MICHELA RICCIARDI CELSI **1580884**

ORBIT INJECTION OF A SATELLITE IN MINIMUM TIME

OPTIMAL CONTROL, CONTROL ENGINEERING

INDEX

1. Introduction

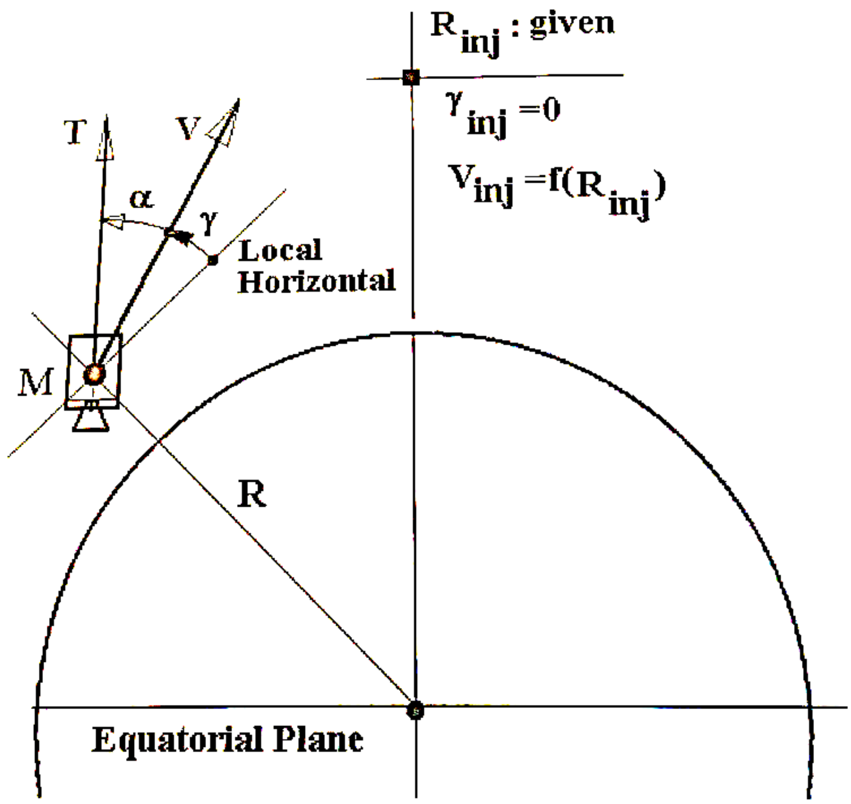

Figure 1. Orbit injection.

The purpose of the project is to study the orbit injection of a satellite and to compute the guidance law $\alpha(t)$ which leads such a satellite to reach the final orbit in minimum time. It is required to insert into a circular orbit – at a height h_{fin} with respect to the Earth – a body having mass M_0 released by a rocket at a height h_0 from the Earth, velocity V_0 and slope of the trajectory γ_0.

So, the aim is to find $\alpha(t)$ such that the maximum payload is delivered into a prescribed circular orbit. This is the reason we why need to maximize the final mass, which means, for a constant burn rate, to minimize the time to get into orbit. Hence, minimizing the time for a given burn rate, the burned propellant is minimized too. So, the infinite-dimensional problem consists in finding the steering law $\alpha(t)$, which is a function of time (an infinite number of points), that minimizes the final time to orbit, represented by the t_f variable.

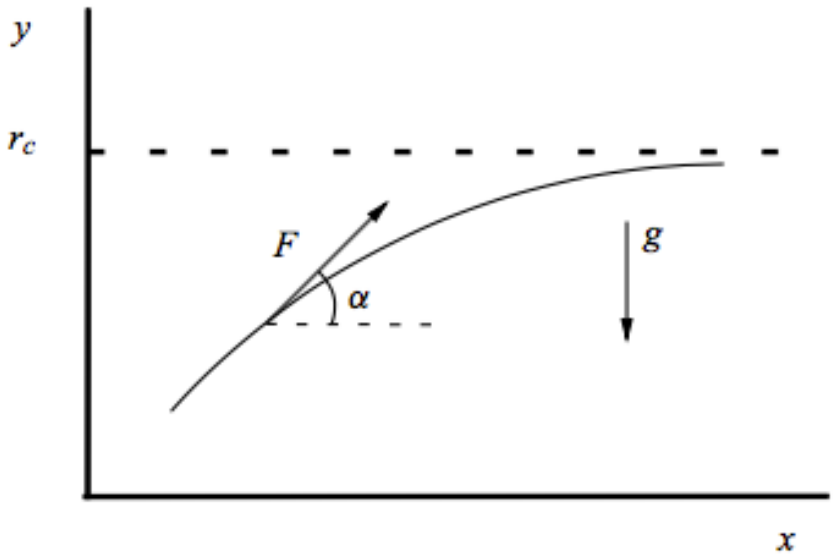

Figure 2. Problem of finding the guidance law to maximize satellite mass.

Traditional launch vehicle guidance may involve different phases. The first one is an open-loop phase for the atmospheric portion of flight which typically is performed by relying on a non-optimal piecewise-linear attitude program. The second one is a closed-loop guidance phase for the exo-atmospheric portion of flight and it yields an analytic solution under particular assumptions. In the end, the third one is a closed-loop guidance phase possibly required when the vehicle is approaching orbital conditions for final precision orbit injection.

Numerical approaches to optimal guidance usually refer to nonlinear programming or multiple shooting. Nonlinear programming is a direct method formulation in which the optimization problem is transformed into a parametric optimization problem and, in this way, the original infinite-dimensional problem is approximated by a finite-dimensional problem in the reduced space of the control parameters and gradient techniques are used to search for a solution that optimizes the performance index.

