A mirror symmetry conjecture: *The fundamental group of the Stringy Kähler Moduli Space acts on $D^b(X)$ via spherical twists*

Michela Barbieri

The B-side: Toric Geometric Invariant Theory

We start with algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $\mathbb{C}^n$:

$$(\lambda_1, \ldots, \lambda_r) \cdot (z_1, \ldots, z_n) = (\lambda_1{}^{q_{11}} \lambda_2{}^{q_{12}} \ldots \lambda_r{}^{q_{1r}} z_1, \ldots, \lambda_1{}^{q_{n1}} \lambda_2{}^{q_{n2}} \ldots \lambda_r{}^{q_{nr}} z_1)$$

We get an $n \times r$ integer matrix $Q = (q_{ij})$, called the weight matrix.

We want to construct an algebraic variety that parametrises the action's orbits. **Geometric Invariant Theory** (GIT) is the theory that tells **the unstable locus** to throw away before we quotient so that we get a good/geometric quotient [7]. GIT quotients are often denoted $X//G$.

Example 1 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ linearly, i.e. $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. If we take the quotient space, we see that the orbit of the origin cannot be separated from any other orbit. So we have to remove the origin and as expected, we get GIT quotient*

$$\mathbb{C}^2//\mathbb{C}^* = \mathbb{C}^2 \setminus (0,0)/\mathbb{C}^* = \mathbb{P}^1$$

In general GIT quotients are not unique and depend on a choice of **stability condition** $\phi \in \mathbb{Z}^r$, where for a given stability condition we denote the GIT quotient $X//_\phi G$. Different choices of ϕ have us remove different unstable loci and give us non-isomorphic (but birational) quotients.

Example 2 *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^3$ via $\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$. The stability conditions space is $\mathbb{Z}$. For $\phi > 0$, we have unstable locus $Z_+ = \{x = y = 0\}$. We have GIT quotient*

$$\mathbb{C}^3//_\phi \mathbb{C}^* = \mathbb{C}^3 \setminus \{x = y = 0\}/\mathbb{C}^* = \mathcal{O}(-1)_{\mathbb{P}^1_{x:y}}.$$

For $\phi < 0$, we have unstable locus $Z_- = \{z = 0\}$.

$$\mathbb{C}^3//_\phi \mathbb{C}^* = \mathbb{C}^3 \setminus \{z = 0\}/\mathbb{C}^* = \mathbb{A}^1_{x,y}.$$

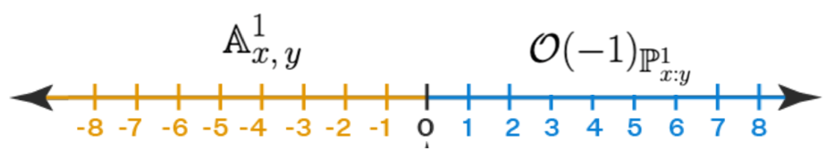

Figure 1: Secondary fan of GIT problem $\mathbb{C}^3_{(1,1,-1)}$

Mirror Symmetry Conjecture Heuristics

Mirror symmetry is a series of mysterious relationships between complex and symplectic geometry. Given a *complex geometry* X, there exists a mirror *symplectic geometry* $\hat{X}$ such that we have equivalence of categories

$$D^b(X) \cong \mathsf{Fuk}(\hat{X})$$

Actually, we have a whole family of mirrors, each of whom is symplectomorphic to $\hat{X}$, but has a different complex structure. We call the **Stringy Kähler moduli space** the complex structure moduli space $\mathcal{M}_{CS}$ of the symplectic manifold $\hat{X}$.

Proposition 1 *There is a monodromy action of $\pi_1(\mathcal{M}_{CS})$ on $\hat{X}$ via symplectomorphism, and hence an action of $\pi_1(\mathcal{M}_{CS})$ on $\mathsf{Fuk}(\hat{X})$ via autoequivalence.*

By mirror symmetry, the action of $\pi_1(\mathcal{M}_{CS})$ carries to an action of $D^b(X)$ via autoequilvance. In the context of Calabi Yau toric GIT, the conjecture is that there is a particular way that $\pi_1(\mathcal{M}_{CS})$ acts, via spherical twists.

The program above has been done in certain cases e.g. in [6].

Calabi-Yau Toric GIT

Definition 1 *A GIT problem $(\mathbb{C}^*)^r$ acting on $\mathbb{C}^n$ is* **Calabi-Yau** *if the rows of the weight matrix Q add up to 0.*

Suppose X and Y are two GIT quotients. Via wall-crossing you can find a semi-orthogonal decomposition $D^b(X) \cong \langle D^b(Y), \mathcal{C}\rangle$ [5]. In the Calabi-Yau setting $D^b(X) \cong D^b(Y)$ for any quotients.

