



Functional Programming An Introduction

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Functional Programming Overview

What is functional programming?

- Functions are first class (objects).
 - That is, everything you can do with "data" can be done with functions themselves (such as passing a function to another function).
- Recursion is used as a primary control structure.
 - In some languages, no other "loop" construct exists.
- There is a focus on list processing.
 - Lists are often used with recursion on sub-lists as a substitute for loops.
- "Pure" functional languages eschew side-effects.
 - This excludes assignments to track the program state.
 - This discourages the use of statements in favor of expression evaluations.

Whys

- All these characteristics make for more rapidly developed, shorter, and less bug-prone code.
- A lot easier to prove formal properties of functional languages and programs than of imperative languages and programs.



Functional Programming Overview

The basic idea is to model everything as a "mathematical function".

There are only **two** linguistic constructs:

- abstraction, used to define the function;
- application, used to call it.

No **state** concept

- this means no assignments are allowed
- variables are just names.

E.g., in $f(x) = x + 1$ the name f is irrelevant,

- the function $g(x) = x + 1$ represents the same function;
- it can be referred as $x \mapsto x + 1$.



Functional Programming λ -Calculus [Church and Kleene ~1930]

λ -expressions are made of **constants**, **variables**, **λ** , **.** and **parenthesis**

1. if x is a variable or a constant then x is a λ -expression;
2. if x is a variable and M is a λ -expression then $\lambda x.M$ is a λ -expression;
3. if M, N are λ -expressions then (MN) is a λ -expression.

Abstraction \neq Application

λ -calculus provides only two basic operations: abstraction and application

- $\lambda x.x + 1$ is an example of abstraction that defines the successor;
- $(\lambda x.x + 1)7$ is an example of application that calculates the successor of 7;
 - application is left-associative, i.e., $MNP \equiv (MN)P$.

Binding, Free and Bound Variables

- in $\lambda x.xy$ x is a bound variable whereas y is unbound (free)
- in $\lambda x.\lambda y.xy$ (for short $\lambda xy.xy$) both variables are bound;
- in $(\lambda x.M)y$, all the occurrences of x in M are replaced by y (denoted as $M[x/y]$) and brings to $M[x/y]$ as a result
 - e.g., $(\lambda x.x + 1)7 \rightarrow x + 1[x/7] \rightarrow 7 + 1 \rightarrow 8$.





Functional Programming

ML [Milner et al. ~1970]

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ML is a general-purpose functional programming language developed by Robin Milner et al. in the 70ies.

- ML is the acronym for metalanguage, since it is an abstraction on polymorphic λ -calculus.

Features of ML include:

- a call-by-value evaluation strategy, first-class functions, parametric polymorphism,
- static typing, type inference, algebraic data types, pattern matching, and exception handling.

ML uses eager evaluation, which means that all sub-expressions are always evaluated.

- lazy evaluation can be achieved through the use of closures.

We will use **OCaML** (<http://caml.inria.fr>).



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ML/OCaML [Leroy et al. ~1980]

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OCaML is an implementation of ML with extra functionality (object-orientation, modules, imperative statements, ...).

OCaML comes with

- an interpreter (ocaml) and
- a compiler (ocamlc).

```
let main() = print_string("Hello World in ML Style\n");;
main();;
```

```
[12:28]cazzola@surtur:~/lp/ml>ocamlc -o helloworld helloworld.ml
[12:28]cazzola@surtur:~/lp/ml>ls
helloworld* helloworld.cmi helloworld.cmo helloworld.ml
[12:28]cazzola@surtur:~/lp/ml>helloworld
Hello World in ML Style.
[12:28]cazzola@surtur:~/lp/ml>rlwrap ocaml
Objective Caml version 4.12.0
```

```
# let main() = print_string("Hello World in ML Style\n");;
val main : unit -> unit = <fun>
# main();;
Hello World in ML Style.
- : unit = ()
# ^D
[12:29]cazzola@surtur:~/lp/ml>
```



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Functional Programming

ML Functions

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ML derives directly from λ -calculus:

- functions are defined independently of their name

```
let succ = fun x -> x+1;;
let succ x = x+1;;
```

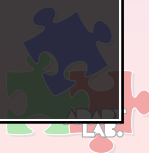
Functions can be aliased

```
let succ' = succ;;
```

- calls are simply the application of the arguments to the function

```
succ 2;;
(fun x -> x+1) 2;;
```

```
[16:19]cazzola@surtur:~/lp/ml>ocaml
Objective Caml version 4.12.0
# let succ = fun x -> x+1;;
val succ : int -> int = <fun>
# succ 7;;
- : int = 8
# succ -1;;
Error: This expression has type int -> int
but an expression was expected of type int
```



