Theory of Schemes: First Exercise Session

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1 Sheaf of homomorphisms

Let $\mathscr{F}, \mathscr{G}$ be two presheaves of sets on a topological space X.

(a) Let us define for every open subset U of X

$$s\,\mathrm{Hom}(\mathscr{F},\mathscr{G})(U) := \mathrm{Hom}(\mathscr{F}(U),\mathscr{G}(U));$$

prove that $s\,\mathrm{Hom}(\mathscr{F},\mathscr{G})$ cannot be promoted to a presheaf in a natural way, i.e. there are no natural restriction functions for $V \subset U$.

(b) Let us define for every open subset U of X

$$\mathscr{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$$

where $\mathscr{F}|_U$ is the presheaf $\mathscr{F}$ restricted to the open U (the same for $\mathscr{G}$) and the Hom in the right hand side is the set of morphisms of presheaves. Prove that $\mathscr{H}om(\mathscr{F},\mathscr{G})$ can be promoted naturally to a presheaf. Moreover, prove that if G is a sheaf, then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is a sheaf as well.

2 Sum and product of sheaves

Let $\{\mathscr{F}_i\}_{i\in I}$ be a family of presheaves of abelian groups on the topological space X. We define the two presheaves:

$$\Big(\bigoplus_{i\in I}\mathscr{F}_i\Big)(U) := \bigoplus_{i\in I}\mathscr{F}_i(U)$$

and

$$\Big(\prod_{i\in I}\mathscr{F}_i\Big)(U) := \prod_{i\in I}\mathscr{F}_i(U)$$

with the obvious restriction functions. Prove that if $\mathscr{F}_i$ is a sheaf for every $i \in I$, then $\prod_{i\in I}\mathscr{F}_i$ is a sheaf. The same is not true for $\bigoplus_{i\in I}\mathscr{F}_i$: find a counterexample. Hint: Remember that $\bigoplus$ and $\prod$ are equal if I is finite

3 Support of sheaves and sections

Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space of X. We define the subset of X

$$\mathrm{Supp}(\mathscr{F}) := \{x \in X | \mathscr{F}_x \neq 0\}$$

where $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at the point x in X. Let $s \in \mathscr{F}(X)$ be a (global) section, then we define the subset of X

$$\mathrm{Supp}(s) := \{x \in X | s_x \neq 0\}$$

where s_x is the image of s through the natural morphism $\mathscr{F}(X) \to \mathscr{F}_x$.

(a) Prove that $\mathrm{Supp}(s)$ is always a closed subset of X;

(b) prove that if $\mathscr{F}$ is a sheaf of rings, then $\mathrm{Supp}(\mathscr{F})$ is always a closed subset of X;

(c) Find a topological space X and a sheaf $\mathscr{F}$ of abelian groups such that $\mathrm{Supp}(\mathscr{F})$ is not a closed subset of X.

4 Left exactness of global sections

Let

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

be an exact sequence of sheaves of abelian groups on a topological space X. Prove that for every U open subset of X, the sequence

$$0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U)$$

is still exact. Find a topological space X and a surjective morphism of sheaves $\mathscr{F} \to \mathscr{G}$ such that $\mathscr{F}(X) \to \mathscr{G}(X)$ is not surjective.

5 Sections of affine varieties

Let X be an affine variety as defined in class and $A(X)$ be the ring of functions of X. We have also defined the sheaf of functions $\mathscr{O}_X$ on the topological space X.

(a) Prove that there an isomorphism $A \simeq \mathscr{O}_X(X)$;

(b) prove that the open subset $\mathbb{A}^n \setminus 0$ of $\mathbb{A}^n$ is not an affine variety for $n \geq 2$. Hint: compute $\mathscr{O}_X(X)$ in the case $n = 2$ to get the idea