SNOM HW2

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1 Problem 1

1.1

To prove the first claim I have build a counterexample: I have five item and three agents, each one with her own valuation function v(·).

	item 1	item 2	item 3	item 4	item 5
v_1	50	6	10	20	45
v_2	45	10	15	30	35
v_3	40	5	20	25	40

At the beginning I have assigned arbitrarily the first three item to agent one, two and three respectively since when an agent has her allocation empty is not envied by anyone. Then I had to assign the fourth item to second agent that is not envied by anyone because item 2 has a small value (smaller than their items) for the other two players. Finally the last item is assigned to the third agent.

The final allocation is:

- A_1 = {item 1}
- A_2 = {item 2, item 4}
- A_3 = {item 3, item 5}

Allocation A=(A_1,A_2,A_3) is **not** envy-free because for example $v_1(A_1) \leq v_1(A_3) \Rightarrow 50 \leq 55$ so it is proved that this mechanism does not always produce envy-free allocations! (Also agent 2 envies agent 3)

1.2

To prove this second claim I have build another counterexample: I have five item and three agents, each one with her own valuation function v(·).

	item 1	item 2	item 3	item 4	item 5
v_1	50	6	20	20	45
v_2	45	12	15	30	35
v_3	40	1	10	15	1

At the beginning I have assigned arbitrarily the first three item to agent one, two and three respectively since when an agent has her allocation empty is not envied by anyone. Then I had to assign the fourth item to second agent that is not envied by anyone because item 2 has a small value (smaller than their items) for the other two players. Finally the last item is assigned to the third agent.

The final allocation is:

- $A_1 = \{\text{item 1}\}$
- $A_2 = \{\text{item 2, item 4}\}$
- $A_3 = \{\text{item 3, item 5}\}$

The allocation A = $(A_1,\ A_2,\ A_3)$ is **not** EFX allocation because for example $v_3(A_3) \leq v_3(A_2 \setminus \{\text{item 2}\}) \Rightarrow 11 \leq 15$ so it is proved that this mechanism does not always produce EFX allocations! (However this is an EF1 allocation).

1.3

An agent j is envied by an agent i if $v_i(A_j) > v_i(A_i)$. By contradiction $v_i(A_j \setminus \{g\}) > v_i(A_i),\ \forall g \in A_j$. Since the valuation function is additive I can rewrite my expression as: $v_i(A_j) - v_i(g) > v_i(A_i),\ \forall g \in A_j$. This means that if I removed at random an item from A_j it is still true that $v_i(A_j) > v_i(A_i)$. This means that j is envied by i, but nevertheless I have assigned g to j, and not to i! This cannot be possible because it contradicts the fourth point of the mechanism: I cannot assign an element to an agent that is envied.

1.4

I have considered an instance where all the agents have the same ordinal preference over the items. Then I have modified second point of the mechanism: I have ordered in a decreasing way the items.

	item 1	item 2	item 3	item 4	item 5
v_1	60	55	35	20	10
v_2	55	50	45	35	15
v_3	50	45	40	30	5

The final allocation is:

- $A_1 = \{\text{item 1}\}$
- $A_2 = \{\text{item 2, item 5}\}$
- $A_3 = \{\text{item 3, item 4}\}$

This is an EFX allocation because it is verifiable that:
$v_i(A_i) \geq v_i(A_j \setminus \{g\})$ holds for every pair $i, j \in N$, with $A_j \neq \emptyset, and\ \forall g \in A_j$.
Indeed for:

