Theory of Schemes: Second Exercise Session

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1 Decomposition in irreducible components

(a) Let X be a topological space. Prove that the following are equivalent:

  (i) X is a noetherian topological space;

  (ii) every open subset of X is quasi-compact;

  (iii) every non-empty family of closed subset of X has a minimal member.

(b) Let Y be a noetherian topological space and $Z \subset X$ be a closed subset. Prove that there exists an irredundant decomposition $Z = \bigcup Z_i$ where Z_i are closed and irreducible subsets of X. The decomposition is unique (up to the order). Solved during the exercise session

2 Topological properties of schemes

Let X be a scheme.

(a) Prove that any irreducible and closed subset $Z \subset X$ has a unique generic point. Such topological spaces are called *sober*.

(b) Schemes are usually not Hausdoorf. Describe all Hausdoorf affine schemes.

(c) Prove that X is T_0, i.e. for every pair of distinct points x, y of X there exists an open U of X which contains exactly one of them. Solved during the exercise session

(d) Prove that every quasi-compact sober topological space has a closed point. In particular, every quasi-compact scheme has a closed point. Solved during the exercise session

3 Local rings

Recall that we say that a commutative unitary ring A is local if it has only one maximal ideal m. We say that a morphism $\phi : A \to B$ of local rings is *local* is the preimage of the maximal ideal of B is equal to the maximal ideal of A.

(a) A is a local ring if and only if $\operatorname{Spec} A$ has a unique closed point;

(b) ϕ is local if and only if the image of the maximal ideal of A is cointaned inside the maximal ideal of B;

(c) ϕ is local if and only if $\operatorname{Spec} \phi$ sends the closed point of $\operatorname{Spec} B$ to the closed point of $\operatorname{Spec} A$;

(d) find a morphism of local rings which is not local;

(e) prove that $\operatorname{Spec} A$ is a point if and only if A is local and $\sqrt{0}$ is maximal. Notation: $\sqrt{I}$ is the radical ideal associated to the ideal I

4 Noetherian rings and topological space

Let A be commutative unitary ring.

- prove that if A is a noetherian ring then $\operatorname{Spec} A$ is a noetherian topological space,
- find an example where $\operatorname{Spec} A$ is a noetherian topological space but A is not noetherian. Hint: try to find a non-noetherian ring whose topological space is a point...

5 Reduced subscheme

Let A be the quotient rings of the polynomial algebra $k[x, y, z, w]$ by the ideal $I := (x^2, xy, y^2, xw - yz)$. Prove that $X_{\mathrm{red}} \simeq \mathbb{A}^2_k$.

6 Sheaf of units

Let $(X, \mathscr{O}_X)$ be a scheme. We say that a section $s \in \mathscr{O}_X(U)$ is a unit if there exists an element $t \in \mathscr{O}_X(U)$ such that $st = 1$ for some U open in X.

(a) Prove that $s \in \mathscr{O}_X(U)$ is a unit if and only if $s_x \in \mathscr{O}_{X,x}$ is a unit for every $x \in U$.

(b) Let $\mathscr{O}_X^\times(U) \subset \mathscr{O}_X(U)$ be the subgroup of units. Show that $\mathscr{O}_X^\times$ is a subsheaf of $\mathscr{O}_X$.