Digital health and computational epidemiology

Lesson 9

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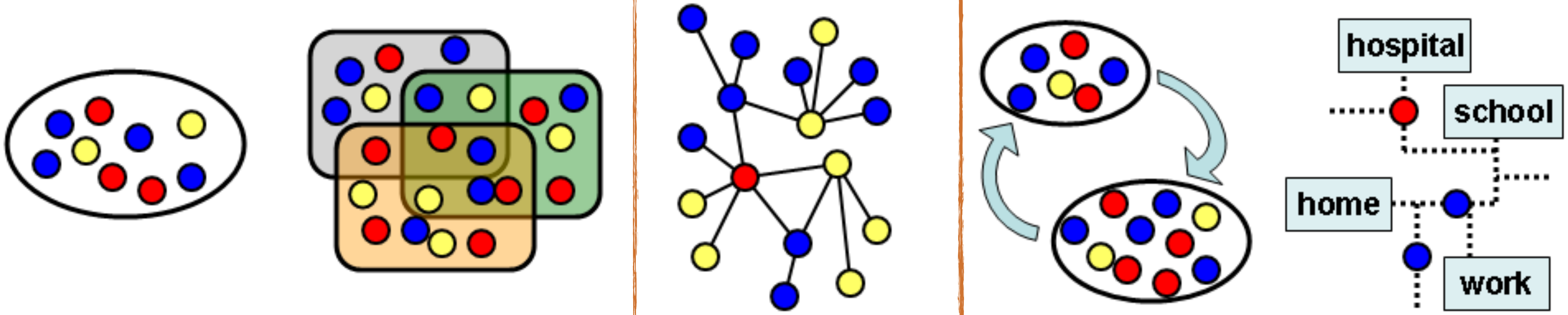
Epidemics on networks