- pair 1,2: $v_1(A_1) = 60$, $v_1(A_1) \geq v_1(A_2 \setminus item5) = 55$ and $v_1(A_1) \geq v_1(A_2 \setminus item2) = 10$.
- pair 1,3: $v_1(A_1) \geq v_1(A_3 \setminus item3) = 20$ and $v_1(A_1) \geq v_1(A_3 \setminus item4) = 35$
- pair 2,3: $v_2(A_2) = 65$, $v_2(A_2) \geq v_2(A_3 \setminus item3) = 35$ and $v_2(A_2) \geq v_2(A_3 \setminus item4) = 45$
- pair 2,1: $v_2(A_2) \geq v_2(A_1 \setminus item1) = 0$.
- pair 3,1: $v_3(A_3) = 70$, $v_3(A_3) \geq v_3(A_1 \setminus item1) = 0$
- pair 3,2: $v_3(A_3) \geq v_3(A_2 \setminus item2) = 5$ and $v_3(A_3) \geq v_3(A_2 \setminus item5) = 45$

If I ordered the items in arbitrary way the final allocation could not be EFX anymore. It is not generally true that if all the agents have the same ordinal preference over the items the result is an EFX allocation, but this just happens for some instances.

2 Problem 2

2.1

"A mechanism is called truthful if for every bidder i, and for every profile b_i of the other bidders, it is a dominant strategy for i to declare her real value v_i". Moreover for the Myerson's Lemma I know that "Allocation rule $\mathbf{x}$ is truthful $\Rightarrow$ Allocation rule $\mathbf{x}$ is monotone". So I only have to demonstrate that the mechanism is truthful (Intuitively that sharing mechanism is monotone is clear because larger bid gives more stuff). To prove that sharing mechanism is truthful I have to show that $u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b', \mathbf{b}_{-i})$, $\forall b' \neq v_i$:

- **agent i does not get the digital good** $u_i(v_i, \mathbf{b}_{-i}) = 0$. If agent i bidded a price smaller than v_i, u_i would remain 0. While if agent i bidded a price p^* bigger than v_i and she achieved the digital good her utility would be $u_i = v_i - p^* < 0$.
- **agent i get the digital good** $u_i(v_i, \mathbf{b}_{-i}) = v_i - p$. If agent i bidded a price p^* larger than v_i, she still get the good and her utility does not change. While if agent i bidded a price p^* smaller than v_i or she would

not get the good anymore and her utility would become 0 or she still get the good and her utility does not change.

In conclusion agent i has not reasons to bid anything different from her valuation so the sharing mechanism is truthful!

2.2

To prove it I have considered the case in which at each step just one agent is removed because he offers just an epsilon less than the set price (ϵ is small as you like) and at the end no one get the digital good. So I have:
HS $= 0 + \sum_{i=1}^{n} \frac{1-\epsilon}{n-i} = \frac{1-\epsilon}{n} + \frac{1-\epsilon}{n-1} + ... + \frac{1-\epsilon}{1} = \sum_{i=1}^{n} \frac{1-\epsilon}{i} \approx log(n)$
The final approximation is due to the fact that I have obtained the harmonic series that diverges but the sum of the first n terms is approximated through the logarithm of n [1].

2.3

To prove that under this procedure, it is not always the best strategy for an agent to truthfully report his values I have made a counterexample in which there are 2 agents and 2 digital goods. $B_1 = 0.2$ and $B_2 = 0.4$

	good 1	good 2
v_1	0.3	1.7
v_2	1.3	0.4

Running the sharing mechanism for the good 1, agent 1 is removed from the game because her value is less than 0.5 while agent 2 get the good for a price equals to 1. Instead running the sharing mechanism for the good 2, agent 2 is removed from the game because her value is less than 0.5 while agent 1 get the good for a price equals to 1. The utilities of the two agents are:

- $v_1(R) = min\{\sum_{j\in R} v_i(j), B_i\} = min\{1.3, 0.2\} = 0.2$ and so $u_1 = v_1(R) - 1 = -0.8$.

- $v_2(R) = min\{\sum_{j\in R} v_i(j), B_i\} = min\{1.7, 0.4\} = 0.4$ and so $u_2 = v_2(R) - 1 = -0.6$.

It is clear that with these values of B_i (smaller than $\sum_{j\in R} v_i(j)$) the agents should not report their values truthfully but bid a smaller values in such a way to not get the good. Indeed in that way their utilities would be zero and not negative! If instead B_i is higher than $\sum_{j\in R} v_i(j)$ report truthfully the value is still the best strategy.

