Can we do without the hairdresser? A mathematical solution

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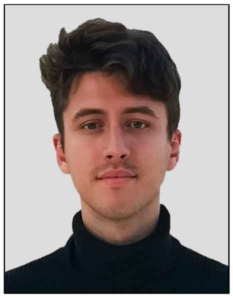

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The Germanic folk tale “Rapunzel”, popularized by the Brothers Grimm in 1812, tells the story of a young woman imprisoned in a tower, who lets down her long hair from a window, allowing a prince to climb up and visit her. Is this a realistic situation? Could someone’s hair actually grow that long?

This article investigates, using mathematics, what would happen if a person decided to stop cutting their hair. Specifically, we introduce a stochastic model for hair growth and examine how the average length evolves as time tends to infinity.

We will use only mathematical tools from an undergraduate probability course, apart from some results concerning the Poisson process, whose proofs can be found in the provided references.

Hair growth model

Let us assume that each hair grows at a constant rate v and that the lifetime of a hair is a random variable with exponential distribution. The exponential distribution is a well-known continuous distribution with density function:

$$f(x) = \lambda e^{-\lambda x}\, \mathbf{1}_{\{x \geq 0\}}$$

where λ is a positive parameter, and $\mathbf{1}_{\{x \geq 0\}}$ denotes the indicator function, which is equal to 1 if $x \geq 0$, and 0 otherwise. The exponential distribution is commonly used to model the lifetimes of processes that do not age, meaning that a process that is still alive (active) has a constant instantaneous probability of death (or failure) per unit of time, regardless of its age. Formally, if D is a random lifetime distributed as an exponential distribution,

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}(t \leq D < t + \Delta t \mid D \geq t)}{\Delta t} = \lambda$$

where λ is the parameter of the distribution. Therefore, in the context of this article, λ can be interpreted as the shedding rate, that is, the probability per unit time that a hair, still present at time t, will fall out in the next instant.

As soon as a hair falls out, another one immediately starts growing in its place, always at the same rate v. We assume the lifetimes of hairs are independent and identically distributed (i.i.d.). Finally, let us assume that at time $t_0 = 0$ the person is completely bald (probably, before making the fateful decision, the person decided to go for one last drastic haircut, or maybe the person in question is a newborn).

Let us focus on a single hair follicle. Let T be the time elapsed since $t_0 = 0$, and let $T_1, \dots, T_n \leq T$ be the times at which the hair growing from that follicle has fallen out. The length of the hair at time T is given by

$$L = v \cdot (T - \max\{0, T_1, \dots, T_n\})$$

where the 0 accounts for the case in which the hair has never fallen out between 0 and T.

Let us now study the probability distribution of the random variable L. Since, by hypothesis, the times between consecutive falls are i.i.d. exponential random variables, $T_1, \dots, T_n$ represent realizations of a Poisson process on the interval $[0, T]$ (for an introduction to Poisson processes, please refer to [**1**]).

The Poisson process has some convenient proprieties:

- the sum P_T of events occurring between 0 and T is distributed as a Poisson random variable with parameter λT:

$$\mathbb{P}(P_T = k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}, \quad k \in \mathbb{N} \; ;$$

- the process is stationary, meaning that the distribution of the number of events on an interval depends only by the length of the interval

$$\mathbb{P}(P_{T+s} - P_T = k) = \mathbb{P}(P_s = k), \quad s > 0 \; .$$

Let $A(T)$ be the random variable that represents the age of the hair at time T

$$A = T - \max\{0, T_1, \dots, T_n\}, \quad L = v\,A \; .$$

The law of A can be derived by considering that, if $A > s$, then no event of the hair falling out has occurred in the interval $[T - s,\ T]$. Therefore

$$\mathbb{P}(A > s) = \mathbb{P}(P_T - P_{T-s} = 0) = \mathbb{P}(P_s = 0), \quad 0 \leq s < T$$

where in the last equality we have used the stationarity property of the Poisson process.

Let $F_A(s)$ be the cumulative distribution function (CDF) of A:

$$F_A(s) = \mathbb{P}(A \leq s) = 1 - \mathbb{P}(A > s) \; .$$

Recalling the distribution of a Poisson process, we write

$$F_A(s) = \begin{cases} 0 & \text{if } s < 0, \\ 1 - \mathbb{P}(P_s = 0) = 1 - e^{-\lambda s} & \text{if } 0 \leq s < T, \\ 1 & \text{if } s \geq T. \end{cases}$$

Note that

$$\lim_{s \to T^-} F_A(s) = 1 - e^{-\lambda T} \neq 1$$

meaning that the CDF has a discontinuity due to a probability mass in $s = T$, that is, a single point to which a non-zero probability is assigned. This discontinuity arises from the fact that there is a $e^{-\lambda T} > 0$ probability that a hair has not fallen out between time 0 and T, and therefore that its age is exactly T.

