

BRANE MONODROMY FOR TORIC GIT

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ABSTRACT. Given a toric Calabi-Yau GIT problem, we have a construction for the parameter space over which mirror Landau Ginzburg models live, called the Fayet-Iliopoulos Parameter Space (*FIPS*). It is conjectured that the fundamental groupoid of *FIPS* acts on the derived categories of the GIT quotients via spherical twists about spherical functors and line bundle twists. This is an expository report on this conjecture, where we compute a rank 2 example in detail.

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1. INTRODUCTION

When mathematicians talk about mirror symmetry, they refer to a collection of mysterious relationships between different geometric objects spanning differential, symplectic, and algebraic geometry. The story starts in 1991 when string theorists Candelas, de la Ossa, Green, and Parkes published their famous paper [3]. This paper caught the attention of mathematicians because the physicists used mysterious methods to carry out calculations that they had been unable to do.

The question was, what was the number n_d of degree d curves on the quintic threefold $X := \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4$? Mathematicians had only been able to work out n_d for $d < 4$ and were unsure whether the number was

even finite for higher d . In [3], physicists computed n_d for $d < 11$, with the first three numbers matching those the mathematicians had worked out. The physicists' techniques seemed like magic to the mathematicians, and the study of their work since then has resulted in a plethora of new and interesting mathematics.

The rough physics is as follows. Given a Calabi-Yau manifold (X, J, ω) ¹ (we sometimes just write X for short), string theory constructs a $\mathcal{N} = (2, 2)$ *super conformal field theory* (SCFT). This construction was first introduced in [8, Section 6] in 1960. It is called a string theory because point particles in X are replaced with 1-real-dimensional strings in X . As the strings propagate, they trace out 2-dimensional surfaces. $\mathcal{N} = (2, 2)$ is short-hand for the supersymmetry algebra which acts on our SCFT. Applying an automorphism to the supersymmetry algebra somehow constructs a 'different' but 'isomorphic' $\mathcal{N} = (2, 2)$ SCFT. We call two SCFTs related in this way *mirror symmetric*.

Conjecture 1.1 (Physics mirror symmetry). Given a Calabi-Yau threefold (X, J, ω) there exists another Calabi-Yau threefold (X^v, J^v, ω^v) whose corresponding $\mathcal{N} = (2, 2)$ SCFT is mirror symmetric to that of (X, J, ω) . We call (X^v, J^v, ω^v) mirror to (X, J, ω) .

This is not a precise mathematical conjecture as the mathematical language to define and relate QFTs is not rigorous: for example, it is not clear what *isomorphic* SCFT really means mathematically. We are also far from knowing how to construct mirrors in general, however there are some constructions such as the SYZ mirror conjecture [18]. In simple cases, such as for toric varieties, there are proven recipes for the mirror constructions called *Hori-Vafa* mirrors [12]. This will be the context of this report and we will discuss toric mirror symmetry in more detail later on.

Though imprecise as stated, the physics mirror symmetry conjecture implies some actual mathematical relationships between (X, J, ω) and a mirror (X^v, J^v, ω^v) . The most classical one of these is a Hodge-theoretic claim about the Dolbeault cohomology groups of the manifolds. A more modern statement that is a big area of active research right now is the homological mirror symmetry conjecture, which relates some puzzlingly different categories of X and X^v . This was first presented by Maxim Kontsevich at the 1994 International Congress of Mathematicians. The categories are called the Fukaya category and the derived category, the former depending only on the symplectic structure and the latter depending only on the complex structure.

¹The $J : TX \rightarrow TX$ here is an almost complex structure on the manifold X . This introduces a multiplication by i to tangent spaces of X ($J^2 = -Id$). ω is a symplectic form, which is a non-degenerate 2-form. We require ω to be compatible with J (in which case we call ω a Kähler form) and the Calabi-Yau condition means that *canonical bundle* of X is trivial.

The way this relates to the physics mirror symmetry is the following. Given a SCFT coming from (X, J, ω) one can construct an associate A-model and B-model via *topological twisting*, where the A-model depends only on the symplectic structure and the B-model depends only on the complex structure. If we consider open strings, then what the strings attach to are objects called ‘branes’. In the A-model, these branes are Lagrangian submanifolds (with local systems)², while in the B-model, they are algebraic subvarieties. Via this perspective it is understood that the data of a *triangulated category*³ is carried in both models. For the A-model this is called the Fukaya Category, denoted $\text{Fuk}(X)$, and for the B-model, this is called the derived category, denoted $D^b(X)$. Very roughly, the objects and morphisms in these categories are the following:

$\text{Ob}(\text{Fuk}(X)) =$ Chain complexes of Lagrangian submanifolds $L \subset X$.

$\text{Hom}_{\text{Fuk}(X)}(L, L') =$ A chain complex made from free linear combinations of intersection points $L \cap L'$, with differential counting holomorphic discs between intersection points. It is called the Floer complex and denoted $CF(L, L')$.

$\text{Ob}(D^b(X)) =$ Chain complexes of coherent sheaves on X . These are sheaves that are locally finitely generated modules. Examples include vector bundles, often called locally free sheaves in this context.

$\text{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) =$ Chain maps $\mathcal{F}^\bullet \rightarrow \mathcal{E}^\bullet$ where we invert quasi-isomorphisms.

Quasi-isomorphisms are chain maps that induce isomorphisms on the level of homology.

See [13, Chapter 2] for a good exposition on the bounded derived category of coherent sheaves. The mirror operation relates the A-model of (X, J, ω) to the B-model (X^v, J^v, ω^v) , and the B-model of (X, J, ω) to the A-model of (X^v, J^v, ω^v) , implying equivalences of categories

$$\begin{aligned}\text{Fuk}(X) &\cong D^b(X^v), \\ D^b(X) &\cong \text{Fuk}(X^v).\end{aligned}$$

²A half dimensional submanifold $L \subset X$ is Lagrangian if $\omega|_L = 0$.

³A triangulated category \mathcal{A} is an additive category with a shift functor $T : \mathcal{A} \rightarrow \mathcal{A}$, and a set of *distinguished triangles* $A \rightarrow B \rightarrow C \rightarrow T(A)$, $A, B, C \in \mathcal{A}$, which satisfy some conditions, see [13, Chapter 1]

Given this, if we simply start with a *complex geometry*⁴ X , we can ask if there is a symplectic geometry X^v such that

$$D^b(X) \cong \mathrm{Fuk}(X^v) \quad (1)$$

From now on, we will call an X^v that satisfies (1) a mirror to X . Because we didn't specify the symplectic structure of X , we have a whole family of mirrors, each of which is symplectomorphic to X^v , but has a different complex structure. We denote the complex structure moduli space of the symplectic manifold X^v by $\mathcal{M}_{\mathrm{CS}}$. Likewise we can consider all the Kähler forms (symplectic structures) on X , and we call this moduli space *the String Kähler Moduli Space* $\mathcal{M}_{\mathrm{SKMS}}$ ⁵.

There is a monodromy action of $\pi_1(\mathcal{M}_{\mathrm{SKMS}})$ on $D^b(X)$ via autoequivalence, which is the B-side 'brane monodromy'. There is also a monodromy action of $\pi_1(\mathcal{M}_{\mathrm{CS}})$ on $\mathrm{Fuk}(X^v)$ via autoequivalence, which is the A-side 'brane monodromy'. The homological mirror symmetry conjecture implies $\mathcal{M}_{\mathrm{CS}} \cong \mathcal{M}_{\mathrm{SKMS}}$. Since it's easier to understand $\mathcal{M}_{\mathrm{CS}}$ rather than $\mathcal{M}_{\mathrm{SKMS}}$, we will study how the A-side brane monodromy carries over to the B-side.

The way this action should carry over is not clear entirely. This is because mirror symmetry proofs don't necessarily give explicit mirror maps that we can use. The main objective of this report is to make sense of a conjecture on how this action carries over in a specific context, which is **toric Calabi Yau Geometric Invariant Theory (GIT)**. The reason we work in this context is threefold: firstly, toric geometry is combinatorial and allows for the construction of informative yet hands-on examples; secondly, we have a heuristic recipe a parameter space called the Fayet-Iliopoulos parameter space (*FIPS*), which is closely related to $\mathcal{M}_{\mathrm{CS}}$; and lastly, the GIT context provides us with interesting autoequivalences, which come from so called *spherical functors*.

Toric GIT is a theory of how to take quotients when you have an algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$. It turns out that to get something nice, you need to remove a certain zero locus from V before taking the naive topological quotient. It also turns out that this process is not unique and depends on a *stability condition*, which is just a choice of character $\theta : T \rightarrow \mathbb{C}^*$. Different characters will tell you to remove different loci, and therefore give birational but not isomorphic quotients (in general). We denote the quotient corresponding to stability condition θ by $X_\theta = V //_\theta T$. We call the locus we remove *the unstable locus*. We will explain in more detail how a stability condition determines an unstable locus in Section

⁴We are deliberately being vague here but just think of an algebraic variety for now.

⁵In $\mathcal{M}_{\mathrm{SKMS}}$ of X we include complexified Kahler forms and other birational models of X .

2.1. By some representation theory, any character $\theta : T \rightarrow \mathbb{C}^*$ is of the following form

$$\theta(\lambda_1, \dots, \lambda_r) = \lambda_1^{\theta_1} \dots \lambda_r^{\theta_r}$$

where $(\theta_1, \dots, \theta_r) \in \mathbb{Z}^r$. So the characters live in a rank r lattice L^v , called *the character lattice*.

The character lattice L^v splits into positive cones where any two stability conditions lying in the interior of the same cone have the same unstable locus. The top dimensional cones correspond to good quotients in the sense that they are smooth Deligne-Mumford stacks (the stabilisers are finite groups). The GIT quotients corresponding to the top dimensional cones are therefore the ones we mean when we discuss the collection of GIT quotients. We call the decomposition of L^v *a wall and chamber decomposition*, where note two GIT quotients are said to be separated by a wall if their cones share a codimension 1 face.

We note that the wall and chamber decomposition of L^v gives us the data of a toric fan, which we call *the secondary toric fan*. The geometric realisation of this fan is a toric variety \mathcal{F} which we call the *secondary toric variety*. By toric orbit-cone correspondence, fixed points in \mathcal{F} under the dual torus T^v action correspond to GIT quotients.

Let us consider a wall W in the secondary fan separating two GIT quotients X and X' . The *wall-crossing formula*, which we explain in more detail in Section 2.2, tells us that

$$D^b(X) = \langle D^b(X'), D^b(Z), \dots, D^b(Z) \rangle_{\text{SOD}}$$

where Z is the T -quotient of the fixed locus of the action of the one-parameter family perpendicular to the wall, and the number of copies of Z can be easily computed combinatorially⁶. We may call X' *minimal* with respect to X , and X *maximal* with respect to X' .

A toric Calabi-Yau GIT problem is a toric GIT problem where the sum of the weights is 0. This condition on the action of the torus T that ensures all the quotients are Calabi-Yau i.e. that they have trivial canonical bundle. This ensures that whenever you wall cross the number of copies of $D^b(Z)$ is 0, so that all your GIT quotients are derived equivalent through wall crossing. If we fix a GIT quotient X , wall-crossing through a wall W actually gives us countably-infinite derived equivalences between X and the GIT quotient on the other side of the wall. Going

⁶In fact, Z comes from a GIT problem called a *Higgs GIT problem*, whose data is a subset of the data of our original GIT problem. We will explain this in more detail in Section 2.2.

through one equivalence and back through another produces a non-trivial autoequivalence of $D^b(X)$. For any wall we therefore have countably-infinitely-many non-trivial autoequivalences of $D^b(X)$.

Though the number of copies of $D^b(Z)$ in the SOD is 0, the toric variety Z is still very important. For any wall W and corresponding Z we get a *spherical functor*

$$F : D^b(Z) \rightarrow D^b(X).$$

This means that the *spherical twist* $T_F : D^b(X) \rightarrow D^b(X)$ is an autoequivalence. We will explain what F and T_F are in Section 2.3. In [11], Halpern-Leistner and Shipman proved that the autoequivalences coming from wall crossing are the same as those coming from these spherical functors.