Furthermore, multiple shooting is used in indirect methods. In this case, optimization is obtained by satisfying a set of necessary conditions which are expressed in the form of a Two-Point Boundary Value Problem (TPBVP), instead of evaluating the performance index directly. As a consequence, for a constrained case this may lead to a Multi-Point Boundary Value Problem (MPBVP).

Before proceeding with the mathematical model, it is important to discuss in general about the launch of the satellite and the controlling support services. They represent a very critical point in the creation of space communications and the most expensive phase of the total system cost. At the same time, the need to make a satellite body capable of surviving the stresses of the launch stages is a major element in their design phase. Satellites are also designed to be compatible with more than one model of launch vehicle and launching type. In a more determined sense, there are multi-stage expendable and, manned or unmanned, reusable launchers. Owing to location and type of site, there are land-based and sea-based launch systems. Additional rocket motors, such as perigee and apogee kick propulsion systems, may also be required. The process of launching a satellite is based mostly on launching into an equatorial circular orbit and after into GEO, but similar processes or phases are used for all types of orbits. The processes involved in the launching technique depend on the type of satellite launcher, the geographical position of the launching site, and constraints associated with the payload. In order to successfully put the satellite into the transfer and drift orbit, the launcher must operate with great precision with regard to the magnitude and orientation of the velocity vector. Moreover, launching a satellite into orbit about the Earth is very expensive due to the high speed required to achieve a low circular orbit. In fact, the lowest energy orbit must be high enough above the sensible atmosphere to avoid immediate dissipation of its energy and orbit decay and re-entry. For this reason, the lowest sustainable orbit is about 160 km above the Earth's surface. At this altitude, the satellite must travel at $7.9\ km/s$ to stay in circular orbit. For example, the cost of reaching circular orbit with the United States Space Shuttle in 2011 was about $22000\ \$/kg$ [1][2][3][4].

2. Mathematical Model

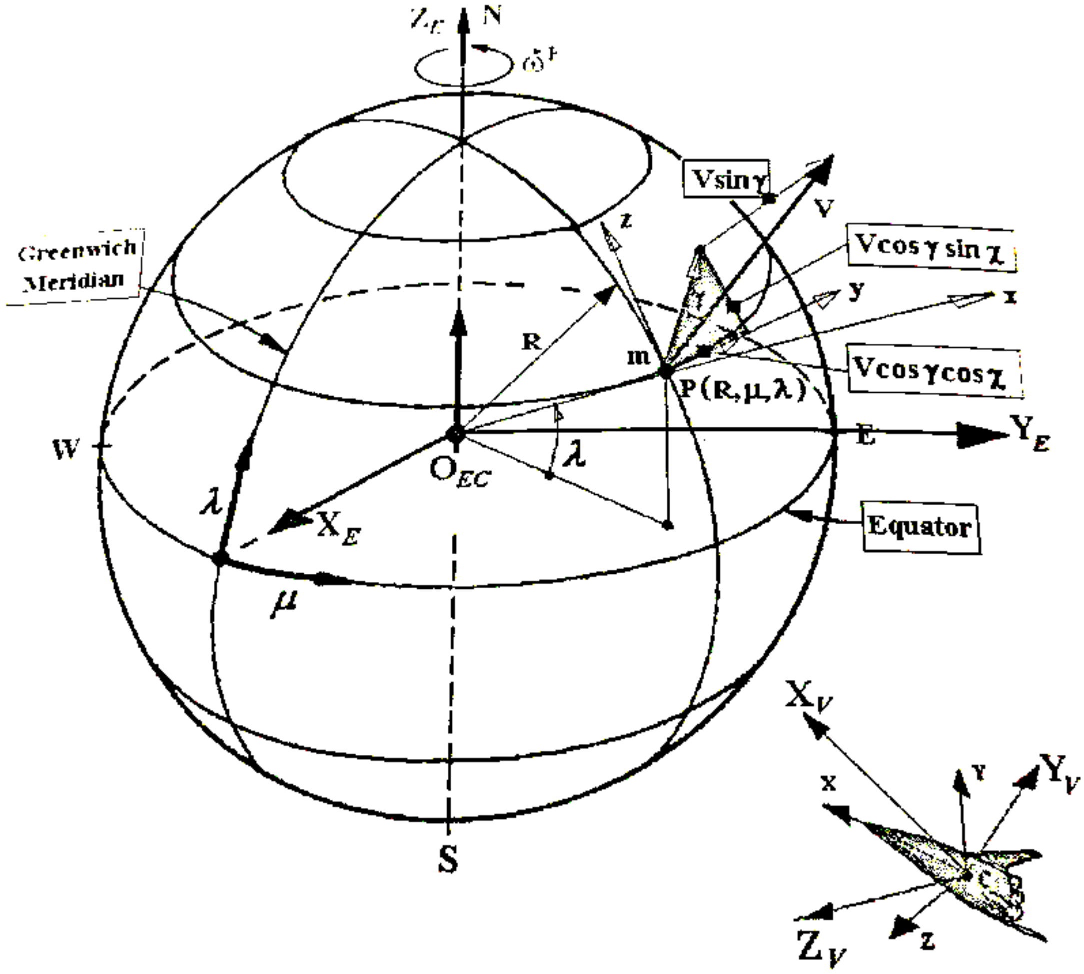

Figure 3. Coordinate reference system.

It is necessary to write the equations of motion in order to study the trajectories around the Earth and in general a constant angular velocity $\omega = \frac{2\pi}{T_r} \cong \frac{2\pi}{86164} = 72.921 \times 10^{-6}\ rad/sec$ is assumed, where T_r corresponds to the duration of a Sideral Day. Given a material point of mass m at a distance R (which is a vector) from the origin (which is the Earth's centre) with a linear velocity V (which is a vector), then the equation of the forces can be written as:

$$m\left[\frac{d\bar{V}}{dt} + 2\bar{\omega}\times\bar{V} + \bar{V} + \bar{\omega}\times\bar{\omega}\times\bar{R}\right] = \bar{F},$$

knowing that $\frac{d\bar{R}}{dt} = \bar{V}$, that $\bar{R} = [R \quad 0 \quad 0]^T$.