Let $A = (a_{ij}) \in M_{(n-r)\times n}(\mathbb{Z})$ be the cokernel of Q. Using the CY condition we choose the first row of A to be 1's. The column vectors in $\mathbb{Z}^{n-r}$, called **rays**, and span a $(n - r - 1)$ dimensional polytope $\mathcal{P}(A)$, called the **Primary Polytope**. Triangulations of $\mathcal{P}(A)$ give us the toric fans for our GIT quotients [2, Section 4].

The mirror family is a family of Landau Ginsburg Models [3] $\left((C^*)^{n-r}, W_{\underline{\mathbf{a}}}\right)$ where $W_{\underline{\mathbf{a}}} : (\mathbb{C}^*)^{n-r} \to \mathbb{C}$:

$$\begin{aligned} W_{\underline{\mathbf{a}}}(X_1, \ldots, X_{n-r}) &= \text{polynomial with coefficients } \underline{\mathbf{a}} \text{ in variables } X_1, \\ &\quad \text{where the powers are given by the entries of } A = (a_{ij}). \end{aligned}$$

The mirror symmetry statement here says for any GIT quotient X, $D^b(X) \cong FS\left((C^*)^{n-r}, W_{\underline{\mathbf{a}}}\right)$, where FS is for Fukaya-Seidel category [8].

$\underline{\mathbf{a}}$ parametrises our mirror family and we have

$$\mathcal{M}_{CS} = \left((\mathbb{C}^*)^n \setminus \Delta_A\right)/A \cong (\mathbb{C}^*)^r \setminus \nabla_A$$

where the hypersurface $\Delta_A \subset (\mathbb{C}^*)^n$ is called the **A-discriminant** [4].

Example 3 *Suppose $A = (2\,1\,0)$. Then $W_{a,b,c}(X) = aX^2 + bX + c$. Then $\Delta_A = \{b^2 = 4ac\} \subset (\mathbb{C}^*)^3$, and using A to scale out b, we get $\left((\mathbb{C}^*)^3 \setminus \Delta_A\right)/A \cong \left((\mathbb{C}^*)^2\right) \setminus \{ac = 1/4\}$.*

The main observation now is that the discriminant locus arises as the union of components:

$$\Delta_A = \bigcup_{\Gamma \text{ min face}} \Delta_\Gamma$$

The components correspond to **minimal faces** of $\mathcal{P}(A)$, that is faces whose rays are **linearly dependent**.

Γ corresponds to a 'sub GIT problems' Q_Γ with non-trivial minimal GIT quotients Z_Γ, where by minimal GIT quotient we mean a GIT quotient that cannot be decomposed further by the wall-crossing. There are spherical functors $F_\Gamma : D^b(Z_\Gamma) \to D^b(X)$ [1], where by spherical we mean that the twist

$$T_{F_\Gamma} = C(\, FR \xrightarrow{\text{counit}} I_{D^b(X)}\,)$$

is an autoequivalence of $D^b(X)$, where $R_\Gamma : D^b(X) \to D^b(Z_\Gamma)$ is the right adjoint.

Meridians of the Δ_Γ in $\pi_1(\mathcal{M}_{CS})$ acts on $D^b(X)$ via the spherical twists T_{F_Γ}.

Toric Calabi Yau 3-folds of Picard Rank 2

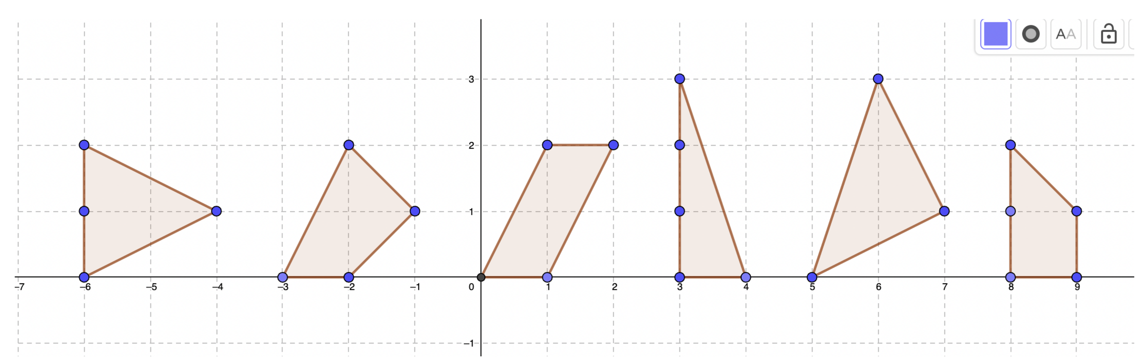

Figure 2: All the primary polytopes of Toric Calabi Yau 3-folds of Picard Rank 2

Rank 2 Example

Consider GIT problem

$$Q = \begin{array}{c} \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \qquad A^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}$$

