Lecture I

Oscillation probability in JUNO

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Starting formula for $\bar{\nu}_e \to \bar{\nu}_e$ oscillations (assumed to be known):

$$P_{ee} = P(\bar{\nu}_e \to \bar{\nu}_e) = \left| \sum_{i=1}^{3} U_{ei} U_{ei}^{*} e^{-i \frac{m_i^2}{2E} x} \right|^2$$

where:

$$\begin{cases} x = \text{baseline} \\ E = \bar{\nu}_e \text{ energy} \\ m_i^2 = \text{squared mass of } \nu_i \\ U_{ei} = \text{PMNS mixing matrix elements} \\ |U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1 \quad \text{by } 3\nu \text{ unitarity} \end{cases}$$

In standard (PDG) conventions:

$$\begin{cases} |U_{e1}|^2 = \cos^2\theta_{13} \cos^2\theta_{12} \\ |U_{e2}|^2 = \cos^2\theta_{13} \sin^2\theta_{12} \\ |U_{e1}|^2 = \sin^2\theta_{13} \end{cases}$$

Let us work out the external square modulus:

$$P_{ee} = \left| \sum_i |U_{ei}|^2 e^{-i\frac{m_i^2}{2E}x} \right|^2$$

$$= \left(\sum_i |U_{ei}|^2 e^{-i\frac{m_i^2}{2E}x} \right) \left(\sum_j |U_{ej}|^2 e^{+i\frac{m_j^2}{2E}x} \right) \qquad \leftarrow |\ |^2 = (\ )\cdot(\ )^*$$

$$= \sum_{ij} |U_{ei}|^2 |U_{ej}|^2 e^{i\frac{m_j^2 - m_i^2}{2E}x} \qquad \leftarrow \text{group}$$

$$= \sum_{ij} |U_{ei}|^2 |U_{ej}|^2 \left(+1 - 1 + e^{i\frac{m_j^2 - m_i^2}{2E}x} \right) \qquad \leftarrow \text{add } +1-1$$

$$= \sum_{ij} |U_{ei}|^2 |U_{ej}|^2 + \left(\sum_{i<j} + \sum_{i>j} \right) |U_{ei}|^2 |U_{ej}|^2 \left(e^{i\frac{m_j^2 - m_i^2}{2E}} - 1 \right) \qquad \leftarrow \begin{cases} \text{break } \sum_{ij} = \sum_{i<j} + \sum_{i>j} + \sum_{i=j} \\ \text{note } \sum_{i=j} (\cdots) = 0 \end{cases}$$

$$= 1 + \sum_{i<j} |U_{ei}|^2 |U_{ej}|^2 \left(e^{i\frac{m_j^2 - m_i^2}{2E}} + e^{-i\frac{m_j^2 - m_i^2}{2E}} - 2 \right) \qquad \leftarrow \begin{cases} \text{use } \sum_i |U_{ei}|^2 = 1 \\ \text{and } \sum_{i>j} = \sum_{i<j} \ (i \leftrightarrow j) \end{cases}$$

$$= 1 + 2 \sum_{i<j} |U_{ei}|^2 |U_{ej}|^2 \left(\cos\left(\frac{m_j^2 - m_i^2}{2E} \right) - 1 \right) \qquad \leftarrow \cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$$

$$= 1 - 4 \sum_{i<j} |U_{ei}|^2 |U_{ej}|^2 \sin^2\left(\frac{m_j^2 - m_i^2}{4E} \right) \qquad \leftarrow \cos\alpha - 1 = -2\sin^2\left(\frac{\alpha}{2}\right)$$

In standard conventions:

$$\underset{i<j}{|U_{ei}|^2 |U_{ej}|^2} = \begin{cases} \cos^4\theta_{13} \cos^2\theta_{12} \sin^2\theta_{12} & , \quad ij = 12 \\ \cos^2\theta_{13} \sin^2\theta_{13} \sin^2\theta_{12} & , \quad ij = 23 \\ \cos^2\theta_{13} \sin^2\theta_{13} \cos^2\theta_{12} & , \quad ij = 13 \end{cases}$$

$$\Delta m^2_{ji} = m_j^2 - m_i^2$$

Thus:

$$\boxed{\begin{aligned} P_{ee} = 1 &- \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2\left(\frac{\Delta m^2_{21}}{4E}\right) \\ &- \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2_{32}}{4E}\right) \\ &- \cos^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2_{31}}{4E}\right) \end{aligned}}$$

(JUNO P_{ee} in vacuum)

Intermezzo on conversion factors in vacuum

Previous P_{ee} is in natural units ($\hbar = c = 1$). Usual textbook conversion:

$$\frac{\Delta m^2_{ji}\, x}{4E} = \underset{\substack{\uparrow \\ \text{3-digit conversion factor.}}}{1.27} \left(\frac{\Delta m^2_{ji}}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

However: 3 digits not appropriate to JUNO subpercent accuracy!

More digits:

$$\hbar c = 197.327 \text{ MeV}\cdot\text{fm} = 1 \quad \text{in natural units}$$

$$\rightarrow 1\,\text{MeV} \cdot 1\,\text{m} = 5.06773 \times 10^{12}$$

$$\rightarrow \frac{\Delta m^2_{ji}\, x}{4E} = \frac{1}{4}\left(\frac{\Delta m^2_{ji}}{eV^2}\cdot eV^2\right)\left(\frac{x}{m}\cdot m\right)\left(\frac{MeV}{E}\cdot\frac{1}{MeV}\right)$$

$$= \frac{1}{4}\left(\frac{1\,eV^2\cdot 1\,m}{1\,MeV}\right)\left(\frac{\Delta m^2_{ji}}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

$$= \frac{10^{-12}}{4}(MeV\cdot m)\left(\frac{\Delta m^2_{ji}}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

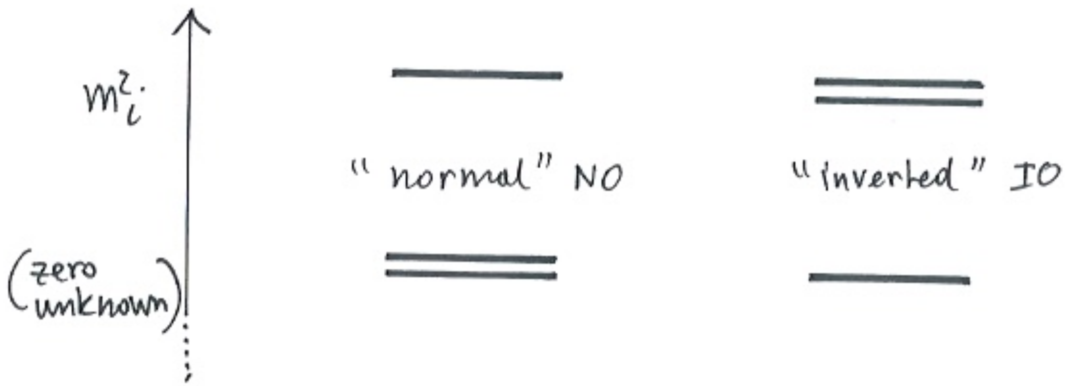
$$= 1.26693\left(\frac{\Delta m^2_{ji}}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

Conversion factor $\simeq 1.267$ with better than 10^{-4} accuracy.

Never use 1.27 in JUNO!

Mass ordering ("hierarchy") and other conventions

Experimentally, the spectrum of m_i^2 has one small and one large splitting:



NO = normal ordering
IO = inverted "

- Conventionally, the close mass states are denoted as (ν_1, ν_2).
- You may conventionally choose $m_1 < m_2$ without loss of generality.
- The remaining state is ν_3, with either $m_3 > m_{1,2}$ (NO) or $m_3 < m_{1,2}$ (IO)

→ Standard labels for mass states:

with $\Delta m^2_{21} > 0$ <u>always</u>

and Δm^2_{31} (or Δm^2_{32}) > 0 in NO and < 0 in IO

We have further introduce a convention that typically helps to show symmetries (or their absence) in the two mass orderings (Fogli + hep-ph/0105080):

$$\begin{cases} \delta m^2 = m_2^2 - m_1^2 = \Delta m^2_{21} > 0 \\ \Delta m^2 = m_3^2 - \dfrac{m_2^2 + m_1^2}{2} = \dfrac{1}{2}\left(\Delta m^2_{31} + \Delta m^2_{32}\right) \gtrless 0 \end{cases}$$

To avoid carrying out also the factor $\frac{x}{4E}$ we further define the phases:

$$\begin{cases} \delta = \dfrac{\delta m^2 x}{4E} > 0 \\ \Delta = \dfrac{\Delta m^2}{4E} \gtrless 0 \end{cases}$$

(not to be confused with δ_{CP}!)