Other than having a good understanding of lots of non-trivial autoequivalences of derived categories of Calabi-Yau toric GIT quotients, we also have another important tool which is that we have a recipe for the mirror. Consider a rank r dimension n toric Calabi-Yau GIT problem $T \cong (\mathbb{C}^*)^r$ acting on $V \cong \mathbb{C}^n$. The data of T acting on V determines a Gauged Linear Sigma Model (GLSM), a particular kind of *supersymmetric 2-dimensional QFT* (SQFT). The mirror is given by a *Landau Ginzburg*⁷ model $((\mathbb{C}^*)^{n-r}, W)$ where $W : (\mathbb{C}^*)^{n-r} \rightarrow \mathbb{C}$ is a Laurent polynomial called the *superpotential*. This mirror symmetry was described by physicists Hori-Vafa [12] and a corresponding homological mirror symmetry statement has been proved mathematically. The Landau Ginzburg model is a mirror in the homological sense via the equivalence of categories

$$D^b(X) \cong \text{FS}((\mathbb{C}^*)^{n-r}, W)$$

where X is a GIT quotient and by FS we mean the *Fukaya-Seidel Category*, or the *Fukaya category with stops* [17]. In fact we have a family of these mirror Landau Ginzburg models, which vary by choice of coefficients of the Laurent polynomial W . There's a combinatorial recipe to construct the parameter space over which the mirror Landau Ginzburg models live and we call this parameter space the Fayet-Iliopoulos parameter space (*FIPS*). It is the complement of the *GKZ discriminant locus* Δ of W inside the dual torus T^\vee [7]. This discriminant generalises the well-known discriminant of the quadratic to Laurent polynomials of any type in any number of variables, which we will explain in more detail in Section 2.4.

An important remark is that *FIPS* is not the same as \mathcal{M}_{CS} or $\mathcal{M}_{\text{SKMS}}$, see Remark 2.8 in [6] for an explanation of this. That said, there should be a map from $\text{FIPS} \rightarrow \mathcal{M}_{\text{CS}}$. Since *FIPS* is the only thing we have a heuristic recipe for, we

⁷Given a Landau Ginzburg model, one can also build a SQFT.

study a monodromy action of $FIPS$ on the derived categories of the GIT quotients. It is helpful to think of $FIPS$ as living inside the secondary toric variety \mathcal{F} . Recall that there are points in the boundary of \mathcal{F} that correspond to GIT quotients. This makes the \mathcal{F} an interesting compactification of $FIPS$ because we can then see limiting regions of $FIPS$ that correspond to specific GIT quotients. We can consider the fundamental groupoid of $FIPS$ $\Pi_1(FIPS)$ with a base point for every limiting region. The conjecture is that a path from one quotient X to another quotient X' acts on the set of derived categories of the GIT quotients via composition of wall-crossings (or spherical functor twists) and line bundle twists. Non-homotopic paths will correspond to different equivalences.

In Section 2.7 we will explain more details of the theory and put into plain sight how we exactly match the paths/loops in $\Pi_1(FIPS)$ to certain equivalences between the derived categories of the GIT quotients. In Section 3 we prove the conjecture for a rank 2 example.

Finally, in Section 4 we lay out our plan to prove this conjecture for toric Calabi-Yau threefolds of Picard rank 2.

2. BACKGROUND ON TORIC CY GIT

2.1. General Toric Geometric Invariant Theory. We start with the algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$. We call this a rank r dimension n toric geometric invariant theory (GIT) problem. We would like to construct a space that parametrises the actions orbits. *Geometric Invariant Theory* (GIT) is the theory that tells us *the unstable locus* to throw away before we quotient so that we get a good/geometric quotient [15]. GIT quotients are often denoted $X // G$. By the representation theory of the algebraic torus, for a choice of coordinates on T and V the actions must be of the form:

$$(\lambda_1, \dots, \lambda_r) \cdot (z_1, \dots, z_n) = (\lambda_1^{q_{11}} \lambda_2^{q_{21}} \dots \lambda_r^{q_{r1}} z_1, \dots, \lambda_1^{q_{1n}} \lambda_2^{q_{2n}} \dots \lambda_r^{q_{rn}} z_n)$$

We get an $r \times n$ integer matrix $Q = (q_{ij})$, called the *weight matrix*. The column vectors $q_i = Q(e_i)$ of Q are hence called the *weights*.

Example 2.1. Consider \mathbb{C}^* acting on \mathbb{C}^2 linearly, i.e. $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. If we take the quotient space, we see that the orbit of the origin cannot be separated from any other orbit. So we have to remove the origin and as expected, we get GIT quotient

$$\mathbb{C}^2 // \mathbb{C}^* = \mathbb{C}^2 \setminus \{(0, 0)\} / \mathbb{C}^* = \mathbb{P}^1$$

In general GIT quotients are not unique and depend on a choice of *stability condition*. As we explained in the introduction, in the affine GIT problem a stability

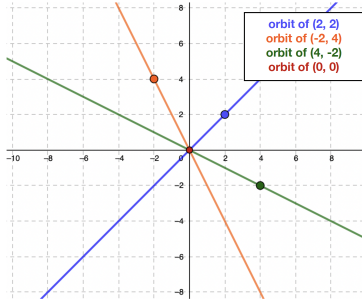


FIGURE 1. Orbits of the \mathbb{C}^* action on \mathbb{C}^2 .

condition is just a choice of character $\theta : T \rightarrow \mathbb{C}^*$. Since the character lattice $\text{Hom}(T, \mathbb{C}^*)$ is isomorphic to \mathbb{Z}^r , we can see $\theta \in \mathbb{Z}^r$. Different choices of θ have us remove different unstable loci and give us non-isomorphic (but birational) quotients. This amounts to just looking at the positive cones of collections of weights q_i who contain θ . We will spell exactly how next, where we use Section 4 of [4] as our main reference.

We will reformulate in a non-coordinate dependent way. The following is our GIT data:

- $T \cong (\mathbb{C}^*)^r$ a connected algebraic torus of rank r ,
- $L := \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^r$, the *cocharacter lattice*,
- $q_1, \dots, q_n \in L^v = \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^r$, a collection of characters we call *weights*.

We call L^v the *character lattice*. The weights q_1, \dots, q_n define a map from T to $(\mathbb{C}^*)^n$ and hence define an action on \mathbb{C}^n through the action of the torus. For a subset $I \subset \{1, 2, \dots, n\}$, and write \bar{I} for the complement of I , and set

$$\angle_I = \left\{ \sum_{i \in I} a_i q_i \mid a_i \in \mathbb{R}_{>0} \right\} \in L^v \otimes \mathbb{R} \quad (2)$$

where we call \angle_I the *strictly positive cone on weights* $\{q_i \mid i \in I\}$. We set $\angle_\emptyset = \{0\}$. Consider a *stability condition* $\theta \in L^v$. Set

$$\mathcal{A}_\theta = \{I \subset \{1, 2, \dots, n\} \mid \theta \in \angle_I\}.$$

We make the following assumptions on θ :

- (1) $\{1, 2, \dots, n\} \in \mathcal{A}_\theta$,
- (2) for any $I \in \mathcal{A}_\theta$, $\{q_i \mid i \in I\}$ spans $L^v \otimes \mathbb{R}$ over \mathbb{R} .

The first ensures that our quotient won't be empty and the second ensures that we are picking a good stability condition, in that our quotient will be a smooth Deligne

Mumford stack. Consider coordinates z_1, \dots, z_n on \mathbb{C}^n . We define the ideal

$$\mathcal{I}_\theta = \left\langle \prod_{i \in I} z_i \mid I \in \mathcal{A}_\theta \right\rangle$$

We define *the unstable locus of θ* to be

$$S_\theta = V(\mathcal{I}_\theta)$$

so that our GIT quotient is toric smooth DM stack:

$$X_\theta = [\mathbb{C}^n \setminus S_\theta / T].$$

The space of stability conditions satisfying our assumptions has a wall and chamber structure. The chamber C_θ to which θ belongs is given by

$$C_\theta = \bigcap_{I \in \mathcal{A}_\theta} \angle_I$$

and $X_\theta = X_{\theta'}$ if and only if $\theta' \in C_\theta$. The GIT quotient $X_{\theta'}$ changes when we cross a codimension one boundary of C_θ . This wall and chamber structure gives us a fan in L^v , which we call *the secondary fan*. We note that the 1-dimensional cones may be more than just the weights of the action. Corresponding to the secondary fan is a toric variety, the secondary toric variety \mathcal{F} .

Example 2.2. Consider \mathbb{C}^* acting on \mathbb{C}^3 via $\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$. The character lattice is isomorphic to \mathbb{Z} . For $\theta > 0$, we have unstable locus $Z_+ = \{x = y = 0\}$. We have GIT quotient

$$\mathbb{C}^3 //_\theta \mathbb{C}^* = \mathbb{C}^3 \setminus \{x = y = 0\} / \mathbb{C}^* = \mathcal{O}(-1)_{\mathbb{P}_{x,y}^1}.$$

For $\theta < 0$, we have unstable locus $Z_- = \{z = 0\}$.

$$\mathbb{C}^3 //_\theta \mathbb{C}^* = \mathbb{C}^3 \setminus \{z = 0\} / \mathbb{C}^* = \mathbb{A}_{x,y}^2.$$



FIGURE 2. Secondary fan of GIT problem $\mathbb{C}^3_{(1,1,-1)}$

Given the weights $q_i \in L^v$ we can also define weight map $Q : \mathbb{Z}^n \rightarrow L^v$ in the obvious way, and we consider the kernel $A : M \rightarrow \mathbb{Z}^n$ of Q . We assume that our weight map is surjective up to torsion, so that our GIT quotient stacks do not have infinite isotropy groups. By definition A is injective and by the assumption on Q ,

$M \cong \mathbb{Z}^{n-r}$. We call $a_i := A^v(e_i) \in M^v$ rays. We package this information in the complex

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} L^v \rightarrow 0$$

Just as we can see Q as an $r \times n$ dimensional matrix, we will see A as a $n \times (n-r)$ dimensional matrix. We will often give our GIT problems as packaged in a complex, as this notation makes it easier for us to define other GIT problems using data from our original GIT problem. We will discuss this more in Section 2.2.

2.2. Wall-Crossing Formula. Let us consider a rank r torus T acting on a n -dimensional vector space V , with GIT problem given by complex

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} L^v \rightarrow 0.$$

In [9], Daniel Halpern-Leistner relates the derived categories of GIT quotients to the equivariant derived category⁸ $D_T^b(V)$. In particular, given any reductive GIT problem and choice of stability condition θ , he proves the existence of a *semi-orthogonal decomposition*⁹ (SOD) of the equivariant derived category where one of the components is equivalent to $D^b(X_\theta)$. In the case of toric GIT we can take this further and explicitly relate the derived categories of GIT quotients separated by a wall.

By analysing [9] in the toric case, we see that given a toric GIT quotient X_θ we can find a finite collection \mathcal{W}_θ of T -equivariant line bundles¹⁰ on V and infinite collection \mathcal{C} of T -equivariant sheaves supported on the unstable locus S_θ such that

- (1) \mathcal{W}_θ and \mathcal{C} generate (under shifts and cones) $D_T^b(V)$,
- (2) The triangulated category generated (under shifts and cones) by \mathcal{W}_θ is equivalent to $D^b(X_\theta)$.

We call \mathcal{W}_θ a *window*. We emphasize that the choice of \mathcal{W}_θ is not unique, in particular we can always twist a window by an equivariant line bundle. Moreover, given two stability condition θ and ϕ separated by a wall W , we can find a countable infinite collection of windows \mathcal{W}_θ and \mathcal{W}_ϕ such that (say) $\mathcal{W}_\phi \subset \mathcal{W}_\theta$, and each of these gives us a semi-orthogonal decomposition (ignoring ordering)

$$D^b(X_\theta) = \langle D^b(X_\phi), D^b(Z), \dots, D^b(Z) \rangle_{\text{SOD}} \quad (3)$$

⁸This is just the derived category of T -equivariant coherent sheaves on V .