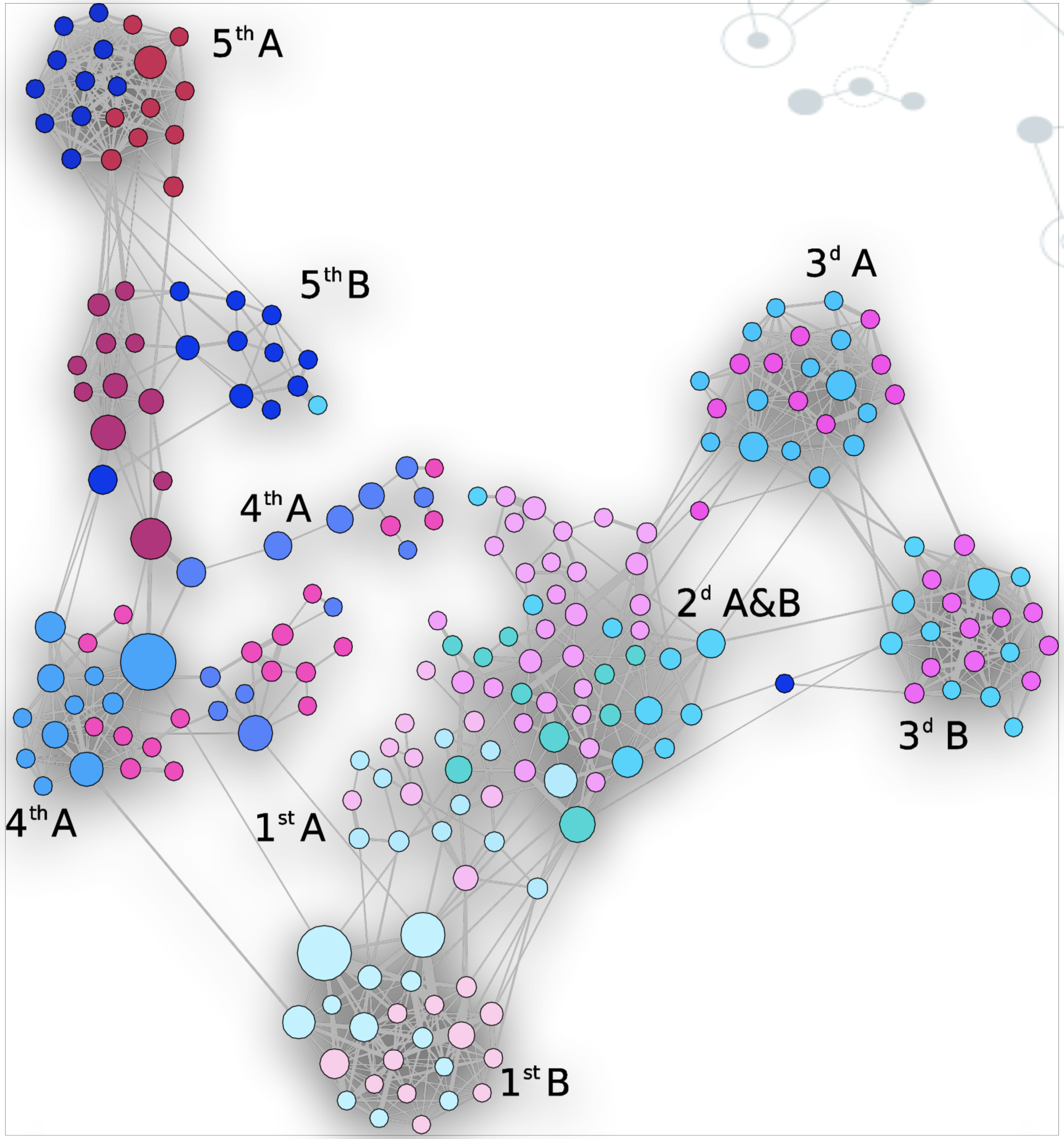
Models



Homogeneous mixing | **Social structure** | **Contact network models** | **Multi-scale models** | **Agent Based models**