Then:

$$\boxed{\begin{aligned} P_{ee} = 1 &- \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2(\delta) \\ &- \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\Delta + \frac{\delta}{2}\right) \\ &- \cos^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\Delta - \frac{\delta}{2}\right) \end{aligned}}$$

← "slow" oscillations

← "fast" and close oscill. frequencies

Note:

- P_{ee} not invariant under $+\Delta m^2 \to -\Delta m^2$ (unless $\theta_{12} = \frac{\pi}{4}$, excluded) ← Fogli + hep-ph/0105080
- Can be used to determine sign($\pm\Delta m^2$) at reactors, Petcov + hep-ph/0112074 <u>provided that you see both slow and fast oscillations</u>

Remember that, experimentally:

$$\begin{cases} \delta m^2 \sim 7.4 \times 10^{-5}\,\mathrm{eV}^2 \\ \Delta m^2 \sim 2.5 \times 10^{-3}\,\mathrm{eV}^2 \\ \sin^2\theta_{12} \sim 0.30 \\ \sin^2\theta_{13} \sim 0.022 \end{cases} \longrightarrow \text{two small numbers:} \begin{cases} \delta m^2/\Delta m^2 \sim 3 \times 10^{-2} \\ \sin^2\theta_{13} \sim 2 \times 10^{-2} \end{cases}$$

Well-known limits of Pee (work them out!):

1) $\delta = \frac{\delta m^2 x}{4E} \ll 1$ $\quad [\sin^2\delta \sim 0$ and $\sin^2\Delta \sim \mathcal{O}(1)]$

$\rightarrow$ can only see fast oscillations

$$\boxed{P_{ee} \cong 1 - \sin^2 2\theta_{13} \sin^2\Delta}$$

(Daya Bay, RENO, DC)

2) $\Delta = \frac{\Delta m^2 x}{4E} \gg 1$ $\quad [\sin^2\delta \sim \mathcal{O}(1)$ and $\langle\sin^2\Delta\rangle \sim \frac{1}{2}$, averaged oscill.$]$

$\rightarrow$ can only see slow oscillations

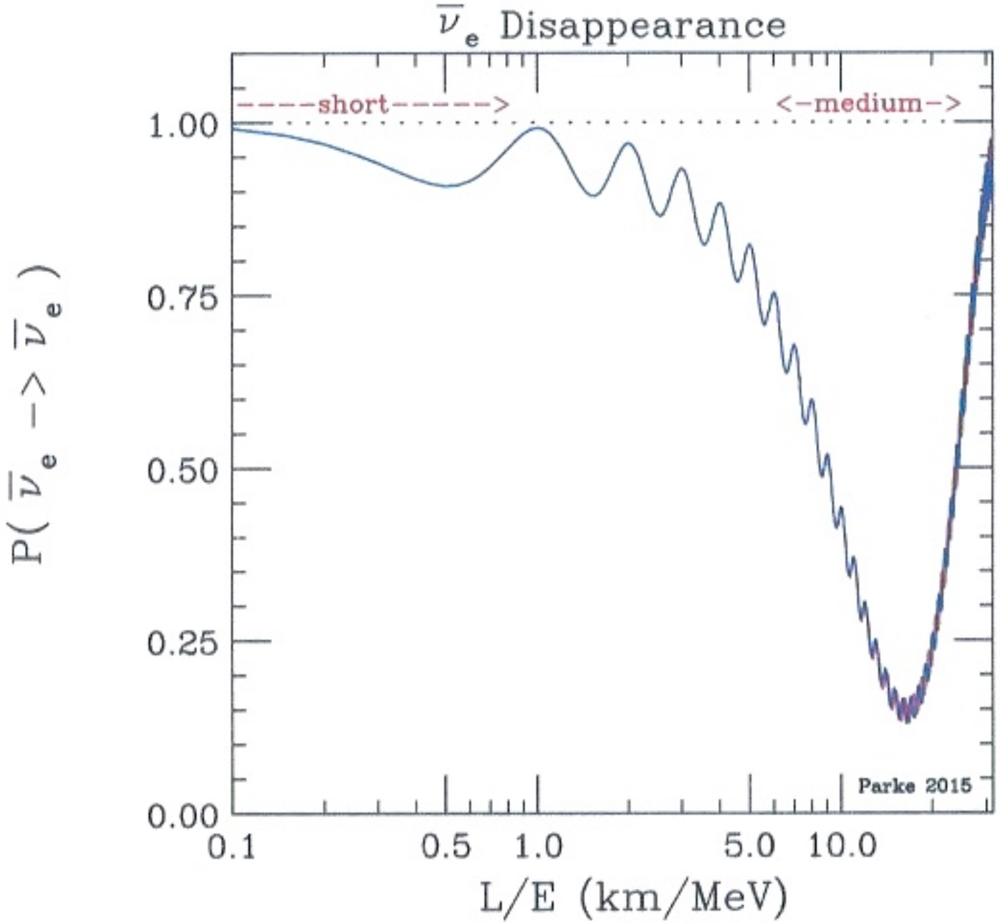
$$\boxed{P_{ee} = \cos^4\theta_{13}\,(1 - \sin^2 2\theta_{12} \sin^2\delta) + \sin^4\theta_{13}}$$

(KamLAND)

Note: this Pee has the form

$$P_{ee}^{3\nu} = \cos^4\theta_{13}\, P_{ee}^{2\nu} + \sin^4\theta_{13} \quad \text{with} \quad P_{ee}^{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2\delta$$

Fast and slow oscillations at the same time :



Pee (L/E) from
Parke 1601.07464

L = x

(blue = NO
red = IO)

FIG. 1: The vacuum survival probability for $\bar{\nu}_e$ as a function of L/E. Blue is for the normal mass ordering (NO) and red is the inverted mass ordering (IO) with Δm^2_{31} and Δm^2_{32} chosen in such a fashion that the two survival probabilities are identical at small L/E, ie. $\Delta m^2_{31}(IO) = -\Delta m^2_{31}(NO) + 2\sin^2\theta_{12}\Delta m^2_{21}$. Near the solar oscillation minimum, $L/E \sim 15$ km/MeV, the phase of the θ_{13} oscillations advances (retards) for the normal (inverted) mass ordering and the two oscillation probabilities are distinguishable, in principle. Also near the solar minimum, the amplitude of the θ_{13} oscillations is significantly reduced compared to smaller values of L/E. The short baseline experiments, Daya Bay, RENO and Double Chooz, probe $L/E < 0.8$ km/MeV and the medium baseline, JUNO and RENO 50, probe $6 < L/E < 25$ km/MeV, as indicated.

What does it mean to compare NO and IO in JUNO ?

Since $P_{ee} = P_{ee}(\Delta m^2, \delta m^2, \theta_{12}, \theta_{13})$ and these 4 parameters are <u>not</u> fixed numbers, but are endowed with uncertainties,
the <u>CORRECT WAY</u> is :

1) Take NO (sign $\Delta m^2 = +1$). Let $(\Delta m^2, \delta m^2, \theta_{12}, \theta_{13})$ float and constrain them with JUNO data. Get best fits $\overline{\Delta m^2}$, $\overline{\delta m^2}$, $\overline{\theta_{12}}$, $\overline{\theta_{13}}$ and an estimator of the overall goodness-of-fit at best fit (e.g., $\overline{\chi}^2$ or others)

2) Take IO (sign $\Delta m^2 = -1$). Repeat. Get different best fits $\overline{\overline{\Delta m^2}}$, $\overline{\overline{\delta m^2}}$, $\overline{\overline{\theta_{12}}}$, $\overline{\overline{\theta_{13}}}$ and a different g.o.f. estimator value (e.g. $\overline{\overline{\chi}}^2$)

3) Compare $\overline{\chi}^2$ and $\overline{\overline{\chi}}^2$:
If $\overline{\chi}^2 < \overline{\overline{\chi}}^2$ then NO is favored over IO at some C.L. given by $\Delta\chi^2$
If $\overline{\chi}^2 > \overline{\overline{\chi}}^2$ then IO " " " NO " " " " " "

If you don't like χ^2s, you may choose any other estimator, but you <u>must</u> leave $\Delta m^2, \delta m^2, \theta_{12}, \theta_{13}$ free to float <u>independently</u> in normal and inverted ordering.

NOTES:

(a) Actually JUNO will not improve upon θ_{13} w.r.t. Daya Bay + RENO+DC. If these experiments measure

$$\sin^2\theta_{13} = \widetilde{\sin^2}\theta_{13} \pm \sigma_{13}$$

then one can account for this constraint ~~m~~ through a penalty

$$\chi^2_{13} = \left(\frac{\sin^2\theta_{13} - \widetilde{\sin^2}\theta_{13}}{\sigma_{13}}\right)^2$$

in the JUNO data fit, for both NO and IO.

(b) All studies based on the correct way to compare NO and IO converge towards a baseline around ~50 km to be sensitive to the mass ordering.

INCORRECT WAYS to compare NO and IO assume that some mass-mixing parameters (or combinations) are fixed in both orderings, and then compare $P_{ee}(NO) - P_{ee}(IO)$.

This may be used for illustrative purposes only but not for fits.
E.g. in the previous figure by Parke, it was imposed that

$$-\Delta m^2_{31}(IO) = \Delta m^2_{31}(NO) - 2\sin^2\theta_{12}\,\delta m^2$$

in order to match the blue and red curves at small L/E.

The resulting difference $P_{ee}(NO) - P_{ee}(IO)$ at larger L/E is illustrative of mass-ordering effects, but is not an observable that can be fit by data.