⁹A semi-orthogonal decomposition of a triangulated category \mathcal{T} is a decomposition of \mathcal{T} into triangulated subcategories

$$\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle$$

such that the categories together generate \mathcal{T} under shifting (think of shifting chain complexes) and taking mapping cones [13, Chapter 1].

¹⁰ T -equivariant coherent sheaves on V are just graded modules of the graded polynomial ring R (the ring of regular functions on V), where the \mathbb{Z}^r grading on R is given by the action of T .

where Z is constructed by a ‘sub-GIT problem’ called a *Higgs GIT problem*, which we will explain shortly, and the number of copies of Z can be easily computed combinatorially. We will refer to equation (3) as the *wall crossing formula*.

The number of copies of $D^b(Z)$ is

$$\kappa = (\det V)(\lambda_W), \quad (4)$$

where $\det V := q_1 + \cdots + q_n \in L^v$ is called the *character*, and $\lambda_W \in L$ is the one-parameter subgroup perpendicular to the wall, oriented so that $\kappa \geq 0$. The GIT chamber the oriented λ_W points to will be the maximal one (the one with the bigger derived category, i.e. X_θ).

Definition 2.1. Consider a subset $\mathcal{T} \subset \{1, \dots, n\}$ and consider the corresponding set of weights $Q(\mathcal{T})$. We define $L_{\mathcal{T}}^v$ as the primitive sublattice generated by these weights

$$L_{\mathcal{T}}^v = L^v \cap \langle Q(\mathcal{T}) \rangle_{\mathbb{R}} \subset L_{\mathbb{R}}^v.$$

We define a GIT problem

$$0 \rightarrow M_{\mathcal{T}} \xrightarrow{A_{\mathcal{T}}} \mathbb{Z}^{\mathcal{T}} \xrightarrow{Q_{\mathcal{T}}} L_{\mathcal{T}}^v \rightarrow 0,$$

which we call *Higgs GIT problem* associated to \mathcal{T} .

We have a Higgs GIT problem associated to a wall W by taking the subset of weights to be the weights lying on the subspace spanned by the wall. We denote this subset of weights $Q(W)$, and the GIT problem as Q_W . This gives a rank $r - 1$, dimension $|Q(W)|$ toric GIT problem. Z is a GIT quotient of this new GIT problem, where we choose the stability condition to be on the wall W . In fact, Z is a quotient of the fixed locus of the action of λ_W by the rest of the action of T .

2.3. Spherical Functors. The goal of this section will be to construct *spherical functors* from the data of a toric *Calabi-Yau* GIT problem. We start first by explaining some general theory about them.

General Theory. The first type of spherical functors that were studied were *spherical objects*. Let X be a smooth projective variety over a field k .

Definition 2.2. [13, Chapter 8] An object $\mathcal{E}^\bullet \in D^b(X)$ is called *spherical* if

- (1) $\mathcal{E}^\bullet \otimes \omega_X \cong \mathcal{E}^\bullet$ and,
- (2)

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) \cong \begin{cases} k & \text{if } i = 0, \dim(X), \\ 0 & \text{otherwise.} \end{cases}$$

These objects are called spherical because condition (2) can be rephrased

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[*]) \cong H^*(S^{\dim(X)}, k)$$

where $S^{\dim(X)}$ denotes the $\dim(X)$ -dimensional sphere. We define the *spherical twist* $T_{\mathcal{E}^\bullet} : D^b(X) \rightarrow D^b(X)$ about a spherical object to be

$$T_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet) = C(\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \otimes \mathcal{E}^\bullet \rightarrow \mathcal{F}) \quad (5)$$

where the map is given by evaluation and note all the functors are derived¹¹. By [13, Proposition 8.6], the spherical twist $T_{\mathcal{E}^\bullet} : D^b(X) \rightarrow D^b(X)$ about any spherical object $\mathcal{E}^\bullet \in D^b(X)$ is an autoequivalence. The formula for the spherical twist may look familiar, and this is because at the level of K -theory it acts as reflection through the hyperplane orthogonal to the Mukai vector $v(\mathcal{E}^\bullet)$. So in a sense the spherical twist is a categorification of reflection.

Spherical twists about *spherical functors* are generalisations of spherical twists about spherical objects. Our main reference for spherical functors is [1, Section 1]. Consider an exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between triangulated categories \mathcal{A} and \mathcal{B} . with left and right adjoints $L, R : \mathcal{B} \rightarrow \mathcal{A}$. We define the *twist* $T : \mathcal{B} \rightarrow \mathcal{B}$ to be the cone on the counit $FR \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{B}}$ of the adjunction, so that there is an exact triangle

$$FR \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{B}} \rightarrow T, \quad (6)$$

and the *cotwist*

$$\mathrm{id}_{\mathcal{A}} \xrightarrow{\eta} RF \rightarrow C, \quad (7)$$

We say that F is spherical if C is an equivalence and $R \cong C_T L$.

Theorem 2.1 (Rouquier and Anno). *The spherical twist $T_F : \mathcal{B} \rightarrow \mathcal{B}$ about a spherical functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is an autoequivalence.*

Proof. Though attributed to Rouquier and Anno, Nick Addington gives a proof [1, Theorem 1]. \square

How is a spherical twist about a spherical functor a generalisation of the spherical twist about a spherical object? Suppose we have a spherical object $\mathcal{E}^\bullet \in D^b(X)$ with X a smooth projective variety over k . We define:

$$F = \mathcal{E}^\bullet \otimes - : D^b(pt) \rightarrow D^b(X).$$

We can prove that F is a spherical functor, and since $R = \mathrm{RHom}(\mathcal{E}^\bullet, -)$, the twist $T_F : D^b(X) \rightarrow D^b(X)$ is exactly like the twist $T_{\mathcal{E}^\bullet}$ we defined earlier. Note that $R \cong C_T L$ just amounts to condition (1) in Definition 2.2.

¹¹There are some technical issues with the definition of the spherical twist because taking cones is not functorial. We bypass any such issues because in our geometric context, where our triangulated category is the derived category of an algebraic variety, we can actually describe any spherical twist (even about a spherical functor) as a Fourier Mukai transform.

Spherical functors in toric CY GIT. Let us consider a rank r torus T acting on a n -dimensional vector space V . Let X and X' be two GIT quotients separated by a wall W . In this subsection we will be defining a spherical functor $F : D^b(Z) \rightarrow D^b(X)$ for *Calabi-Yau* toric GIT problems, where Z is the same variety showing up in the wall-crossing formula (3). Note that in [11], Halpern-Leistner and Shipman proved that the autoequivalences coming from wall crossing give the autoequivalences coming from these spherical functors.

We start by explaining what we mean by a Calabi-Yau toric GIT problem. Our GIT problem is given by a complex

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} L^v \rightarrow 0.$$

Definition 2.3. We say a torus action T on V is *Calabi-Yau* if T acts through $SL(V)$.

If T acts through $SL(V)$, then for all $(\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$, we require the matrix by which the torus acts

$$\begin{pmatrix} \lambda_1^{q_{11}} \lambda_2^{q_{21}} \dots \lambda_r^{q_{r1}} & 0 & \dots & 0 \\ 0 & \lambda_1^{q_{12}} \lambda_2^{q_{22}} \dots \lambda_r^{q_{r2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_1^{q_{1n}} \lambda_2^{q_{2n}} \dots \lambda_r^{q_{rn}} \end{pmatrix}$$

to have determinant 1. This means

$$\lambda_1^{q_{11} + \dots + q_{1n}} \lambda_2^{q_{21} + \dots + q_{2n}} \dots \lambda_r^{q_{r1} + \dots + q_{rn}} = 1$$

for all $\lambda_i \in \mathbb{C}^*$. So we need the sums $q_{m1} + \dots + q_{mn} = 0$ for all $m \in \{1, \dots, r\}$, in other words we need the sum of the entries in each row of our weight matrix Q to be 0. We call this the Calabi-Yau case, because our GIT quotients are Calabi-Yau if and only if the torus acts through $SL(V)$.

Let us consider a wall W in our GIT problem. Because of the Calabi-Yau condition, the number of copies of $D^b(Z)$ in the wall crossing formula is 0. This is because $\det V = 0$ in formula (4). We instead use the wall-crossing formula on $D^b(Z)$ itself in the Higgs problem Q_W , which may not be Calabi-Yau. Upon applying the wall crossing formula once we find

$$D^b(Z) = \langle D^b(Z'), D^b(Y), \dots, D^b(Y) \rangle_{SOD},$$

where Z' is a GIT quotient in Q_W that's separated by a wall W' from Z , and Y is a subvariety coming from a the Higgs GIT problem of Q_W associated to W' . Notice this GIT problem will also be a Higgs GIT problem of our original GIT problem Q . We imagine iterating the wall crossing formula on $D^b(Z')$ and $D^b(Y)$ until we can no longer cross to a more minimal quotient. This always terminates after a finite

number of steps as each step strictly decreases the rank of the GIT problems. We get SOD (ignoring ordering):

$$D^b(Z) = \langle D^b(Z_0), \dots, D^b(Z_0), \dots, D^b(Z_k), \dots, D^b(Z_k) \rangle_{SOD} \quad (8)$$

where each Z_i is a minimal GIT quotient for a Higgs problem of our original GIT problem. In [14] Kite-Segal prove that no matter how we choose to wall-cross, the subcategories occurring in the final decomposition and their multiplicities are independent of all choices. In the same paper, they show that the irreducible factors Z_i come from Higgs problems associated to *relevant subspaces*.

Definition 2.4. We call a subspace $H \subset L_{\mathbb{R}}^v$ *relevant* if the cone spanned by the weights lying in H is the whole of H .

A relevant subspace H defines a Higgs GIT problem by considering the weights lying on H . These Higgs GIT problems will have a non-empty minimal phase Z_H . The important point is that Z_H are precisely the factors Z_0, \dots, Z_k in the SOD (8) for $D^b(Z)$. In other words, the relevant subspaces H index the irreducible factors.

So how is the spherical functor F defined? In Section 2.1 we presented a method to analyse the stability of points in a toric GIT problem. An equivalent and more general method is the *Hilbert-Mumford Criterion*, which essentially says it's enough to analyse the stability of points in the GIT problems coming from one-parameter subgroups [16]. A point will then be (semi-)stable if it is (semi-)stable for all one-parameter subgroups. In practice in toric GIT problems we only need to consider a finite number of one-parameter subgroups.

We now note that the stability condition for X and X' only differ by one parameter subgroup λ_W normal to the wall. This means only one irreducible component of the unstable for points will different. We look at the component that we remove to obtain X' and see it as a subset of X . We denote it Y . Y is a bundle over Z by [2] so we have maps

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow \pi & & \\ Z & & \end{array}$$

and $F := i_* \pi^*$ where all of the functors are derived. Note the right adjoint is $R = \pi_* i^!$.

Recall that the wall-crossing algorithm produced a semi-orthogonal decomposition:

$$D^b(Z) = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$$

By Theorem 4.14 in [11]:

- (1) The restriction of F to each piece gives a spherical functor $F_i : \mathcal{C}_i \rightarrow D^b(X)$, where we mean restriction in the sense of [13, Chapter 1],
- (2) The spherical twist T_F factors as:

$$T_F = T_{F_1} \circ \cdots \circ T_{F_m}, \quad (9)$$

So for each wall and relevant subspace H , we get some spherical functors $F_H : D^b(Z_H) \rightarrow D^b(X)$.

In the next section we discuss how to construct the *FIPS* and in particular describe the discriminant locus Δ . The *FIPS* will be the complement of Δ , and the upshot is that Δ decomposes into a union of discriminant loci

$$\Delta = \bigcup_{i=0}^k \Delta_i,$$

and each such component will correspond to a variety Z_i from the SOD (8). In our conjecture we will therefore want meridians in $\Pi_1(\text{FIPS})$ about Δ_i to act by twists by spherical functors $F_i : D^b(Z_i) \rightarrow D^b(X)$.