3 Problem 3

3.1

I want to show that the following function is a potential function:
$\phi(s) = \frac{1}{2}\sum_{j=1}^{m} l_j(\mathbf{s})^2 = \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{n_j(s)} w_i\right)^2$.

I compute the difference of the potential function:
$\phi(s) - \phi(s') = \frac{1}{2}\left(\sum_{j\in s_i \cap s_i'} l_j(\mathbf{s})^2 - l_j(\mathbf{s}')^2 + \sum_{j\in s_i \smallsetminus s_i'} l_j(\mathbf{s})^2 - \sum_{j\in s_i' \smallsetminus s_i} l_j(\mathbf{s}')^2\right)$
$\phi(s) - \phi(s') = \frac{1}{2}\left(\sum_{j\in s_i} l_j(\mathbf{s})^2 - \sum_{j\in s_i'} l_j(\mathbf{s}')^2\right)$

I compute the difference of the cost: $c_i(s) - c_i(s') = l_{s_i}(\mathbf{s}) - l_{s_i'}(\mathbf{s}')$

So finally: $(c_i(s) - c_i(s')) \cdot (\phi(s) - \phi(s')) = (l_{s_i}(\mathbf{s}) - l_{s_i'}(\mathbf{s}')) \cdot (l_{s_i}(\mathbf{s})^2 - l_{s_i'}(\mathbf{s}')^2)$

These two differences that I have computed have the same sign indeed their product is greater than zero because if $l_{s_i}(\mathbf{s}) > l_{s_i'}(\mathbf{s}')$ also $l_{s_i}(\mathbf{s})^2 > l_{s_i'}(\mathbf{s}')^2$ and on the other hand if $l_{s_i}(\mathbf{s}) < l_{s_i'}(\mathbf{s}')$ also $l_{s_i}(\mathbf{s})^2 < l_{s_i'}(\mathbf{s}')^2$.
In conclusion I can claim that his game has at least a pure Nash equilibrium!

3.2

In this game each agent has just two possible strategies $s_i \in \{-1, 1\}$, so when I consider agent i that varies her strategy, she moves from 1 to -1 or vice versa. The candidate as potential function is: $\phi(\mathbf{s}) = \sum_{j<i\in V} w_e \cdot s_i \cdot s_j$

The utility of an agent is $u_i = \sum_{e=(i,j)\in E} w_e \cdot s_i \cdot s_j = \sum_{i\neq j} w_{i,j} \cdot s_i \cdot s_j$

I compute the difference of the utility: $u_i(\mathbf{s}) - u_i(\mathbf{s}') = \sum_{i\neq j} w_{i,j} \cdot s_i \cdot s_j - \sum_{i\neq j} w_{i,j} \cdot s_i' \cdot s_j = 2\sum_{i\neq j} w_{i,j} \cdot s_i \cdot s_j$ since $s_i' = -s_i$

The index $j \neq i$ allows to consider all the edges between two different nodes. But $j \neq i$ are all the $j : j < i$ plus all the $j : i < j$ so:

$u_i(\mathbf{s}) - u_i(\mathbf{s}') = 2\sum_{j:j<i} w_{i,j} \cdot s_i \cdot s_j + 2\sum_{j:i<j} w_{i,j} \cdot s_i \cdot s_j$. But this is equal to $\phi(\mathbf{s}) - \phi(\mathbf{s}')$ so in this case this potential function is an exact potential! (To do this part I have looked at [2])

4 Problem 4

4.1 Borda

Firstly I have reported in the two tables below an example of preference profile and for each alternative its Borda score.

Table 1: Preference profile

Agent	Ranking			
1	b	c	a	d
2	a	b	d	c
3	d	c	b	a
4	a	d	c	b

Table 2: Results

Alternative	Borda Score
a	7
b	6
c	5
d	6

The third agent hates the alternative a and she wish that a is not winner. To achieve her desire she misreports her preference ordering in such a way that a is not anymore the winner. Indeed the third agent moves b in the first position and down-shifts d and c. In this way b is the winner!

