

Digital epidemiology

Lesson 9

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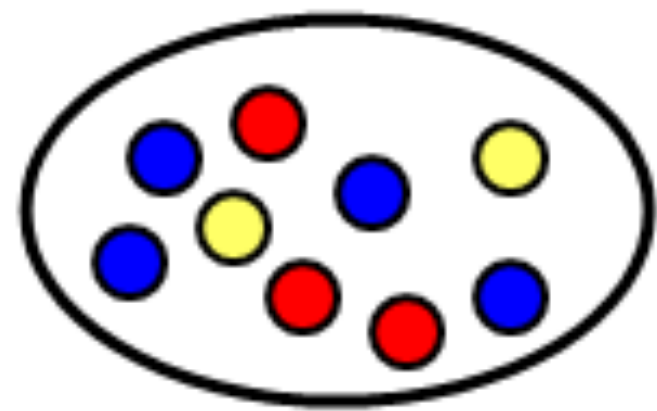
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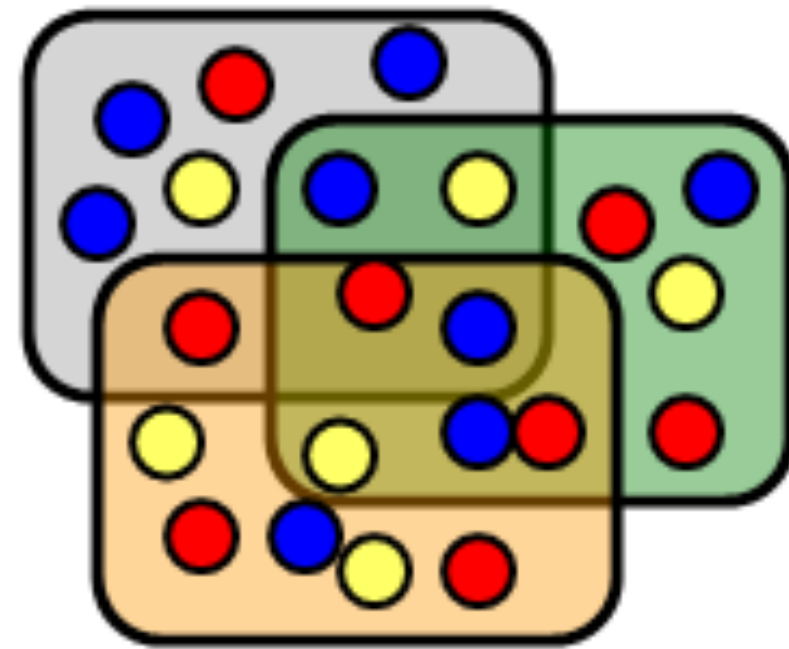
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Epidemics on networks