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Name Scope

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Scoping

- a new binding to a name hides the old bind;
- static binding is used in function definition (closure).
 - i.e., a triplet: args list, function body and environment (x, x+y, [5/y]).

```
[17:01]cazzola@surtur:~/lp/ml>ocaml
OCaML version 4.12.0
# let f x = 5;;
val f : 'a -> int = <fun>
# let f x = 7;;
val f : 'a -> int = <fun>
# f 1;;
- : int = 7
# let y = 5;;
val y : int = 5
# let addy = fun x -> x+y;;
val addy : int -> int = <fun>
# addy 8;;
- : int = 13
# let y=10;;
val y : int = 10
# addy 8;;
- : int = 13
# (fun x -> x+y) 8;;
- : int = 18
[17:57]cazzola@surtur:~/lp/ml>
```



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High-Order Functions

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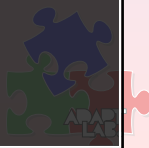
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In ML functions are first class citizens

- i.e., they can be used as values;
- when passed to a function this is an high-order function.

```
let compose f g x = f (g x);;
let compose' (f, g) x = f (g x);;
```

```
[15:30]cazzola@surtur:~/lp/ml>ocaml
# let compose f g x = f (g x);;
val compose : ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b = <fun>
# let compose' (f,g) x = f (g x);;
val compose' : ('a -> 'b) * ('c -> 'a) -> 'c -> 'b = <fun>
# let succ = fun x -> x + 1;;
val succ : int -> int = <fun>
# let plus1 = compose succ;;
val plus1 : ('a -> int) -> 'a -> int = <fun>
# let plus1' = compose' succ;;
Error: This expression has type int -> int
      but an expression was expected of type ('a -> 'b) * ('c -> 'a)
# let plus2 = plus1 succ;;
val plus2 : int -> int = <fun>
# let plus2' = compose' (succ, succ);;
val plus2' : int -> int = <fun>
# plus2 7;;
- : int = 9
# plus2' 7;;
- : int = 9
```



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Functions & Pattern Matching

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Functions can be defined by pattern matching.

```
match expression with
| pattern when boolean expression -> expression
| pattern when boolean expression -> expression
```

Patterns can contain

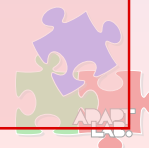
- constants, tuples, records, variant constructors and variable names;
- a catchall pattern denoted `_` that matches any value; and
- sub-patterns containing alternatives, denoted `pat1 | pat2`.

When a pattern matches

- the corresponding expression is returned.
- the (optional) when clause is a guard on the matching; it filters out undesired matchings.

```
let invert x =
  match x with
  | true -> false
  | false -> true ;;

let invert' = function
  true -> false | false -> true ;;
```



Recursion

Definition: Recursive Function

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A function is called recursive when it is defined through itself.

Example: Factorial.

- $5! = 5 * 4 * 3 * 2 * 1$
- Note that: $5! = 5 * 4!$, $4! = 4 * 3!$ and so on

Potentially a recursive computation.

From the mathematical definition:

$$n! = \begin{cases} 1 & \text{if } n=0, \\ n * (n-1)! & \text{otherwise.} \end{cases}$$

When $n=0$ is the base of the recursive computation (axiom) whereas the second step is the inductive step.



Recursion

What in ML?

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Still, a function is recursive when its execution implies another invocation to itself.

- directly, i.e. in the function body there is an explicit call to itself;
- indirectly, i.e. the function calls another function that calls the function itself (mutual recursion).

```
let rec fact(n) = if n<=1 then 1 else n*fact(n-1);;

let main() =
  print_endline("fact( 5) : - " ^ string_of_int(fact(5)));
  print_endline("fact( 7) : - " ^ string_of_int(fact(7)));
  print_endline("fact(15) : - " ^ string_of_int(fact(15)));
  print_endline("the largest admissible integer is :- " ^ string_of_int(max_int));
  print_endline("fact(25) : - " ^ string_of_int(fact(25)));;

main();;
```

```
[11:31]cazzola@surtur:~/lp/ml>ocamlc -o fact fact.ml
[11:31]cazzola@surtur:~/lp/ml>fact
fact( 5) : - 120
fact( 7) : - 5040
fact(15) : - 1307674368000
the largest admissible integer is :- 4611686018427387903
fact(25) : - 2188836759280812032
[11:31]cazzola@surtur:~/lp/ml>
```





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Execution: What's Happen?