2.4. FIPS. Let us consider a rank r torus T acting on a n -dimensional vector space V , with GIT problem given by complex

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} L^v \rightarrow 0.$$

If we choose a basis for M , A is represented by $n \times (n-r)$ dimensional integer matrix. Let $P \in \mathbb{Z}^k \otimes_{\mathbb{Z}} \mathbb{R}$ be the convex hull of the rays $a_i = A^T(e_i) \in \mathbb{Z}^{n-r}$. We call P the primary polytope. Since we could have chosen a different basis for M , we have an action by $GL_k(\mathbb{Z})$ on this polytope. If we assume the Calabi-Yau condition, we can choose a basis for M so that the column of A is a list of 1s. This shows that the primary polytope P lives at height 1 and is $k-1$ dimensional. The different triangulations of the primary polytope correspond precisely to the different GIT quotients, in that the cone on the triangulated polytope gives you the toric fan of your quotient.

The *superpotential* W_a associated to our GIT problem is the Laurent Polynomial with Newton Polytope given by the rows of A . Explicitly, we assign independent variables X_1, \dots, X_{n-r} to each column and for each row of A we consider the monomial in X_1, \dots, X_{n-r} given by the entries of A in that row. The *superpotential* W_a associated to our GIT problem is a linear combination of these monomials:

$$W_a(X_1, \dots, X_{n-r}) = \sum_{i=1}^n a_i \prod_{j=1}^{n-r} X_j^{A_{ij}} \quad (10)$$

Landau Ginzburg models $((\mathbb{C}^*)^{n-r}, W_a)$, where we see the coefficients $a \in (\mathbb{C}^*)^n$ as being fixed, are Hori-Vafa mirrors to our GIT problem [12]. Not all coefficients

$a \in (\mathbb{C}^*)^n$ give a mirror though. There are non-generic coefficients for which W_a does not give a mirror. These non-generic coefficients live on a hypersurface called *the discriminant locus* of W_a . Recall the discriminant for polynomials in one variable (say quadratics or cubics) captures the non-generic behaviour of having roots with multiplicity greater than 1. The discriminant locus we refer to is a generalisation of this discriminant but for any Laurent polynomial in any number of variables. The theory for this generalised discriminant locus can be found in book [7] by Gelfand-Kapranov-Zelevinski. For this reason the discriminant locus is sometimes referred to as the GKZ discriminant locus.

The discriminant locus Δ of W_a decomposes into a union of other irreducible discriminant loci:

$$\Delta = \bigcup_{i=0}^k \Delta_i,$$

and we will start by explaining how to compute the *principal discriminant component*, which is one of our discriminant components Δ_i . We consider the non-generic a :

$$D_A = \{a \in (\mathbb{C}^*)^n \mid \exists x \in (\mathbb{C}^*)^{n-r} \text{ such that } W_a(x) = 0 \text{ and } dW_a(x) = 0.\}$$

The closure of D_A is an affine variety, which is always irreducible [7, Ch. 9]. In nice examples we can directly compute D_A by solving the system of equations, however in generality this will not work. For example, even the discriminant of the cubic requires more work to deduce. There is a method which in principle allows one to always compute D_A called *The Cayley Method* [7, Ch. 2], although it quickly becomes computationally involved. In the interest of brevity we won't explain the details of it in this report, so we will restrict ourselves to examples where the computation is straight-forward.

We note that D_A is invariant under a rank $n - r$ torus action with weight map A^v (in matrix form this is just taking the transpose). This follows because D_A is invariant under rescaling the X_i variables and an action by A^v on the coefficients $a \in (\mathbb{C}^*)^n$ can be absorbed by the variables by rescaling. We can hence quotient $D_A \subset (\mathbb{C}^*)^n$ by $(\mathbb{C}^*)^{n-r}$ and get $\Delta_A \subset (\mathbb{C}^*)^r$. We call Δ_A the *principal discriminant* of A . Note we did not have to use GIT theory when we took the quotient here as the action is free and hence the naive topological quotient is geometric.

What about the other discriminant components? If we pick a subset $\mathcal{S} \subset \{1, \dots, n\}$, we can consider the collection of rows of A corresponding to \mathcal{S} . We write $A^{\mathcal{S}}$ for the matrix which is the collection of such rows. We define the discriminant component $\Delta_{\mathcal{S}}$ associated to \mathcal{S} to be essentially the principal discriminant component of $A^{\mathcal{S}}$ and see it as living $(\mathbb{C}^*)^r$ by seeing $D_{A^{\mathcal{S}}}$ as living in $(\mathbb{C}^*)^{|\mathcal{S}|} \subset (\mathbb{C}^*)^n$. The

discriminant locus will be a union of these discriminants $\Delta_{\mathcal{S}}$ for some \mathcal{S} . Note $\Delta_A = \Delta_{\{1, \dots, n\}}$. We compute an example below to make this all clear.

Example 2.3. Consider GIT problem

$$0 \rightarrow \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{Q} \mathbb{Z} \rightarrow 0.$$

with

$$Q = \begin{pmatrix} 2 & -1 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} a \\ b \\ c \\ d \end{matrix}$$

We assign independent variables X, Y and Z to the columns of A in order. By definition (10), we have superpotential

$$\begin{aligned} W_{a,b,c,d}(X, Y, Z) &= aX + bXZ + cXZ^{-1} + dXY, \\ &= X(a + bZ + cZ^{-1} + dY). \end{aligned}$$

We compute partial differentials

$$\frac{\partial W_{a,b,c,d}}{\partial X} = a + bZ + cZ^{-1} + dY, \quad (11)$$

$$\frac{\partial W_{a,b,c,d}}{\partial Y} = dX, \quad (12)$$

$$\frac{\partial W_{a,b,c,d}}{\partial Z} = X(b - cZ^{-2}). \quad (13)$$

There is no $X \in \mathbb{C}^*$ for which (12)=0, so the principal discriminant locus is empty. Consider $\mathcal{S} = \{1, 2, 3\}$. We have

$$A^{\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

. We have associated superpotential and differentials

$$\begin{aligned} W_{a,b,c}^{\mathcal{S}}(X, Y, Z) &= X(a + bZ + cZ^{-1}), \\ \frac{\partial W_{a,b,c}^{\mathcal{S}}}{\partial X} &= a + bZ + cZ^{-1}, \end{aligned} \quad (14)$$

$$\frac{\partial W_{a,b,c}^{\mathcal{S}}}{\partial Y} = 0, \quad (15)$$

$$\frac{\partial W_{a,b,c}^{\mathcal{S}}}{\partial Z} = X(b - cZ^{-2}). \quad (16)$$

(16)=0 implies $Z = \pm \sqrt{\frac{c}{b}}$. Plugging this into (14)=0 we find $D_{A^{\mathcal{S}}} = \{a^2 = 4cb\} \subset (\mathbb{C}^*)_{a,b,c,d}^4$. We can use the rank 3 torus action to scale out a, c , and d and we get discriminant component $\{c = 1/4\} \in \mathbb{C}_c^*$. We haven't shown this yet but it turns out that this gives us the only non-trivial component of the discriminant locus Δ of A , so that $FIPS$ is $\mathbb{C}^* \setminus \{1/4\}$.

Not all subsets \mathcal{S} give us discriminant components. Only a certain type of subset will be important for us. These are the *minimal faces* of our primary polytope P .

Definition 2.5. A face $\Gamma \subset P$ is called *minimal* if every ray in Γ is linearly dependent on the other rays in Γ .

The only subsets \mathcal{S} we care about are those that correspond to minimal faces of the primary polytope P . This leads us to the following definitions.

Definition 2.6. The discriminant locus Δ is the union of the subvarieties Δ_Γ , for each minimal face $\Gamma \subset P$ such that Δ_Γ is a hypersurface.

Definition 2.7. We define the Fayet-Iliopoulos Parameter Space (*FIPS*) to be the complement of the discriminant locus Δ in $(\mathbb{C}^*)^r$.

Example 2.4 (Example 2.3 continued). In Example 2.3, the primary polygon is as in Figure 3 below. We only have two minimal faces: the whole polygon P itself and

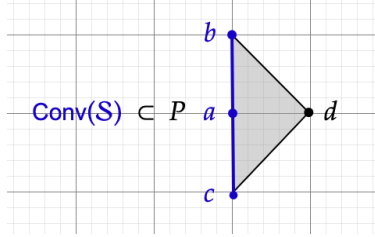


FIGURE 3. The primary polygon P and minimal face \mathcal{S} .

the face that is the convex hull of \mathcal{S} , denoted $\text{Conv}(\mathcal{S})$. This is why the principal component and the component associated to \mathcal{S} gave us the entire discriminant locus.

The following proposition is what allows us to relate discriminant components to components in the wall-crossing SOD (8) for Z_W associated to a wall W in the secondary fan. We just discussed that the former components correspond to minimal faces, and we recall that the components of the latter correspond to relevant subspaces.

Proposition 2.1. *Let $\Gamma \subset P$ be a minimal face of the primary polytope and consider its indexing subset in $\{1, \dots, n\}$, which by abuse of notation we denote Γ too. We can consider subspace H_{Γ^c} spanned by the weights associated to the indexing subset Γ^c . H_{Γ^c} is relevant, and moreover the map $\Gamma \rightarrow H_{\Gamma^c}$ is a bijection between the minimal faces of P and the relevant subspaces of $L_{\mathbb{R}}^v$.*

Proposition 2.1 may inspire us to make some (possibly vague) guesses on how the fundamental group of *FIPS* acts on the derived categories of our GIT quotients.

By the correspondence, we may roughly postulate that loop around a discriminant component Δ_Γ should correspond to a spherical twist about a spherical functor $F_\Gamma : D^b(Z_{\Gamma^c}) \rightarrow D^b(X)$.

Let us first describe an insightful way to think about $FIPS$. Since $FIP = (\mathbb{C}^*)^r \backslash \Delta$, we can see $FIPS$ as living inside the secondary toric variety \mathcal{F} , which is r -dimensional too. It turns out that the secondary toric variety is a particularly nice compactification of $FIPS$. We can compactify the discriminant locus $\bar{\Delta}$ in \mathcal{F} and we see $FIPS$ now as $\mathcal{F} \backslash \bar{\Delta} \cup \{\text{toric boundary}\}$.

Recall that the fixed points under the torus action in \mathcal{F} correspond to GIT quotients. These clearly live in the toric boundary $\mathcal{F} \backslash (\mathbb{C}^*)^r$ and hence not in $FIPS$. The toric boundary is the union of the divisors corresponding to the 1-dimensional cones in the secondary fan via orbit-cone correspondence. Suppose there are m 1-dimensional cones and denote the corresponding divisors D_i . We can consider the fundamental groupoid of $FIP = \mathcal{F} \backslash \bar{\Delta} \cup D_1 \cup \dots \cup D_m$ with base point set being made up of one base point per GIT quotient, chosen to be ‘close’ to the point in \mathcal{F} corresponding to the GIT quotient.

Conjecture 2.1 (Main Conjecture). Let us consider toric Calabi-Yau GIT problem T acting on V . We consider the fundamental groupoid of $FIPS = \mathcal{F} \backslash \bar{\Delta} \cup D_1 \cup \dots \cup D_m$ with base point set being made up of one base point per GIT quotient, chosen to be ‘close’ to the point in \mathcal{F} corresponding to the GIT quotient. We denote the fundamental groupoid $\Pi_1(FIPS)$. Then $\Pi_1(FIPS)$ acts faithfully on the set S of the derived categories of our GIT quotients via equivalences. Moreover this action also satisfies:

- (1) Given two GIT quotients X and Y separated by a wall and a derived equivalence through wall-crossing, there is a path in $\Pi_1(FIPS)$ that acts on S via that derived equivalence.
- (2) Given a GIT quotient X , a loop around a discriminant component Δ_Γ should correspond to a spherical twist about a spherical functor $F_\Gamma : D^b(Z_{\Gamma^c}) \rightarrow D^b(X)$.
- (3) Given a GIT quotient X , recall that there is a corresponding chamber in the secondary fan. Then loops around the toric boundary divisors from 1-dimensional cones in the chamber correspond to twists by $\text{Pic}X$.

We will be proving conditions (2) and (3) of this conjecture in a rank 2 GIT problem in Section 3.