Indeed, other choices for fixed parameters (e.g. the same absolute value of Δm^2_{31}, or of Δm^2_{32}, or of Δm^2, in NO and IO) would give different results for $P_{ee}(NO) - P_{ee}(IO)$.

For instance:

Lo Secco 1306.0845 fixed $|\Delta m^2_{32}|$ to be identical in NO and IO, obtaining (try it!):

$$\boxed{\Delta P_{ee} = P_{ee}(NO) - P_{ee}(IO) \propto \sin 2\delta}$$

He then argued that $\sin 2\delta \sim 1$ optimizes the sensitivity to the mass ordering, resulting in an optimal baseline ~30 km.

Wrong! If you fix $|\Delta m^2|$ to be identical in NO and IO, you would get (try it):

$$\boxed{\Delta P_{ee} \propto \sin\delta} \quad (\text{instead of } \sin 2\delta)$$

and thus a different "optimal" baseline. Fixing is misleading!

In general, it is not possible to separate unambiguously an "odd" term in P_{ee} under NO/IO swap, see the discussion in

Capozzi + 1309.1638
Xing 1808.02256

Papers making this assumption (including recent ones) are methodologically flawed.

A more profound and useful way to understand the effect of NO/IO swap on P_{ee} is to recast the effect in terms of the oscillation phase rather than amplitude.

We shall show that P_{ee} can be rewritten exactly in a form where the NO/IO swap amounts to swap the sign of a phase that does not scale as L/E

Evidence for NO or IO amounts to find evidence for this non-L/E phase, and to determine its $\pm$ sign.

Basic papers:	Nunokawa+	hep-ph/0503283
	Minakata+	hep-ph/0701151
revisited in:	Giuffoli+	1208.1991
	Giuffoli+	1302.0624

See also the review Parke 1601.07464
where the solution of a similar problem (how to express two close sound tones with a single tone frequency plus a modulated phase) is traced back to ~~the~~ Helmholtz 1863!

More precisely, we want to prove that one can write $P_{ee} = P_{ee}^{3\nu}$ as:

$$\boxed{P_{ee}^{3\nu} = c_{13}^4 \, P_{ee}^{2\nu} + s_{13}^4 + 2 s_{13}^2 c_{13}^2 \sqrt{P_{ee}^{2\nu}} \cos\left(\frac{\Delta m_{ee}^2 x}{2E} \pm \varphi\right) \quad \begin{cases} + \text{ NO} \\ - \text{ IO} \end{cases}}$$

where $P_{ee}^{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2 \delta$

and $\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 - s_{12}^2 \Delta m_{32}^2 = \Delta m^2 \pm \frac{1}{2}(c_{12}^2 - s_{12}^2)\, \delta m^2$

with $\begin{cases} \cos\varphi = \left[c_{12}^2 \cos(2 s_{12}^2 \delta) + s_{12}^2 \cos(2 c_{12}^2 \delta)\right] / \sqrt{P_{ee}^{2\nu}} \\ \sin\varphi = \left[c_{12}^2 \sin(2 s_{12}^2 \delta) - s_{12}^2 \sin(2 c_{12}^2 \delta)\right] / \sqrt{P_{ee}^{2\nu}} \end{cases}$

(and $\delta = \delta m^2 x / 4E$, $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$)

Basically we are summing up two "waves" with close frequencies governed by Δm_{31}^2 and Δm_{32}^2 into a single wave governed by Δm_{ee}^2, augmented with a correction φ whose sign tells the mass ordering. Note that if it were $\varphi \propto x/E$, it could be absorbed into $\Delta m_{ee}^2 x / 2E$ by redefining Δm_{ee}^2.
Only the "non L/E" part of φ contributes to NO/IO discrimination.

Proof :

Assume NO for the moment; (for IO, just flip the relative sign of Δm^2 and δm^2).

Define
$$\Delta m^2_{ee} = c^2_{12}\,\Delta m^2_{31} + s^2_{12}\,\Delta m^2_{32} \qquad \text{and} \qquad \Delta_{ee} = \Delta m^2_{ee}\, x/4E$$

So that
$$\begin{cases} \Delta m^2_{31} = \Delta m^2_{ee} + s^2_{12}\,\delta m^2 = \Delta m^2 + \delta m^2/2 \\ \Delta m^2_{32} = \Delta m^2_{ee} - c^2_{12}\,\delta m^2 = \Delta m^2 - \delta m^2/2 \\ \Delta m^2 = \Delta m^2_{ee} - \frac{1}{2}\left(c^2_{12} - s^2_{12}\right)\delta m^2 \end{cases}$$

Then the $P_{ee} = P^{3\nu}_{ee}$ at page 6 can be re-written as:

$$\begin{aligned} P^{3\nu}_{ee} &= 1 - c^4_{13}\left(1 - P^{2\nu}_{ee}\right) - \sin^2 2\theta_{13}\left[s^2_{12}\sin^2(\Delta_{ee} - c^2_{12}\delta) + c^2_{12}\sin^2(\Delta_{ee} + s^2_{12}\delta)\right] \\ &= 1 - c^4_{13} + c^4_{13} P^{2\nu}_{ee} + \frac{1}{2}\sin^2 2\theta_{13}\left[s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\delta) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\delta) - 1\right] \\ &= c^4_{13} P^{2\nu}_{ee} + s^4_{13} + 2\, s^2_{13} c^2_{13}\left[s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\Delta_{21}) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\delta)\right] \end{aligned}$$

We now force (Helmoltz!) the term in [...] to have the single-waveform :
$$s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\delta) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\delta) = \eta\cos(2\Delta_{ee} + \varphi)$$
with the amplitude η and the phase φ to be determined (if they exist).

In order to find a solution (η, φ) we expand the cosines and get:

$$s_{12}^2 \cos(2\Delta_{ee}) \cos(2c_{12}^2 \delta) + s_{12}^2 \sin(2\Delta_{ee}) \sin(2c_{12}^2 \delta)$$
$$+ c_{12} \cos(2\Delta_{ee}) \cos(2 s_{12}^2 \delta) - c_{12}^2 \sin(2\Delta_{ee}) \sin(2 s_{12}^2 \delta)$$
$$= \eta \cos(2\Delta_{ee}) \cos\varphi - \eta \sin(2\Delta_{ee}) \sin\varphi$$

namely,

$$\cos(2\Delta_{ee}) [s_{12}^2 \cos(2c_{12}^2 \delta) + c_{12}^2 \cos(2 s_{12}^2 \delta) - \eta \cos\varphi]$$
$$= \sin(2\Delta_{ee}) [- s_{12}^2 \sin(2c_{12}^2 \delta) + c_{12}^2 \sin(2 s_{12}^2 \delta) - \eta \sin\varphi]$$

that, since $\cos(2\Delta_{ee}) \neq \sin(2\Delta_{ee})$ in general, can be solved only if the terms in [....] are both vanishing:

$$\rightarrow \begin{cases} s_{12}^2 \cos(2 c_{12}^2 \delta) + c_{12}^2 (2 s_{12}^2 \delta) = \eta \cos\varphi \\ s_{12}^2 \sin(2 c_{12}^2 \delta) - c_{12}^2 (2 s_{12}^2 \delta) = -\eta \sin\varphi \end{cases}$$

If we square ~~the~~ and sum, we get:

$$\eta^2 = s_{12}^4 + c_{12}^4 + 2 s_{12}^2 c_{12}^2 \left[\cos(2c_{12}^2 \delta) \cos(2 s_{12}^2 \delta) - \sin(2 s_{12}^2 \delta) \sin(2 c_{12}^2 \delta)\right]$$

The term in [...] can be simplified by noticing that:

$$\sin^2\delta = \sin^2(\delta(c_{12}^2+s_{12}^2))$$
$$= \frac{1}{2}\left(1-\cos 2\left(\delta(c_{12}^2+s_{12}^2)\right)\right)$$
$$= \frac{1}{2} - \frac{1}{2}\left[\cos(2c_{12}^2\delta)\cos(2s_{12}^2\delta) - \sin(2c_{12}^2\delta)\sin(2s_{12}^2\delta)\right]$$

Hence it is

$$\eta^2 = s_{12}^4 + c_{12}^4 + 4s_{12}^2c_{12}^2\left[\frac{1}{2}-\sin^2\delta\right]$$
$$= 1-4s_{12}^2c_{12}^2\sin^2\delta$$
$$= P_{ee}^{2\nu}$$

and also

$$\begin{cases} \cos\varphi = \frac{1}{\sqrt{P_{ee}^{2\nu}}}\left(c_{12}^2\cos(2s_{12}^2\delta) + s_{12}^2\cos(2c_{12}^2\delta)\right) \\ \sin\varphi = \frac{1}{\sqrt{P_{ee}^{2\nu}}}\left(c_{12}^2\sin(2s_{12}^2\delta) - s_{12}^2\sin(2c_{12}^2\delta)\right) \end{cases}$$

that define φ parametrically. This completes the proof for NO.

For inverted ordering: one may change the sign of Δm_{ee}^2 within the oscillating term $\cos(2\Delta_{ee}+\varphi)$ or equivalently keep $\mathrm{sign}(\Delta m_{ee}^2)=+$ and change $\varphi\to-\varphi$.

