Theory of Schemes: Fourth Exercise Session

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1 Basic properties of the fiber product

Let X, Y, Z be three schemes over a base scheme S. Notation: we recall a scheme X over S is the datum of a morphism $X \to S$ of schemes Construct the following natural isomorphisms

(a) $X \times_S S \simeq X$; Notation: here S is considered as an S-scheme with the identity morphism

(b) $X \times_S Y \simeq Y \times_S X$;

(c) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.

Suppose we have the following diagram of schemes

$$\begin{tikzcd} & & X \arrow[d] \\ Y \arrow[r] & T \arrow[r] & S \end{tikzcd}$$

then construct the natural isomorphism

(d) $(X \times_S T) \times_T Y \simeq X \times_S Y$. Solved during exercise session

2 Properties not preserved by fiber product

Let X, Y two schemes over $\operatorname{Spec} k$ with k any field. Find a counterexample to the following statements:

(a) X, Y connected implies $X \times_k Y$ is connected;

(b) X, Y reduced implies $X \times_k Y$ is reduced;

(c) X, Y irreducible implies $X \times_k Y$ is irreducible (but connected). Solved during the exercise session

Notation: by $X \times_k Y$ we mean $X \times_{\operatorname{Spec} k} Y$
Hint: if k is algebraically closed, these statements are actually true, so try with some non-alg. closed fields...

3 Fibers of a morphism

Let $\phi : X \to Y$ be a morphism of schemes and $y \in Y$ a point. We denote by $X_y := X \times_Y \operatorname{Spec} k(y)$ the schematic fiber of ϕ at y, where $\operatorname{Spec} k(y) \to Y$ is the natural morphism induced by y.

(a) Describe X_y in the case $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and y some prime ideal in A.

(b) Prove that the morphism $X_y \to X$ induced by the fiber product construction is an homeomorphism onto the set-theoretic fiber $\phi^{-1}(y) \subset X$. Solved during the exercise session

4 Free extensions

Let $A \subset B$ be an extension of rings such that B is free of rank n as a A-module. Let $\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced morphism of schemes and y be a prime ideal of A.

(a) $\phi^{-1}(y)$ has at most n points (set-theoretically);

(b) if $\phi^{-1}(y)$ has exactly n points (set-theoretically), then $(\operatorname{Spec} B)_y$ is isomorphic to $\operatorname{Spec} k(y)^{\oplus n}$.

5 Almost isomorphisms

Find a morphism $f : X \to Y$ of schemes such that the morphism $X_y \to \operatorname{Spec} k(y)$ (induced by the construction of the fiber product) is an isomorphism for every point $y \in Y$ but f is not an isomorphism. Can you find an example with Y reduced?