Remark 2.1 (Important Remark). We make a remark with regards to Definition 2.7 of $FIPS$. It turns out that if we want condition (3) in Conjecture 2.1 to be true, we may need to consider a bigger class of coefficients of the superpotential W_a . A

priori we considered $a \in (\mathbb{C}^*)^n$, but we actually need to be more careful. If a ray (row of A) is not a vertex of the primary polytope P , then it turns out we should allow the corresponding coefficient to be 0 in $FIPS$ as otherwise we lose stacky information. So in fact we consider $a \in (\mathbb{C})^n \setminus \{\prod_{a_i \in \text{Vert } P} a_i\}$. We will illustrate this in the following example.

Example 2.5 (Example 2.3 continued). The GIT problem in Example 2.3 is \mathbb{C}^* acting on \mathbb{C}^4 by $\lambda \cdot (x, y, z, w) = (\lambda^2 x, \lambda^{-1} y, \lambda^{-1} z, w)$. The secondary fan is

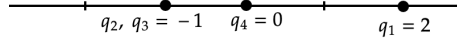


FIGURE 4. The secondary fan.

The secondary toric variety is the smooth stack $\mathcal{F} = \mathcal{P}_{q_1, q_2}^1(2, 1)$. A positive stability condition will have us remove $\{x = 0\}$. We scale out the x coordinate but end up with a leftover \mathbb{Z}_2 action with weight 1 on the y and z coordinates, so we have quotient $X_+ = [\mathbb{C}_{y,z}^2 / \mathbb{Z}_2] \times \mathbb{C}_w$. A negative stability condition will have us remove $\{y = z = 0\}$, and so we have quotient $X_- = \mathbb{C}_{x,w}^2 \times \mathbb{P}_{[y:z]}^1$. So $\text{Pic} X_+ = \mathbb{Z}_2$ and $\text{Pic} X_- = \mathbb{Z}$.

To $\mathcal{S} = \{1, 2, 3\}$ we have corresponding relevant subspace $H_{\{4\}} = 0$ via Proposition 2.1. The GIT problem for this relevant subspace is \mathbb{C}^* acting on \mathbb{C} with weight 0. The quotient is obviously just \mathbb{C} (with infinite isotropy group though?).

Recall we computed that the discriminant locus (before quotienting by the action of A) was $\{a^2 = 4ab\} \subset (\mathbb{C}^*)_{a,b,c,d}^4$. Note though in Figure 3, a is not one of the vertices of the primary polytope. By Remark 2.1, $FIPS$ should actually be the complement of $\{a^2 = 4ab\}$ in $\mathbb{C}_a \times (\mathbb{C}^*)_{b,c,d}^3$ instead. We use A to scale out b, c, d and we end up with a \mathbb{Z}_2 action on a , so $FIPS = [\mathbb{C}_a \setminus \{a = \pm 2\} / \mathbb{Z}_2]$. We see $FIPS$ as a subset of \mathcal{F} via embedding $a \mapsto [a : 1]$. $[0 : 1]$ in \mathcal{F} corresponds to X_+ , which corresponds to the origin in $FIPS$. If we pick base point in $FIPS$ close to 0 we get orbifold fundamental group $\pi_1(FIPS) = \langle \alpha, \beta | \alpha^2, [\alpha, \beta] \rangle$ where α corresponds to a loop around the origin and β is a loop around $a = \pm 2$. $\pi_1(FIPS)$ should act on $D^b(X_+)$ where α is twisting by the generator of $\text{Pic} X_+ = \mathbb{Z}_2$ and β is the spherical twist about spherical functor $F : D^b(\mathbb{C}_w) \rightarrow D^b(X_+)$ given by inclusion. Note that if we didn't have relation α^2 in $\pi_1(FIPS)$ this wouldn't work, which is why Remark 2.1 is important.

3. RANK 2 EXAMPLE

Let us consider rank 2 toric GIT problem $(\mathbb{C}^*)^2$ acting on \mathbb{C}^5 via weight matrix $Q : \mathbb{Z}^5 \rightarrow \mathbb{Z}^2$:

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} & a & b & c & d & e \end{matrix}$$

We have kernel $A : \mathbb{Z}^3 \rightarrow \mathbb{Z}^5$:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix}$$

We start by computing what our GIT quotients are. It turns out when the character lattice is 2-dimensional, the wall and chamber decomposition described in Section 2.1 is produced immediately by taking the positive cones on the weights.

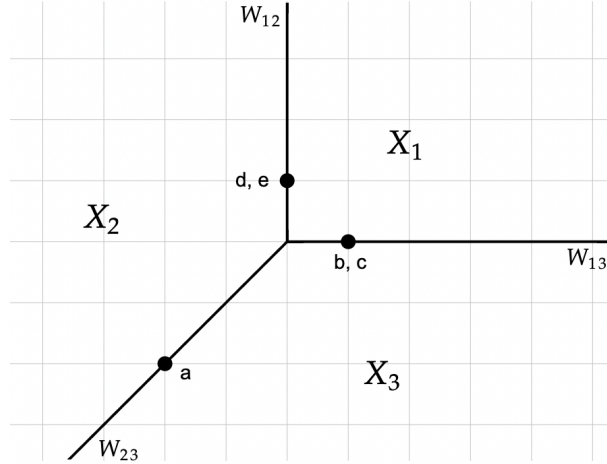


FIGURE 5. The secondary fan of the rank 2 example.

In Figure 5, labelled by X_1, X_2 and X_3 , and between them, walls W_{12} , W_{13} , and W_{23} . As per Section 2.1 we compute the ideals:

$$\mathcal{I}_{X_1} = \langle db, dc, eb, ec \rangle$$

$$\mathcal{I}_{X_2} = \langle da, ea \rangle$$

$$\mathcal{I}_{X_3} = \langle ba, ca \rangle$$

Using toric geometric invariant theory we get that the unstable loci for each chamber X_i is given by the vanishing of the ideal \mathcal{I}_{X_i}

$$S_1 = V(\mathcal{I}_{X_1}) = \{d = e = 0\} \cup \{b = c = 0\}$$

$$S_2 = V(\mathcal{I}_{X_2}) = \{a = 0\} \cup \{d = e = 0\}$$

$$S_3 = V(\mathcal{I}_{X_3}) = \{a = 0\} \cup \{b = c = 0\}$$

and so our GIT quotients are

$$X_1 = \frac{\mathbb{C}^5 \setminus \{d = e = 0\} \cup \{b = c = 0\}}{(\mathbb{C}^*)^2}$$

$$X_2 = \frac{\mathbb{C}^5 \setminus \{a = 0\} \cup \{d = e = 0\}}{(\mathbb{C}^*)^2}$$

$$X_3 = \frac{\mathbb{C}^5 \setminus \{a = 0\} \cup \{b = c = 0\}}{(\mathbb{C}^*)^2}$$

It's not hard to see that our quotients are in fact the smooth Deligne-Mumford stacks

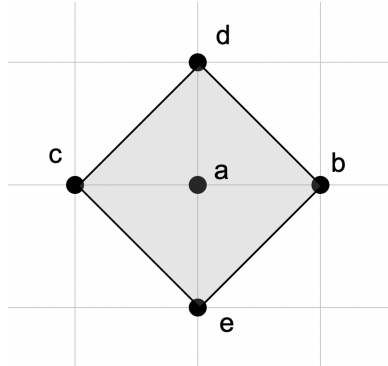
$$X_1 = \mathcal{O}(-2, -2)_{\mathbb{P}_{b:c}^1 \times \mathbb{P}_{d:e}^1}$$

$$X_2 = \left[\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}_{d:e}^1} / \mathbb{Z}_2 \right]$$

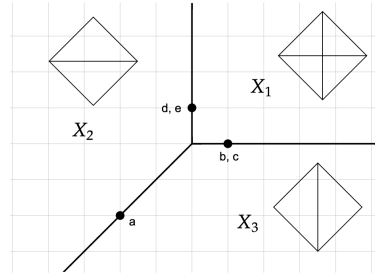
$$X_3 = \left[\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}_{b:c}^1} / \mathbb{Z}_2 \right]$$

where \mathbb{Z}_2 acts on the fibres by -1 for both X_2 and X_3 , so that we have a stacky \mathbb{P}^1 in both cases (the zero sections of the rank 2 vector bundles).

We can also look at the primary polytope $P \subset \mathbb{Z}^2$, which in this case is just a polygon. Recall that triangulations of the P give us toric fans for our GIT quotients, see Figure 6a. For a detailed explanation of how to find which triangulation corresponds to which chamber, see Section 4 of [4]. By Section 2.4, we have only



(A) The primary polytope.



(B) The secondary polytope with the triangulations of the primary polygon.

FIGURE 6

one discriminant component, the principal one, because our P has no minimal faces other than P itself. So $\Delta = \Delta_P$. We compute the non-generic coefficients of the Laurent polynomial $W_{a,b,c,d,e} : (\mathbb{C}^*)^3 \rightarrow \mathbb{C}$

$$W_{a,b,c,d,e}(X, Y, Z) = X(a + bY + cY^{-1} + dZ + eZ^{-1})$$

where $(a, b, c, d, e) \in \mathbb{C}^5 \setminus \{bcde = 0\}$. Note we don't remove $\{a = 0\}$ as the corresponding ray in Figure 6a of the primary polytope P is not a vertex of P . Non-generic coefficients D_A of $W_{a,b,c,d,e}$ are $(a, b, c, d, e) \in \mathbb{C}^5 \setminus \{bcde = 0\}$ where $W_{a,b,c,d,e}$ has critical values. By construction we have an action of the 3-dimensional torus $(\mathbb{C}^*)^3$ on $\mathbb{C}^5 \setminus \{bcde D_A = 0\}$ with weights prescribed by the matrix A^T . We quotient by $(\mathbb{C}^*)^3$ and get that our $FIPS$ is the smooth DM stack

$$FIPS = \mathcal{P}_{1:2:2}^2 \setminus \{bd\Delta = 0\}$$

where we scaled c and e to 1 and are in coordinates $[a : b : d]$, and Δ is the image of D_A after quotienting.

The secondary toric stacky fan is $\mathcal{F} = (\mathbb{Z}^2, \Sigma_{\mathbb{P}^2}, \beta : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2)$ where $\Sigma_{\mathbb{P}^2}$ is the standard toric fan for \mathbb{P}^2 , and β is given by

$$\beta(e_1) = (-2, -2), \beta(e_2) = (1, 0), \beta(e_3) = (0, 1)$$

The realisation of this stacky fan is the smooth DM stack $\mathcal{F} := \mathbb{P}_{1:2:2}^2$, with coordinates $[a : b : d]$. We can see directly how $FIPS$ naturally embeds into \mathcal{F} via inclusion.

We now explicitly compute the discriminant locus. Recall we scaled out c and e . We need to find the a, b and d for which the superpotential

$$W_{a,b,d}(X, Y, Z) = X(a + bY + Y^{-1} + dZ + Z^{-1})$$

has critical values. As this problem is simple and our polygon is very symmetric, we can actually do this directly without appealing to more complicated techniques described in [7]. We just need to find the a, b, d for which the partial differential equations have a common zero:

$$\frac{\partial W_{a,b,d}}{\partial X} = a + bY + Y^{-1} + dZ + Z^{-1}, \quad (17)$$

$$\frac{\partial W_{a,b,d}}{\partial Y} = b - Y^{-2}, \quad (18)$$

$$\frac{\partial W_{a,b,d}}{\partial Z} = d - Z^{-2}. \quad (19)$$

Equation (18) and (19) imply $Y = \pm \frac{1}{\sqrt{b}}$ and $Z = \pm \frac{1}{\sqrt{d}}$, and plugging this into (17) we get

$$\Delta = \{d^2 + b^2 + a^4 - 2bd - 2a^2b - 2a^2d = 0\} \quad (20)$$

We will be computing the fundamental group of $FIPS$, where we focus our attention to GIT quotient X_3 by picking our base point in $FIPS$ close to the toric

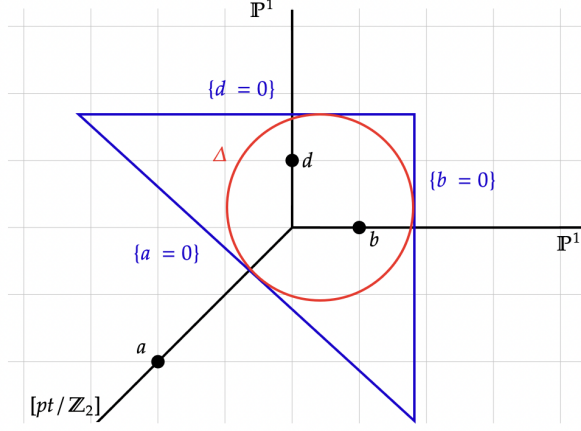


FIGURE 7. A picture of $FIPS$ inside the secondary toric variety \mathcal{F} . The blue is the toric boundary of \mathcal{F} via orbit-cone correspondence. The discriminant Δ is red and sits inside \mathcal{F} . We see that the discriminant locus meets each curve in one point with multiplicity two by equation (20). The variety labels to the walls W are the Z_W from the wall-crossing formula, see Section 2.2.

fixed point $[0, 0, 1] \in \mathcal{F}$. Following condition (3) in Conjecture 2.1, the meridians about $\{a = 0\}$ and $\{b = 0\}$ should correspond to twisting by line bundles of X_3 . Recall $X_3 = [\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1_{b:c}}/\mathbb{Z}_2]$ with local coordinates on the fibres d, e , and we have Picard group

$$\text{Pic} X_3 \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad (21)$$

where the torsion free part comes from $\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1_{b:c}}$ and \mathbb{Z}_2 comes from a choice of representation of the action of \mathbb{Z}_2 ¹². We represent our line bundles by $\mathcal{O}(n, m)$ where $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}_2$.