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```
[11:45]cazzola@surtur:~/lp/ml>ocaml
OCaML version 4.12.0

# let rec fact(n) =
  if n<=1
  then 1
  else n*fact(n-1);;
val fact : int -> int = <fun>
# fact 4;;
- : int = 24
[11:46]cazzola@surtur:~/lp/ml>
```

It runs fact(4):

- a new frame with $n = 4$ is pushed on the stack;
- n is greater than 1;
- it calculates $4 * \text{fact}(3)$, it returns 24

It runs fact(3):

- a new frame with $n = 3$ is pushed on the stack;
- n is greater than 1;
- it calculates $3 * \text{fact}(2)$, it returns 6

It runs fact(2):

- a new frame with $n = 2$ is pushed on the stack;
- n is greater than 1;
- it calculates $2 * \text{fact}(1)$, it returns 2

It runs fact(1):

- a new frame with $n = 1$ is pushed on the stack;
- n is equal to 1;
- it returns 1



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Side Notes on the Execution

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At any invocation the run-time environment creates an **activation record** or **frame** used to store the current values of:

- local variables, parameters and the location for the return value.

To have a frame for any invocation permits to:

- trace the execution flow;
- store the current state and restore it after the execution;
- avoid interferences on the local calculated values.

Warning:

Without any stopping rule, the inductive step will be applied "for-ever".

- Actually, the inductive step is applied until the memory reserved by the virtual machine is full.



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Case Study: Fibonacci Numbers

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Leonardo Pisano, known as Fibonacci, in 1202 in his book "Liber Abaci" faced the (quite unrealistic) problem of determining:

"how many pairs of rabbits can be produced from a single pair if each pair begets a new pair each month and every new pair becomes productive from the second month on, supposing that no pair dies"

To introduce a sequence whose i -th member is the sum of the 2 previous elements in the sequence. The sequence will be soon known as the **Fibonacci numbers**.



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Case Study: Fibonacci Numbers (Cont'd)

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Fibonacci numbers are recursively defined:

$$f(n) = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1 \text{ or } n=2, \\ f(n-1) + f(n-2) & \text{otherwise.} \end{cases}$$

The implementation comes forth from the definition:

```
open List;;
let rec fibo(n) = if n<=1 then n else fibo(n-1) + fibo(n-2);;

let main() =
  let in's = [5; 7; 15; 25; 30] in
  for i=0 to List.length in's -1 do
    print_endline(
      "fibo(" ^ string_of_int(nth in's i) ^ ") :- " ^ string_of_int(fibo(nth in's i)));
  done;;
main();;
```

```
[16:08]cazzola@surtur:~/lp/ml>ocamlc -o fibo fibo.ml
[16:14]cazzola@surtur:~/lp/ml>fibo
fibo(5) :- 5
fibo(7) :- 13
fibo(15) :- 610
fibo(25) :- 75025
fibo(30) :- 832040
[16:14]cazzola@surtur:~/lp/ml>
```





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Recursion Easier & More Elegant

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The recursive solution is more intuitive:

```
let rec fibo(n) = if n<=1 then n else fibo(n-1) + fibo(n-2);;
```

The iterative solution is more cryptic:

```
let fibo(n) =
  let fib' = ref 0 and fib'' = ref 1 and fib = ref 1 in
  if n<=1 then n
  else
    (for i=2 to n do
      fib := !fib' + !fib'';
      fib' := !fib'';
      fib'' := !fib;
    done;
    !fib);;
```

But ...



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Tail Recursion

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The iterative implementation is more efficient:

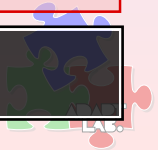
```
[18:22]cazzola@surtur:~/lp/ml>time time_ifibo 50
fibonacci(50) :- 12586269025
0.000u 0.006s 0:00.00 0.0%      0+0k 0+0io 0pf+0w
[18:22]cazzola@surtur:~/lp/ml>time time_rfibo 50
fibonacci(50) :- 12586269025
1605.211u 1.688s 26:48.62 99.8% 0+0k 0+0io 0pf+0w
[18:49]cazzola@surtur:~/lp/ml>
```

The overhead is mainly due to the creation of the frame but this also affects the occupied memory.

This can be avoided with a tail recursive solution:

```
let rec trfibonacci n m fib_m' fib_m =
  if (n=m) then fib_m
  else (trfibonacci n (m+1) fib_m (fib_m'+fib_m));;
let fibo n = if n<=1 then 1 else trfibonacci n 1 0 1;;
```