Referring to Figure 3 and considering γ (the slope) as the angle between $\bar{V}$ and the tangent plane in P and χ (the heading) as the angle between $V cos\gamma$ and the axis y of the tangent reference frame (taken with positive sign around axis x), the vector $\bar{V}$ can be defined as:

$$\bar{V} = \begin{bmatrix} V \cdot \sin\gamma \\ V \cdot \cos\gamma \cdot \cos\chi \\ V \cdot \cos\gamma \cdot \sin\chi \end{bmatrix}.$$

Introducing the longitude μ and the latitude λ of P with respect to the reference with the Earth's centre as the origin, the following differential cinematic equations are obtained:

$$\begin{cases} \dot{R} = V \cdot \sin\gamma \\ \dot{\mu} = \dfrac{V \cdot \cos\gamma \cdot \cos\chi}{R \cdot \cos\lambda} \\ \dot{\lambda} = \dfrac{V \cdot \cos\gamma \cdot \sin\chi}{R} \end{cases}.$$

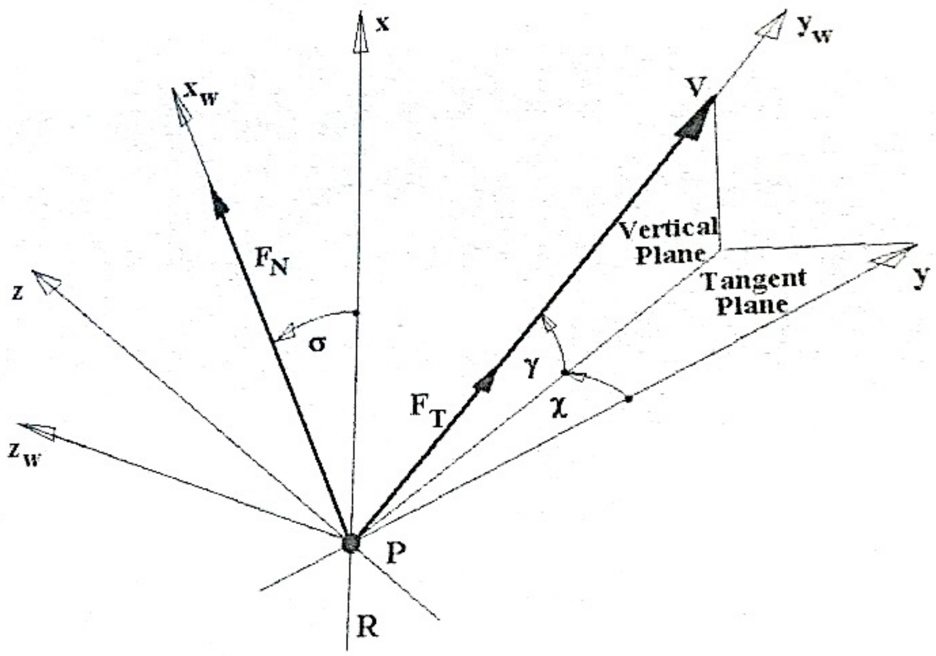

Figure 4. Forces.

Moreover, let the Earth's angular velocity be given by the vector $\bar{\omega} = [\sin\lambda \quad 0 \quad \cos\lambda]^T \cdot \omega$. Taking into account the acting forces (see Figure 4), the final form of the equations is obtained, considering σ as the rotation around Oy_w:

Heading

$$V\frac{d\chi}{dt} = \frac{1}{m}\frac{F_N \sin\sigma}{\cos\gamma} - \frac{V^2}{R}\cos\gamma \cdot \tan\lambda \cdot \cos\chi + 2\omega V[\tan\gamma \cdot \cos\lambda \cdot \sin\chi - \sin\lambda] - \omega^2 R\,\frac{\sin\lambda \cdot \cos\lambda \cdot \cos\chi}{\cos\gamma},$$

Velocity

$$\cos\gamma \cdot \frac{dV}{dt} - V \cdot \sin\gamma \cdot \frac{d\gamma}{dt} = \frac{1}{m}\left[\frac{F_z}{\sin\chi} - \frac{F_N \cdot \sin\sigma}{\tan\chi}\right] - \frac{V^2}{R}\cos\gamma \cdot \sin\gamma - 2\omega V\,\frac{\cos\lambda \cdot \sin\gamma \cdot \sin\chi}{\tan\chi} - \omega^2 R \cdot \sin\lambda \cdot \sin\chi,$$

Slope

$$V\frac{d\gamma}{dt} = \frac{1}{m}F_N \cdot \cos\sigma - g(R) \cdot \cos\gamma + \frac{V^2}{R} \cdot \cos\gamma + 2\omega V \cdot \cos\lambda \cdot \cos\chi + \omega^2 R \cdot \cos\lambda\,(\cos\gamma \cdot \cos\lambda + \sin\gamma \cdot \lambda \cdot \sin\chi).$$

It is more convenient to choose adimensional variables when the state variables assume extremely different values, as in the considered case. In fact, the radius reaches millions of meters while, on the other side, the slope of the trajectory remains around fractions of radians.