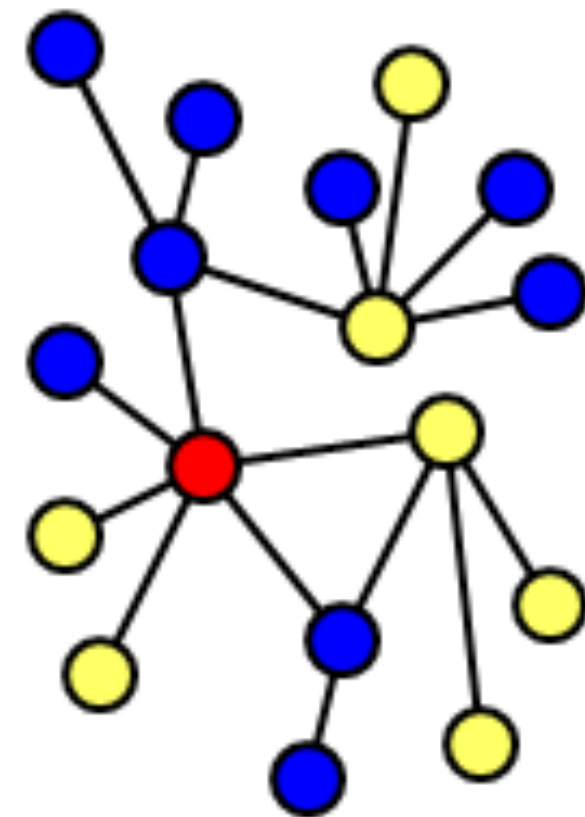
Models



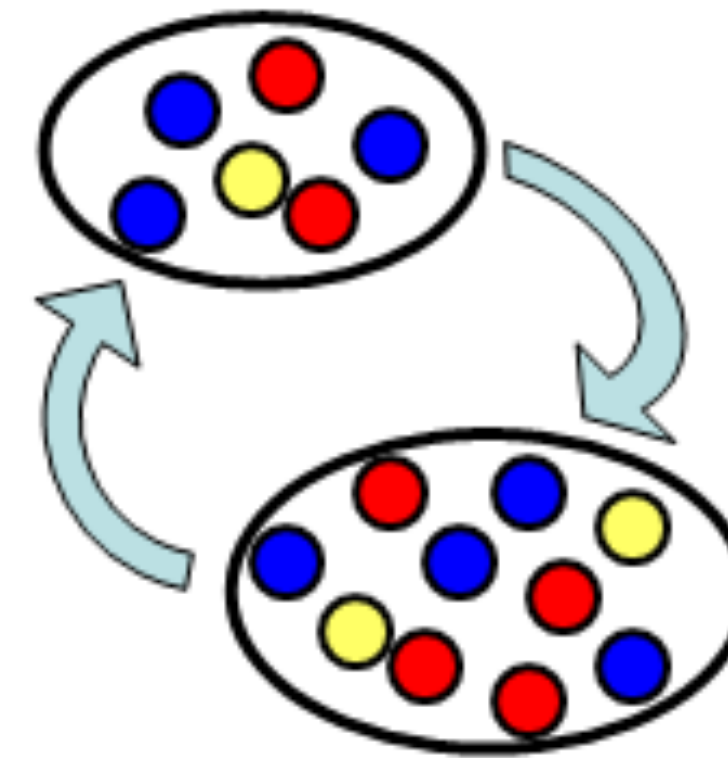
**Homogeneous
mixing**



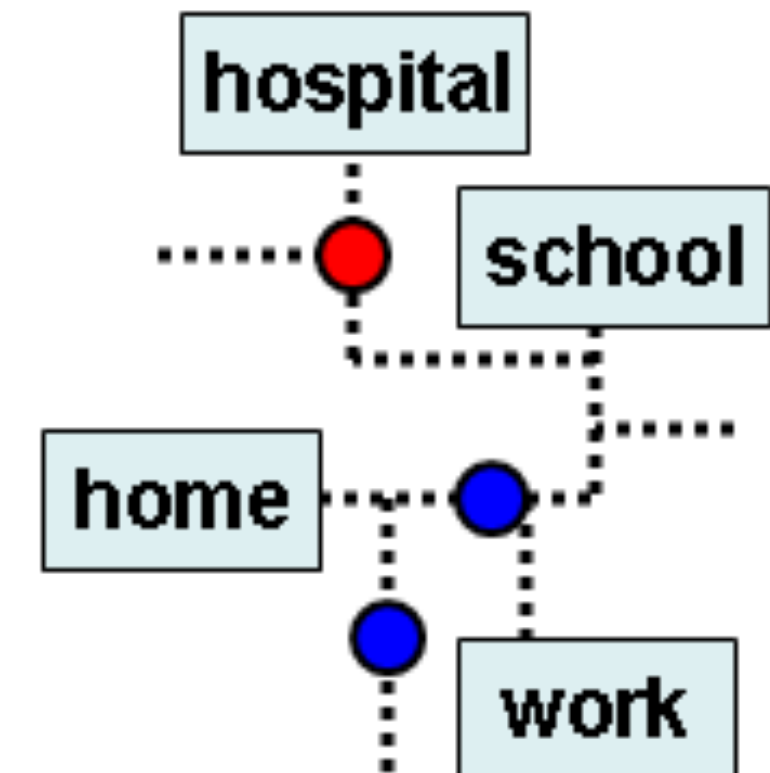
**Social structure
mixing**



**Contact network
models**



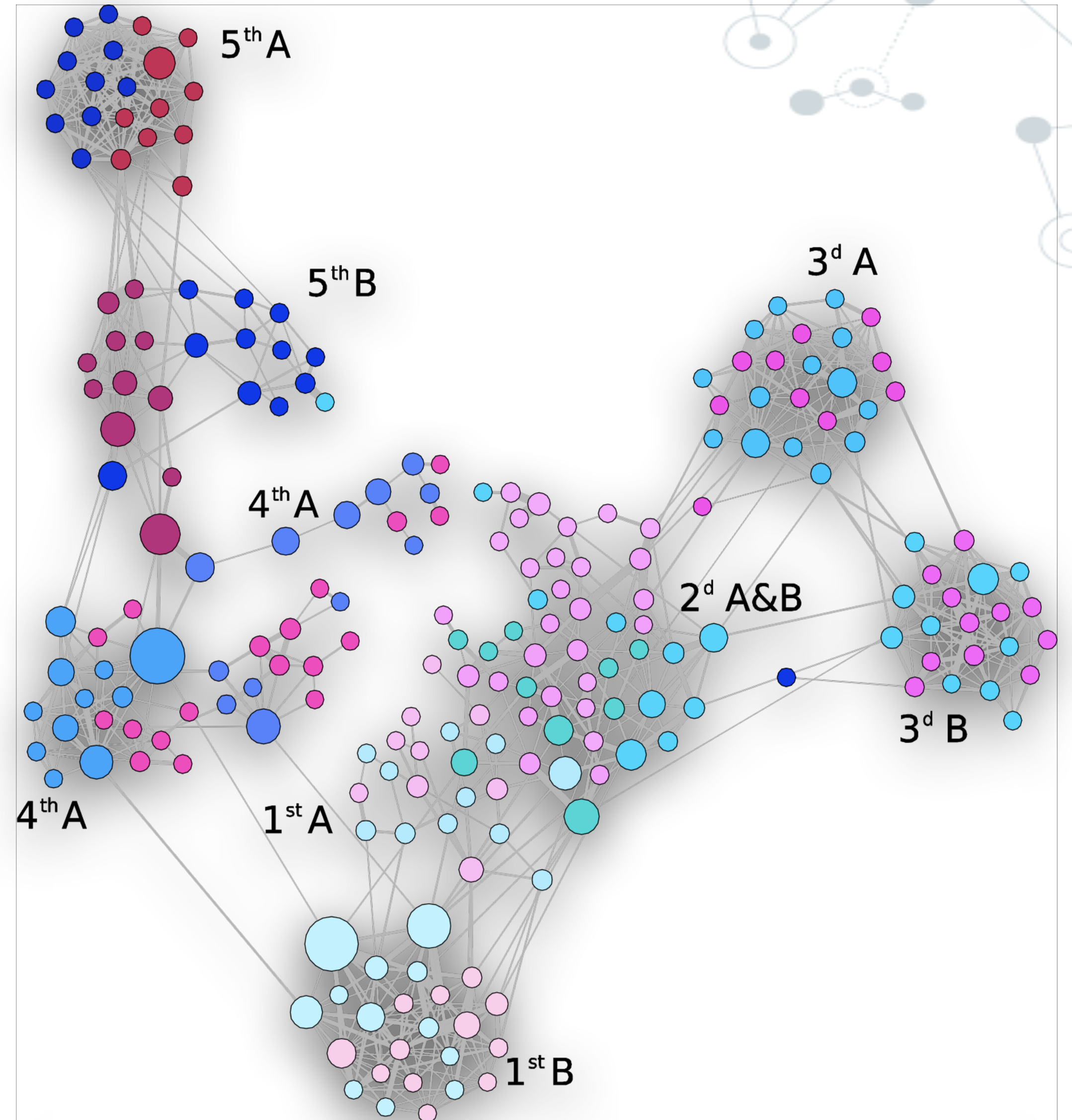
**Multi-scale
models**



**Agent Based
models**

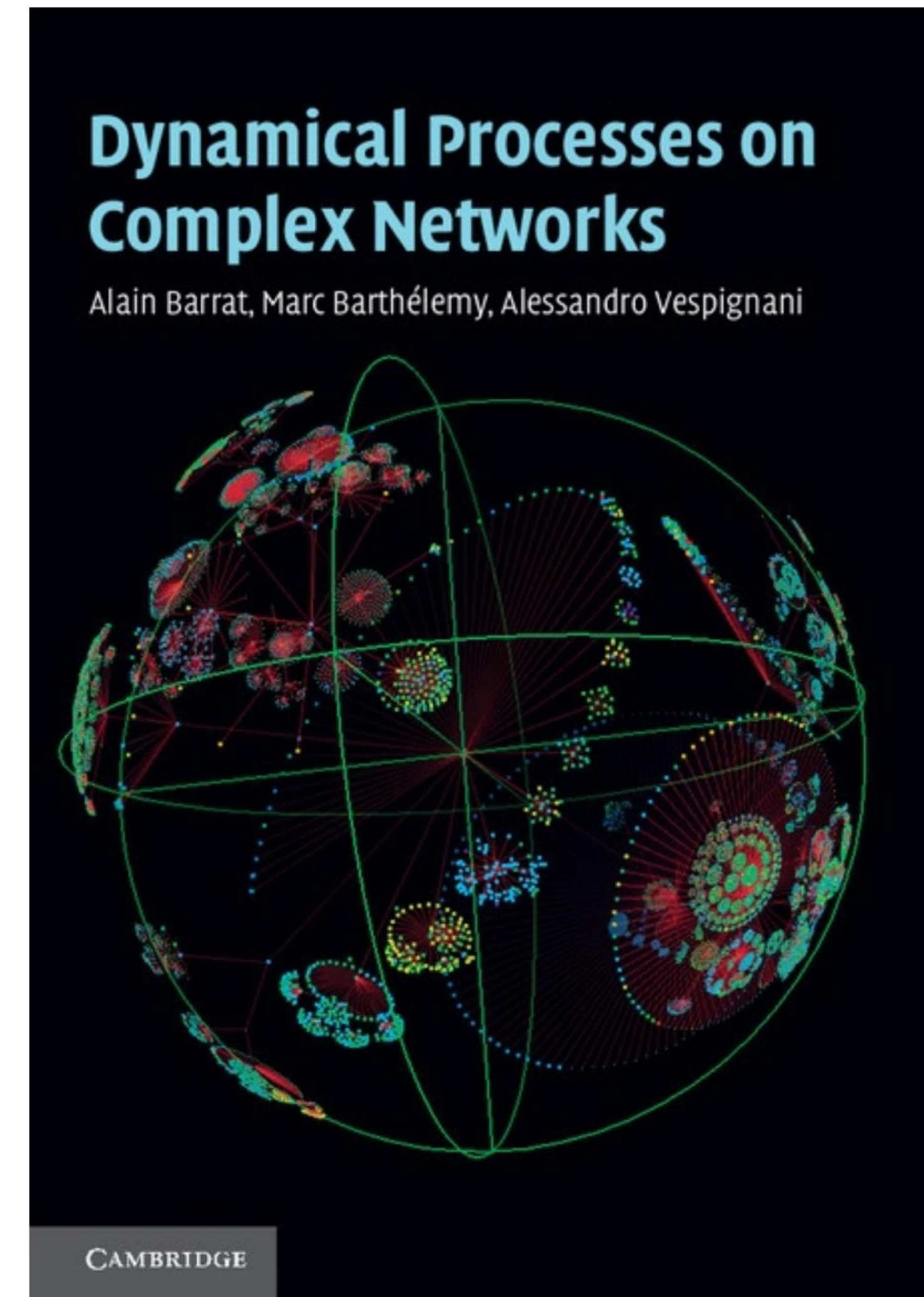
Epidemics on networks

- Homogeneous mixing is not always realistic
- Contacts are not equal and not constant across groups.
- Real contact networks display high heterogeneities
- **Pastor-Satorras et al. Epidemic processes in complex networks. Rev. Mod. Phys. 87, 925**



References

- Pastor-Satorras et al. Epidemic processes in complex networks. Rev. Mod. Phys. 87, 925 (2015)
- Pastor-Satorras, and Vespignani. Epidemic spreading in scale-free networks. Phys. Rev. Lett. 86, 14 (2000)
