Theory of Schemes: Third Exercise Session

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1 Sections of projective spaces

We recall the definition of $\mathbb{P}^n_k$ as the scheme defined by gluing the affine schemes

$$U_i := \operatorname{Spec} k[x_0/x_i, \ldots, x_n/x_i]$$

for every $i = 0, \ldots, n$. For every $j \neq i$ let $U_{i,j}$ be the distinguished open in U_i defined by the section x_j/x_i, then we have gluing isomorphisms

$$\tau_{ji} : U_{i,j} \longrightarrow U_{j,i}$$

defined as the "identity", namely $x_k/x_i \mapsto (x_k/x_j)(x_i/x_j)^{-1}$. The morphisms $\{\tau_{ji}\}$ satisfy the cocycle condition for the gluing.

(a) Compute the ring $\mathscr{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k)$ of global sections. Solved during the exercise session

Consider now the two (isomorphic) affine schemes: $U_2 = U_1 := \operatorname{Spec} k[t] = \mathbb{A}^1_k$ and the two opens $U_{1,2} \subset U_1$ and $U_{2,1} \subset U_2$ defined as $\operatorname{Spec} k[t]_t = \mathbb{A}^1_k \setminus 0$. We can construct (at least) two gluing isomorphisms:

$$\tau : U_{1,2} \longrightarrow U_{2,1}$$

which is defined by the association $t \mapsto t^{-1}$, and

$$\tau' : U_{1,2} \longrightarrow U_{2,1}$$

which is defined by the association $t \mapsto t$. These two data gives us two schemes which will be denoted respectively X and X'. Convince yourself that $X \simeq \mathbb{P}^1_k$.

(b) Compute the ring $\mathscr{O}_{X'}(X')$ of global sections of X' and deduce that X' is not isomorphic to $\mathbb{P}^1$.Solved during the exercise session

2 Some natural sheaves over $\mathbb{P}^n$

In the notation of the gluing data of $\mathbb{P}^n$ as in exercise 1, consider the sheaf of abelian groups (in fact of $\mathscr{O}_{U_i}$-modules) $\mathscr{F}_i$ over U_i defined as $\mathscr{O}_{U_i}$. For every integer $m \in \mathbb{Z}$, consider the following gluing functions

$$\tau^m_{ji} : \mathscr{F}_i|_{U_{i,j}} \longrightarrow \mathscr{F}_j|_{U_{j,i}}$$

defined by the association $f \mapsto (x_i/x_j)^{-m}\tau_{ji}(f)$. Prove that the gluing data $(U_i, \tau^m_{i,j})$ satisfy the cocycle condition and denote by $\mathscr{O}_{\mathbb{P}^n}(m)$ the resulting sheaves of abelian groups (in fact of $\mathscr{O}_{\mathbb{P}^n}$-modules).

(a) Compute the global sections $\mathscr{O}_{\mathbb{P}^n_k}(m)(\mathbb{P}^n_k)$.

3 Morphisms to affine schemes

Let X be a scheme and A be a commutative unitary ring. We have a natural map of sets

$$\eta : \operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathrm{Rng}}(A, \mathscr{O}_X(X))$$

defined by the association $(f, f^\sharp) \mapsto f^\sharp(X)$. Notation: the left-hand side is the set of morphisms of schemes and the right-hand side is the set of morphisms of rings

(a) Prove that η is a bijection. Solved during the exercise session

Suppose we are given the following commutative diagram

$$\begin{tikzcd} \mathbb{P}^n_k \arrow[r, "f"] \arrow[dr, "\pi"] & \operatorname{Spec} A \arrow[d, "g"] \\ & \operatorname{Spec} k \end{tikzcd}$$

where π is the natural structural morphism of $\mathbb{P}^n_k$.

(b) Prove that there exists a morphism $s : \operatorname{Spec} k \to \operatorname{Spec} A$ such that $s \circ \pi = f$ and $g \circ s = \mathrm{id}_{\operatorname{Spec} k}$.

Therefore there no non-trivial morphisms from a projective space to an affine scheme.

4 Sections and morphisms

Let $(X, \mathscr{O}_X)$ be a scheme.

(a) Prove that every section $s \in \mathscr{O}_X(X)$ gives rise to a morphism of schemes $X \to \mathbb{A}^1$ and viceversa;

(b) prove that every section $s \in \mathscr{O}_X^\times(X)$ gives rise to a morphisms of schemes $X \to \mathbb{A}^1 \setminus 0$ and viceversa.

5 Intersection of affine schemes in not affine

Find a scheme X and two open affine subscheme U_1 and U_2 of X such that $U_1 \cap U_2$ is not affine. Hint: We will see that this cannot happen if X is affine. Try to work with the non-affine schemes we know.