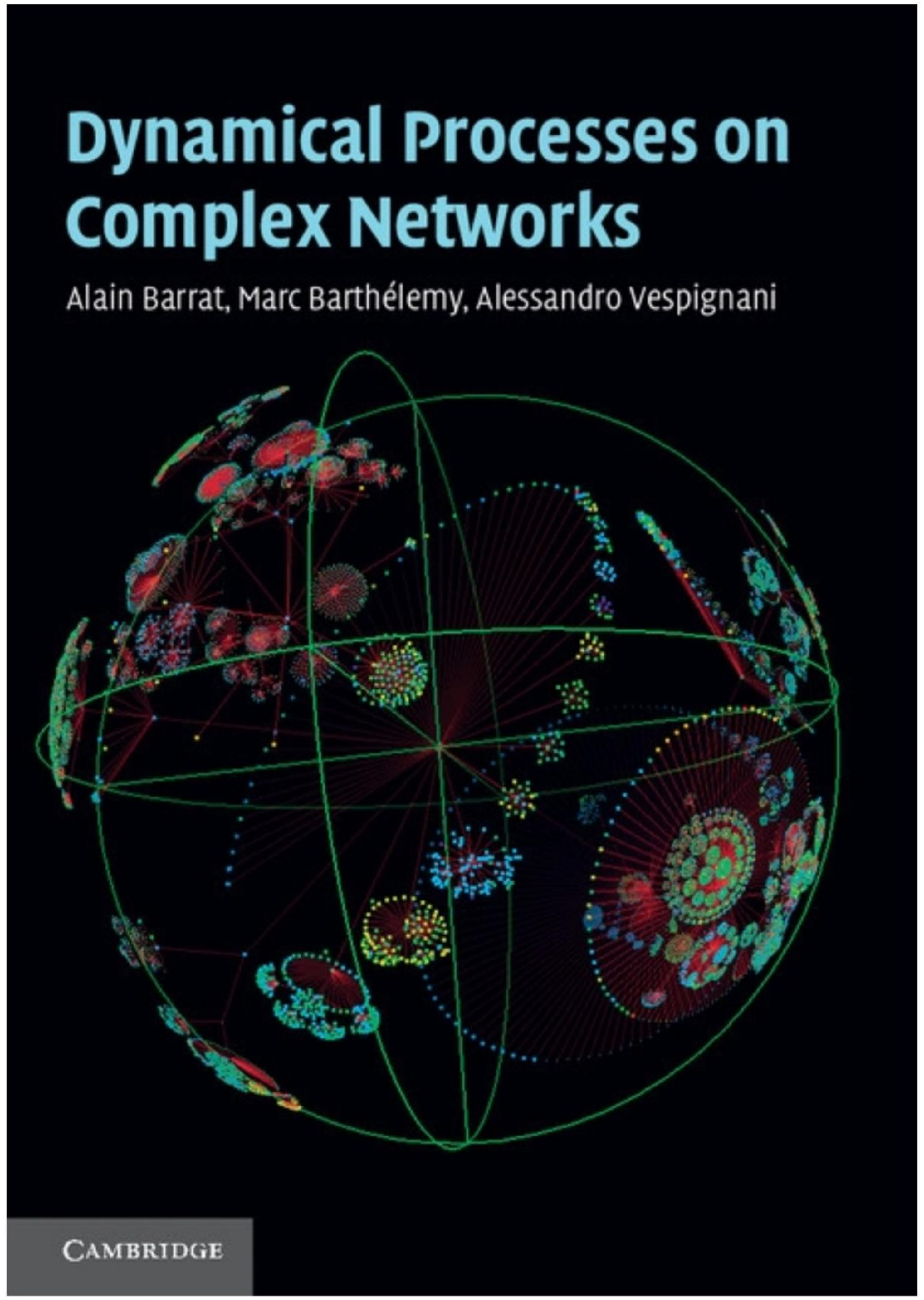
Epidemics on networks

- Homogeneous mixing is not always realistic
- Contacts are not equal and not constant across groups.
- Real contact networks display high heterogeneities
- **Pastor-Satorras et al. Epidemic processes in complex networks. Rev. Mod. Phys. 87, 925**



References

- Pastor-Satorras et al. Epidemic processes in complex networks. Rev. Mod. Phys. 87, 925 (2015)
- Pastor-Satorras, and Vespignani. Epidemic spreading in scale-free networks. Phys. Rev. Lett. 86, 14 (2000)
- Barrat, Barthelemy, Vespignani. Dynamical processes on complex networks. Cambridge University Press



Epidemics on networks

- We consider a network of N nodes where each node can be in an epidemic state, S, I or R
- We define the density of nodes in a given state, as:

$$\rho^S(t) = \frac{S(t)}{N}, \rho^I(t) = \frac{I(t)}{N}, \rho^R(t) = \frac{R(t)}{N}$$

Degree-based mean field

- Nodes with the **same degree** k are considered as **statistically equivalent**
- We focus on partial densities for each compartment: $\rho_k^\alpha, \alpha = S, I, R$
- These variables are not independent: $\sum_\alpha \rho_k^\alpha = 1$
- The total fraction of individuals in compartment α is equal to $\rho^\alpha(t) = \sum_k P(k)\rho_k^\alpha(t)$

Degree-based mean field

- The network is considered in a mean-field perspective (**annealed network** approximation).
- The adjacency matrix is completely destroyed. Only the degree and the two-vertex correlations of each node are preserved.
- The adjacency matrix is replaced by its ensemble average.

$$\bar{A}_{ij} = \frac{k_j P(k_i \,|\, k_j)}{N P(k_i)}$$

The DBMF SIS model

$$\frac{d\rho_k^I(t)}{dt} = \lambda k[1 - \rho_k^I(t)] \sum_{k'} P(k' \mid k)\rho_{k'}^I(t) - \mu\rho_k^I(t)$$

Transmission happens over k links

Density of S nodes

Probability of a node of degree k to be connected to an infected of degree k’

The DBMF SIS model

$$\frac{d\rho_k^I(t)}{dt} = \lambda k[1 - \rho_k^I(t)] \sum_{k'} P(k' \mid k)\rho_{k'}^I(t) - \mu\rho_k^I(t)$$