- Barrat, Barthélemy, Vespignani. Dynamical processes on complex networks. Cambridge University Press



Epidemics on networks

- We consider a network of N nodes where each node can be in an epidemic state, S , I or R
- We define the density of nodes in a given state, as:

$$\rho^S(t) = \frac{S(t)}{N}, \rho^I(t) = \frac{I(t)}{N}, \rho^R(t) = \frac{R(t)}{N}$$

Degree-based mean field

- ▶ Nodes with the **same degree** k are considered as **statistically equivalent**
- ▶ We focus on partial densities for each compartment: ρ_k^α , $\alpha = S, I, R$
- ▶ These variables are not independent: $\sum_{\alpha} \rho_k^\alpha = 1$
- ▶ The total fraction of individuals in compartment α is equal to $\rho^\alpha(t) = \sum_k P(k) \rho_k^\alpha(t)$

Degree-based mean field

- The network is considered in a mean-field perspective (**annealed network** approximation).
- The adjacency matrix is completely “destroyed”. Only the degree and the two-vertex correlations of each node are preserved.
- The adjacency matrix is replaced by its ensemble average:

$$\bar{A}_{ij} = \frac{k_j P(k_i | k_j)}{NP(k_i)}$$

The DBMF SIS model

$$\frac{d\rho_k^I(t)}{dt} = \lambda k [1 - \rho_k^I(t)] \sum_{k'} P(k' | k) \rho_{k'}^I(t) - \mu \rho_k^I(t)$$

Transmission happens over k links

Density of S nodes

Probability of a node of degree k to be connected to an infected of degree k'

The DBMF SIS model

$$\frac{d\rho_k^I(t)}{dt} = \lambda k[1 - \rho_k^I(t)] \sum_{k'} P(k' | k) \rho_{k'}^I(t) - \mu \rho_k^I(t)$$

If we assume the network to be **uncorrelated**: $P(k' | k) = \frac{k' P(k')}{\langle k \rangle}$

then $\frac{d\rho_k^I(t)}{dt} = \lambda k[1 - \rho_k^I(t)] \Theta - \mu \rho_k^I(t)$ where $\Theta = \sum_{k'} \frac{k' P(k')}{\langle k \rangle} \rho_{k'}^I(t)$

Solution 1

Early stage approximation: $\rho_k^I(t) \ll 1$

then
$$\frac{d\Theta}{dt} = \left(\frac{\lambda}{\mu} \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right) \Theta$$

which implies that Θ will grow if:

$$\frac{\lambda}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

**Epidemic
threshold**

The DBMF threshold

$$\frac{\lambda}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

- ▶ In an infinite scale-free network, with $P(k) \sim k^{-\gamma}$, and $2 \leq \gamma \leq 3$, $\langle k^2 \rangle \rightarrow \infty$ which implies that **the epidemic threshold vanishes** in the thermodynamic limit.
- ▶ There is a finite prevalence for any value of the spreading parameters.