Thus the oscillating factor is $\cos(2\Delta_{ee}\pm\varphi)$ $\begin{cases}+\text{NO}\\-\text{IO}\end{cases}$. End of proof.

[Note: $\pm\varphi$ is an advancement/retardation phase for Δ_{ee} oscillations]

- <u>Empirical parametrization for the phase φ</u>

In principle we may get φ from the ratio of the previous $\sin\varphi$ and $\cos\varphi$, namely

$$\boxed{\varphi = \arctan \frac{c_{12}^2 \sin(2 s_{12}^2 \delta) - s_{12}^2 \sin(2 c_{12}^2 \delta)}{c_{12}^2 \cos(2 s_{12}^2 \delta) + s_{12}^2 \cos(2 c_{12}^2 \delta)}} \, .$$

However, this is not practical as it leads to a quadrant ambiguity.

In Capozzi+ 1309.1638* we have found an empirical approximation to φ in closed form,

$$\boxed{\varphi \simeq 2 s_{12}^2 \delta \left(1 - \frac{\sin 2\delta}{2\delta \sqrt{P_{ee}^{2\nu}}}\right)}$$

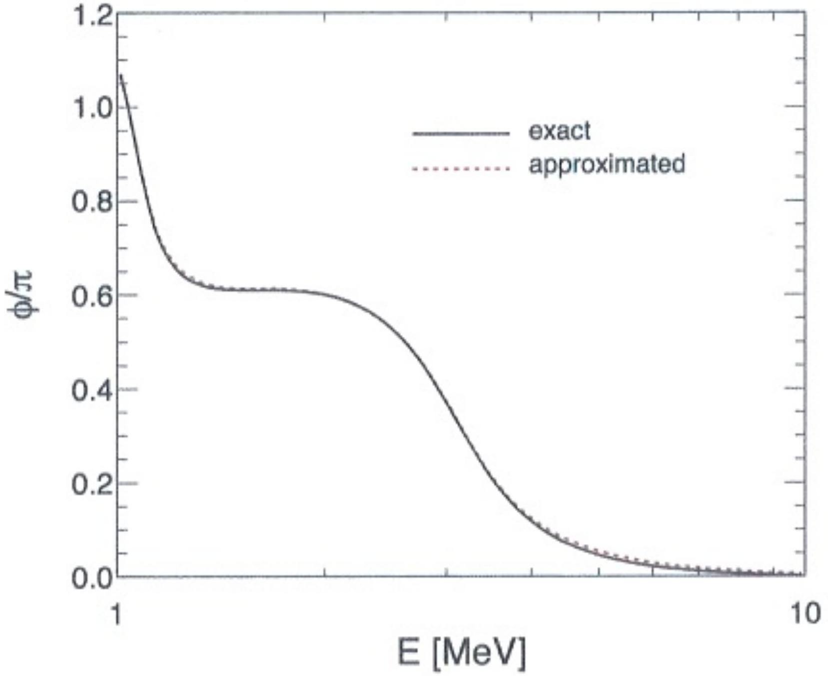
which is particularly useful. It also shares two properties of the exact parametric definition of φ:

a) it periodically increases with δ as $\varphi(\delta+\pi) = \varphi(\delta) + 2\pi s_{12}^2$ (hep-ph/0701151)

b) it starts with a cubic term $\propto \delta^3$ in a power expansion. (1208.1991)

* There was a typo in Eq. (45) therein, one should read $\sin 2\delta$ and not $\sin\delta$.

How does φ look like ?



Capozzi + 1309.1638
exact vs approximated φ
in a JUNO-like expt.

Note $L = 52.5$ km fixed
Note $\log(E)$ scale.

Observable only ~~at~~
for E above IBD threshold:

$E > 1.806$ MeV

FIG. 4: Comparison of exact and approximate values (in units of π) of the phase contribution φ embedding hierarchy effects, as a function of neutrino energy E, for $s_{12}^2 = 0.307$, $\delta m^2 = 7.54 \times 10^{-5}$ eV^2, and $L = 52.5$ km. See the text for details.

Determining the hierarchy amounts to prove that φ does not scale as L/E, namely as $1/E$ in this plot (slanted line). There is only a restricted range $E \sim 2 \div 4$ MeV where φ is above threshold energy and is not too small: very challenging task! One should then also determine $\text{sign}(\varphi) = \pm \binom{NO}{IO}$.

Another way of looking at the same challenge (Capozzi+ 2006.01648)

$P_{ee}(E)$ ratio of phases $\varphi/2\Delta_{ee}$

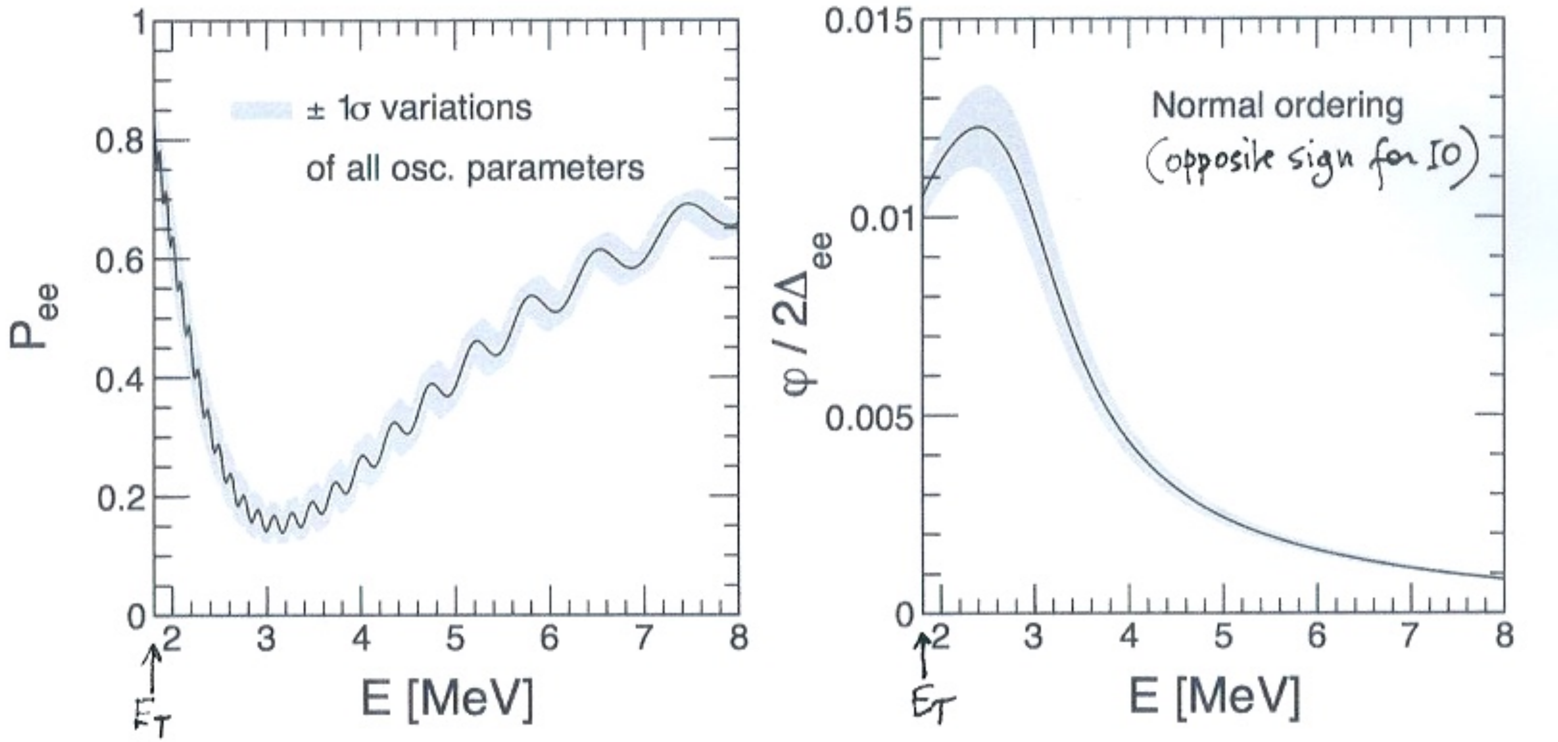


FIG. 4: Survival probability P_{ee} (left panel) and oscillation phase ratio $\varphi/2\Delta_{ee}$ (right panel) for electron antineutrinos with energy E in JUNO. Solid lines are computed for central values of the oscillation parameters, while the gray bands correspond to the envelope of $\leq 1\sigma$ variations in the prior ranges (see the text). Normal ordering is assumed. For inverted ordering, P_{ee} would be similar while $\varphi/2\Delta_{ee}$ would reverse its sign (not shown).

Once more, effects of a non L/E phase φ are small ($\sim$ percent of $2\Delta_{ee}$) and confined in the $\bar{\nu}_e$ energy range $E \sim 2 \div 4$ MeV, where the fast oscillation peaks are very close → need high resolution!

Intermezzo on energies (never be sloppy on energies!)