If we assume the network to be **uncorrelated:** $P(k' \mid k) = \dfrac{k'P(k')}{\langle k \rangle}$

then $\dfrac{d\rho_k^I(t)}{dt} = \lambda k[1 - \rho_k^I(t)]\Theta - \mu\rho_k^I(t)$ where $\Theta = \displaystyle\sum_{k'} \frac{k'P(k')}{\langle k \rangle}\rho_{k'}^I(t)$

Solution 1

Early stage approximation: $\rho_k^I(t) \ll 1$

then
$$\frac{d\Theta}{dt} = \left(\frac{\lambda}{\mu}\frac{\langle k^2\rangle}{\langle k\rangle} - 1\right)\Theta$$

which implies that Θ will grow if:

$$\boxed{\frac{\lambda}{\mu} > \frac{\langle k\rangle}{\langle k^2\rangle}}$$

Epidemic threshold

Solution 2

We look for the **steady state solution** $\rho_k^I(t), t \to \infty$

then: $\dfrac{d\rho_k^I}{dt} = 0$

which implies that: $\rho_k^I = \dfrac{\lambda k \Theta}{\mu + \lambda k \Theta}$

By plugging the above expression in the definition of Θ for uncorrelated networks:

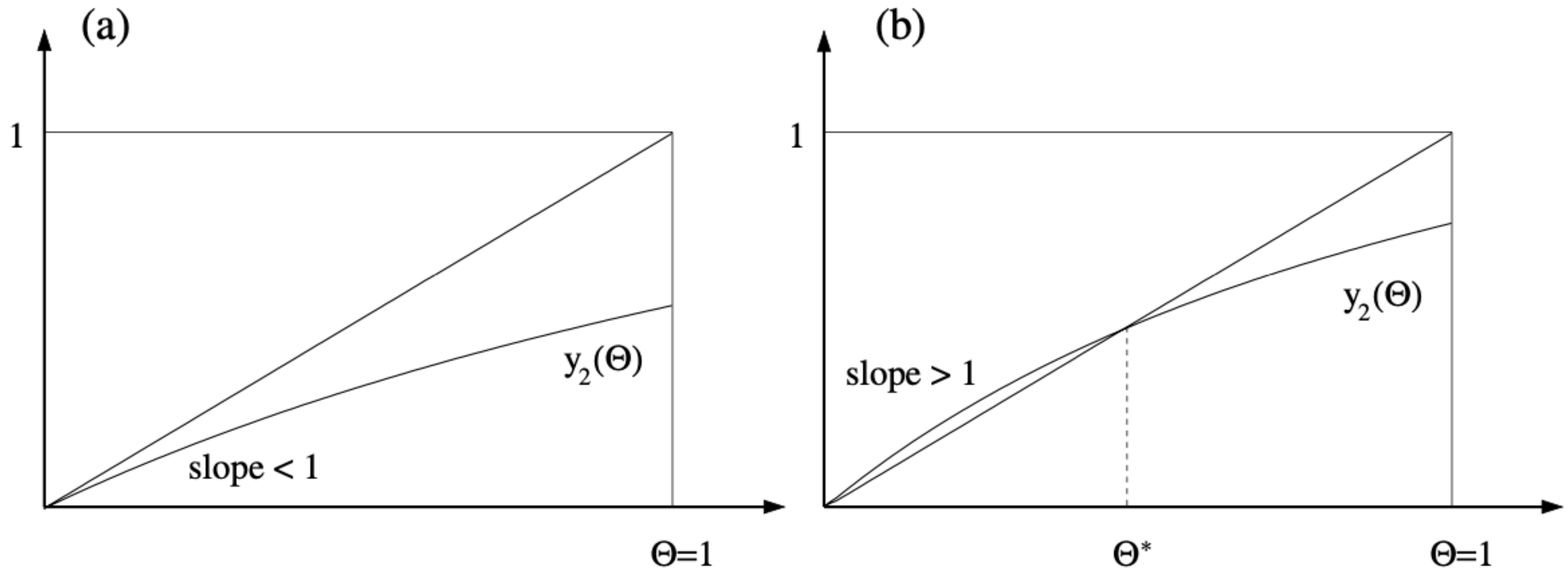
$$\Theta = \sum_{k'} \frac{k' P(k')}{\langle k \rangle} \rho_{k'}^I(t) = \frac{1}{\langle k \rangle} \sum_k \frac{\lambda k^2 P(k) \Theta}{\lambda k \Theta + \mu}$$