The independent variable is represented by the time t. In general, the adimensional parameters used in orbital dynamics problems are:

$\sqrt{\frac{\tilde{\mu}}{R_0}}$ for velocity,

$$\sqrt{\frac{R_0^3}{\tilde{\mu}}} \text{ for time,}$$

$R0$ represents the orbit radius at the initial time and μ is the gravitational parameter of the Earth.
In this way, we can define the new variables as:

- adimensional velocity: $v = \frac{V}{\sqrt{\frac{\tilde{\mu}}{R_0}}}$;
- adimensional time: $\bar{t} = \frac{t}{\sqrt{\frac{R_0^3}{\tilde{\mu}}}}$;
- adimensional radius: $r = \frac{R}{R_0}$;
- adimensional mass: $m = \frac{M}{M_0}$;
- adimensional thrust: $\tau = \frac{T}{\frac{\tilde{\mu}M_0}{R_0^2}} = \frac{T}{g_0 M_0}$.

The differential equations become then:

- with respect to the **Velocity $\boldsymbol{v}$**:

$$\frac{dv}{d\bar{t}} = \frac{\tau}{m} cos\alpha - \frac{1}{r^2} sin\gamma + (\omega t_{orb0})^2 r \cdot \cos\lambda\,(\sin\gamma \cdot \cos\lambda - \cos\gamma \cdot \sin\lambda \cdot \sin\chi)$$

- with respect to the **Slope $\boldsymbol{\gamma}$**:

$$\frac{d\gamma}{d\bar{t}} = \frac{\tau}{mv} \sin\alpha \cdot \cos\sigma + \cos\gamma \left(\frac{v}{r} - \frac{1}{vr^2}\right) + 2\omega t_{orb0} \cdot \cos\lambda \cdot \cos\chi + \frac{(\omega t_{orb0})^2 r}{v} \cos\lambda\,(\cos\gamma \cdot \cos\lambda + \sin\gamma \cdot \sin\lambda \cdot \sin\chi)$$

- with respect to the **Heading $\boldsymbol{\chi}$**:

$$\frac{d\chi}{d\bar{t}} = \frac{\tau}{mv}\frac{\sin\alpha \cdot \sin\sigma}{\cos\gamma} - \frac{v}{r}\cos\gamma \cdot \tan\lambda \cdot \cos\chi + (2\omega t_{orb0})[\tan\gamma \cdot \cos\lambda \cdot \sin\chi - \sin\lambda] - \frac{(\omega t_{orb0})^2 r}{v}\frac{\sin\lambda \cdot \cos\lambda \cdot \cos\chi}{\cos\gamma}.$$

The cinematic equations of the trajectory of the center of mass are:

$$\begin{cases} \dfrac{dr}{d\bar{t}} = v \cdot \sin\gamma \\ \dfrac{d\mu}{d\bar{t}} = \dfrac{v}{r}\dfrac{\cos\gamma \cdot \cos\chi}{\cos\lambda} \\ \dfrac{d\lambda}{d\bar{t}} = \dfrac{v}{r}\cos\gamma \cdot \sin\chi \end{cases}.$$

The mass equation is:

$$\frac{dm}{d\bar{t}} = -\frac{\tau}{v_{exit}}.$$

Having:

- $t_{orb0} = \sqrt{\frac{R_0^3}{\widetilde{\mu}}}$;
- $v_{orb0} = \sqrt{\frac{\widetilde{\mu}}{R_0}}$;
- $v_{exit} = \frac{V_{exit}}{\sqrt{\frac{\widetilde{\mu}}{R_0}}}$.

The initial conditions for the 7 adimensional equations for $\bar{t} = \bar{t}_0 = 0$ are:

- $\gamma = \gamma_0$;
- $v = v_o = \frac{V_0}{V_{orb0}}$;
- $\chi = \chi_0$;
- $r = r_0 = 1$;
- $\mu = \mu_0$;

- $\lambda = \lambda_0$;
- $m = m_0 = 1$.

The conditions related to the circular orbit injection with adimensional variables are:

- $R_{inj} = R_{earth} + h_{inj}$;
- $\gamma_{inj} = 0$;
- $V_{inj} + \omega R_{inj} \cdot \cos i_0 = \sqrt{\tilde{\mu}/R_{inj}}$ where i_0 is the angle of inclination given by $\cos i_0 = \cos \lambda_0 \cdot \cos \chi_0$.

We assume that the body is controlled only on the vertical plane, so it means that there could be heading variations ($\Delta\chi$) of the trajectory and latitude variations ($\Delta\lambda$) when it is outside the equatorial plane [5].

3. Optimal Control Problem

The optimal control problem (e.g., minimum time) can be set as follows:

Given the flight mechanics problem described by the adimensional differential equations introduced above, compute a guidance law $\alpha(t)$ such that the dynamical system is moved from the initial conditions (which are known in advance) to the desired final (or "orbit insertion") conditions in minimum time.

The computation of this guidance law $\alpha(t)$ and of the minimum time can be performed by relying on calculus of variations, more precisely on the solution to the classical Euler-Lagrange equations of the so-called Two-Point Boundary-Value Problem.

Yet, from the computational point of view, the `fmincon.m` routine is adopted, which determines the constrained minimum of a scalar function of several variables, starting from an initial guess estimation of the unknown variables, that is,

$$\min_{\alpha(t),\ t_{min}} f(x) = t_{fin}, \quad s.t.\ g(x) \leq 0.$$

In this case, the function to be minimized is represented by the very amount of time (i.e., t_{min}) required for driving the dynamical system from the initial conditions to the desired final ones. Hence, the problem constraints are represented by the differential equations themselves, together with the set of initial and final conditions. The unknown variables are represented by the final time itself and from the one-dimensional vector $\alpha(t_i)$ which constitutes the guidance law discretized at the time instants t_i over a time interval of amplitude $[0, t_f]$.

