Numerical Optimization Exam

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Exercise 2.9

Consider the function $f(x_1, x_2) = (x_1 + x_2^2)^2$. At the point $x^T = (1, 0)$ we consider the search direction $p^T = (-1, 1)$. Show that p is a descent direction and find all minimizers of the problem (2.10).

Solution:

We have to show that

$$\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T p < 0.$$

The gradient of f is given by

$$\nabla f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 2x_2^2 \\ 4x_1 \cdot x_2 + 4x_2^3 \end{pmatrix}.$$

Thus $\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T p = -2 < 0.$$

To find the minimizers of problem (2.10), we compute

$$\begin{aligned} \bar{\alpha} &= \min_{\alpha > 0} f(x + \alpha p) \\ &= \min_{\alpha > 0} f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \\ &= \min_{\alpha > 0} (1 - \alpha + \alpha^2)^2. \end{aligned}$$

We differentiate in α and impose that the derivative is zero, to obtain that

$$2(1 - \alpha + \alpha^2)(2\alpha - 1) = 0 \Leftrightarrow \alpha = \frac{1}{2}$$

This is a minimum because the second derivative of the function evaluated at point $\bar{\alpha} = \frac{1}{2}$ is greater than zero:

$$6\bar{\alpha}^2 - 6\bar{\alpha} + 3 = \frac{1}{4} > 0.$$

Exercise 3.7

Prove the result (3.28) by working through the following steps. First, use (3.26) to show that

$$\|x_k - x^*\|_Q^2 - \|x_{k+1} - x^*\|_Q^2 \;=\; 2\alpha_k \,\nabla f_k^{\mathsf{T}} Q\,(x_k - x^*) \;-\; \alpha_k^2\,\nabla f_k^{\mathsf{T}} Q\,\nabla f_k,$$

where $\|\cdot\|_Q$ is defined by (3.27). Second, use the fact that $\nabla f_k = Q(x_k - x^*)$ to obtain

$$\|x_k - x^*\|_Q^2 - \|x_{k+1} - x^*\|_Q^2 \;=\; \frac{2\left(\nabla f_k^{\mathsf{T}} \nabla f_k\right)^2}{\nabla f_k^{\mathsf{T}} Q\,\nabla f_k} \;-\; \frac{\left(\nabla f_k^{\mathsf{T}} \nabla f_k\right)^2}{\nabla f_k^{\mathsf{T}} Q\,\nabla f_k},$$

and

$$\|x_k - x^*\|_Q^2 \;=\; \nabla f_k^{\mathsf{T}} Q^{-1} \nabla f_k.$$

Solution:

Recall that $x_{k+1} = x_k - \alpha_k p_k$ where $p_k := \nabla f_k$. Hence (step 1)

$$\begin{aligned}
\|x_{k+1} - x^*\|_Q^2 &= (x_{k+1} - x^*)^T Q (x_{k+1} - x^*) \\
&= (x_k - x^* - \alpha_k p_k)^T Q (x_k - x^* - \alpha_k p_k) \\
&= (x_k - x^*)^T Q (x_k - x^* - \alpha_k p_k) - (\alpha_k p_k)^T Q (x_k - x^* - \alpha_k p_k) \\
&= (x_k - x^*)^T Q (x_k - x^*) - 2(\alpha_k p_k)^T Q (x_k - x^*) + (\alpha_k p_k)^T Q (\alpha_k p_k) \\
&= \|x_k - x^*\|_Q^2 - 2(\alpha_k p_k)^T Q (x_k - x^*) + \alpha_k^2 p_k^T Q p_k.
\end{aligned}$$

Then we substitute $\alpha_k = \frac{p_k^T p_k}{p_k^T Q p_k}$ and $p_k = Q(x_k - x^*)$ to obtain that (step 2)

$$\begin{aligned}
\|x_{k+1} - x^*\|_Q^2 &= \|x_k - x^*\|_Q^2 - 2\frac{p_k^T p_k}{p_k^T Q p_k} p_k^T p_k + \left(\frac{p_k^T p_k}{p_k^T Q p_k}\right)^2 p_k^T Q p_k \\
&= \|x_k - x^*\|_Q^2 - \frac{(p_k^T p_k)^2}{p_k^T Q p_k}.
\end{aligned}$$

But the weighted norm of the error can be written as (step 3)

$$(x_k - x^*)^T Q (x_k - x^*) = (x_k - x^*)^T Q Q^{-1} Q (x_k - x^*) = p_k^T Q^{-1} p_k$$