Table 3: Phony preference profile

Agent	Ranking			
1	b	c	a	d
2	a	b	d	c
3	b	d	c	a
4	a	d	c	b

Table 4: Manipulated Results

Alternative	Borda Score
a	7
b	8
c	4
d	5

4.1 Veto

Firstly I have reported in the two tables below an example of preference profile and for each alternative its Veto score.

Table 5: Preference profile

Agent	Ranking			
1	a	c	b	d
2	d	b	a	c
3	b	d	a	c
4	b	c	a	d

Table 6: Results

Alternative	Veto Score
a	4
b	4
c	2
d	2

With the initial preference profile b would be the winner (she ties a, but since I have decided to break the ties with a pairwise comparison, b is the winner). Nevertheless the first agent prefer a by far respect to b so she misreports her preference profile swapping b with d and in this way b has just three points and a is the winner!

Table 7: Phony preference profile

Agent	Ranking			
1	a	c	d	b
2	d	b	a	c
3	b	d	a	c
4	b	c	a	d

Table 8: Manipulated Results

Alternative	Veto Score
a	4
b	3
c	2
d	2

4.2

I have N voters and M candidates. Each voter i assigns to each candidate a numerical score according its valuation function $v_i : M \to \mathbb{R}$.
Following the hint I have to show that a deterministic truthful mechanism is ordinal. By contradiction I can claim that my truthful mechanism, even if $v_i(j) > v_i(k)$, put first the candidate k and then candidate j (not ordinal). But in this way voter i would want to manipulate her order declaring for j a smaller vote until j will be before k in the order. So mechanism is not truthful anymore! Consequently a deterministic truthful mechanism is ordinal! The Gibbard-Satterthwaite theorem claims that "for at least three alternatives, any strategy-proof and onto the set of alternatives social choice function must be a dictatorship".

5 Problem 5

5.1

After some numerical examples of list of preferred temperatures (preference profile) **l** (like **l**=[15,25,37]), I have found that the temperature that minimizes the *total displeasure* is the **median** of **l** (and not its mean as I have initially thought). If the length of **l**, n, is even, I can choose as its median the temperature in position $l_{\frac{n}{2}}$. Through numerical examples (like **l**=[15,20,35]), I have found that the temperature that minimizes the *maximum displeasure* is the **centre** of **l**, $c = \frac{t_{min}+t_{max}}{2}$. The mean and the median are less robust to outliers (case in which t_{max} or t_{min} are very far from other temperatures).

5.2

I assume that the "dictator" has as preferred temperature the median of the preferred temperature $t_{dict} = t_{min}$. The displeasure of the dictator is zero, while the displeasure of all other n - 1 agents is at least 1 so, summing all the others displeasure, the total displeasure is at least n-1. (I have assumed that all the temperatures are integers). An example with n = 7 could be **l** = [20, 21, 21, 21, 21, 21, 21] with t_{dict} = 20 and the total displeasure is equals to n - 1 = 6 (t_{opt} = 21 and the optimal total displeasure is 1).

I assume that the "dictator" has as preferred temperature the maximum of the preferred temperature $t_{dict} = t_{max}$. The approximation ratio for the maximum displeasure is: $\frac{|t_{max}-t_{min}|}{|\frac{t_{min}+t_{max}}{2}-t_{min}|} = \frac{t_{max}-t_{min}}{\frac{t_{max}-t_{min}}{2}} = 2$. At the denominator I have put the optimal maximum displeasure while at the numerator the maximum displeasure for my dictator. Also in this case I have assumed that temperatures are integer numbers. An example with n = 3 could be $\mathbf{l}$ = [20, 21, 22] with t_{dict} = 22 and the maximum displeasure is equals to 2 (t_{opt} = 21) and the optimal maximum displeasure is 1).

