Chaotic time series prediction with Reservoir computing

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***Abstract*— Chaotic systems can be completely deterministic and yet still be inherently unpredictable over long periods of time. Because chaos continually increases in systems, it is impossible to predict the future of systems well. For instance, even the small flap of a butterfly's wings could set the world on a vastly different trajectory, such as by causing a hurricane.**

I. Introduction

The aim of this project was to predict chaotic time series with reservoir computing. Three chaotic problems were taken into account.

The problems taken into account are

- Mackey-Glass equations
- Lorenz attractor with standard parameters
- Restricted Three Body Problem

The reservoir used is an Echo State Network and the read out is performed via a ridge regression

II. Dataset

The data for the project was generated synthetically using an integrator to solve the differential equations of the problems. A standard RK45 algorithm is used, implemented via *sklearn* and *JiTCDDE* packages.

A. Mackey-Glass

Mackey–Glass equations refer to a family of delay differential equations whose behaviour manages to mimic both healthy and pathological behaviour in certain biological contexts, controlled by the equation's parameters. Originally, they were used to model the variation in the relative quantity of mature cells in the blood. The Equation is the following:

$$\frac{dx}{dt} = \beta \frac{x_\tau}{1 + x_\tau^n} - \gamma x \quad \gamma, \beta, n > 0 \tag{1}$$

where:

- $\beta = 0.2$
- $\gamma = 0.1$
- $\tau = 17$
- $n = 10$

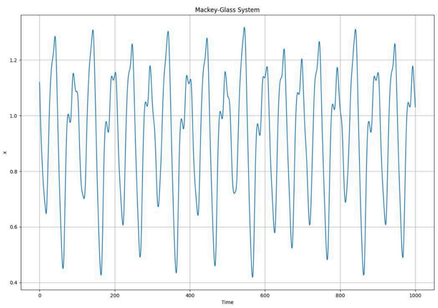

Figure 1: The Mackey-Glass time series

B. Lorenz

The Lorenz system is composed of three ordinal differential equations describing atmospheric convection. It displays chaotic behaviour using:

- $\sigma = 10$
- $\rho = 28$
- $\beta = \frac{8}{3}$

The equations of motion are:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned} \tag{2}$$

In Figure 2 the numerical integration result is shown.

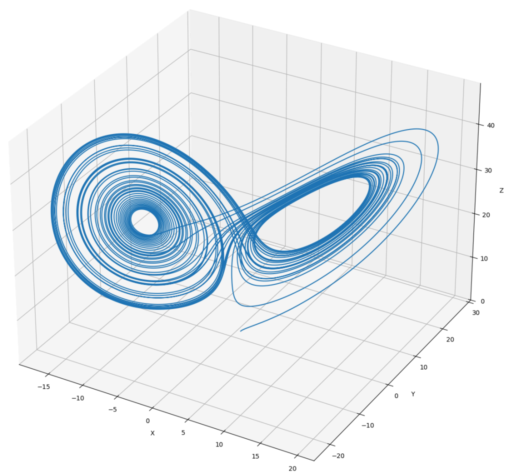

Figure 2: Lorenz attractor starting from coordinates (1, 1, 1)

C. *Restricted Three Body Problem*

In the restricted three-body problem, a body of negligible mass moves under the influence of two massive bodies. Having negligible mass, the planetoid exerts force on the two massive bodies that may be neglected; therefore the resulting system can be analyzed and described as a two-body motion problem. By setting the reference frame into the rotating center of mass of the two bigger bodies, the motion of the planetoid is the following:

$$\begin{aligned} \frac{d^2x}{d}t^2 &= -m_1\frac{x-x_1}{r_1^3} - m_2\frac{x-x_2}{r_2^3} \\ \frac{d^2y}{d}t^2 &= -m_1\frac{y-y_1}{r_1^3} - m_2\frac{y-y_2}{r_2^3} \end{aligned} \tag{3}$$

where $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}$

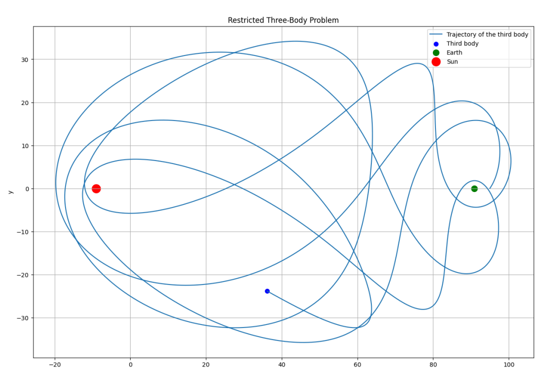

Figure 3: Trajectory of the Three Body Problem

III. Model

The reservoir used was an Echo State Network defined as follows:

$$h_t = (1-\mathrm{lr})\cdot h_{t-1} + \mathrm{lr}\cdot(\tanh(W_{\mathrm{in}}x_t + W_r h_{t-1})) \tag{4}$$

where *lr* is the leaking rate, which describe how much of the previous step information is passed onto the current one.

