Theory of Schemes: Fifth Exercise Session

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1 Basic properties of separated morphisms

Let X, Y, Z be three schemes.

(a) Prove that if $f : X \to Y$ and $g : Y \to Z$ are separated morphisms, then $g \circ f$ is separated. Solved during the exercise session

(b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are morphisms and $g \circ f$ is separated, then f is separated.

(c) Prove that if $\pi : X \to Y$ is a separated morphism and $s : Y \to X$ is a section of π (i.e. $\pi \circ s = \mathrm{Id}_Y$), then s is a closed embedding.

2 Affine diagonal

Let X be scheme. We say that a scheme X has affine diagonal if for every affine opens U and V of X the intersection $U \cap V$ is affine.

(a) Prove that X has affine diagonal if and only if the diagonal morphism $\Delta : X \to X \times_{\operatorname{Spec} \mathbb{Z}} X$ is an affine morphism. In particular, separated schemes (over $\operatorname{Spec} \mathbb{Z}$) have affine diagonal. Solved during the exercise session

(b) Find a scheme X with affine diagonal which is not separated.

3 When do morphisms of schemes coincide?

Let $f, g : X \to Y$ be two morphism of schemes and suppose there exists a dense open subset U of X such that $f|_U = g|_U$.

(a) Prove that if X is reduced and Y is separated, then $f = g$. Solved during the exercise session

(b) Find counterexamples to the previous statement if we remove one of the two hypothesis, namely either X reduced or Y separated.

4 Trivial projective space

Let A be a commutative unitary ring and $R := A[t]$. We define a grading on R setting $\deg t = 1$ and $\deg a = 0$ for every $a \in A$. Prove that the structural morphism

$$\operatorname{Proj} R \longrightarrow \operatorname{Spec} A$$

is an isomorphism.

5 Functoriality of the Proj construction

Let $\phi : R \to S$ be a graded morphism of graded rings. We recall that we denote by R_+ (respectively S_+) the irrelevant ideal of R (respectively S), i.e. all the elements of positive degree. We denote by $G(\phi)$ the open subset of $\operatorname{Proj} S$ defined as the complement of the closed subset $\mathbb{V}(\phi(R_+))$, i.e.

$$G(\phi) := \{p \in \operatorname{Proj} S | \phi(R_+) \not\subset p\}.$$

We can consider $G(\phi)$ as a scheme with the structure given as an open subset of $\operatorname{Proj} S$.

(a) Prove that ϕ induces a morphisms of schemes $G(\phi) \to \operatorname{Proj} R$. Solved during the exercise session

(b) Prove that if ϕ is surjective, then $G(\phi) = \operatorname{Proj} S$ and the morphism $\operatorname{Proj} S \to \operatorname{Proj} R$ is a closed embedding.

6 Some separated (or not) schemes

Find which one of the following schemes (or morphisms of schemes) is separated and motivate the answer:

(a) $X = \operatorname{Spec} A$ with $A := \frac{k[x_1, x_2, \ldots, x_n, \ldots]}{(x_1^2, x_3^3, x_4^4, \ldots)}$;

(b) $\operatorname{Proj} R$ with $R := \frac{k[x,y,z]}{(xz-y^2)}$ where $\deg x = \deg y = \deg z = 1$;

(c) X' as in exercise 1 of the second exercise session. In the same notation, is the morphism $U_1 \to X'$ separated?

7 Monomorphisms are separated

Let $f : X \to Y$ be a morphism of schemes. We say that f is a monomorphism if the composition function

$$\operatorname{Hom}(S, X) \longrightarrow \operatorname{Hom}(S, Y)$$

defined by $\alpha \mapsto f \circ \alpha$ is injective for any scheme S.

(a) Prove that if f is a closed immersion then it is a monomorphism.

(b) Prove that f is a monomorphism if and only if the diagonal $\Delta_f : X \to X \times_Y X$ of the morphism f is an isomorphism.

(c) Prove that if f is a monomorphism then f is separated.