In line with this we will scale out d and now think of $FIPS$ as $[(\mathbb{C}^2 \setminus \{bf(a, b) = 0\})/\mathbb{Z}_2]$ where $f(a, b) = (a^2 + b)^2 - 4a^2b - 2(a^2 + b) + 1$ and \mathbb{Z}_2 acts on a by -1 and it acts on b trivially. Our choice of base point in the fundamental group will now be a point close to the origin.

What should a meridian around $\{f(a, b) = 0\}$ correspond to? By condition (2) in Conjecture 2.1, we want a meridian about discriminant component Δ_Γ for $\Gamma \subset P$ a minimal face to correspond to a spherical twisting about a spherical

¹²The reader may find it helpful to sort of think of coherent sheaves on X_3 as finitely generated $\mathbb{Z} \oplus \mathbb{Z}_2$ graded $C[b, c, d, e]$ modules where d, e are in graded piece $(-1, -1)$ and b, c in graded piece $(1, 0)$. This is imprecise.

functor $F_\Gamma : Z_{\Gamma^c} \rightarrow D^b(X_3)$. In this case we only have one non-trivial discriminant component which is $\Gamma = P$. We have $\Gamma^c = \emptyset$ and hence $Z_{\Gamma^c} = pt$. So we want meridians about Δ to correspond to spherical twists about spherical functors $F : D^b(pt) \rightarrow D^b(X_3)$. By our discussion in Section 2.3 on spherical functors, this is just a spherical twist about spherical object $F(\mathcal{O}_{pt})$.

A priori it seems like we have several spherical functors $F : D^b(pt) \rightarrow D^b(X_3)$ coming from the two walls W_{13} and W_{23} . For W_{13} the fixed locus under $(0, 1)$ is $\{a = d = e = 0\}$, and we can quotient by the remaining action and find $Z_{13} = \mathbb{P}_{b,c}^1$. Similarly the fixed locus under $(1, -1)$ is $\{a = 0\}$, and we can quotient by the remaining action and find $Z_{23} = [pt/\mathbb{Z}_2]$. Via wall crossing we have two SODs $D^b(Z_{13}) = \langle D^b(pt), D^b(pt) \rangle_{SOD}$ and $D^b(Z_{23}) = \langle D^b(pt), D^b(pt) \rangle_{SOD}$. Looking at the GIT quotient X_1 on the other side of W_{13} , we see we remove $\{d = e = 0\}$ instead of $\{a = 0\}$ for X_3 . In X_3 , $\{d = e = 0\}$ is just the stacky zero section $[\mathbb{P}_{b,c}^1/\mathbb{Z}_2]$.

$$\begin{array}{ccc} [\mathbb{P}_{b,c}^1/\mathbb{Z}_2] & \xrightarrow{i_{13}} & X_3 \\ \downarrow \pi_{13} & & \\ Z_{13} = [pt/\mathbb{Z}_2] & & \end{array}$$

Looking at the GIT quotient X_2 on the other side of W_{23} , we see we remove $\{d = e = 0\}$ instead of $\{b = c = 0\}$ for X_3 . Again $\{d = e = 0\}$ is just the stacky zero section $[\mathbb{P}_{b,c}^1/\mathbb{Z}_2]$.

$$\begin{array}{ccc} [\mathbb{P}_{b,c}^1/\mathbb{Z}_2] & \xrightarrow{i_{23}} & X_3 \\ \downarrow \pi_{23} & & \\ Z_{23} = \mathbb{P}_{b,c}^1 & & \end{array}$$

We have two spherical functors $F_{13} = i_{13*}\pi_{13}^*$ and $F_{23} = i_{23*}\pi_{23}^*$ coming from the walls W_{13} and W_{23} , where all these functors are derived. If we restrict these both to the SOD components of $D^b(Z_{13})$ and $D^b(Z_{23})$ we just get spherical object $\mathcal{O}_{\mathbb{P}^1}$ twisted by some line bundle, depending on the choice of SOD component. This answers our question: we will want a meridian around $\{f(a, b) = 0\}$ to correspond to twisting about $\mathcal{O}_{\mathbb{P}^1}$ (or a twist of it).

Now that we've explained what meridians should correspond to it's time to compute the fundamental group of $FIPS = [(\mathbb{C}^2 \setminus \{bf(a, b) = 0\})/\mathbb{Z}_2]$ with base point near the origin. This is a smooth DM stack and to compute the orbifold fundamental group, we first remove the \mathbb{Z}_2 orbifold curve $\{a = 0\}$ and compute the fundamental group of this non-stacky space. In the group presentation there will be a representative γ for the meridian around the stacky locus of the orbifold, and to obtain the orbifold fundamental group we just need to quotient out by γ to the power of the

order of the finite group acting i.e. γ^2 .

So we compute the fundamental group of $[\mathbb{C}^2 \setminus \{abf(a, b) = 0\} / \mathbb{Z}_2]$. We have a homeomorphism via to $X = \mathbb{C}^2 \setminus \{abg(a, b) = 0\}$ where $g(a, b) = (a+b)^2 - 4ab - 2(a+b) + 1$. We use the Zariski-Van-Kampen method and braid monodromy techniques for computing the fundamental group of curve complements. The first thing we need to do this is to choose a projection to \mathbb{C}^1 which is almost everywhere a locally trivial fibration. This is explained well in [5]. In Figure 8 we see the real picture of X , where the parabola intersects both axes at tacnodes $(0, 1)$ and $(1, 0)$.

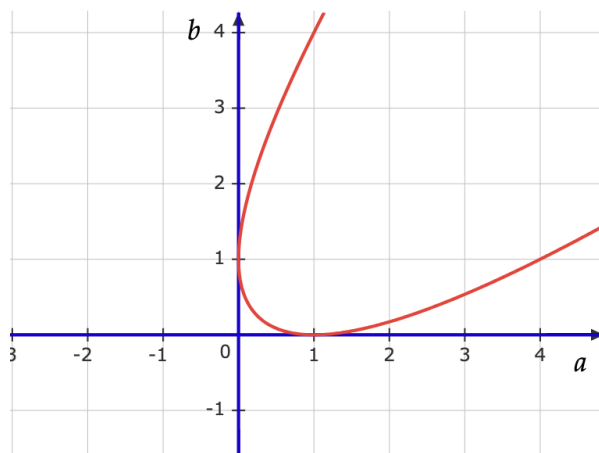


FIGURE 8. Real picture of X .

We have a choice of which projection to use, but the best projection is one where we can maximally exploit what we see in the real picture. For example, ideally we want a projection where we can see all non-generic fibres in the real picture. We also want a projection where all the points missing from generic real fibres are in the real picture. This is not necessarily always possible, but in this example it is by projecting onto $\{a = -b\}$. We therefore consider the projection $X \xrightarrow{\pi} \mathbb{C}^1$, defined as $(a, b) \mapsto a - b$, where the fibres are lines in $\mathbb{C}_{a,b}^2$ of gradient 1. We can see that π is a locally trivial fibration outside of the three special fibers:

- $L_1 = \pi^{-1}(z_1)$, the fiber containing the tacnode at $(0, 1)$,
- $L_2 = \pi^{-1}(0)$, the fiber containing the node at $(0, 0)$,
- $L_3 = \pi^{-1}(1)$, the fiber containing the tacnode at $(1, 0)$.

Our first step will be to compute the fundamental group of X without the special fibres. We will do this by the following theorem.

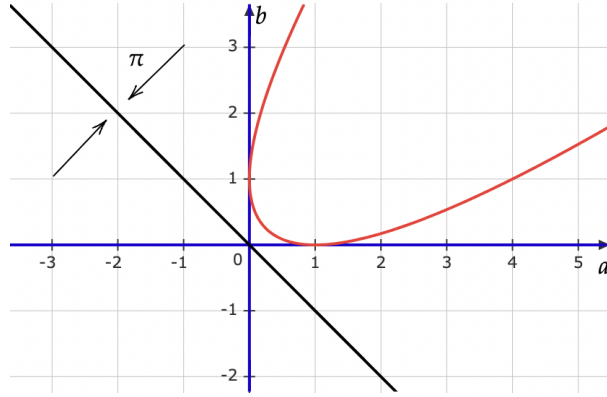


FIGURE 9. The projection π of X in the real picture.

Theorem 3.1. [5, Theorem 2.1] *Suppose you have a locally trivial fibration $\pi : M \rightarrow N$ with section $s : N \rightarrow M$ and take $p \in N$. Let $F = \pi^{-1}(p)$ be the fibre over p , and $x \in F \subset X$. Then $\pi_1(M, x) = \pi_1(F, x) \rtimes \pi_1(N, p)$, where the action of $\pi_1(N, p)$ on $\pi_1(F, x)$ is given by monodromy.*

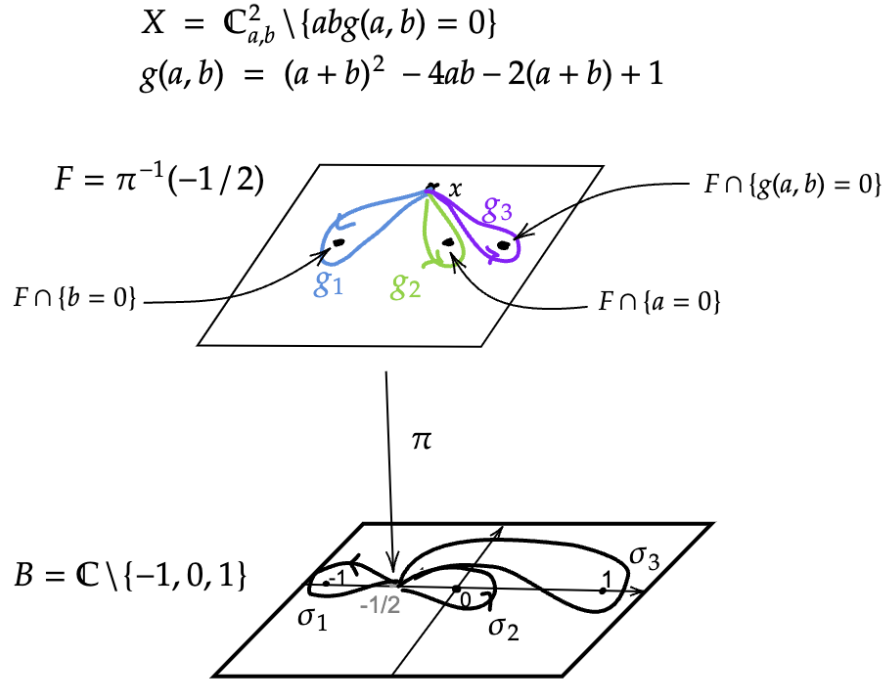


FIGURE 10. Loops in Y .

