now is that the discriminant locus is non-irreducible and splits up into components:

$$\Delta_A = \bigcup_{\Gamma \text{ min face}} \Delta_\Gamma$$

Each of these components corresponds to a minimal face of $\mathcal{P}(A)$, that is faces whose rays are linearly **dependent**. The minimal faces are in 1-1 correspondence with the 'sub GIT problems' with non-trivial minimal GIT quotients Z_Γ, where by minimal GIT quotient we mean a GIT quotient that cannot be decomposed further by the wall-crossing algorithm described in [5]. There are spherical functors $F_\Gamma : D^b(Z_\Gamma) \to D^b(X)$ [1], where by spherical we mean that the twist

$$T_{F_\Gamma} = C(\, FR \xrightarrow{\text{counit}} I_{D^b(X)}\,)$$

is an autoequivalence of $D^b(X)$, where $R_\Gamma : D^b(X) \to D^b(Z_\Gamma)$ is the right adjoint.

Meridians of the Δ_Γ in $\pi_1(\mathcal{M}_{CS})$ acts on $D^b(X)$ via the spherical twists T_{F_Γ}.

Rank 2 Example

Consider GIT problem

$$Q = \begin{array}{cc} \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array} \qquad A^T = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array}$$

We have

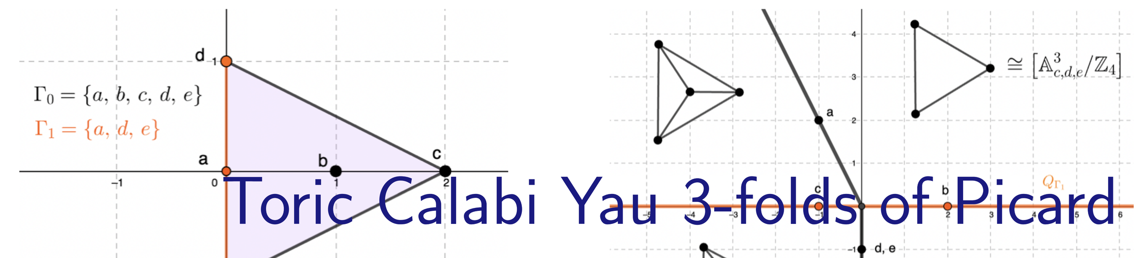
$$\mathcal{M}_{CS} \cong (\mathbb{C}^*)^2_{a,b} \backslash \left(\nabla_{\Gamma_0} \cup \nabla_{\Gamma_1}\right)$$

where we can compute

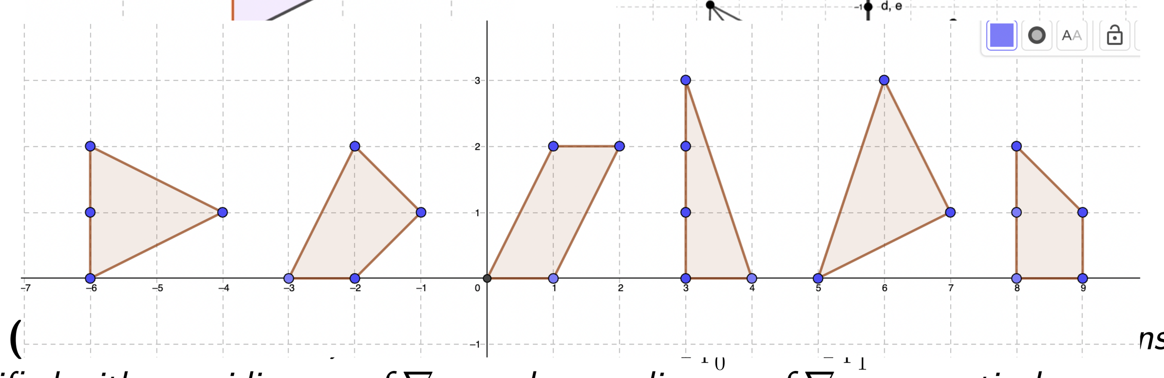
$$\nabla_{\Gamma_0} = \{(b^2 - a)^2 = 4\} \qquad \nabla_{\Gamma_1} = \{a^2 = 4\}.$$

The 'sub GIT problems' Q_{Γ_0} and Q_{Γ_1}, they have minimal GIT quotients $Z_{\Gamma_0} = \text{pt}$ and $Z_{\Gamma_1} = \mathbb{A}^1$, and spherical functors $F_{\Gamma_0} \mathcal{O}_{\mathsf{pt}} = \mathcal{O}_0 \in D^b([\mathbb{A}^3/\mathbb{Z}_4])$ and $F_{\Gamma_0} = i_* \pi^*$

$$\begin{array}{ccc} [\mathbb{A}^1_c/\mathbb{Z}_4] & \xrightarrow{i} & [\mathbb{A}^3_{c,d,e}/\mathbb{Z}_4] \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathbb{A}^1_c & & \end{array}$$



Toric Calabi Yau 3-folds of Picard Rank 2



Lemma 1 (... *ns in $\pi_1(\mathcal{M}_{CS})$ when identified witha meridian μ_0 of ∇_{Γ_0} and a merdian μ_1 of ∇_{Γ_1} respectively.*

Figure 3: All the primary polytopes of Toric Calabi Yau 3-folds of Picard Rank 2

References

[1] N. Addington. New derived symmetries of some hyperkähler varieties. *arXiv preprint arXiv:1112.0487*, 2011.

[2] T. Coates, H. Iritani, and Y. Jiang. The crepant transformation conjecture for toric complete intersections. *Advances in Mathematics*, 329:1002–1087, 2018.

[3] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*, volume 68. American Mathematical Society Providence, RI, 1999.

[4] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *A-discriminants*. Springer, 1994.

[5] D. Halpern-Leistner. The derived category of a git quotient. *Journal of the American Mathematical Society*, 28(3):871–912, 2015.

[6] D. Halpern-Leistner and S. Sam. Combinatorial constructions of derived equivalences. *Journal of the American Mathematical Society*, 33(3):735–773, 2020.

[7] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[8] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*, volume 10. European Mathematical Society, 2008.