5.3

A social choice function (SCF) will take as input the preference profile $\mathbf{l} = [t_1, t_2, ..., t_n]$, and outputs a winning alternative. My social choice function is the median of l defined as:

$$\begin{cases} f(\mathbf{l}) = t_{\frac{n}{2}}, & \text{if n is even} \\ f(\mathbf{l}) = t_{\frac{n+1}{2}}, & \text{if n is odd} \end{cases}$$

I choose this function to be sure that the approximation ratio for the total displeasure is 1 (its output is equal to the optimal output for the total displeasure) and consequently it always guarantees an optimal outcome for the total displeasure. Now I have to show that this function is truthful so I have to show that $c_i(t_i, \mathbf{l}_{-i}) \leq c_i(l', \mathbf{l}_{-i}),\ \forall l' \neq t_i$:

- **person i has $\mathbf{t_i} > \mathbf{median(l)}$.** If she declare a t^* such that $t^* > t_i$ the median does not change so her displeasure remains the same. Also if she declare a t^* such that $median(l) < t^* < t_i$ nothing changes. If she declare a t^* such that $t^* < median(l) < t_i$ the median decreases and so her displeasure increases.

- **person i has $\mathbf{t_i} < \mathbf{median(l)}$.** If she declare a t^* such that $t^* < t_i$ the median does not change so her displeasure remains the same. Also if she declare a t^* such that $t_i < t^* < median(l)$ nothing changes. If she declare a t^* such that $t_i < median(l) < t^*$ the median increases and so her displeasure increases.

So I have shown that person i has not interested to declare something to different from her preferred temperature and this means that social choice function is truthful.

5.4

I consider a profile with only two people: $\mathbf{l}$ = [0,5] I would like to take a social choice that if it had an approximation ratio smaller than 2, it would not be truthful. By contradiction I claim that a truthful mechanism that has an approximation ratio smaller than 2 exists and it is for example $f(\mathbf{l}) = 5$. The optimum is the centre of $\mathbf{l}$, 2.5, which has a maximum displeasure of 2.5. To achieve an approximation ratio better than 2, chosen temperature must be in (0,5). But for the second person should be convenient to deviate and declare as her temperature 10 so to shift the optimal solution to 5 that is her preferred temperature but this deviation contradicts the truthfulness!

5.5

5.5.2

(I have made firstly this point because I will exploit the computation of the expected displeasure in the other proof.)

The expected displeasure of the algorithm is:
$\frac{1}{4} \cdot (t_r - t_l) + \frac{1}{4} \cdot (t_r - t_l) + \frac{1}{2} \cdot \frac{t_l + t_r}{2} = \frac{3t_r - t_l}{4}$

The optimum cost of the algorithm is: $\frac{t_l + t_r}{2}$

So the approximation ratio is: $\frac{3t_r - t_l}{2(t_l + t_r)}$. If I assume for example that $t_r = 10$ and $t_l = 0$, I obtain as approximation ratio $\mathbf{\frac{3}{2}}$.

5.5.1

The outcome of this randomized algorithm depends only on the minimum and the maximum preferred temperatures, there are three possible cases:

- person i declares a temperature t^* such that $t^* < t_l$ and so the expected displeasure is: $\frac{3t_r - t^*}{4}$. The numerator is larger ($t^* < t_l$) so the expected displeasure is increased.

- person i declares a temperature t^* such that $t^* > t_r$ and so the expected displeasure is: $\frac{3t^* - t_l}{4}$. The numerator is larger ($t^* > t_r$) so the expected displeasure is increased.

- person i declares a temperature t^* such that $t_l < t^* < t_r$ and the expected displeasure is: $\frac{3t_r - t_l}{4}$. As I have already written in this case since t_l and t_r have remained the same the expected displeasure does not change.

So I have shown that person i has not interested to misreport her true preferred temperature and this means that the mechanism is *truthful-in-expectation*

References

[1] Wikipedia, "https://it.wikipedia.org/wiki/Serie_armonicaApprossimazioni_della_serie_armonica"

[2] $http : //www.math.tau.ac.il/\ mansour/course_games/scribe/lecture6.pdf$