# Digital health and computational epidemiology

## Lesson 8

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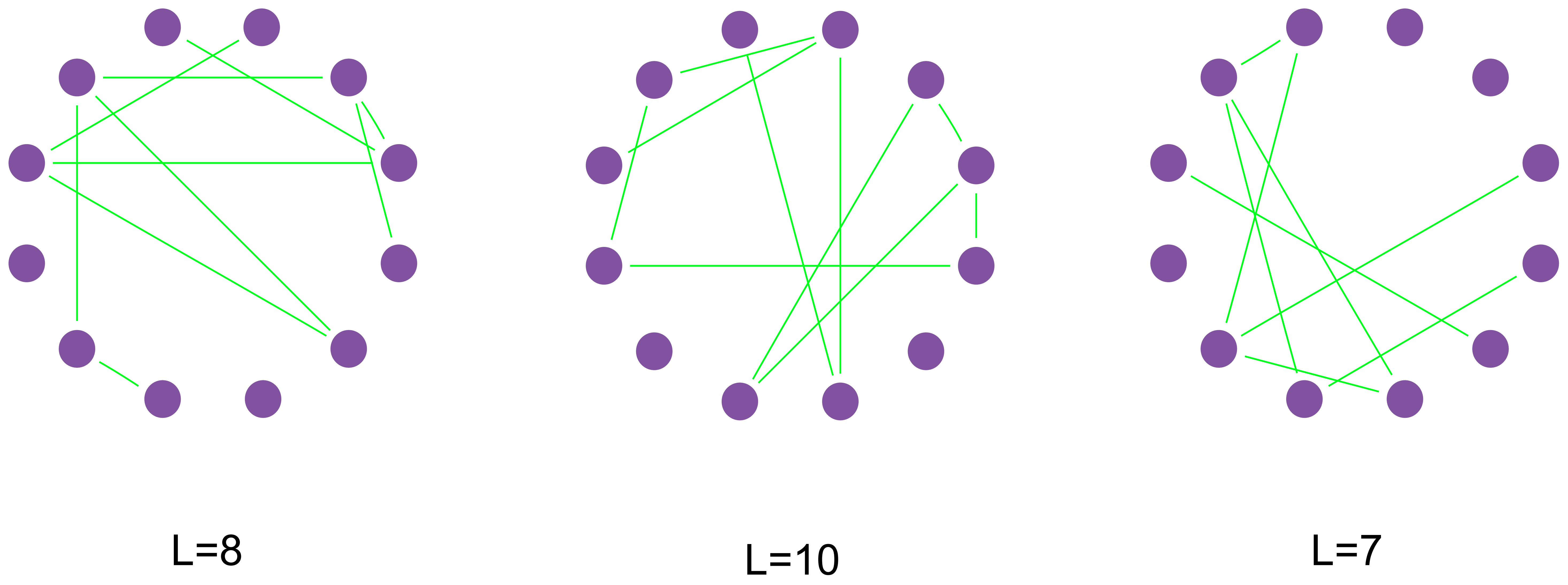
# Network theory: models