From the distribution of $A(T)$, we derive the distribution of $L(T)$:

$$F_L(l) = \mathbb{P}(L \leq l) = \mathbb{P}\left(A \leq \frac{l}{v}\right) = F_A(l/v) = \begin{cases} 0 & \text{if } l < 0, \\ 1 - e^{-\frac{\lambda}{v}l} & \text{if } 0 \leq l < vT, \\ 1 & \text{if } l \geq vT. \end{cases}$$

The cumulative distribution function $F_L(l)$ inherits from $F_A(s)$ the discontinuity at $l = vT$, while for $0 \leq l < vT$ it is a continuous distribution, from which we can derive the probability density $f_L(l)$:

$$f_L(l) = \frac{d}{dl}F_L(l) = \frac{\lambda}{v}e^{-\frac{\lambda}{v}l} \quad \text{for } 0 \leq l < vT \; .$$

We finally derive the average hair length as a function of T by calculating the expected value of the random variable L:

$$\begin{aligned}
\mathbb{E}[L] &= \int_0^{vT} l\, f_L(l)\, dl + vT\, \mathbb{P}(l = vT) \\
&= [l\, F_L(l)]_0^{vT} - \int_0^{vT} F_L(l)\, dl + vT\, \mathbb{P}(l = vT) \\
&= \left[l\,(1 - e^{-\frac{\lambda}{v}l})\right]_0^{vT} - \int_0^{vT} (1 - e^{-\frac{\lambda}{v}l})\, dl + vT\, e^{-\lambda T} \\
&= \left[l - l\, e^{-\frac{\lambda}{v}l})\right]_0^{vT} - \left[l + \frac{v}{\lambda}e^{-\frac{\lambda}{v}l}\right]_0^{vT} + vT\, e^{-\lambda T} \\
&= vT - vTe^{-\lambda T} - vT - \frac{v}{\lambda}e^{-\lambda T} + \frac{v}{\lambda} + vTe^{-\lambda T} \\
&= \frac{v}{\lambda}(1 - e^{-\lambda T}) \; .
\end{aligned}$$

Let's study the behavior $\mathbb{E}[L]$ as a function of $T \geq 0$. By taking the first and second derivatives:

$$\frac{d}{dT}\mathbb{E}[L] = ve^{-\lambda T}, \quad \frac{d^2}{dT^2}\mathbb{E}[L] = -v\lambda e^{-\lambda T} \; .$$

Since $\frac{d^2}{dT^2}\mathbb{E}[L] < 0$, the function $\mathbb{E}[L]$ is concave down. This means that, even though the growth rate of each individual hair is constant and equal to v, the average hair length grows more slowly as T progresses.

The probability of a hair not having fallen out before time T is the height of the jump discontinuity in the CDF, i.e., $e^{-\lambda T}$. As $T \to \infty$, this probability tends to 0, and the distribution of L tends to an exponential distribution with parameter $\frac{\lambda}{v}$. Regarding the mean length:

$$\lim_{T \to \infty} \mathbb{E}[L] = \frac{v}{\lambda} \; .$$

This means that, even if a person decided to never cut their hair again and lived for an infinite amount of time with their hair continuously growing, the length of the hair would not grow indefinitely. Instead, it would converge to a constant average length, equal to the ratio of the growth rate v and the shedding rate λ.

A priori, this average length would only be reached for $T \to +\infty$. However, by observing the shape of $\mathbb{E}[L]$ as a function of T, we can recognize an exponential transient, a phenomenon common in many areas of physics, such as in electrical engineering [**2**]. Typically the transient is considered finished after a time equal to 5τ, with τ being the time constant of the transient, in this case equal to $\frac{1}{\lambda}$. Therefore, we can consider that after a time $T = \frac{5}{\lambda}$, the average hair length has reached convergence. Note that this time depends only on the shedding rate λ, and not on the hair growth speed v.

On average, hair grows between 0.5 and 1.5 centimeters per month (therefore between 6 and 18 cm per year). Moreover, the average lifespan of a hair is 2-4 years for men and 3-6 years for women [**3**]. The mean of an exponential distribution is the reciprocal of the parameter λ. This means that, according to this model, the time required to reach the maximum hair length starting from zero and without ever cutting it is between 10-20 years for men and 15-30 years for women. Furthermore, the maximum achievable length is between 12 cm and 72 cm for men and between 18 cm and 108 cm for women. These are indicative ranges that provide valid values for the majority of the population. Of course, throughout history, there have been exceptions. For example, there are documented cases of women with hair longer than 2 meters, as well as many men whose hair length remains zero, no matter how long they wait.

Model improvements

The model we introduced involves several simplifications of the physical reality.

First of all, the number of cycles per hair follicle is not infinite, but approximately 15-20 for men and 20-25 for women. However, this is not a major limitation for the model, as the number of cycles is theoretically sufficient for the length to reach convergence both in men and women.

Another relevant point is that a hair does not grow at the same speed throughout its life but goes through three distinct cycles:

- Anagen phase, the hair growth phase, lasting about 2-4 years for men and 3-6 years for women;
- Catagen phase, lasting about 7-21 days, during which the hair's rate of growth slows until it stops;
- Telogen phase, lasting about 3 months, during which the hair remains attached to the follicle without growing. At the end of this phase, the hair detaches, and a new cycle begins.