Hence,

$$\begin{aligned}
\|x_{k+1} - x^*\|_Q^2 &= \|x_k - x^*\|_Q^2 \left(1 - \frac{(p_k^T p_k)^2}{(p_k^T Q p_k)\|x_k - x^*\|_Q^2}\right) \\
&= \|x_k - x^*\|_Q^2 \left(1 - \frac{(p_k^T p_k)^2}{(p_k^T Q p_k)(p_k^T Q^{-1} p_k)}\right).
\end{aligned}$$

Exercise 4.7

When B is positive-definite, the *double-dogleg method* constructs a path with three line segments from the origin to the full Newton step. The four points that define the path are

- the origin;
- the unconstrained Cauchy step $p^c = -(g^\mathsf{T} g)/(g^\mathsf{T} Bg)\, g$;
- a fraction of the full step $\tilde{\gamma}\, p^B = -\tilde{\gamma} B^{-1} g$, for some $\tilde{\gamma} \in (\gamma, 1]$ where γ is defined in Exercise 4.6; and
- the full step $p^B = -B^{-1} g$.

Show that $\|p\|$ increases monotonically along this path. (Note: the double-dogleg method, as discussed in Dennis and Schnabel [92, Section 6.4.2], was for some time thought to be superior to the standard dogleg method, but later testing has not shown much difference in performance.)

Solution:

We do believe that Exercise 4.6 is mandatory to solve Exercise 4.7. Moreover, it was funny. Given a symmetric matrix $B \succeq 0$, the matrix $B^{\frac{1}{2}}$ is well posed. Thus, we have that

$$\|g\|^4 = \langle g, g \rangle^2 = \langle B^{\frac{1}{2}} g, B^{-\frac{1}{2}} g \rangle^2 \leq \|B^{\frac{1}{2}} g\|^2 \|B^{-\frac{1}{2}} g\|^2 = (g^T B g)(g^T B^{-1} g).$$

Therefore $\gamma \leq 1$.

We now have to prove that the segment $0 - p_j$ has increasing norm for increasing step j, that is

$$\|p_{j+1}\| \geq \|p_j\|$$

Clearly, it is true for $j = 1$ since $p_1 = 0$, and for $j = 3$, since

$$\|p_4\| = \frac{1}{\gamma} \|p_3\|.$$

and $\gamma \leq 1$. It remains to prove that for the previous γ it holds that $\|p_3\| \geq \|p_2\|$. By direct computation we have that

$$\|p_3\| = \frac{\|g\|^4 \ \|B^{-1} g\|}{(g^T B g)(g^T B^{-1} g)} = \frac{\|g\|^3}{(g^T B g)} \frac{\|g\| \ \|B^{-1} g\|}{(g^T B^{-1} g)} \geq \frac{\|g\|^3}{(g^T B g)} = \|p_2\|.$$

The last inequality holds since $\|AB\| \leq \|A\| \|B\|$ and since

$$\|p_2\| = \|g\| \frac{\|g\|^2}{(g^T B g)}.$$

Exercise 5.1

Implement Algorithm 5.2 and use to it solve linear systems in which A is the Hilbert matrix, whose elements are $A_{i,j} = 1/(i+j-1)$. Set the right-hand-side to $b = (1, 1, ..., 1)^T$ and the initial point to $x_0 = 0$. Try dimensions $n = 5, 8, 12, 20$ and report the number of iterations required to reduce the residual below 10^{-6}.

Solution:

The solution of the exercise is available here.

Exercise 12.18

Consider the problem of finding the point on the parabola $y = \frac{1}{5}(x-1)^2$ that is closest to $(x, y) = (1, 2)$ in the Euclidean norm sense. We can formulate this task as

$$\min_{x,y} f(x, y) = (x-1)^2 + (y-2)^2 \quad \text{subject to} \quad (x-1)^2 = 5y.$$

(a) Find all the KKT points for this problem. Is the LICQ satisfied at these points?

(b) Which of the KKT points are actually solutions of the constrained problem?

(c) By directly substituting the constraint into the objective and eliminating the variable x, we obtain an unconstrained optimisation problem. Show that the solutions of this reduced problem cannot be solutions of the original constrained problem.