IBD : $\bar{\nu}_e + p \to e^+ + n$

$E(\bar{\nu}_e) \gtrsim 1.806 \text{ MeV} = E_T$

$m_e = 0.5110 \text{ MeV}$

$m_n - m_p = 1.293 \text{ MeV}$

$E = \bar{\nu}_e$ energy

$E_e =$ true positron energy (total: $E_e = T_e + m_e$)

$E_e + m_e =$ true visible energy of the event (not observable)

$E_{vis} =$ observed " " " " (= prompt measured energy)

For no recoil : $E = E_e + (m_n - m_p) = E_e + 1.293 \text{ MeV}$

With recoil : $\mathcal{O}(E/m_p)$ corrections + smearing

For perfect resol. : $E_{vis} = E_e + m_e$

With finite resol. : $\mathcal{O}(\%)$ smearing

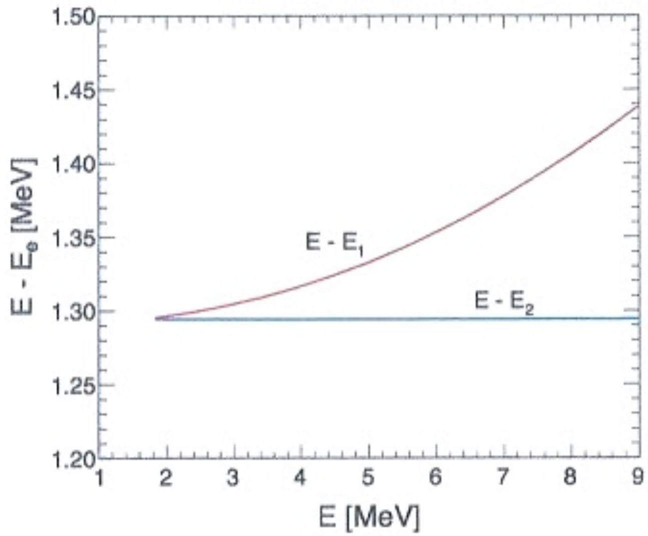
If you forget about recoil & resolution:

$$E_{vis} = E_e + m_e = E - (m_n - m_p) + m_e \simeq E - 0.78 \text{ MeV}$$

Often quoted ... but too approximate for JUNO purposes.

From Capozzi+ 1309.1638 . (See strumia+ astroph/0302055 for IBD Xsec.)

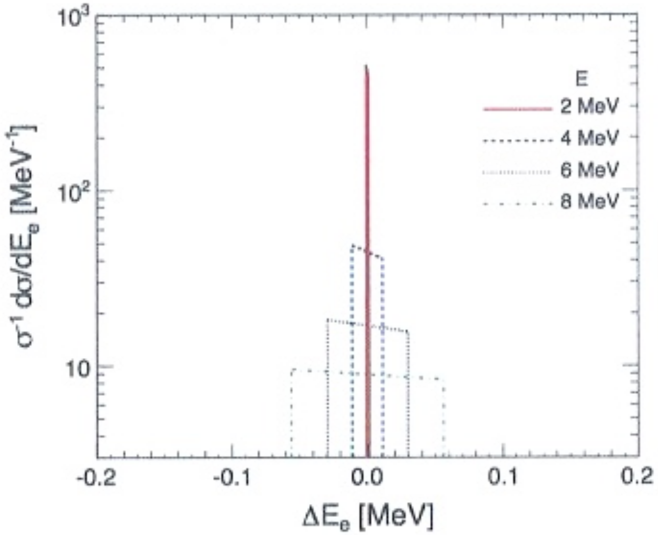
NO RECOIL
$E - E_e = 1.293$ →



WITH RECOIL :
median energy shifted
+ smearing within (E_1, E_2) range

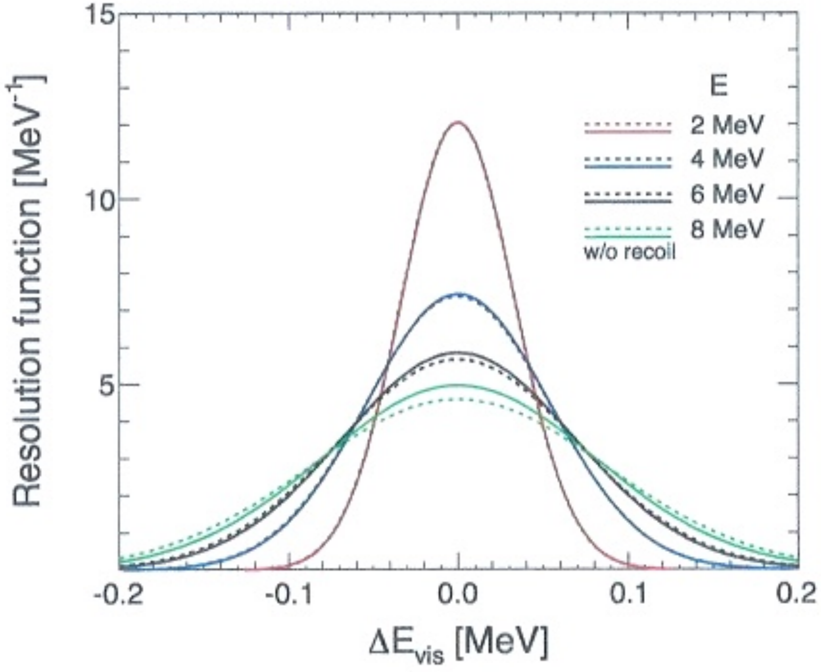
How is E_e distributed in (E_1, E_2) ?

RECOIL
DISTRIBUTION



... it is distributed like a ~ top-hat function in (E_1, E_2)

→ becomes a ~ Dirac delta only towards threshold.



Energy resolution introduces another (Gaussian) smearing.

—— ≡ resolution $= \Lambda$

---- ≡ resolution $\otimes$ recoil $= \Lambda \otimes \sqcap$

It turns out that the convolution of a gaussian with a top-hat function (resolution $\otimes$ recoil) is an analytic function (involving erf's). Useful.

(See Capozzi + 1309.1638).

These smearings are relevant for event spectra.

Smearing the probability P_{ee}: effects of multiple reactor core distances

We have obtained: $P^{3\nu}_{ee} = C^4_{13} P^{2\nu}_{ee} + S^4_{13} + 2 S^2_{13} C^2_{13} \sqrt{P^{2\nu}_{ee}} \cos(2\Delta_{ee} \pm \varphi)$

with: $P^{2\nu}_{ee} = 1 - \sin^2 2\theta_{12} \sin^2 \delta$

and: $\Delta_{ee} = \Delta m^2_{ee} L/4E$, $\delta = \delta m^2 L/4E$

Let us consider N reactors at close distances L_n, each contributing to the total flux Φ with its own flux Φ_n.

Define weights: $w_n = \dfrac{\Phi_n}{\Phi} = \dfrac{\Phi_n}{\sum_n \Phi_n} \rightarrow \sum_{n=1}^{N} w_n = 1$

Define flux-weighted distance $L = \sum_{n=1}^{N} w_n L_n$

so that $\boxed{L_n = L + \Delta L_n} \rightarrow \sum_n w_n \Delta L_n = 0$

We want to calculate the effects of $L_n \neq 0$ $(n = 1 \ldots N)$ on $P^{3\nu}_{ee}$, in the hypothesis that $\Delta L_n / L \ll 1$.

Note that such variations affect mainly the fast oscillations driven by Δ_{ee}, and much less the slow ones driven by δ.
We can take both $P_{ee}^{2\nu}$ and $\pm\varphi$ as constants, and focus on the oscillating term $\cos(2\Delta_{ee} \pm \varphi)$. The relevant effect is:

$$\boxed{\cos\left(\frac{\Delta m^2_{ee} L}{2E} \pm \varphi\right) \longrightarrow \sum_{n=1}^{N} w_n \cos\left(\frac{\Delta m^2_{ee} L_n}{2E} \pm \varphi\right)}$$

To calculate this weighted mean we need the trigonometric identities:

$$\sum_n w_n \cos(x + \epsilon_n) = w \cos(x + \epsilon)$$

where $w^2 = \sum_{nm} w_n w_m \cos(\epsilon_n - \epsilon_m)$

and $\tan\epsilon = \left(\sum_n w_n \sin\epsilon_n\right) / \left(\sum_n w_n \cos\epsilon_n\right)$

In our case it will be
$$\begin{cases} x \longleftrightarrow \dfrac{\Delta m^2_{ee} L}{2E} \pm \varphi \\[2ex] \epsilon_n \longleftrightarrow \dfrac{\Delta m^2_{ee} \Delta L_n}{2E} \end{cases}$$

$\Rightarrow$

$$\frac{\Delta m^2_{ee} L_n}{2E} \pm \varphi = \underbrace{\left(\frac{\Delta m^2_{ee} L}{2E} \pm \varphi\right)}_{x} + \underbrace{\left(\frac{\Delta m^2_{ee} \Delta L_n}{2E}\right)}_{\epsilon_n}$$