The following MATLAB scripts perform the computations described above:

- the main program (**`main.m`**) which provides all the data that are required by the problem and calls the `fmincon.m` routine;

- a function (**`objfun.m`**), which is necessary for formulating the optimal problem and contains the definition of the function that has to be minimized;
- a function (**`constraints.m`**) which is necessary for setting some linear and nonlinear constraints that the optimized should satisfy;
- a function (**`orbequa3d`**) that contains the differential equations that have to be integrated at each time step.

4. Simulations

Two cases are analysed in this project:

1. The first one is an **equatorial launch heading East**:

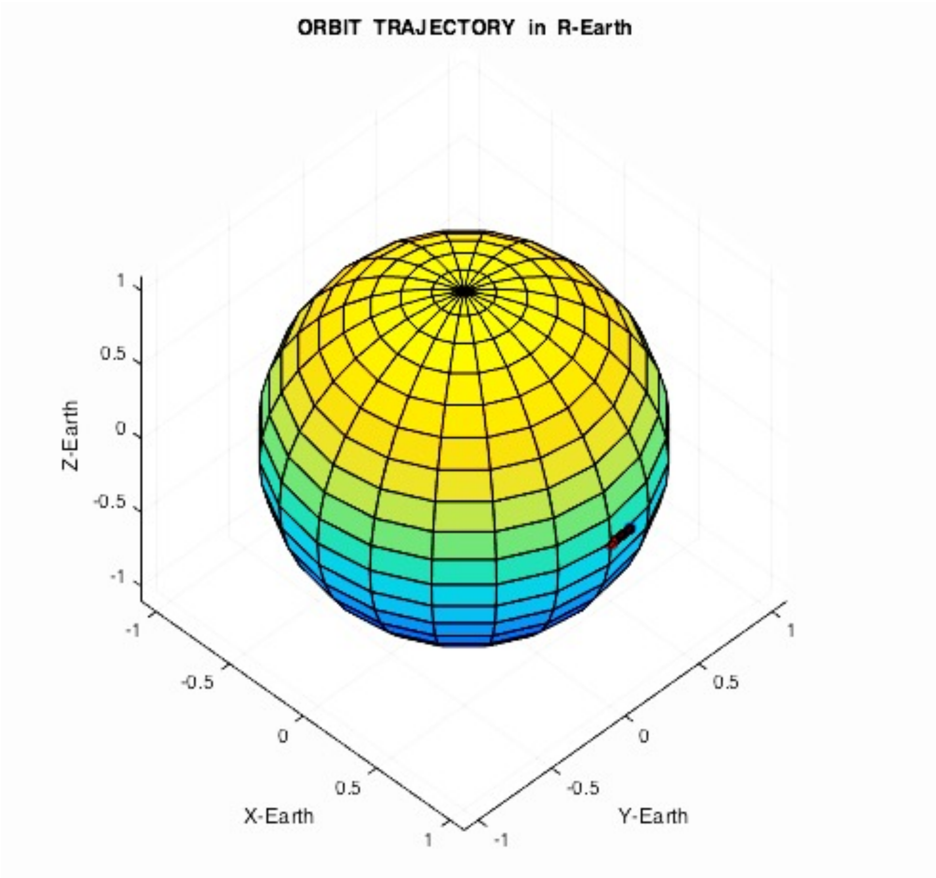

Figure 5. East Orbit Trajectory.

The following initial conditions are assumed:

- velocity with respect to the Earth: $V_0 = 6000\ m/sec$;
- trajectory's slope: $\gamma_0 = 5°$;
- trajectory's heading: $\chi_0 = 0$;
- height: $h_0 = 4000\ m$;
- longitude: $\lambda_0 = 0$;
- latitude: $\mu_0 = 0$;

- mass: $M_0 = 430 \frac{kg}{\frac{m}{sec^2}}$.

The explicit conditions for injection are given by:

- trajectory's slope: $\gamma_{inj} = 5°$;
- height: $h_{inj} = 100000\ m$;
- velocity: $V_{inj} = 7383\ m/sec$.

In this way, the satellite is surely heading East due to the fact the heading and the longitude are set to zero. In the next figure, velocity, slope, radius and mass are shown:

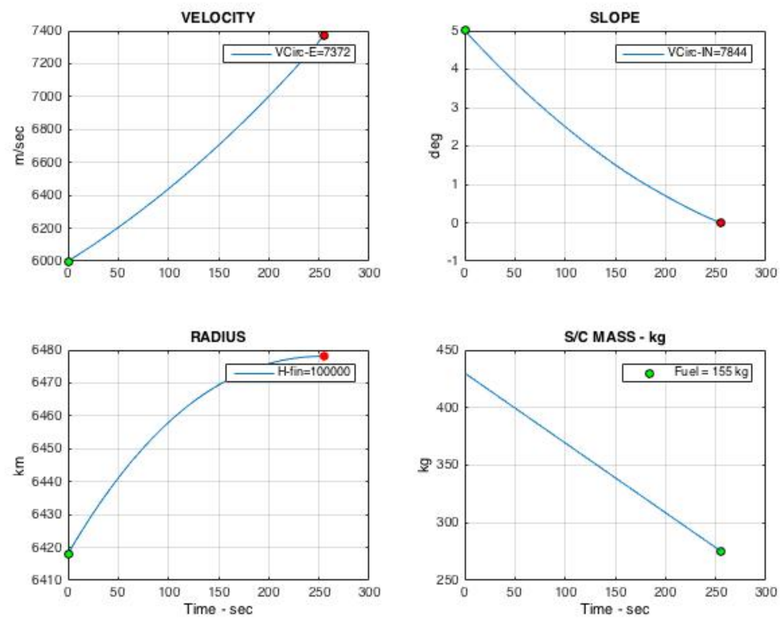

Figure 6. East: Velocity, Slope, Radius, Mass.