Solution:

(a)

We have to find the Lagrangian

$$\mathcal{L} = f - \lambda c$$

This is given by

$$\mathcal{L}(x, y, \lambda) = (1+\lambda)(x-1)^2 + (y-2)^2 - 5\lambda y$$

Therefore

$$\nabla_{x,y}\mathcal{L} = \nabla_{x,y} f - \lambda \nabla_{x,y} c = \begin{pmatrix} 2(1+\lambda)(x-1) \\ 2(y-2) - 5\lambda \end{pmatrix}$$

Thus we have to find the points (x^*, y^*, λ^*) satisfying KKT. The first condition is that

$$\nabla_{x,y}\mathcal{L}(x^*, y^*, \lambda^*) = \begin{pmatrix} 2(1+\lambda^*)(x^*-1) \\ 2(y^*-2) - 5\lambda^* \end{pmatrix} = 0$$

The first equation tells that either $\lambda^* = -1$, either $x^* = 1$; the second one tells that $y^* = \frac{5}{2}\lambda^* - 2$. Hence the three possible points are $(1, \frac{5}{2}\lambda^* - 2, \lambda^*)$, $(x^*, -\frac{9}{2}, -1)$ and $(1, -\frac{9}{2}, -1)$. But the second and the third one do not satisfy the second condition of the KKT:

$$c(x^*, y^*) = 5y^* - (x^* - 1)^2 = 0$$

Thus the only possible KKT point is given by the first one, where y^* satisfies the second condition of the KKT:

$$5\left(\frac{5}{2}\lambda^* - 2\right) = 0 \Leftrightarrow \lambda^* = \frac{4}{5}$$

That is $(x^*, y^*, \lambda^*) = (1, 0, \frac{4}{5})$.

LICQ are satisfied since we have only one active constraint and

$$\nabla_{x,y} c = \begin{pmatrix} -2(x-1) \\ 5 \end{pmatrix} \neq 0.$$

(b)

Being $f(x, y)$ convex and differentiable over the entire domain, any stationary point (x^*, y^*) is a global minimizer of f.

(c)

We remark that the constraint implies also that $y \geq 0$. Thus we would take in account this condition when we directly substitute the constraint into the objective function... However, if we forget this condition, we obtain

$$\min_{y \in \mathbb{R}} f(y) = (y-2)^2 + 5y$$

which is obtained in the point given by

$$f'(y^*) = 0 \Leftrightarrow 2(y^* - 2) + 5 = 0 \Leftrightarrow y^* = -\frac{1}{2}.$$

Exercise 13.9

Consider the following linear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & -5x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 2x_1 + \tfrac{1}{2}x_2 \leq 8, \\ & x \geq 0. \end{aligned}$$

(a) Add slack variables x_3 and x_4 to convert this problem to standard form.

(b) Using Procedure 13.1, solve this problem with the simplex method, showing at each step the basis and the vectors λ, s_N, $\bar{x}_B$ and the value of the objective function. (The initial choice of B for which $\bar{x}_B \geq 0$ should be obvious once you have added the slacks in part (a).)

Solution:

(a)

The slack variables s_3, s_4 satisfy

$$x_1 + x_2 + s_3 = 5$$

$$2x_1 + \frac{1}{2}x_2 + s_4 = 8.$$

(b)

We start by computing the matrices used for the steps.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \qquad c = \begin{pmatrix} -5 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad B = [A_i]_{i \in \mathcal{B}}, \qquad N = [A_i]_{i \in \mathcal{N}}.$$

The first step is given by $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ so that $x_1 = 0 = x_2$. Then

$$B = I_2$$

$$x_B = B^{-1}b = b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\lambda = B^{-T}c_B = c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$s_N = c_N - N^T\lambda = c_N = \begin{pmatrix} -5 \\ -1 \end{pmatrix} < 0.$$

Hence we have to compute another step. We follow the Dantzig rule and pick $q = 1$. Thus

$$Bd = d = A_q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$x_q^+ = \min_{i|d_i>0}(x_B)_i/d_i = \min\left\{\frac{5}{1}, \frac{8}{2}\right\} = 4, \qquad p = 2.$$

Moreover

$$x_B^+ = x_B - dx_q^+ = \begin{pmatrix} 5 \\ 8 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We now remove the index $p=2$ from $\mathcal{B}$ and we replace with the entering index $q=1$ from $\mathcal{N}$. Then we have the new set of indeces $\mathcal{B}=\{3,1\}$, $\mathcal{N}=\{4,2\}$. Thus

$$B=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}, \qquad B^{-1}=\begin{pmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{pmatrix}, \qquad N=\begin{pmatrix}0 & 1\\ 1 & \frac{1}{2}\end{pmatrix}.$$

Moreover,

$$c_B=\begin{pmatrix}0\\ -5\end{pmatrix}, \qquad c_N=\begin{pmatrix}0\\ -1\end{pmatrix}$$

Thus,

$$x_B=B^{-1}b=\begin{pmatrix}1\\ 4\end{pmatrix}$$

$$\lambda=B^{-T}c_B=\begin{pmatrix}0\\ -\frac{5}{2}\end{pmatrix}$$

and

$$s_N=c_N-N^T\lambda=\begin{pmatrix}0\\ -1\end{pmatrix}+\begin{pmatrix}0 & 1\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\ -\frac{5}{2}\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\\ \frac{1}{4}\end{pmatrix}>0$$

which ends the iterations.