Solution 2

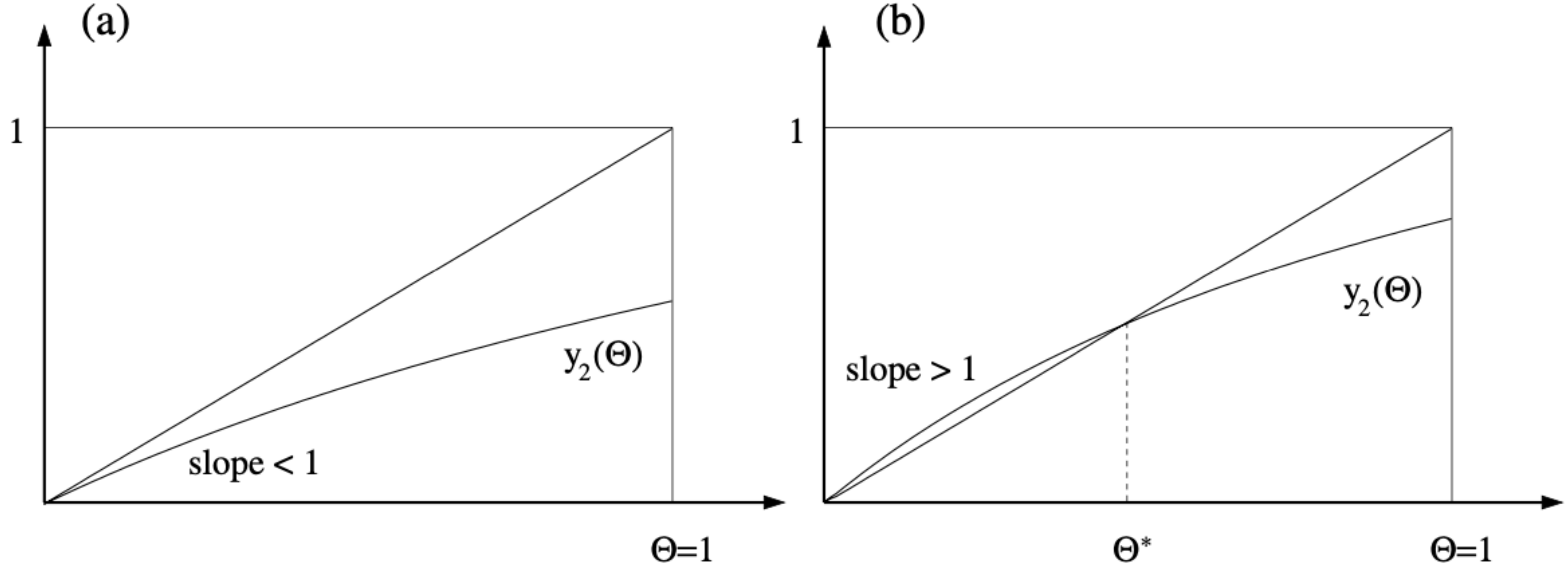
$$\Theta = \frac{1}{\langle k \rangle} \sum_k \frac{\lambda k^2 P(k) \Theta}{\lambda k \Theta + \mu}$$

Self-consistent equation of the form x = F(x)
With the following conditions:
F(0) = 0
F' > 0
F'' < 0 (concave)

Solution 2



Solution 2



There is a non-zero solution if and only if F'(0)>1 which means

$$\frac{1}{\langle k \rangle} \sum_k \frac{\lambda k^2 P(k)}{\mu} > 1$$

$$\boxed{\frac{\lambda}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle}}$$

Epidemic threshold

The DBMF threshold

$$\frac{\lambda}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

- In an infinite scale-free network, with $P(k) \sim k^{-\gamma}$, and $2 \leq \gamma \leq 3$, $\langle k^2 \rangle \rightarrow \infty$ which implies that **the epidemic threshold vanishes** in the thermodynamic limit.