We restrict our π to the aforementioned locally trivial fibration. Let $Y = X \setminus L_1 \cup L_2 \cup L_3$. $\pi : Y \rightarrow \mathbb{C}^1 \setminus \{-1, 0, 1\}$. It is easy to see that this now a locally trivial fibration (see Figure 10). We denote the base $\mathbb{C}^1 \setminus \{-1, 0, 1\}$ by B . Each fibre of π is now homeomorphic to $\mathbb{C}^1 \setminus \{3pts\}$. We consider loops $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(B, -1/2)$ that freely generate the fundamental group on the the base B with base point $-1/2 \in B$. Let $F = \pi^{-1}(-1/2) \cong \mathbb{C}^1 \setminus \{3pts\}$ be the fibre of π above this basepoint. We choose a basepoint $x \in F \subset X$ in the fibre to be quite far from the three missing points¹³. The loops $g_1, g_2, g_3 \in \pi_1(F, x)$ around the missing points in the fibre coming from $\{b = 0\}$, $\{a = 0\}$, and $\{g(a, b) = 0\}$ respectively freely generate the fundamental group of the fibre with base point $x \in X$. The missing points are precisely the points of intersection of the curves in Figure 8 with the line $b = a - 1/2$. As we move along the loops σ_i on the base we have a monodromy action on g_1, g_2, g_3 . This monodromy action arises from the continuous trajectory of the missing points in the fibres as we move along loops on the base through the homotopy lifting property of locally trivial fibrations. We shall denote the loops after after monodromy $g_j^{\sigma_i}$. By Theorem 3.1 we have group presentation for the fundamental group of Y :

$$\begin{aligned} \pi_1(Y, x) &\cong \pi_1(F, x) \rtimes \pi_1(B, -1/2) \\ &= \langle g_1, g_2, g_3, \sigma_1, \sigma_2, \sigma_3 \mid \forall i, j \in \{1, 2, 3\}, \quad g_j^{\sigma_i} = \sigma_i^{-1} g_j \sigma_i \rangle \end{aligned}$$

So we now need to understand $g_j^{\sigma_i}$. As we move around σ_i the three points missing in the fibre move. Let us parametrise σ_i by $t \in [0, 1]$. Braid monodromy techniques become useful because in $\pi^{-1}(\sigma_i([0, 1]))$, the missing points actually form 3 strands of a braid. The local monodromy can be worked out from the braid.

We work this out explicitly for the loop $\sigma_1 \in \pi_1(B, -1/2)$. Locally near $-1 \in B$, our fibration is equivalent to the locally trivial fibration

$$\tau : \mathbb{C}_{x,y}^2 \setminus \{y = -2, y = 0, y = x^2\} \rightarrow \mathbb{C}_x^*.$$

Here we can take $\sigma_1(t) = e^{2\pi i t} \in \mathbb{C}_x^*$ and $\{y = -2\}$, $\{y = 0\}$ and $\{y = x^2\}$ correspond to the curves $\{b = 0\}$, $\{a = 0\}$ and $\{g(a, b) = 0\}$.

We can see the associated braid on three strands in Figure 12. The loops g_1, g_2, g_3 in the fibre F move along the strands and where they end up is precisely the braid monodromy action of σ_1 . We draw the movement of the loops along the braids in Figure 13. In the same way as we see exhibited in Figure 13, any braid on 3 strands acts on $\pi_1(F, x)$. Also, any loop on the base corresponds to a braid up to homotopy. We recall that B_3 is generated by braids b_1 and b_2 and has presentation $B_3 = \langle b_1, b_2 \mid b_1 b_2 b_1 = b_2 b_1 b_2 \rangle$.

¹³More specifically we need to pick $x \in X$ far enough so that if $A \subset F$ is a bounded closed subset containing the paths traced out by the missing points in the fibres as we move along the loops σ_i on the base, then $x \in B \setminus A$. We want this so that later we can easily argue that gluing the special fibres back in and computing $\pi_1(X)$ just amounts to trivialising the σ_i .

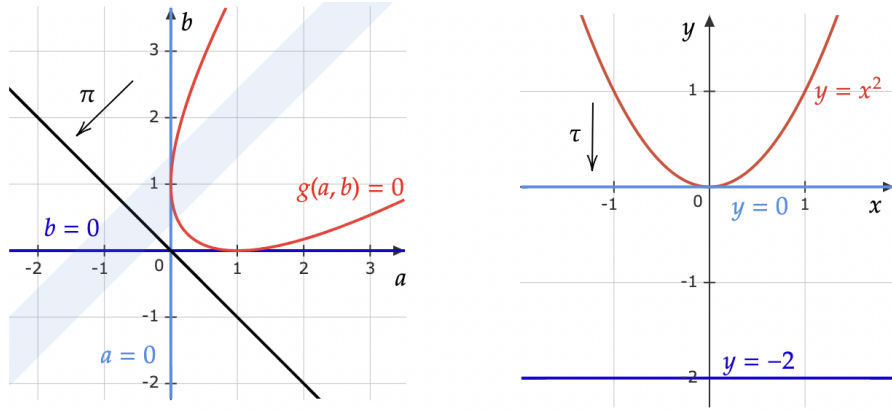


FIGURE 11. The locally trivial fibration $\pi : Y \rightarrow B$ locally around $-1 \in B$.

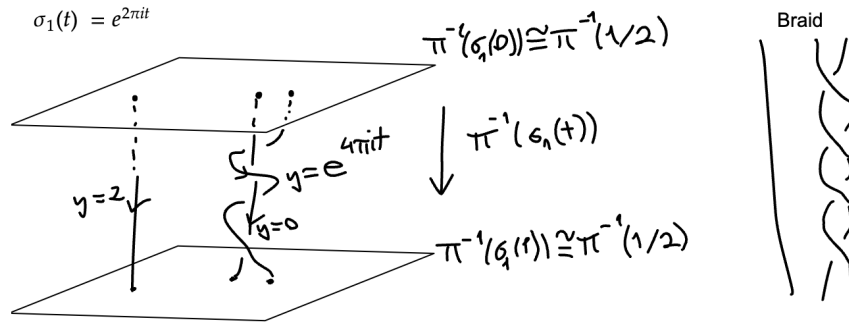


FIGURE 12. The braid associated to σ_1 .

In Figure 12 we see that the braid for σ_1 is $(b_1)^4$. With reference to Figure 15 the braid b_1 acts on g_1 , g_2 , and g_3 by

$$\begin{aligned} g_1 &\mapsto g_1 \\ g_2 &\mapsto g_2 g_3 g_2^{-1} \\ g_3 &\mapsto g_2 \end{aligned}$$

and the braid b_2 acts on g_1 , g_2 , and g_3 by

$$\begin{aligned} g_1 &\mapsto g_1 g_2 g_1^{-1} \\ g_2 &\mapsto g_1 \\ g_3 &\mapsto g_3 \end{aligned}$$

A quick computation shows that the local action of $\sigma_1 = (b_1)^4$ is

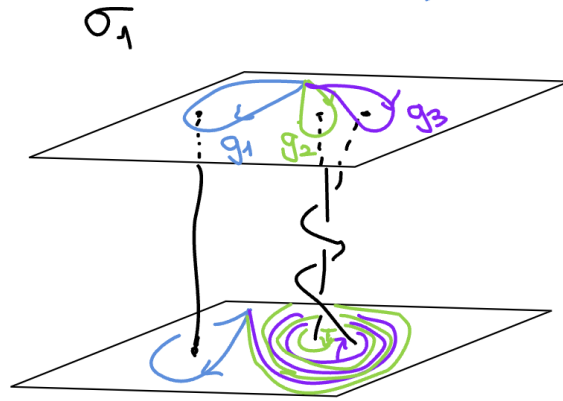


FIGURE 13. The braid monodromy action of σ_1 on g_1, g_2, g_3 .

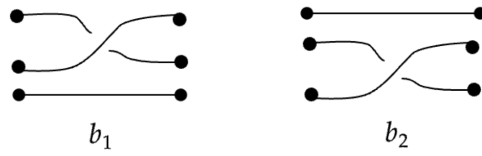


FIGURE 14. The standard generators of the braid group on three stands B_3 .

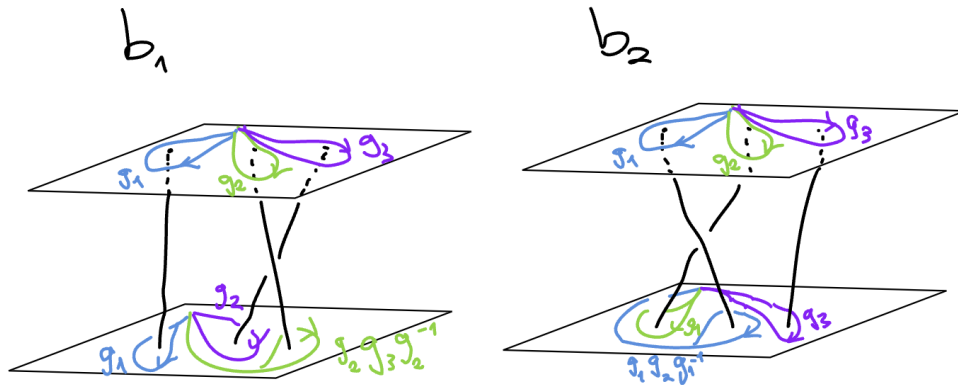


FIGURE 15. The actions of b_1 (left) and b_2 (right) on loops g_1, g_2, g_3

$$\begin{aligned}
g_1 &\mapsto g_1, \\
g_2 &\mapsto (g_2 g_3 g_2 g_3) g_2 (g_2 g_3 g_2 g_3)^{-1}, \\
g_3 &\mapsto (g_2 g_3 g_2) g_3 (g_2 g_3 g_2)^{-1},
\end{aligned}$$

We can compute the local monodromy of σ_2 and σ_3 in the same way, by looking at our locally trivial fibration locally. A crucial point is that looking locally only gives us the local monodromy, but not the global one¹⁴. Given our choice of base point, our computation of the local monodromy σ_1 and σ_2 will be the global one. This is not true for σ_3 .

Repeating the process above we get that the local braid σ_2 is b_2^2 . For σ_2 the local braid is all we need to compute the global monodromy action due to our choice of base point on B . For σ_3 the local monodromy braid is also b_1^4 (just like for σ_1 , which makes sense given the symmetry of X and our fibration). But the local braid is not enough to give us the global monodromy action because to get from our base point $-1/2 \in B$ to a base point in the the local locally trivial fibration picture around $1 \in B$ we need to transverse the loop σ_2 half way, and then we need to bring it back through that segment of σ_2 again. In other words, since $\sigma_2 = b_2^2$, our global braid is going to be $\sigma_3 = b_2 b_1^4 b_2^{-1}$. We can now finally write down what $g_i^{\sigma_j}$ are:

$$\begin{aligned}
g_1^{\sigma_1} &= g_1, \\
g_2^{\sigma_1} &= (g_2 g_3 g_2 g_3) g_2 (g_2 g_3 g_2 g_3)^{-1} \\
g_3^{\sigma_1} &= (g_2 g_3 g_2) g_3 (g_2 g_3 g_2)^{-1}, \\
g_1^{\sigma_2} &= (g_1 g_2) g_1 (g_1 g_2)^{-1}, \\
g_2^{\sigma_2} &= g_1 g_2 g_1^{-1}, \\
g_3^{\sigma_2} &= g_3, \\
g_1^{\sigma_3} &= (g_1 g_3 g_1 g_3) g_1 (g_1 g_3 g_1 g_3)^{-1}, \\
g_2^{\sigma_3} &= (g_1 g_3 g_1 g_3) (g_3 g_1 g_3 g_1)^{-1} g_2 (g_3 g_1 g_3 g_1) (g_1 g_3 g_1 g_3)^{-1}, \\
g_3^{\sigma_3} &= (g_1 g_3 g_1) g_3 (g_1 g_3 g_1)^{-1},
\end{aligned}$$

We are finally now ready to write a presentation for the fundamental group of $\pi_1(X, x)$. By the Zariski-Van Kampen Theorem [5, Theorem 2.6], gluing the special fibres L_i back into Y trivialises the loops on the base σ_i . So we have

$$\pi_1(X, x) = \langle g_1, g_2, g_3 | g_i^{\sigma_j} = g_i, i, j \in \{1, 2, 3\} \rangle.$$

Simplifying the relations we get a simple presentation of $\pi_1(X, x)$,

$$\pi_1(X, x) = \langle g_1, g_2, g_3 | g_1 g_2 = g_2 g_1, (g_1 g_3)^2 = (g_3 g_1)^2, (g_2 g_3)^2 = (g_3 g_2)^2 \rangle.$$

¹⁴Computing the global monodromy is where the real issues with computing the fundamental group of curve complements comes from.