# Erdős-Renyi random model

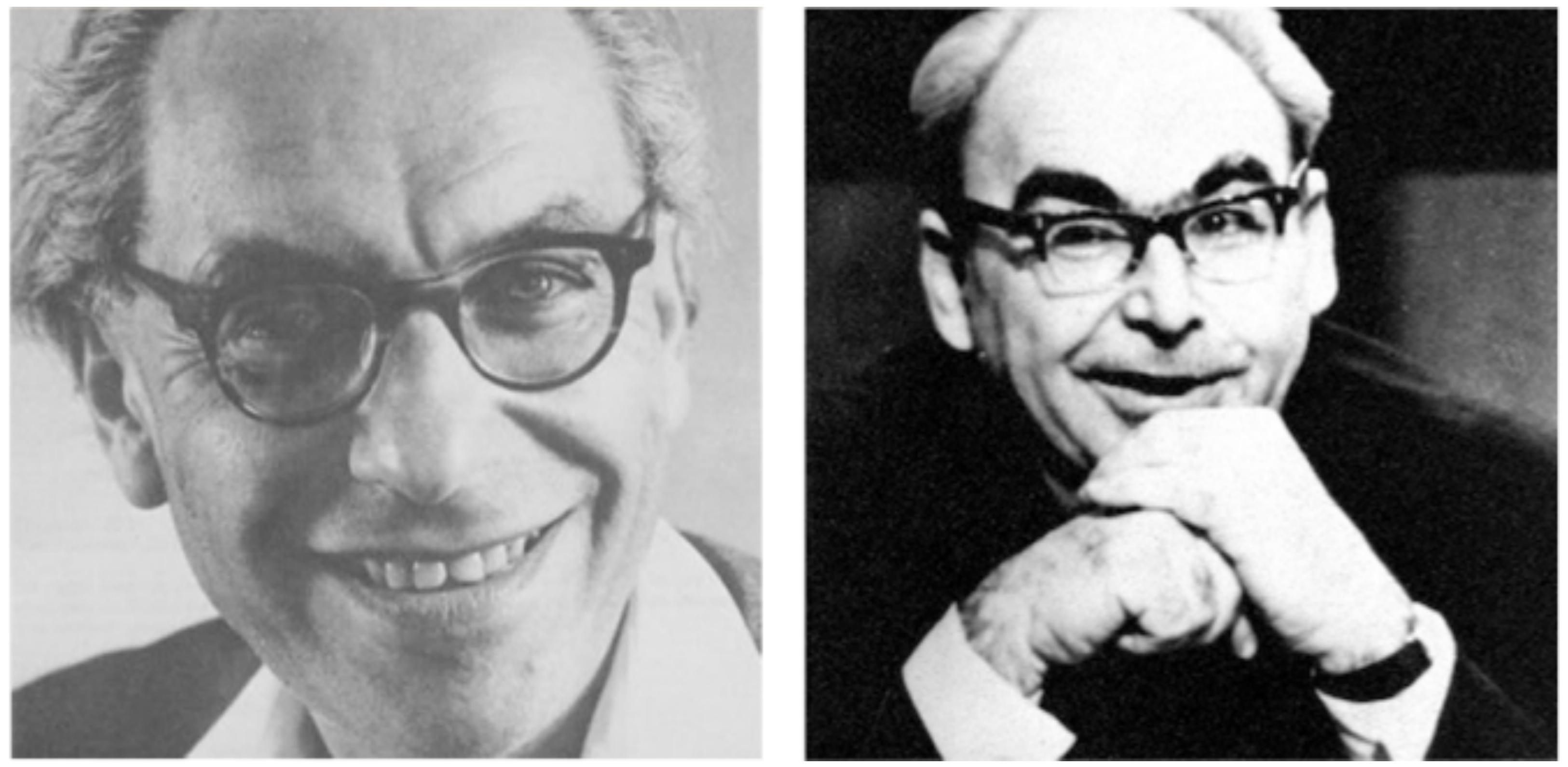
**$G(N, L)$ Model**

N labeled nodes are connected with $L$ randomly placed links. Erdős and Rényi used this definition in their string of papers on random networks [2-9].




$N = 12$

**Pál Erdös**
(1913-1996)



**Alfréd Rényi**
(1921-1970)

Erdös-Rényi model (1960)

# Erdős-Renyi random model

Probability to have exactly *L* links in a network of *N* nodes and probability *p* of placing a link:

$$P(L) = \begin{pmatrix} \binom{N}{2} \\ L \end{pmatrix} p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

- Probability that we have L links with L successful attempts: $p^L$
- Probability that the remaining attempts didn't create a link: $(1-p)^{N(N-1)/2 - L}$
- The binomial coefficient counting the number of different ways we can place L links among N nodes