It is easy to notice that the obtained optimal control law (see Figure 7) satisfies the final imposed conditions and that the satellite is inserted into the right trajectory (in this respect, see Figure 6: green and red points are perfectly matched).

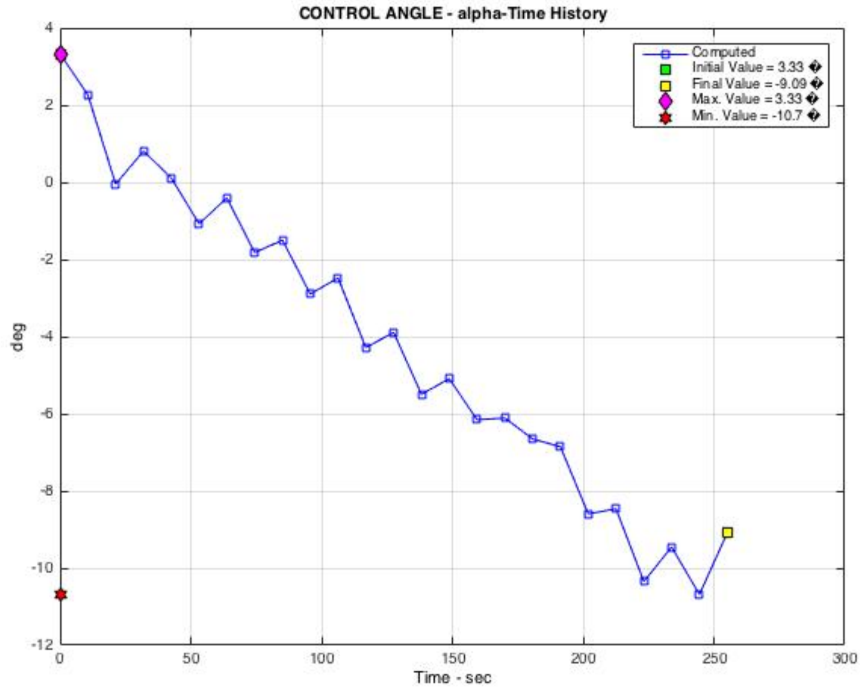

Figure 7. East: Optimal guidance control law $\alpha(t)$.

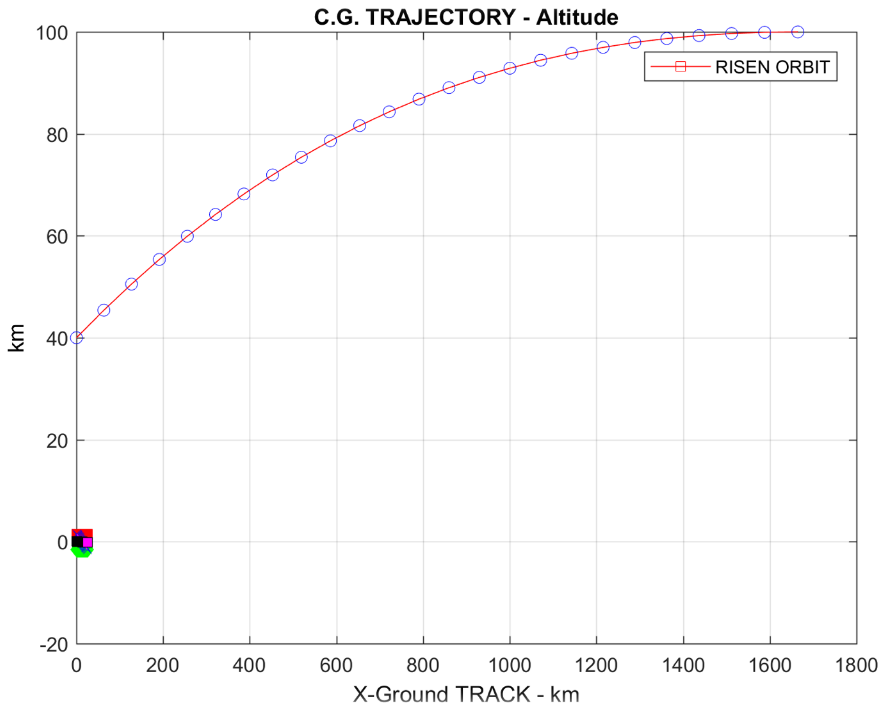

Figure 8. East: Trajectory (Altitude).

Hence, the trajectory required to execute the orbit injection is about $1650\ km$ (as shown in Figure 8) and it takes more or less $250\ sec$ to accomplish the East manoeuvre (see Figure 6).

2. The second case is an **equatorial launch heading West**:

The assumptions made are basically the same, but with a slight modification, because this time we need to impose $\chi_0 = 180^{\circ}$ in order to ensure that the satellite is heading West.

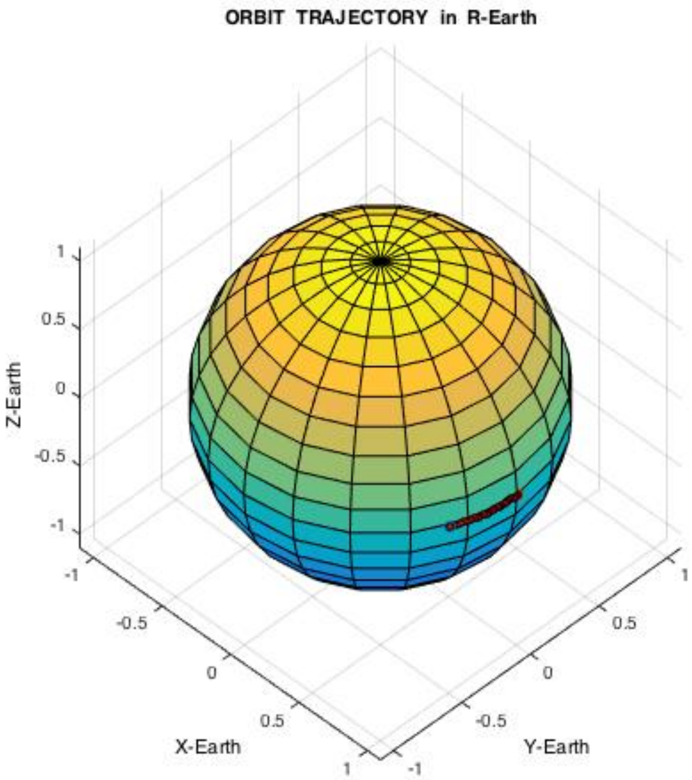

Figure 9. Earth launch heading West.