This is also known as the even Artin group $\mathbb{A}(2, 4, 4)$. To get the orbifold fundamental group of $\pi_1(FIPS)$ recall we have to quotient by the square of the meridian about $\{a = 0\}$, which is represented by g_2 . So

$$\pi_1(FIPS) = \langle g_1, g_2, g_3 | (g_3 g_2)^2 = (g_2 g_3)^2, [g_1, g_2] = 1, (g_1 g_3)^2 = (g_3 g_1)^2, g_2^2 = 1 \rangle. \quad (22)$$

We go back to our story where $\pi_1(FIPS)$ acts via autoequivalence on the bounded derived category of our phase $X_3 = [\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1_{b,c}} / \mathbb{Z}_2]$ with local coordinates on the fibres d, e . Recall that the meridians g_1 and g_2 about $\{b = 0\}$ and $\{a = 0\}$ should correspond to line bundle twists, and the meridian g_3 about the discriminant locus Δ corresponds to doing a spherical twist about the spherical object $S = \mathcal{O}_{\mathbb{P}^1}$. We recall from Equation (21) the structure of $\text{Pic}X_3$, and set

$$\begin{aligned} g_1 &= \otimes \mathcal{O}(1, 0) : D^b(X_3) \rightarrow D^b(X_3), \\ g_2 &= \otimes \mathcal{O}(0, 1) : D^b(X_3) \rightarrow D^b(X_3) \\ g_3 &= T_S : D^b(X_3) \rightarrow D^b(X_3), \end{aligned}$$

where we recall the definition of a spherical twist about a spherical object $S \in D^b(X_3)$:

$$T_S \mathcal{E} = C(\text{Hom}(S, \mathcal{E}) \otimes S \rightarrow \mathcal{E}).$$

To check that the fundamental group does act in this way, we need to verify that the functors assigned satisfy the group relations. We will exploit the following easy lemma.

Lemma 3.1. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a spherical functor between triangulated categories and let $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ be an autoequivalence. Then*

$$\Phi \circ T_F \circ \Phi^{-1} = T_{\Phi \circ F}.$$

Proof. The proof is obvious. \square

The second relation and the fourth relation in the presentation for $\pi_1(FIPS)$ (Equation (22)) are obvious since tensoring by line bundles is commutative and we know that $\mathcal{O}(0, 1)$ is 2-torsion. The first and third relations can be rewritten:

$$\begin{aligned} g_3 g_2 g_3 g_2^{-1} g_3^{-1} &= g_2^{-1} g_3 g_2, \\ g_3 g_1 g_3 g_1^{-1} g_3^{-1} &= g_1^{-1} g_3 g_1. \end{aligned}$$

Substituting in our functors and using Lemma 3.1 the relations read:

$$\begin{aligned} T_{T_S \mathcal{O}_{\mathbb{P}^1}(0, 1)} &= T_{\mathcal{O}_{\mathbb{P}^1}(0, 1)}, \\ T_{T_S \mathcal{O}_{\mathbb{P}^1}(1, 0)} &= T_{\mathcal{O}_{\mathbb{P}^1}(-1, 0)}. \end{aligned}$$

where we used that $\mathcal{O}(0, 1)^{-1} = \mathcal{O}(0, 1)$. Therefore if we show

$$T_S \mathcal{O}_{\mathbb{P}^1}(0, 1) = \mathcal{O}_{\mathbb{P}^1}(0, 1), \quad (23)$$

$$T_S \mathcal{O}_{\mathbb{P}^1}(1, 0) = \mathcal{O}_{\mathbb{P}^1}(-1, 0), \quad (24)$$

then we're done. Let's start by computing $T_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(0, 1)$. We note that¹⁵

$$\begin{aligned} \mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}(k_1, k_2), \mathcal{O}_{\mathbb{P}^1}(m_1, m_2)) &= \mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(m_1 - k_1, m_2 - k_2)), \\ &= R\Gamma(\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})(m_1 - k_1, m_2 - k_2)). \end{aligned}$$

So we start by computing $\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})$, then we twist it accordingly and take deried global sections (sheaf cohomology). To compute the derived functor we start by Koszul resolving $\mathcal{O}_{\mathbb{P}^1}$:

$$0 \rightarrow \mathcal{O}(2, 0) \xrightarrow{(-e, d)} \mathcal{O}(1, 1)^2 \xrightarrow{(d, e)^T} \mathcal{O} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$$

Ignore $\mathcal{O}_{\mathbb{P}^1}$ in the resolution and taking $\mathrm{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^1})$ we get

$$\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = 0 \leftarrow \mathcal{O}_{\mathbb{P}^1}(-2, 0) \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1}(-1, 1)^2 \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1} \leftarrow 0 \quad (25)$$

We twist by $\mathcal{O}(0, 1)$:

$$0 \leftarrow \mathcal{O}_{\mathbb{P}^1}(-2, 1) \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1}(-1, 0)^2 \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1}(0, 1) \leftarrow 0$$

We take derived global sections and get 0 in all degrees. For $\mathcal{O}_{\mathbb{P}^1}(-2, 1)$ we can see this because there is only one extension (of degree 1) of $\mathcal{O}_{\mathbb{P}^1}(-2)$ in \mathbb{P}^1 , and in $[\mathbb{P}^1/\mathbb{Z}_2]$ it is not \mathbb{Z}_2 invariant:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(0, 1) \xrightarrow{\begin{pmatrix} c & -b \end{pmatrix}} \mathcal{O}_{\mathbb{P}^1}(-1, 1)^2 \xrightarrow{\begin{pmatrix} b \\ c \end{pmatrix}} \mathcal{O}_{\mathbb{P}^1}(-2, 1) \rightarrow 0$$

For $\mathcal{O}_{\mathbb{P}^1}(-1, 0)$ we know $\mathcal{O}_{\mathbb{P}^1}(-1)$ in \mathbb{P}^1 has no sheaf cohomology. For $\mathcal{O}_{\mathbb{P}^1}(0, 1)$ the derived global sections come from genuine sections of $\mathcal{O}_{\mathbb{P}^1}$ in \mathbb{P}^1 i.e. degree 0 polynomials in b, c variables (constants), but these are \mathbb{Z}_2 invariant. So $\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(0, 1)) = 0$ and we have

$$T_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(0, 1) = C(0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(0, 1)) = \mathcal{O}_{\mathbb{P}^1}(0, 1),$$

which is precisely the relation (23). To check the relation (24) we twist (25) by $\mathcal{O}(1, 0)$:

$$0 \leftarrow \mathcal{O}_{\mathbb{P}^1}(-1, 0) \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1}(0, 1)^2 \xleftarrow{0} \mathcal{O}_{\mathbb{P}^1}(1, 0) \xleftarrow{0} 0$$

Taking global sections we get $\mathbb{C}_{b,c}^2$ in degree 0. So

$$T_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(1, 0) = C(\mathcal{O}_{\mathbb{P}^1}^2 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1, 0)) = \mathcal{O}_{\mathbb{P}^1}(-1, 0),$$

using the Euler sequence. This is relation (24).

¹⁵In the second equality we use that the local-to-global spectral sequence terminates on the second page, which is easy to argue.

4. TORIC CALABI-YAU THREEFOLDS OF PICARD RANK 2

In the rank 2 example we presented in Section 3, the GIT quotient X_1 is a toric Calabi-Yau threefold of Picard rank 2. Given any toric Calabi-Yau threefold X of Picard rank 2, we can consider the brane monodromy conjecture by constructing a toric Calabi-Yau GIT problem $(\mathbb{C}^*)^2$ acting on \mathbb{C}^5 for which X is a GIT quotient. Note that we know the rank of the GIT problem 2 because of the rank of the Picard group, and we know the dimension of the problem is 5 because the quotient X is three-dimensional. Our problem is therefore given by exact sequence

$$N \xrightarrow{A} \mathbb{Z}^5 \xrightarrow{Q} L^v$$

with Q the weight matrix and $\ker(Q) = A$, where $N \cong \mathbb{Z}^3$ and $L^v \cong \mathbb{Z}^2$. Recall from Section 2.4 the definition of the primary polygon $P \in \mathbb{Z}^3 \otimes_{\mathbb{Z}} \mathbb{R}$, which is the convex hull of the rays $a_i = A^T(e_i) \in \mathbb{Z}^3$. Because of our Calabi-Yau assumption (the weights sum up to 0), we can choose generators for N so that the first column of A is a list of 1s, and we see P living in the rank 2 lattice in \mathbb{Z}^3 where the first integer coordinate is 1. Imposing the first column of A to be 1 does not uniquely determine generators for N . More precisely, these matrix representations of A are unique up to right multiplication by matrices in $GL_3(\mathbb{Z})$ given in block form as

$$B = \left(\begin{array}{c|cc} 1 & t_1 & t_2 \\ \hline 0 & & \\ 0 & & M \end{array} \right)$$

where $t_1, t_2 \in \mathbb{Z}$ and $M \in GL_2(\mathbb{Z})$. This the primary polygon $P \subset \mathbb{Z}^2$ is only unique up to an action of $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ ¹⁶ given by

$$\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, M \right) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because we know A^T is surjective, we know that the primary polytope P has exactly 5 integer points in it. This means that toric Calabi-Yau threefold of Picard rank 2 come from convex lattice polygons in \mathbb{Z}^2 with exactly 5 integer points, up to the $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ action. We have proved that there are only six such classes of polygons, see Figure 16. The third polygon in Figure 16 actually corresponds to our example in Section 3. The fourth and sixth polygon examples have been worked out in [6] as part of another class of examples. I have also done the computation for the first polygon, which we note would give an example where we have more than one discriminant component as there are two minimal faces. To complete this project I need to work out the conjecture for the second and fifth polygons. Though these

¹⁶Multiplication in $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ is given by

$$\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, M \right) \cdot \left(\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, M \right) = \left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + N \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, MN \right).$$

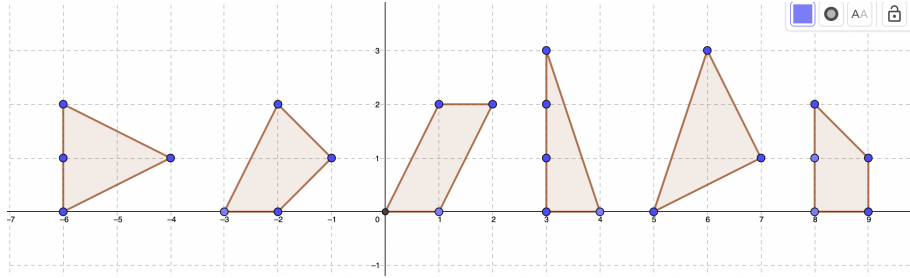


FIGURE 16. The toric Calabi-Yau threefolds of Picard rank 2.

examples only have one discriminant component, they should be harder because we can't compute the discriminant locus directly. We will need to use Cayley method, presented in [7, Chapter 2]. Once we have the curves the fundamental group computation should be harder than those in the examples far, since our curves have been exceptionally nice (the components smooth, with good real pictures). It's my hope that these examples will exhibit some new and interesting behaviour.

We also note that this problem has been solved for a special class of toric CY GIT problems called *quasi-symmetric* [10]. I am thinking about whether this result can be used to generalise to other examples by embedding the data of our GIT problem into a quasi-symmetric one. I am also thinking about whether we can say something about some non-abelian GIT quotients.

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