A mirror symmetry conjecture: *The fundamental group of the Stringy Kähler Moduli Space acts on $D^b(X)$ via spherical twists*

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The B-side: Toric Geometric Invariant Theory

We start with algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $\mathbb{C}^n$. These actions the actions are simple to write out:

$$(\lambda_1, \ldots, \lambda_r) \cdot (z_1, \ldots, z_n) = (\lambda_1{}^{q_{11}}\lambda_2{}^{q_{12}} \ldots \lambda_r{}^{q_{1r}} z_1, \ldots, \lambda_1{}^{q_{n1}}\lambda_2{}^{q_{n2}} \ldots \lambda_r{}^{q_{nr}} z_1)$$

We get an $n \times r$ integer matrix $Q = (q_{ij})$, called the weight matrix.
We construct an algebraic variety that parametrises the actions quotients. *Geometric Invariant Theory* (GIT) is the theory that tells *the unstable locus* to throw away before we quotient so that we get a good/geometric quotient [1]. GIT quotients are often denoted $X//G$.

Example 1 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ linearly, i.e. $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. If we take the quotient space, we see that the orbit of the origin cannot be separated from any other orbit. So we have to remove the origin and as expected, we get GIT quotient*

$$\mathbb{C}^2//\mathbb{C}^* = \mathbb{C}^2\ (0, 0)/\mathbb{C}^* = \mathbb{P}^1$$

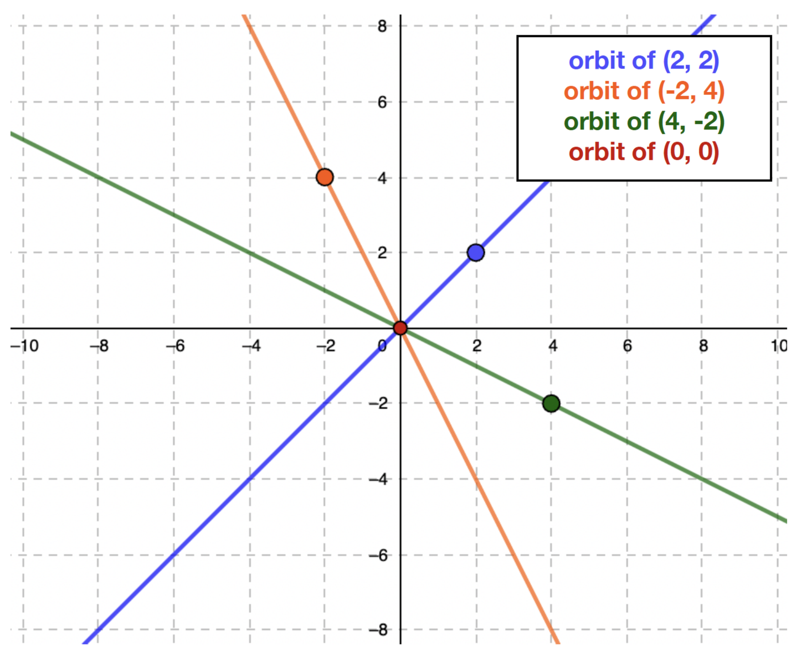


Figure 1: Orbits of the $\mathbb{C}^*$ action on $\mathbb{C}^2$.

In general GIT quotients are not unique and depend on a choice of *stability condition* $\phi \in \mathbb{Z}^r$, where for a given stability condition we denote the GIT quotient $X//_\phi G$. Different choices of ϕ have us remove different unstable loci and give us non-isomorphic (but birational) quotients.

Example 2 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^3$ via $\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$. The stability conditions space is $\mathbb{Z}$. For $\phi > 0$, we have unstable locus $Z_+ = \{x = y = 0\}$. We have GIT quotient*

$$\mathbb{C}^3//_\phi\mathbb{C}^* = \mathbb{C}^3\backslash\{x = y = 0\}/\mathbb{C}^* = \mathcal{O}(-1)_{\mathbb{P}^1_{xy}}.$$

For $\phi < 0$, we have unstable locus $Z_- = \{z = 0\}$.

$$\mathbb{C}^3//_\phi\mathbb{C}^* = \mathbb{C}^3\backslash\{z = 0\}/\mathbb{C}^* = \mathbb{A}^1_{x,y}.$$

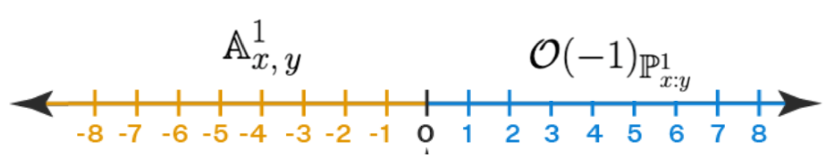


Figure 2: Secondary fan of GIT problem $\mathbb{C}^3_{(1,1,-1)}$

Mirror Symmetry Conjecture Heuristics

Mirror symmetry is a serious of mysterior relationships between complex and symplectic geometry. Its most basic formulation is that given a Kähler manifold X, there exists a mirror Kähler manifold $\hat{X}$ such that

$$D^b(X) \cong \mathrm{Fuk}(\hat{X})$$

Actually, we have a whole family of mirrors, each of whom is symplectomorphic to $\hat{X}$, but has a different complex structure. We call the Stringy Kähler moduli space the complex structure moduli space $\mathcal{M}_{CS}$ of the symplectic manifold $\hat{X}$.