We have defined ΔL_n so that

$$\sum_n w_n \Delta L_n = 0 \quad \rightarrow \quad \sum_n w_n \epsilon_n = 0$$

In this case, since $\tan\epsilon = \dfrac{\sum_n w_n \sin\epsilon_n}{\sum_n w_n \cos\epsilon_n}$,

an expansion in (ϵ, ϵ_n) gives zero at first order for $\sin\epsilon_n$ since $\sum_n w_n \sin\epsilon_n \simeq \sum_n w_n \epsilon_n = 0$, thus $\epsilon \sim \mathcal{O}(\epsilon_n^3)$ and is neglgible. What matters is just the damping factor w:

$$\begin{aligned}
w^2 &= \sum_{nm} w_n w_m \cos(\epsilon_n - \epsilon_m) \\
&\simeq \sum_{nm} w_n w_m \left(1 - \left(\frac{\epsilon_n - \epsilon_m}{2}\right)^2\right) \\
&= \sum_{nm} w_n w_m \left(1 - \frac{\epsilon_n^2}{2} - \frac{\epsilon_m^2}{2} + \epsilon_n \epsilon_m\right) \\
&= \sum_{nm} w_n w_m - \sum_{nm} w_n w_m \epsilon_m^2 + \sum_{nm} w_n w_m \epsilon_n \epsilon_m
\end{aligned}$$

→ cont'd

cont'd

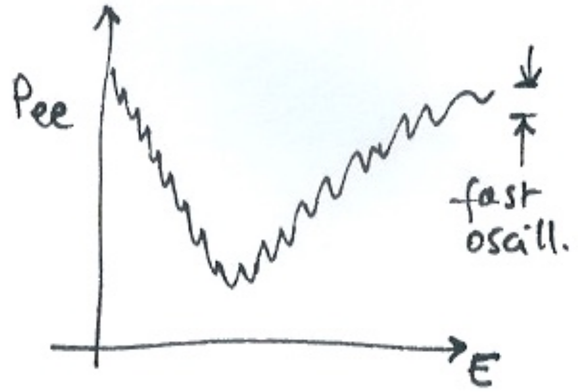
$$W = \Big(\sum_n w_n\Big)^2 - \Big(\sum_n w_n\Big)\Big(\sum_m w_m \epsilon_m^2\Big) + \cancel{\Big(\sum_n w_n \epsilon_n\Big)^2}$$

$$= 1 - \sum_n w_n \epsilon_n^2$$

$$= 1 - \left(\frac{\Delta m^2_{ee} L}{2E}\right)^2 \sum_m w_m \left(\frac{\Delta L_m}{L}\right)^2$$

$$= 1 - 4 (\Delta_{ee})^2 \sum_n w_n \left(\frac{\Delta L_n}{L}\right)^2$$

← Damping factor for fast oscill. amplitude



Final result: in $P_{ee}^{3\nu}$ make the replacement

$$\boxed{\begin{array}{l} \cos(2\Delta_{ee} + \varphi) \rightarrow W \cos(2\Delta_{ee} + \varphi) \\ W \simeq 1 - 4(\Delta_{ee})^2 \sum_n w_n \left(\frac{\Delta L_n}{L}\right) \end{array}}$$

Note that the damping factor goes down as $1/E^2$.
At $E \sim 2$ MeV it is $W \sim 0.7$ in JUNO ($\sim 30\%$ suppression of peaks) → need to be taken into account.

Caveat: in general, $\Phi_n = \Phi_n(t)$

thus also $\Delta L_n = \Delta L_n(t)$ and $W = W(t)$.

Probably it will be inconvenient to sum up time-dependent effects in a single $P_{ee}(t)$, averaged over all distances $L_n(t)$

Better to attach P_{ee} to each core and then sum up the final spectra.

This will represent a problem for external users of the data: how to present results that contain some time-dependence?
What t-integrated quantities are most useful?

Matter effects in JUNO

While multiple reactor distances affect mainly Δm^2_{ee}- driven oscillations (fast), matter effects affect mainly δm^2- driven one's (slow).

To understand why, let us calculate unit conversions for this important quantity related to matter effects:

$A = 2\sqrt{2}\, G_F N_e E$ where N_e is the density of background electrons.

We want to prove that the non-dimensional ratio $A/\Delta m^2_{ij}$ is:

$$\boxed{\frac{A}{\Delta m^2_{ij}} = 1.526\times 10^{-7}\left(\frac{N_e}{\mathrm{mol/cm^3}}\right)\left(\frac{E}{\mathrm{MeV}}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2_{ij}}\right)}$$

Proof: $1\ \mathrm{mol} = 6.022\times 10^{23}$ particles.

$$1\ \frac{\mathrm{mol}}{\mathrm{cm^3}} = \frac{6.022\times 10^{23}}{10^{-6}\ \mathrm{m^3}}\ \frac{\mathrm{MeV^3}}{\mathrm{MeV^3}} = 6.022\times 10^{29}\ \frac{1}{(\mathrm{m\cdot MeV})^3}\ \mathrm{MeV^3}$$

$$= \frac{6.022\times 10^{29}}{(5.0677\times 10^{12})^3}\ \mathrm{MeV^3} = 4.627\times 10^{-9}\ \mathrm{MeV^3}$$

$$\left(\begin{array}{l}\text{reminder: } \hbar c = 1 \\ \rightarrow \quad 1 = 197.327\ \mathrm{MeV\cdot fm}\end{array}\right)$$

$$G_F = 1.16637\times 10^{-5}\ \mathrm{GeV^{-2}} = 1.16637\times 10^{-11}\ \mathrm{MeV^{-2}}$$

$$\frac{2\sqrt{2}\, G_F N_e E}{\Delta m^2_{ij}} = 2\sqrt{2}\left(1.1664\times 10^{-11}\ \mathrm{MeV^{-2}}\right)\left(\frac{N_e}{\mathrm{mol/cm^3}}\ \mathrm{mol/cm^3}\right)\left(\frac{E}{\mathrm{MeV}}\cdot \mathrm{MeV}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2_{ij}}\cdot\frac{1}{\mathrm{eV}^2}\right)$$

$$= 3.299\times 10^{-11}\ \frac{\mathrm{MeV^{-2}\,MeV}}{\mathrm{eV}^2}\ \frac{\mathrm{mol}}{\mathrm{cm^3}}\left(\frac{N_e}{\mathrm{mol/cm^3}}\right)\left(\frac{E}{\mathrm{MeV}}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2_{ij}}\right)$$

$$3.299\times 10^{-11}\ \frac{\mathrm{MeV^{-2}\,MeV}}{\mathrm{eV}^2}\ \frac{\mathrm{mol}}{\mathrm{cm^3}} = 3.299\times 10^{-11}\ \frac{10^{12}}{\mathrm{MeV^3}}\cdot 4.627\times 10^{-9}\ \mathrm{MeV^3} = 1.526\times 10^{-7}$$

The ratio

$$r = \frac{A}{\Delta m^2_{ij}} = 1.526 \times 10^{-7} \left(\frac{N_e}{\text{mol/cm}^3}\right)\left(\frac{E}{\text{MeV}}\right)\left(\frac{\text{eV}^2}{\Delta m^2_{ij}}\right)$$

quantifies the importance of matter effects (A) with respect to vacuum oscillations (Δm^2_{ij}).

Let us consider $\Delta m^2_{ij} \equiv \delta m^2$, $E \sim 4$ MeV (typical at reactor peak) and $N_e \simeq 1.3$ mol/cm³ (typical for crustal density). Then:

$$r = \frac{A}{\delta m^2} \simeq 1.526 \times 10^{-7} \times 1.3 \times 4 \times (7.5 \times 10^{-5})^{-1} \simeq 10^{-2}$$

So the matter effects are already small for δm^2; they are even smaller for Δm^2_{ee}-driven oscillations. Thus we need to consider their effects only on θ_{12} and δm^2, which are replaced by their effective values in constant-density matter, namely $\tilde{\theta}_{12}$ and $\widetilde{\delta m}^2$

matter effects:
$$\begin{cases} \sin 2\tilde{\theta}_{12} = \dfrac{\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} + r)^2 + \sin^2 2\theta_{12}}} \\[2ex] \dfrac{\widetilde{\delta m}^2}{\delta m^2} = \dfrac{\sin 2\theta_{12}}{\sin 2\tilde{\theta}_{12}} \end{cases}$$

note + sign for $\bar{\nu}_e$ (it would be − for ν_e)

Since $r \sim \mathcal{O}(10^{-2})$, we can expand at first order and get:

$$\begin{cases} \sin 2\tilde{\theta}_{12} \simeq \sin 2\theta_{12}\,(1 - r\cos 2\theta_{12}) \\ \delta\tilde{m}^2 \simeq \delta m^2 (1 + r\cos 2\theta_{12}) \end{cases}$$

which shows that matter effects shift these two parameters by a fractional amount $\pm r\cos 2\theta_{12} \sim 4\times 10^{-3}$ at $E \sim 4$ MeV.

This is not negligible w.r.t. the accuracy planned in JUNO for θ_{12} and δm^2.