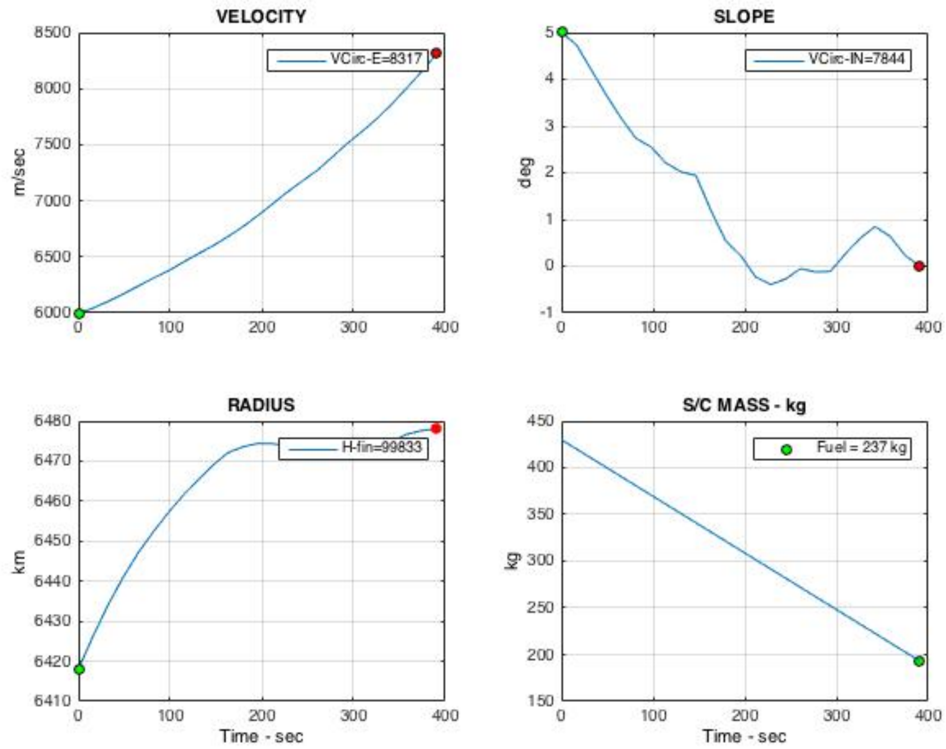

Figure 10. West: Velocity, Slope, Radius and Mass.

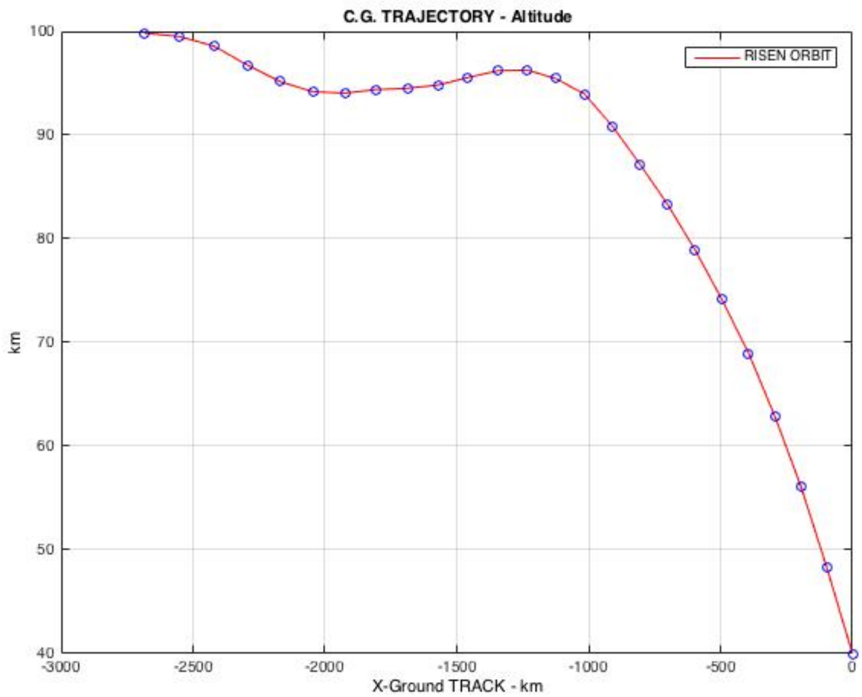

Figure 11. West: Trajectory.

As shown in Figure 10, in this case the satellite needs more time to accomplish the West manoeuvre, in fact it takes about $400\ sec$. From Figure 11, it is possible to notice that the trajectory required to execute the orbit injection is about $2700\ km$. Hence, it is easily concluded that the West manoeuvre is more expensive than the East one.

5.The Bryson Approach

In this section a different and common approach to optimal control problems is proposed using Euler-Lagrange equations and computing the optimal guidance law $\alpha(t)$ with the Newton-Raphson minimization method [6][7][8].