Proposition 1 *There is a monodromy action of $\pi_1(\mathcal{M}_{CS})$ on $\hat{X}$ via symplectomorphism, and hence there is a monodromy action of $\pi_1(\mathcal{M}_{CS})$ on Fuk$(\hat{X})$ via autoequivalence.*

By mirror symmetry, you expect to be able to carry over the action $\pi_1(\mathcal{M}_{CS})$ to an action of $D^b(X)$ via autoequilvance.

Calabi-Yau Toric GIT

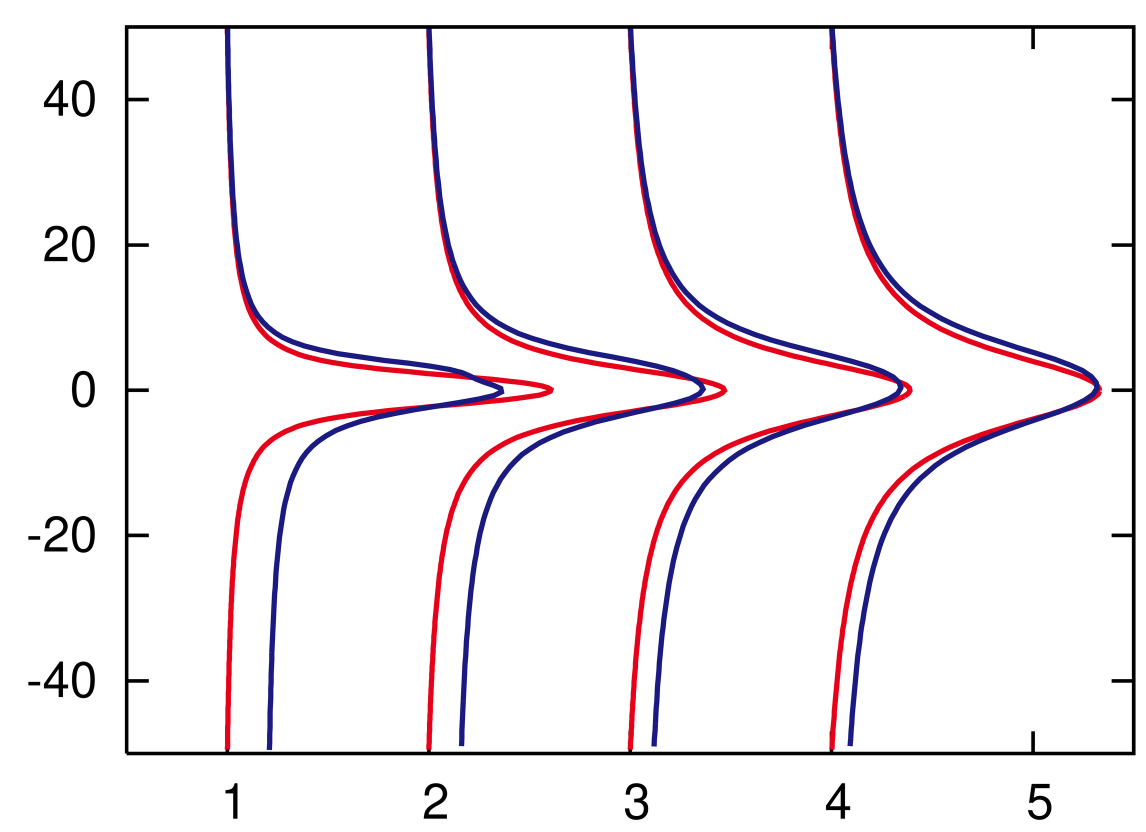
Of course, the principal reason for choosing LaTeX is its ability with equations:

$$U_{\text{doublet}} = \frac{mg}{2\pi\mu a}\left(\int_0^\infty \left\{1 - \frac{2\sinh^2 s - 2s^2}{\sinh 2s + 2s}\right\} \mathrm{d}s\right)^{-1} \approx \frac{1.55mg}{6\pi\mu a},$$

but it's equally easy to include tables:

	U_1		U_2		U_3		U_3
L	MR	SD	MR	SD	MR	SD	error
2.01	0.65528	0.64739	0.63461	0.62691	0.00498	0.00451	9%
2.10	0.73857	0.73126	0.59718	0.58784	0.03517	0.02570	27%
2.50	0.87765	0.87482	0.49545	0.48829	0.07393	0.05853	21%
3.00	0.93905	0.93806	0.41694	0.41356	0.07824	0.06970	11%
4.00	0.97964	0.97945	0.31859	0.31774	0.06925	0.06639	4%
6.00	0.99581	0.99579	0.21586	0.21575	0.05078	0.05019	1%

and, of course, images:



Sectioning

First Subsection

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Second Subsection

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With the `color` package you can use as much colour as you like: but the Red and DarkBlue colours defined here are more useful than the obvious red and blue versions, which tend to seem too bright when printed.

Use spacing as well as colour to highlight a key question or issue:

Who will rid me of this turbulent priest?

Praesent elementum malesuada mauris. Duis aliquet dolor ut nunc. Pellentesque euismod augue a odio.

References

[1] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.