[See Capozzi+ 1309.1638 and later papers Capozzi+ 1508.01392, Li+ 1605.00900, Khan+1910.12900]

Grand total:

$$\boxed{\begin{aligned} P^{3\nu}_{mat} &= C^4_{13} P^{2\nu}_{mat} + S^4_{13} + 2\, S^2_{13} C^2_{13} \sqrt{P^{2\nu}_{mat}} \cdot w \cdot \cos\left(\frac{\Delta m^2_{ee} L}{2E} \pm \varphi\right) \\ \text{with} \quad P^{2\nu}_{mat} &= 1 - \sin^2 2\tilde{\theta}_{12} \sin^2 \tilde{\delta} \end{aligned}}$$

Containing:
- two oscillating phases (Δm^2_{ee} and δ)
- one mass-ordering phase ($\pm\varphi$)
- smearing of distances L_n (w)
- matter effects ($\tilde{\theta}_{12}$, $\tilde{\delta}$)

Why is energy calibration so important?

The mass ordering information is contained in the oscillating term

$$\cos\left(\frac{\Delta m^2_{ee} L}{2E} \pm \varphi\right) \qquad \begin{array}{l} + \text{ NO} \\ - \text{ IO} \end{array}$$

Suppose that you can "engineer" an energy rescaling $E \to E'$, possibly accompanied by a shift $\Delta m^2_{ee} \to \Delta m^{2\prime}_{ee}$, such that:

$$\boxed{\frac{\Delta m^2_{ee} L}{2E} + \varphi(E) = \frac{\Delta m^{2\prime}_{ee} L}{2E'} - \varphi(E)}$$

Then you would swap the spectrum from NO $(+\varphi)$ to IO $(-\varphi)$.
Very dangerous! Noted in Qian + 1208.1551.

The size of these hypothetical mis-calibrations (and their energy profile) can be estimated via the approximation:

$$\varphi \simeq 2 s^2_{12}\, \delta \left(1 - \frac{\sin 2\delta}{2\delta \sqrt{P^{2\nu}_{ee}}}\right)$$

(remember, in 1309.1638 there was a typo, $\sin 2\delta$ was written as $\sin\delta$)

Then the previous equation can be recast in an iterative form for the ratio E'/E, whose trivial (0th) solution is $E'/E=1$, while the second one is:

$$\frac{E'}{E} \simeq \frac{\Delta m^{2\,\prime}_{ee}}{\Delta m^2_{ee}} \mp 2s^2_{12}\,\frac{\delta m^2}{\Delta m^2_{ee}}\left(1-\frac{\sin 2\delta(E)}{2\delta(E)\sqrt{P^{2\nu}_{ee}(E)}}\right)$$

You can find in 1309.1638 examples of energy profiles that can fake a swap of mass ordering, with $E'/E \sim 1+\mathcal{O}(\%)$.

Of course Nature will not be so perverse to engineer a calibration shift $E'=E'(E)$ that exactly swaps $+\varphi \Leftrightarrow -\varphi$ ... but any calibration error will anyway decrease the mass ordering discrimination.

$\rightarrow$ Important to keep $E' \sim E$ at subpercent level!

NOTE: in principle, a miscalibration $E \rightarrow E'(E)$ also modifies the event spectrum $S(E) \rightarrow S'(E')$. One might "undo" the effect of $E \rightarrow E'$ by an opposite modification $S' \rightarrow S$ within S-shape errors. Of course this is extremely unlikely to happen, but shows that "shape" uncertainties are tricky.

Examples from 1309.1638 :

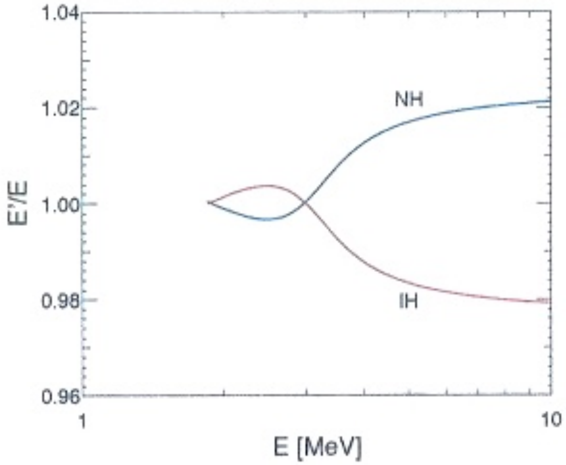


FIG. 17: Profile of the neutrino energy ratio E'/E which flips the sign of the hierarchy-dependent phase φ in the JUNO experiment, for the case $\Delta m^{2\prime}_{ee}/\Delta m^2_{ee} \simeq 1.022$ (0.978) in NH (IH). The profiles are shown for $E \geq E_T = 1.806$ MeV.

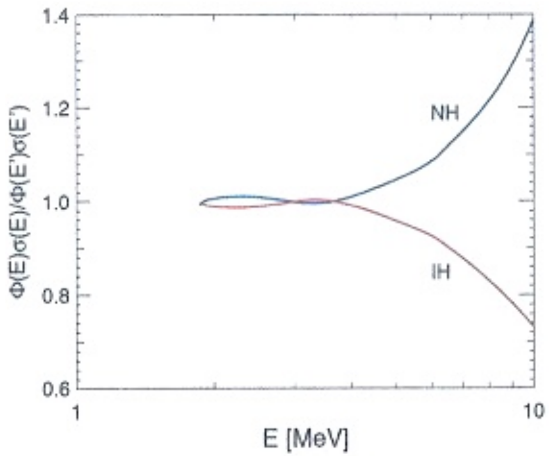


FIG. 18: Energy profiles of the fudge factors which would "undo" the reactor spectral changes induced by the changes $E \to E'$ reported in Fig. 17.

Figures for discussion

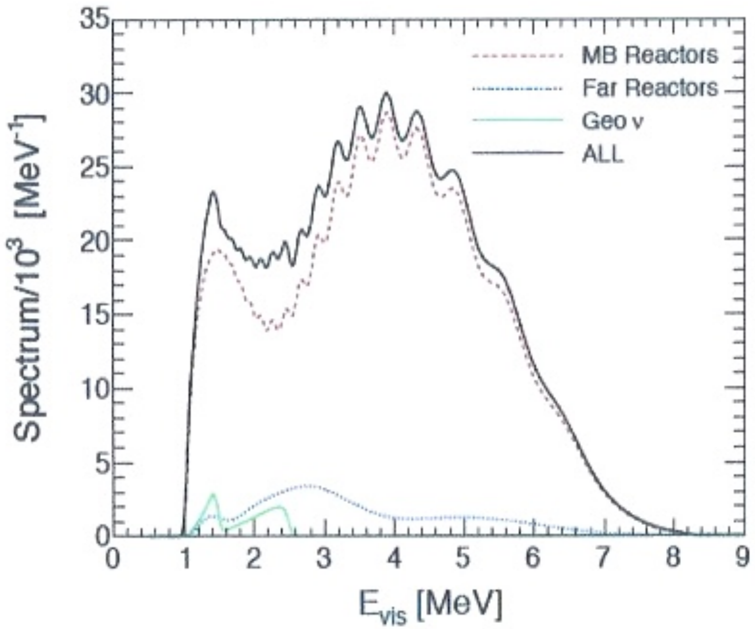


FIG. 5: Absolute energy spectrum of events expected in JUNO for normal hierarchy ($\alpha = +1$) and assuming the central values of the oscillation parameters defined in the text. The breakdown of the total spectrum in its three components (medium baseline reactors, far reactors, geoneutrinos) is also shown

from 1309.1638, Capozzi+

Capozzi+
1508.01392

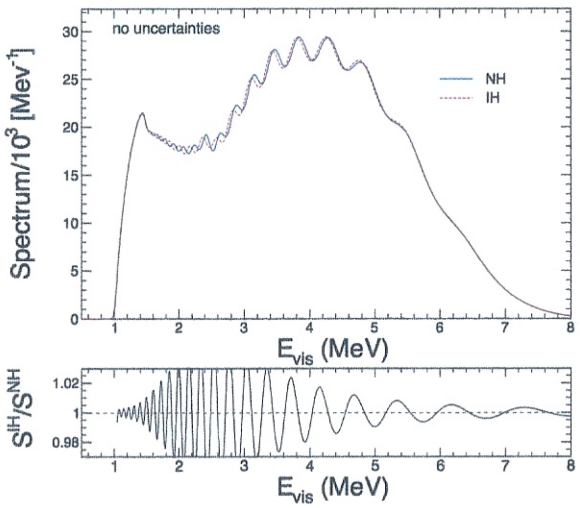


FIG. 2: Comparison of event spectra in NH and IH, as obtained for fixed oscillation parameters and no systematic errors. Top: Absolute spectra for $T = 5$ yr. Bottom: Spectral ratio.

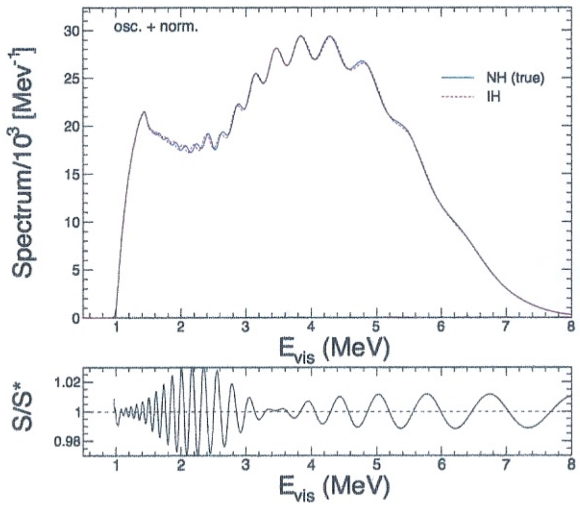


FIG. 3: Comparison of event spectra in NH (true, S^*) and IH (fitted, S), including oscillation and normalization uncertainties.