- There is a finite prevalence for any value of the spreading parameters.

Homogeneous networks

In the case of a **homogeneous network** with a regular (Poisson) degree distribution:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$

$$\langle k^2 \rangle / \langle k \rangle \simeq \langle k \rangle$$

The epidemic threshold then becomes:

$$\frac{\lambda}{\mu} \gtrsim \frac{1}{\langle k \rangle}$$

which is finite and it does only depend on the average connectivity of the network.

Homogeneous networks

In the case of a homogeneous network the solution can be found also by considering the equation:

$$\frac{d\rho_k^I(t)}{dt} = \lambda\langle k\rangle[1 - \rho_k^I(t)]\rho_k^I(t) - \mu\rho_k^I(t)$$

In the **early stage approximation**, the epidemic threshold then becomes:

$$\frac{\lambda}{\mu} > \frac{1}{\langle k\rangle}$$

Immunization

In the case of complex networks, we can consider three different immunization strategies:

- uniform immunization
- proportional immunization
- targeted immunization

Uniform immunization

In the case of uniform immunization, **individuals are randomly chosen to be vaccinated** with a density of immune nodes g.

This corresponds to an effective rescaling of the spreading rate:

$$\lambda \to \lambda(1-g)$$

The threshold is affected in a uniform way:

$$\frac{\lambda}{\mu}(1-g) > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

Uniform immunization

$$\frac{\lambda}{\mu}(1 - g) > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

In an infinite scale-free network, with $P(k) \sim k^{-\gamma}$, and $2 \leq \gamma \leq 3$, $\langle k^2 \rangle \to \infty$

which implies that the **uniform immunization is not effective** unless we

immunize all the network: $g = 1$

Proportional immunization

We can find a better solution through a **proportional immunization**.

Let us define the fraction of immune individuals with connectivity k: g_k

If we impose the condition:

$$\tilde{\lambda} \equiv \lambda k(1 - g_k) = const\,.$$

The system equation becomes:

$$\frac{d\rho_k^I(t)}{dt} = \tilde{\lambda}[1 - \rho_k^I(t)]\Theta - \mu\rho_k^I(t)$$

Proportional immunization

In the case of early stage approximation and low density of infectious individuals, we recover an epidemic threshold:

$$\lambda k(1 - g_k) - \mu > 0$$

which defines a threshold on density of immunized individuals:

$$g_k > 1 - \frac{\mu}{\lambda k}$$

for every class of degree k, to stop the epidemic.

Targeted immunization

Optimum approach: immunize a fraction of all nodes with the largest degree.

This way we introduce a cut-off in the degree distribution.

We need to immunize a fraction of nodes g such that:

$$\frac{\lambda}{\mu} < \frac{\langle k \rangle_g}{\langle k^2 \rangle_g}$$

In the case of the BA network, it is possible to show that: $g_c \simeq e^{-\frac{2\mu}{m\lambda}}$

The fraction of nodes to immunize is exponentially small with λ

How do we find the hubs?

- Targeted immunisation is very hard to achieve in practice, the full network structure is not known
- We need a strategy to find hubs based on a **local knowledge** of the network
- In scale-free networks, this can be done efficiently with the **acquaintance immunisation** (Cohen et al. Phys. Rev. Lett. 2003)
- Instead of immunizing nodes at random, we pick random nodes and for each we immunise one of their neighbours at random.

How do we find the hubs?

- Instead of immunizing nodes at random, we pick random nodes and for each we immunise one of their neighbours at random.

$$\boxed{k_{nn}^{\mathbf{unc}} = \frac{\langle k^2 \rangle}{\langle k \rangle}}$$

- My neighbours are more probably hubs than myself! This is also known as the **friendship paradox**

Next… Spatial epidemic models