A. *Echo State Property & Fading memory*

To ensure the correctness of the model implemented, the Echo State Property and the Fading Memory of the reservoir were verified via the visualization of some components of the reservoir's hidden states throughout the timesteps.

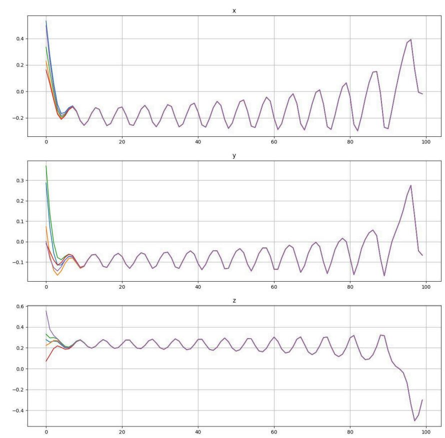

Figure 4: Reservoir components throughout timesteps

We can see that the initial state of the reservoir is forgotten after the first few iterations. Since every component of the reservoir cannot be plotted, a warmup size of 100 was set to be confident that each component of the reservoir reached the thermalization; this number may seem too high but thanks to the abundance of data points it is indeed not much of a lost.

B. *Ridge Regression*

In the first model a Ridge Regression is performed on the output of the Reservoir. The ridge minimizes the following objective function

$$\sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}\beta_j^2 \tag{5}$$

Where α is the shrinking term; for $\alpha = 0$ a Linear regression is obtained.

IV. Results

1) *Mackey Glass:*

Optimal values found for the Mackey-Glass model:

```
reservoir_size = 1024
spectral_radius = 0.99
leaking_rate = 0.15
connectivity = 0.1
ridge_alpha = 5e-09
```

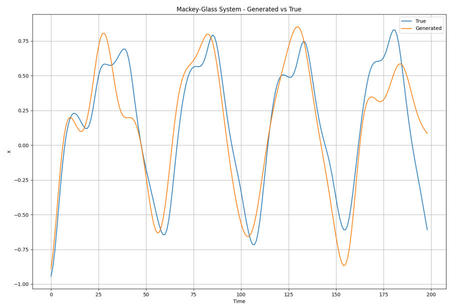

Figure 5: Data generation in orange of the Mackey-Glass trajectory.

2) *Lorenz Attractor:*
Optimal values found for the Ridge regression-based model:

```
reservoir_size = 2048
spectral_radius = 1.1
leaking_rate = 0.51
connectivity = 0.15
ridge_alpha = 1e-06
```

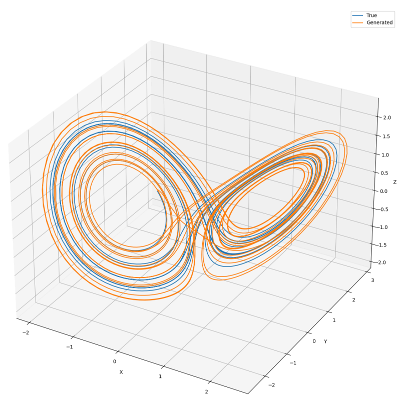

Figure 6: Data generation in orange of the Lorenz Attractor trajectory

3) *Restricted Three Body Problem:*

```
reservoir_size = 2048
spectral_radius = 1.1
leaking_rate = 0.7
connectivity = 0.2
ridge_alpha = 1e-04
```

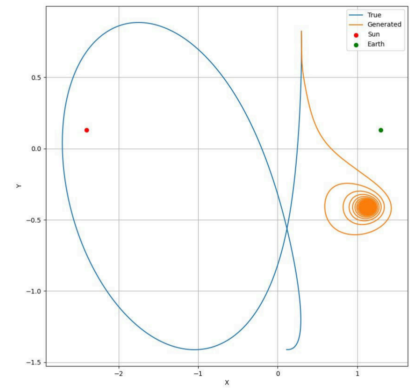

Figure 7: Data generation in orange of the Restricted Three Body problem

V. Conclusions

In the project I explored the use of reservoir computing to predict chaotic time series. Due to the nature of these time series, the generation process is highly unstable and rapidly deviates from the desired trajectory. For what concerns the Lorenz attractor the dynamic is successfully learned by the model and the generation follows a valid orbit even though it is not the correct one. Regarding the three body problem the models were unable to grasp the underlying physics, resulting in a completely wrong trajectory; finally for what concern the Mackey-Glass time series the results were highly unstable and needed a lot of time to find good results. In the end we can say that Reservoir Computing is a valid alternative to recurrent neural network such as LSTM and GRU for time trajectories that are chaotic but still with some kind of regularity (such as for Lorenz attractor). Furthermore reservoir computing allows for faster training and lower energy consumption to fit the data.

VI. Future works

In the future, in order to improve the results of the project, a different type of Reservoir can be taken into account such as the nonlinear vector autoregression (NVAR) since the literature define them as the next generation of Reservoir Computing. Another improvement can be taken in the direction of the architecture, staking multiple Reservoir can benefit the results of the predictions.