Exercise 14.15

Program Algorithm 14.3 in MATLAB. Choose $\eta=0.99$ uniformly in (14.38). Test your code on a linear programming problem (14.1) generated by choosing A randomly, and then setting x, s, b, and c as follows:

$$\begin{aligned}
x_i &= \begin{cases}\text{random positive number}, & i=1,2,\ldots,m,\\ 0, & i=m+1,m+2,\ldots,n,\end{cases}\\
s_i &= \begin{cases}\text{random positive number}, & i=m+1,m+2,\ldots,n,\\ 0, & i=1,2,\ldots,m,\end{cases}\\
\lambda &= \text{random vector},\\
c &= A^{\mathsf{T}}\lambda+s,\\
b &= Ax.
\end{aligned}$$

Choose the starting point (x^0,λ^0,s^0) with the components of x^0 and s^0 set to large positive values.

Solution:

The solution of the exercise is available here.

Exercise 15.3

Do the following problems have solutions? Explain.

(a) $\min_{x\in\mathbb{R}^2} \quad x_1 + x_2 \quad \text{subject to} \quad x_1^2 + x_2^2 = 2,\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1;$

(b) $\min_{x\in\mathbb{R}^2} \quad x_1 + x_2 \quad \text{subject to} \quad x_1^2 + x_2^2 \le 1,\ x_1 + x_2 = 3;$

(c) $\min_{x\in\mathbb{R}^2} \quad x_1 x_2 \quad \text{subject to} \quad x_1 + x_2 = 2.$

Solution:

(a) The set of the points satisfying the constraints is given by the point $x_1 = 1, x_2 = 1$. Therefore the minimum exists and it is equal to

$$\min_{x_1=1=x_2} x_1 + x_2 = 2$$

(b) The set of the points satisfying the constraints is empty :c. Hence the minimum does not exist.

(c) From the equality constraint,

$$x_1(2 - x_1) = -x_1^2 + 2x_1 = -(x_1 - 1)^2 + 1.$$

Setting $y := x_1 - 1$, we can restate (c) as the unconstrained problem

$$\min_{y\in\mathbb{R}} -y^2 + 1$$

which is lower-unbounded. So the minimum does not exist.

Exercise 17.3

Minimize the quadratic penalty function for problem (17.3) for $\mu_k = 1, 10, 100, 1000$ using an unconstrained minimization algorithm. Set $\tau_k = 1/\mu_k$ in Framework 17.1, and choose the starting point x_{k+1}^s for each minimization to be the solution for the previous value of the penalty parameter. Report the approximate solution of each penalty function.

Solution:

The solution of the exercise is available here.

Exercise 19.11

(a) Write the primal-dual system (19.6) for problem (19.53), treating s_1, s_2 as slack variables and denoting the multipliers of (19.53b)-(19.53c) by z_1, z_2. (You should obtain a system of five equations in five unknowns.) Show that the system matrix is singular at any iterate of the form $(x, 0, 0)$.

(b) Show that if the starting point in Example (19.53) lies in the region (19.54), one interior-point step lands on the tangent line to the parabola, as sketched in Figure 19.2. (More precisely, show that the tangent line never lies to the left of the parabola.)

(c) Let $x^{(0)} = -2,\ s_1^{(0)} = 1,\ s_2^{(0)} = 1,\ z_1^{(0)} = z_2^{(0)} = 1$ and set $\mu = 0$. Compute the full Newton step using the system in part (a). Truncate the step, if necessary, with the fraction-to-the-boundary rule using $\tau = 1$. Verify that the new iterate remains inside the region (19.54).

Solution:

(a)

By direct computation,

$$\begin{aligned}
\mathcal{L}(x, \lambda) &= x - \lambda_1(x^2 - s_1 - 1) - \lambda_2 \left(x - s_2 - \frac{1}{2}\right) \\
\nabla_x \mathcal{L} &= 1 - 2\lambda_1 x - \lambda_2 \\
\nabla^2_{xx} \mathcal{L} &= -2\lambda_1
\end{aligned}$$

and

$$A_I = \begin{pmatrix} -2x \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
A &= \begin{pmatrix} \nabla^2_{xx}\mathcal{L} & 0 & -A_I^T(x) \\ 0 & Z & S \\ A_I(x) & -I & 0 \end{pmatrix} \\
&= \begin{pmatrix} -2\lambda_1 & 0 & 0 & -2x & -1 \\ 0 & \lambda_1 & 0 & s_1 & 0 \\ 0 & 0 & \lambda_1 & 0 & s_2 \\ 2x & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

While the RHS is given by

$$b = -\begin{pmatrix} 1 - 2\lambda_1 x - \lambda_2 \\ s_1\lambda_1 \\ s_2\lambda_2 \\ x^2 - s_1 - 1 \\ x - s_2 - \frac{1}{2} \end{pmatrix}$$

Hence, if we take $x = C$ and $s_1 = 0 = s_2$, the fourth and the fifth columns of the matrix A are collinear, thus A is singular.