Homogeneous networks

In the case of a **homogeneous network** with a regular (Poisson) degree distribution:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$

$$\langle k^2 \rangle / \langle k \rangle \simeq \langle k \rangle$$

The epidemic threshold then becomes:

$$\frac{\lambda}{\mu} \gtrsim \frac{1}{\langle k \rangle}$$

which is finite and it does only depend on the average connectivity of the network.

Homogeneous networks

In the case of a homogeneous network the solution can be found also by considering the equation:

$$\frac{d\rho_k^I(t)}{dt} = \lambda \langle k \rangle [1 - \rho_k^I(t)] \rho_k^I(t) - \mu \rho_k^I(t)$$


In the **early stage approximation**, the epidemic threshold then becomes:

$$\frac{\lambda}{\mu} > \frac{1}{\langle k \rangle}$$

Immunization

A decorative network graph in the top right corner, featuring nodes of varying sizes (some solid blue, some hollow white) connected by thin grey lines, forming a complex web.

In the case of complex networks, we can consider three different immunization strategies:

- uniform immunization
 - proportional immunization
 - targeted immunization
- 
- A decorative network graph in the bottom left corner, featuring nodes of varying sizes (some solid blue, some hollow white) connected by thin grey lines, forming a complex web.

Uniform immunization

In the case of uniform immunization, **individuals are randomly chosen to be vaccinated** with a density of immune nodes g .

This corresponds to an effective rescaling of the spreading rate:

$$\lambda \rightarrow \lambda(1 - g)$$

The threshold is affected in a uniform way:

$$\frac{\lambda}{\mu}(1 - g) > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

Uniform immunization

$$\frac{\lambda}{\mu}(1 - g) > \frac{\langle k \rangle}{\langle k^2 \rangle}$$

In an infinite scale-free network, with $P(k) \sim k^{-\gamma}$, and $2 \leq \gamma \leq 3$, $\langle k^2 \rangle \rightarrow \infty$

which implies that the **uniform immunization is not effective** unless we

immunize all the network: $g = 1$

Proportional immunization

We can find a better solution through a **proportional immunization**.

Let us define the fraction of immune individuals with connectivity k : g_k

If we impose the condition:

$$\tilde{\lambda} \equiv \lambda k(1 - g_k) = \text{const}.$$

The system equation becomes:

$$\frac{d\rho_k^I(t)}{dt} = \tilde{\lambda}[1 - \rho_k^I(t)]\Theta - \mu\rho_k^I(t)$$

Proportional immunization

In the case of early stage approximation and low density of infectious individuals, we recover an epidemic threshold:

$$\lambda k(1 - g_k) - \mu > 0$$

which defines a threshold on density of immunized individuals:

$$g_k > 1 - \frac{\mu}{\lambda k}$$

for every class of degree k , to stop the epidemic.

Targeted immunization

Optimum approach: immunize a fraction of all nodes with the largest degree.

This way we introduce a cut-off in the degree distribution.

We need to immunize a fraction of nodes g such that:

$$\frac{\lambda}{\mu} < \frac{\langle k \rangle_g}{\langle k^2 \rangle_g}$$

In the case of the BA network, it is possible to show that: $g_c \simeq e^{-\frac{2\mu}{m\lambda}}$

The fraction of nodes to immunize is exponentially small with λ

How do we find the hubs?

- ▶ Targeted immunisation is very hard to achieve in practice, the full network structure is not known
- ▶ We need a strategy to find hubs based on a **local knowledge** of the network
- ▶ In scale-free networks, this can be done efficiently with the **acquaintance immunisation** (Cohen et al. Phys. Rev. Lett. 2003)
- ▶ Instead of immunizing nodes at random, we pick random nodes and for each we immunise one of their neighbours at random.

How do we find the hubs?

- ▶ Instead of immunizing nodes at random, we pick random nodes and for each we immunise one of their neighbours at random.

$$k_{nn}^{\text{unc}} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

- ▶ My neighbours are more probably hubs than myself! This is also known as the **friendship paradox**

Next... Spatial epidemic models