The aim is to find the control law $\alpha(t)$ in the interval $t_0 \leq t \leq t_f$, minimizing the cost index

$$J = \varphi[x(t_f), t_f]$$

and taking into account the following constraints:

$$\begin{cases} \dot{x} = f(x, u, t) \\ x(t_0) = x_0 \\ \psi[x(t_f), t_f] = 0. \end{cases}$$

The first constraint represents the system of differential equations that describe the system dynamics, the second one represents the initial conditions and the last one represents the final conditions that must be respected in order to find the optimal solution.

So, the necessary conditions such that the solution can be obtained are formulated starting from the "extended" cost index:

$$J = \varphi + v^T\psi + \int_{t_0}^{t_f} \lambda^T\{f[x(t), u(t), t] - \dot{x}(t)\}dt,$$

where v are the Lagrangian multipliers and $\lambda(t)$ are the co-states.
Defining the Hamiltonian as $H(t) = \lambda^T f[x(t), u(t), t]$ and the constraint $\Phi \triangleq \varphi + v^T\psi$, we easily obtain:

$$J = \Phi[\mathrm{x}(t_f), t_f] + \int_{t_0}^{t_f} \{H(t) - \lambda^T\dot{x}(t)\}dt.$$

The Euler-Lagrange equations are:

$$\begin{cases} \dot{\lambda}^T(t) = -H_x(t) \equiv -\lambda^T(t) f_x \\ \lambda^T(t_f) = \Phi_x \equiv \varphi_x + \nu^T \psi_x \end{cases}$$

In order to have a stationary solution, the following transversality conditions are needed:

$$\begin{cases} H_u = \dfrac{\partial H(t)}{\partial u} = 0 \\ \dot{\Phi} = \varphi_{t_f} + H(t_f) = 0. \end{cases}$$

The procedure explained above is a generic one adopted when dealing with this kind of problems. In this particular case, the equations of motion (in adimensional form) are:

$$\begin{cases} \dot{u} = cos\theta \\ \dot{v} = sin\theta \\ \dot{y} = v, \end{cases}$$

with the initial and final conditions:

$$\begin{cases} x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x(t_f) = \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}. \end{cases}$$

The Hamiltonian is:

$$H(t) = \lambda_u(t)\dot{u} + \lambda_v(t)\dot{v} + \lambda_\gamma(t)\dot{\gamma}.$$

The Euler-Lagrange equations are:

$$\begin{cases} \dot{\lambda}_u = -H_u = 0 \\ \dot{\lambda}_v = -H_v = 0 \\ \dot{\lambda}_\gamma = -H_\gamma = 0. \end{cases}$$

Then, the optimal condition which produces the optimal solution is: $H_u = -\lambda_u sin\theta + \lambda_v cos\theta$, which leads to $\tan\theta = \frac{\lambda_v}{\lambda_u}$.
The final conditions on the co-states are:

$$\begin{cases} \lambda_u(t_f) = \nu_u \\ \lambda_v(t_f) = \nu_v \\ \lambda_\gamma(t_f) = \nu_\gamma. \end{cases}$$

From these final conditions it is possible to notice the evolution of the co-states

$$\begin{cases} \lambda_u(t) = \nu_u \\ \lambda_v(t) = \nu_v \\ \lambda_\gamma(t) = \nu_v + \nu_\gamma(t_f - t) \end{cases}$$

Moreover, the transversality condition is:

$$1+H(t_f) = 1 + \nu_u cos\theta(t_f) + \nu_v sin\theta(t_f) = 0.$$

So, it is now easy to find the control law: $\tan\theta = \frac{\nu_v+\nu_\gamma(t_f-t)}{\nu_u}$, from which the optimal solution is formulated:

$$\alpha(t) = tan^{-1}\left(\frac{\lambda_\gamma(t)}{v(t)\lambda_v(t)}\right).$$

6.Conclusions

According to what has been discussed in this project, different approaches can be applied in order to solve the optimal control problem of orbit injection of a satellite in minimum time. Depending on the physical conditions, it is possible to choose the easiest and fastest way to solve such a problem. Surely, the presented approach relying on the `fmincon.m` method is preferred from a computational point of view because no gradients or Hessians are used and in this way energy saving and the risk minimization are pursued, too. Obviously, computations take more time to find the optimal solution but, in the end, such a process turns out to be cheaper.

By the way, some obstacles are encountered when the physical problem becomes more complicated and that is why in this case it is preferred to use the Bryson approach. Indeed, the Bryson procedure could replace the `fmincon.m` routine. Nevertheless, if the problem is not very complicated from a physical point of view, the Bryson approach is not as appropriate as the previous one because it requires hard constraints on the control and on the state variables, adding several other variables, and in this way the computational aspect becomes heavier and heavier, as the mathematical formulation shown in Section 5 implies.

7. Bibliography

[1] A.J. Calise and M.S.K. Leung, *Optimal Guidance Law Development for an Advanced Launch System*, NASA Report, 1995.

[2] J.M. Longuski, J.J. Guzmàn, J.E. Prussing, *Optimal Control With Aerospace Applications*, Springer, 2014.

[3] S.D. Ilcev, *Global Mobile Satellite Communications Theory for Maritime, Land and Aeronautical Applications*, Springer, 2017.

[4] B. Etkin, *Dynamics of Atmospheric Flight*, Dover Publications, 1972.

[5] L. Mangiacasale, *Fondamenti di Automatica e Controllo dei Veicoli Aerospaziali*, ARACNE, 2008.

[6] A.E. Bryson, and Y.C. Ho, *Applied Optimal Control*, CRC Press, 1975.

[7] A.E. Bryson, *Dynamic Optimization*, Addison Wesley, 1999.

[8] N.K. Poulsen, *Dynamic Optimization*, 2012.