(b)

We want to prove that given $(x^i, s_1^i, s_2^i) \in \{(x, s_1, s_2), \quad x^2 - s_1 - 1 \geq 0, \quad x - s_2 - \frac{1}{2} \leq 0, \quad s_1 \geq 0\}$ then the method produces $(x^{i+1}, s_1^{i+1}, s_2^{i+1})$ in the same set. Recall that

$$x^{i+1} = x^i + \alpha_s p_{x^i}$$

and

$$s_j^{i+1} = s_j^i + \alpha_s p_{s_j^i}$$

By using the fourth line of the linear system we obtain that

$$2x^i \cdot p_{x_i} - p_{s_1} = {x^{(i)}}^2 - s_1^i - 1$$

While by using the fifth line of the linear system we have

$$p_{x^i} - p_{s_2} = -(x^i - s_2^i - \frac{1}{2})$$

Hence, since $\alpha_s \leq 1$,

$$\begin{aligned}
{x^{(i+1)}}^2 - s_1^{i+1} - 1 &= {x^{(i)}}^2 + 2\alpha_s x^i \cdot p_{x^i} + \alpha_s^2 p_{x^i}^2 - s_1^i - \alpha_s p_{s_1} - 1 \\
&= \alpha_s^2 p_{x^i}^2 + (1 - \alpha_s)({x^{(i)}}^2 - s_1^i - 1) \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
x^{i+1} - s_2^{i+1} - \frac{1}{2} &= x^i + \alpha_s \cdot p_{x^i} - s_2^i - \alpha_s p_{s_2} - \frac{1}{2} \\
&= (1 - \alpha_s)(x^{(i)} - s_2^i - \frac{1}{2}) \\
&\leq 0
\end{aligned}$$

Moreover, by definition of α_s, we have that $s \geq 0$.

(c)

The system (A) obtained with $x = -2, s_1 = 1 = s_2, \lambda_1 = 1 = \lambda_2$ is given by

$$\begin{pmatrix} -2 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ -4 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_{s_1} \\ p_{s_2} \\ p_{\lambda_1} \\ p_{\lambda_2} \end{pmatrix} = - \begin{pmatrix} 4 \\ 1 \\ 1 \\ 2 \\ -\frac{7}{2} \end{pmatrix}$$

Hence, the solution is given by

$$\begin{pmatrix} p_x \\ p_{s_1} \\ p_{s_2} \\ p_{\lambda_1} \\ p_{\lambda_2} \end{pmatrix} = \begin{pmatrix} \frac{7}{10} \\ -\frac{4}{5} \\ -\frac{14}{5} \\ -\frac{1}{5} \\ +\frac{9}{5} \end{pmatrix}$$

Therefore by direct computation we obtain that

$$\alpha_s = \max_{\alpha \in (0,1)} \{s + \alpha p_s \geq 0\} = \max_{\alpha \in (0,1)} \{\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} \frac{4}{5} \\ \frac{14}{5} \end{pmatrix} \geq 0\} = \min\{\frac{5}{4}, \frac{5}{14}\} = \frac{5}{14}$$

$$\alpha_\lambda = \max_{\alpha \in (0,1)} \{\lambda + \alpha p_\lambda \geq 0\} = \max_{\alpha \in (0,1)} \{\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} -\frac{1}{5} \\ \frac{9}{5} \end{pmatrix} \geq 0\} = \min\{1, 5\} = 1$$

Thus,

$$x^+ = x + \alpha_s p_x = -2 + \frac{1}{4} = -\frac{7}{4}$$

and

$$s^+ = s + \alpha_s p_s = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \alpha_s \begin{pmatrix} \frac{2}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ 0 \end{pmatrix}$$

Hence ${x^{(+)}}^2 - s_1^+ - 1 = \frac{49}{16} - \frac{12}{7} > 3 - \frac{12}{7} > 1 > 0$

Therefore we are in the asked region. Obviously, by point (b) it was possible to conclude without computing the values of the iterate.