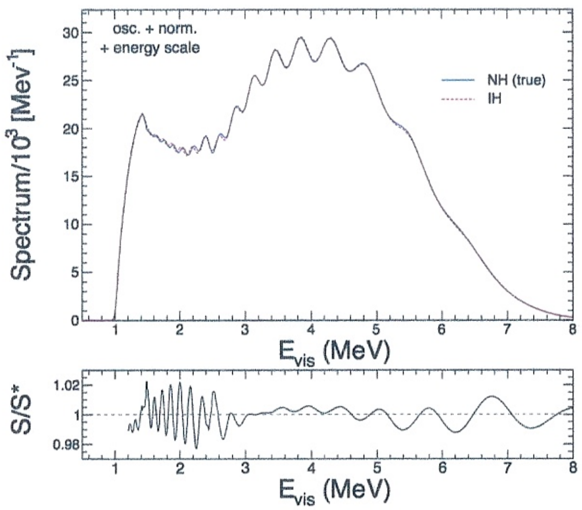


FIG. 4: As in Fig. 3, but including energy-scale systematics.

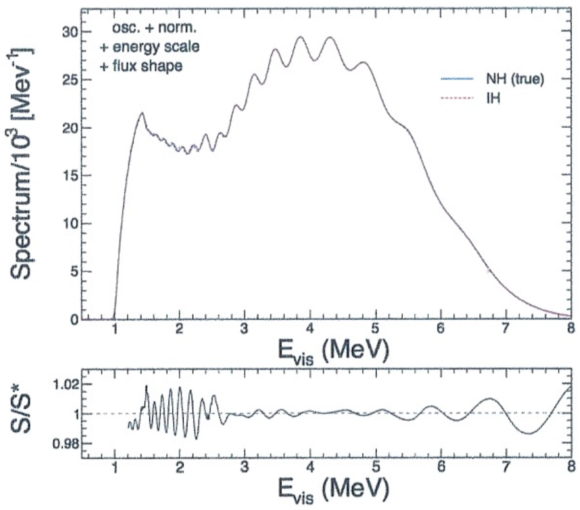


FIG. 5: As in Fig. 4, but including flux-shape systematics.

Fran Capozzi + 1508.01392

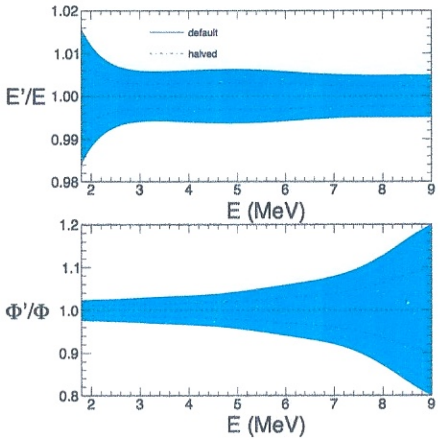


FIG. 1: Default $\pm 1\sigma$ error bands assumed for energy-scale deviations E'/E (top panel) and flux-shape variations Φ'/Φ (bottom panel), in terms of the neutrino energy E. Bands with halved errors are also shown (dot-dashed lines in both panels).

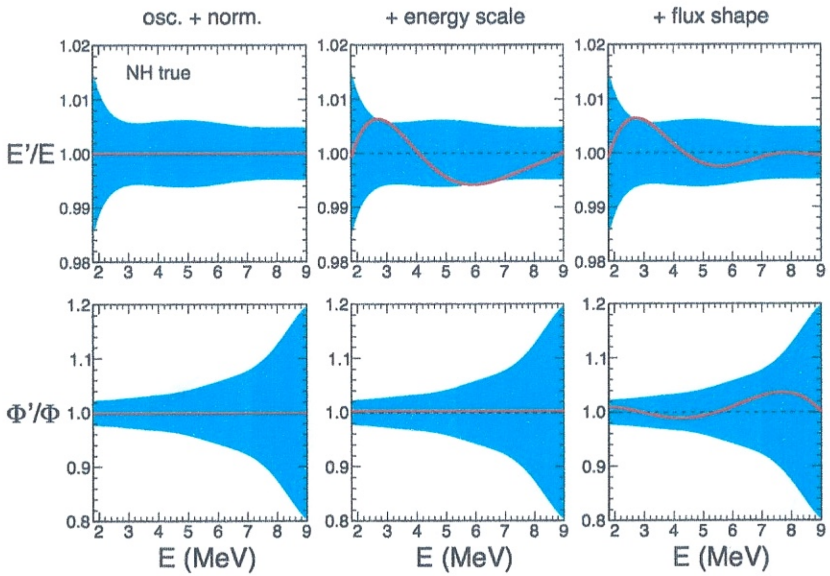


FIG. 6: Energy profile of best-fit deviations E'/E (top panels) and Φ'/Φ (bottom panels), for different sets of systematic uncertainties.

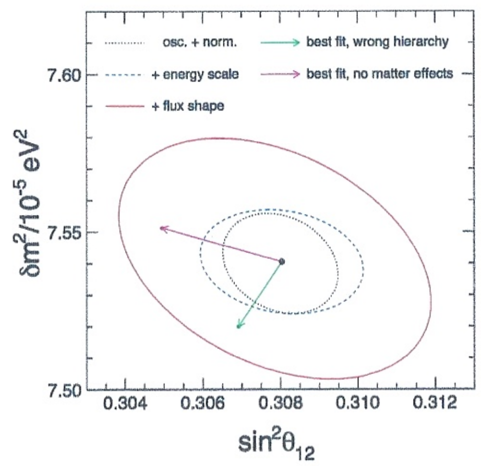


FIG. 9: Mass-mixing parameters (δm^2, s^2_{12}): 1σ contours for true NH and $T = 5$ y, as derived from fits including different systematic uncertainties. The arrows indicate the best-fit displacement in the cases of wrong hierarchy (green) and no matter effects (magenta).

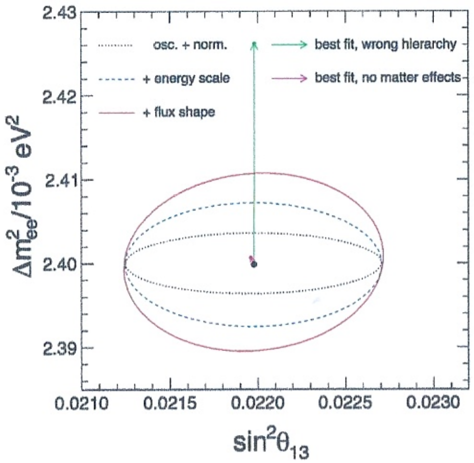


FIG. 10: As in Fig. 9, but for the (Δm^2_{ee}, s^2_{13}) parameters.

From Capozzi+
1508.01392

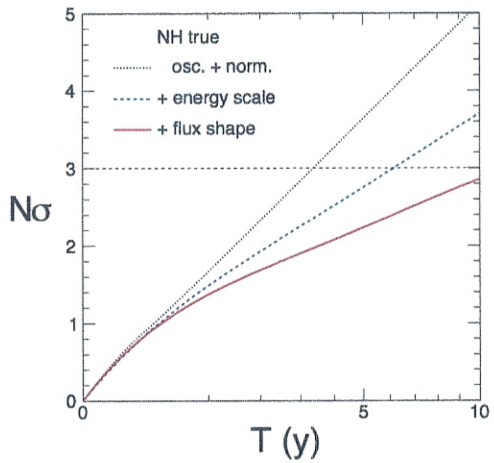


FIG. 7: Case of true NH: Statistical significance of the IH rejection as a function of the detector live time T, as derived from fits including different sets of systematics. Note that the abscissa scales as $\sqrt{T}$. The horizontal 3σ line is shown to guide the eye.

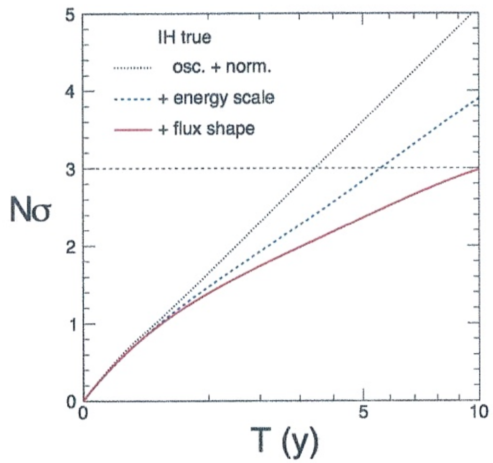


FIG. 8: As in Fig. 7, but for true IH and rejection of NH.

Next lecture:
will be based on
Capozzi+ 2006.01648

END of 1st LECTURE