Our model, on the other hand, considers each hair as always growing, and therefore as if it was always in the Anagen phase. However, since the Catagen and Telogen phases have a much shorter duration compared to the Anagen phase, this simplification does not have a significant impact on the accuracy of the model.

Probably, the most questionable modeling choice from a mathematical point of view is the selection of the exponential distribution to model the lifespan of a hair. This choice assumes that a hair still attached to the scalp always has the same short-term rate of fall, regardless of how long it has been growing (the hair does not age). In reality, as discussed, it is more likely that hair follows a well-defined lifecycle, naturally concluding after a certain period. Therefore, the hair does age, and its lifespan would be more accurately described by a probability distribution like the Weibull [**4**]. The Weibull distribution is a generalization of the exponential distribution and has a probability density function equal to:

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\, \mathbf{1}(x \geq 0)\ .$$

Compared to the exponential distribution, the Weibull distribution has an additional parameter $k > 0$:

- for $0 < k < 1$ the probability of falling out decreases over time;
- For $k = 1$ the distribution is equivalent to an exponential distribution;
- For $k > 1$ the probability of falling out increases over time (the hair ages).

Using the Weibull distribution, an analytical treatment of the model as in the case of the exponential distribution is much more complex, if not impossible. However, the mathematical theory of renewals, which studies the case of stochastic processes with i.i.d. interarrival times following a generic distribution, provides some interesting asymptotic results (for an introduction to renewal theory, refer to [**1**]). In particular, if A is the random variable that represents the age of the hair at time T, Z is the random variable that represents the time between the fall of one hair and the next (interarrival time), and $\mathbb{E}[Z] < \infty$, then:

$$\lim_{T\to+\infty} \mathbb{E}[A(T)] = \frac{\mathbb{E}[Z^2]}{2\mathbb{E}[Z]}\ .$$

The first and the second moments of a Weibull distribution are:

$$\mathbb{E}[Z] = \lambda\,\Gamma(1+\frac{1}{k}), \quad \mathbb{E}[Z^2] = \lambda^2\,\Gamma(1+\frac{2}{k})$$

where $\Gamma(z)$ is the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$, defined for $z > 0$. Using these expressions in our model, we obtain the following limit for the expected hair length:

$$\lim_{T\to+\infty} \mathbb{E}[L] = v\,\frac{\lambda\,\Gamma(\frac{2}{k})}{2\,\Gamma(\frac{1}{k})}$$

Therefore, even in the case of Weibull distribution, the average length of the hair would converge to a finite value. However, in this case, it is not possible to give a numeric estimate of the limit value based on the previous data since, to estimate the additional parameter k, information about the variability of lifetimes is required. Collecting data from various strands of hair, the parameters of the Weibull distribution (and therefore the limit mean length) can be obtained using Maximum Likelihood Estimation (MLE) [**4**].

Conclusion

In this article we have shown that, even if a person decided never to cut their hair again and lived for an infinite amount of time with their hair continuously growing, the length of the hair would not grow indefinitely, but it would converge to a finite maximum value. Using some data from the medical literature, we estimated this limit length and the time necessary to reach it. Finally, we discussed the validity of our model, mentioning possible improvements.

We can now answer the question posed in the title of the paper: yes, theoretically, one could do without a hairdresser. Our estimates suggest that hair would remain at manageable lengths, perhaps a bit inconvenient, but not to the extent of compromising an individual's life. However, we do not recommend such a decision to everyone. Depending on cultural background and personal preferences, skipping haircuts might not be ideal. Additionally, for many people, a visit to the hairdresser is a social ritual, much like attending a soccer match or grabbing a beer with friends, and it would be a shame to give that up.

Summary. The Germanic folk tale "Rapunzel," popularized by the Brothers Grimm in 1812, tells the story of a young woman who lets down her hair from a tower window, allowing a prince to climb up and visit her. But is such a situation realistic? Could someone's hair actually grow that long? This article investigates, using mathematics, what would happen if a person decided to stop cutting their hair. Specifically, we introduce a stochastic model for hair growth and examine how the average length evolves as time tends to infinity. Besides offering curious insights about our hair, the proposed model provides an engaging and unconventional exercise in probability. It can also serve as an innovative way to introduce students and young researchers to the fascinating world of stochastic processes, particularly the Poisson process and renewal theory.

References

1. Ross SM. Introduction to probability models. Academic press; 2014.
2. Hambley AR. Electrical engineering: principles and applications. vol. 4. Prentice Hall; 2017.
3. Buffoli B, Rinaldi F, Labanca M, Sorbellini E, Trink A, Guanziroli E, et al. The human hair: from anatomy to physiology. International journal of dermatology. 2014;53(3):331-41.
4. Montgomery DC, Runger GC. Applied statistics and probability for engineers. John wiley & sons; 2019.