We have

$$\mathcal{M}_{CS} \cong (\mathbb{C}^*)^2_{a,b} \setminus \left(\nabla_{\Gamma_0} \cup \nabla_{\Gamma_1}\right)$$

where we can compute $\nabla_{\Gamma_0} = \{(b^2 - a)^2 = 4\}$ and $\nabla_{\Gamma_1} = \{a^2 = 4\}$. The 'sub GIT problems' Q_{Γ_0} and Q_{Γ_1}, they have minimal GIT quotients $Z_{\Gamma_0} = \text{pt}$ and $Z_{\Gamma_1} = \mathbb{A}^1$, and spherical functors $F_{\Gamma_0}\mathcal{O}_{\text{pt}} = \mathcal{O}_0 \in D^b([\mathbb{A}^3/\mathbb{Z}_4])$ and $F_{\Gamma_0} = i_*\pi^*$:

$$\begin{tikzcd} {[\mathbb{A}^1_c/\mathbb{Z}_4]} \arrow[r, "i"] \arrow[d, "\pi"] & {[\mathbb{A}^3_{c,d,e}/\mathbb{Z}_4]} \\ \mathbb{A}^1_c & \end{tikzcd}$$

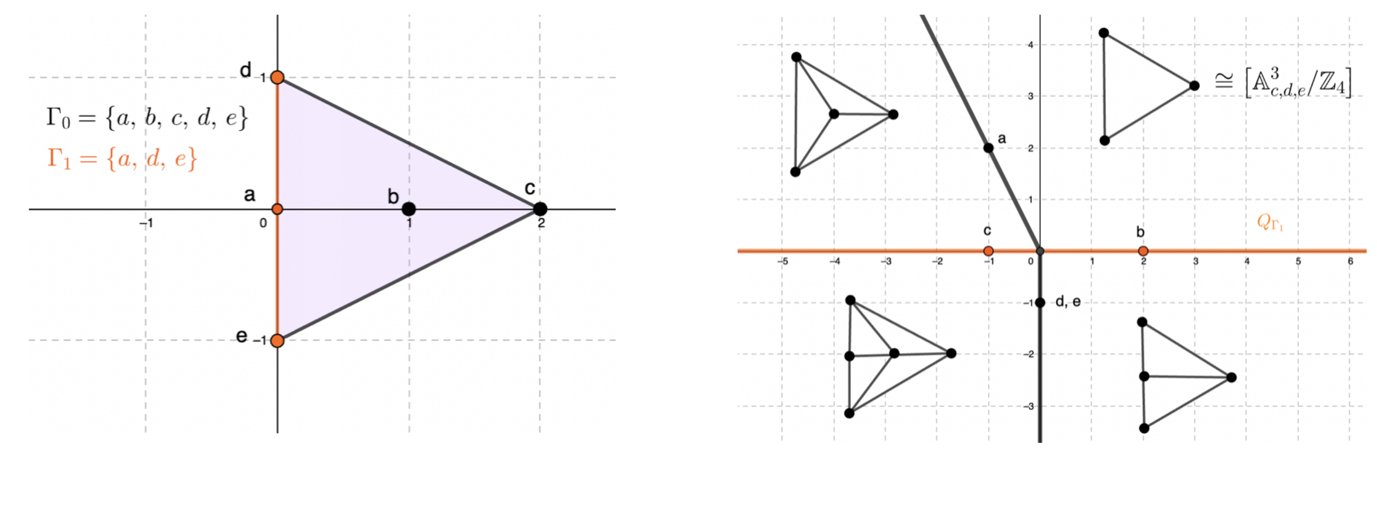

Figure 3: Left: The primary polygon $\mathcal{P}(A)$, Right: The secondary fan

Lemma 1 (M. Barbieri 2023) *The spherical twists $T_{F_{\Gamma_0}}$ and $T_{F_{\Gamma_1}}$ satisfy the relations of a meridian μ_0 of ∇_{Γ_0} and a merdian μ_1 of ∇_{Γ_1} in $\pi_1(\mathcal{M}_{CS})$.*

References

[1] N. Addington. New derived symmetries of some hyperkähler varieties. *arXiv preprint arXiv:1112.0487*, 2011.

[2] T. Coates, H. Iritani, and Y. Jiang. The crepant transformation conjecture for toric complete intersections. *Advances in Mathematics*, 329:1002–1087, 2018.

[3] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*, volume 68. American Mathematical Society Providence, RI, 1999.

[4] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *A-discriminants*. Springer, 1994.

[5] D. Halpern-Leistner. The derived category of a git quotient. *Journal of the American Mathematical Society*, 28(3):871–912, 2015.

[6] D. Halpern-Leistner and S. Sam. Combinatorial constructions of derived equivalences. *Journal of the American Mathematical Society*, 33(3):735–773, 2020.

[7] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[8] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*, volume 10. European Mathematical Society, 2008.