```
[16:59]cazzola@surtur:~/lp/ml>time trfibonacci 50
fibonacci(50) :- 12586269025
0.000u 0.005s 0:00.00 0.0%      0+0k 0+0io 0pf+0w
[16:59]cazzola@surtur:~/lp/ml>
```



The Towers of Hanoi

Definition (Édouard Lucas, 1883)

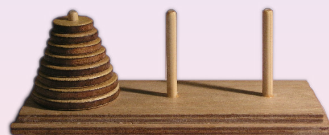
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Problem Description

There are 3 available pegs and several holed disks that should be stacked on the pegs. The diameter of the disks differs from disk to disk each disk can be stacked only on a larger disk.



The goal of the game is to move all the disks, one by one, from the first peg to the last one without **ever** violate the rules.



The Towers of Hanoi

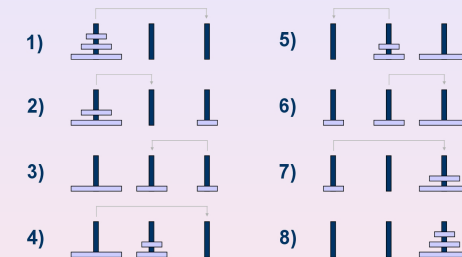
The Recursive Algorithm

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3-Disks Algorithm



n-Disks Algorithm

Base: $n=1$, move the disk from the source (S) to the target (T);

Step: move $n-1$ disks from S to the first free peg (F), move the last disk to the target peg (T), finally move the $n-1$ disks from F to T.





The Towers of Hanoi

ML/OCaML Implementation

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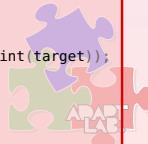
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```
type peg = string * string * string ;;
type pegs = {mutable src: peg; mutable trg: peg; mutable aux: peg} ;;

let nth(x,y,z) n = match n with 1 -> x | 2 -> y | 3 -> z ;;
let set_nth(x,y,z) w n = match n with 1 -> (w,y,z) | 2 -> (x,w,z) | 3 -> (x,y,w) ;;

let set_nth_peg ps p n =
  match n with 1 -> ps.src <- p | 2 -> ps.trg <- p | 3 -> ps.aux <- p ;;
let nth_peg ps n = match n with 1 -> ps.src | 2 -> ps.trg | 3 -> ps.aux ;;

let top(x,y,z) =
  match x,y,z with "0","0","0" -> 3 | "0","0",_ -> 2 | "0",_,_ -> 1 | _,_,_ -> 0 ;;
let p:pegs={src=("1","2","3"); trg=("0","0","0"); aux=("0","0","0")} in
  let rec display ps n =
    if n <= 4 then (
      print_endline(" ^nth ps.src n^" ^nth ps.trg n^" ^nth ps.aux n^");
      display ps (n+1);)
    and move ps source target =
      let s=(top (nth_peg ps source))+1 and t= top (nth_peg ps target) in (
        set_nth_peg ps (set_nth (nth_peg ps target) (nth (nth_peg ps source) s) t) target;
        set_nth_peg ps (set_nth (nth_peg ps source) "0" s) source;
        display ps 1;)
    and move_disks ps disks source target aux =
      if disks <= 1 then (
        print_endline("moving from ^string_of_int(source)^ to ^string_of_int(target)");
        move ps source target;)
      else (
        move_disks ps (disks-1) source aux target;
        print_endline("moving from ^string_of_int(source)^ to ^string_of_int(target)");
        move ps source target;
        move_disks ps (disks-1) aux target source;
        );
  in (print_endline("Start!!!");display p 1; move_disks p 3 1 3 2;) ;;
```



The Towers of Hanoi

3-Disks Run

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```
[16:21]cazzola@surtur:~/lp/ml>ocamlc -o hanoi2 hanoi2.ml
[16:21]cazzola@surtur:~/lp/ml>hanoi2
Start!!!
moving from 1 to 2      moving from 1 to 3      moving from 2 to 3
1 0 0      0 0 0      0 0 0      0 0 0
2 0 0      0 0 0      0 1 0      0 0 2
3 0 0      3 2 1      0 2 3      1 0 3
moving from 1 to 3      moving from 3 to 2      moving from 2 to 1      moving from 1 to 3
0 0 0      0 0 0      0 0 0      0 0 1
2 0 0      0 1 0      0 0 0      0 0 2
3 0 1      3 2 0      1 2 3      0 0 3
[16:21]cazzola@surtur:~/lp/ml>
```



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