# Degree distribution

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2} - L}$$

$$\langle L \rangle = p\,\frac{N(N-1)}{2}$$

$$\langle k \rangle = p\,(N-1)$$

**We are constraining the average degree!**
So if we want SPARSENESS, we need small p

# Degree distribution

$$p(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

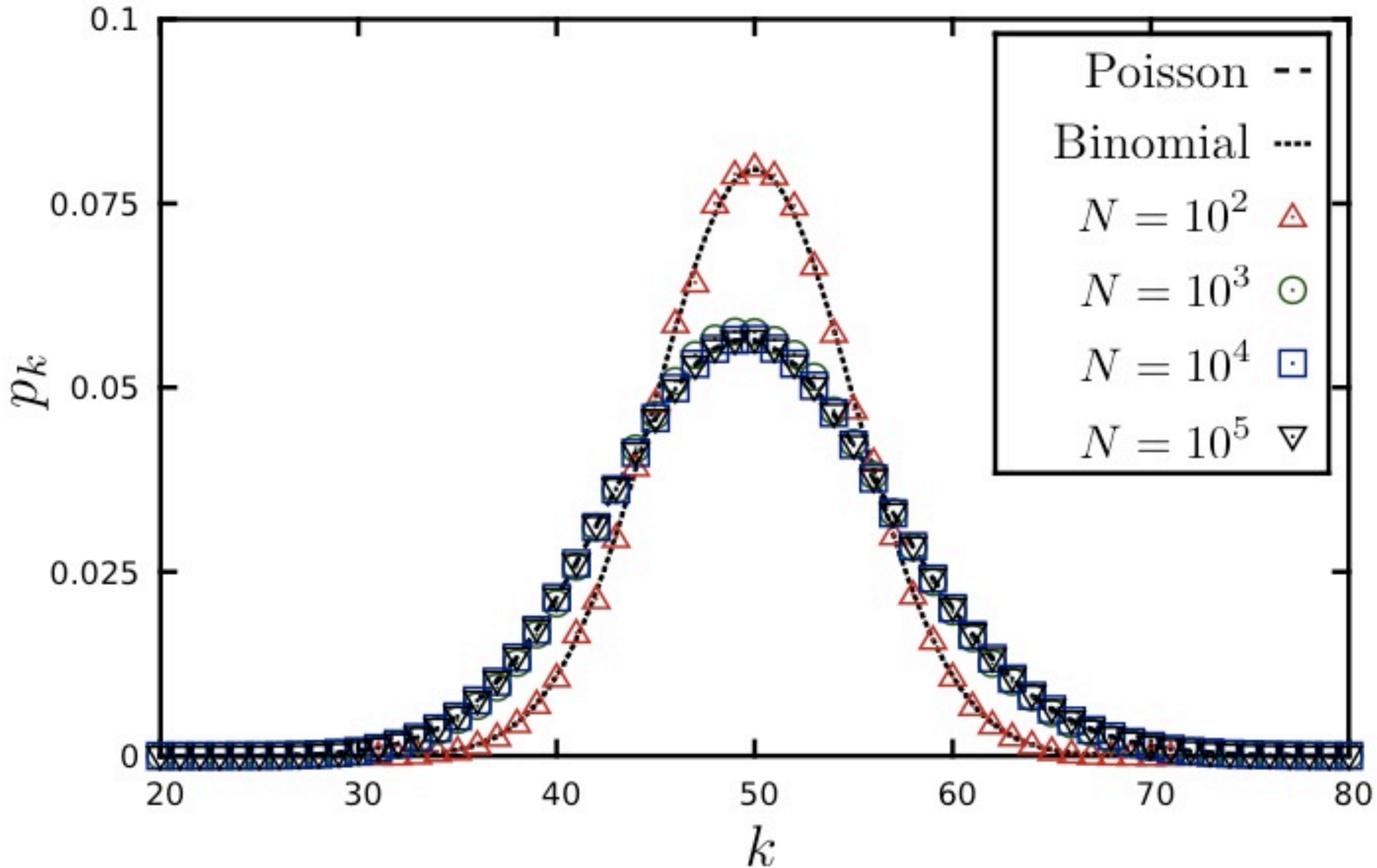
- Probability that we have degree k with k successful attempts: $p^k$
- Probability that the remaining attempts didn't create a link: $(1-p)^{N-1-k}$
- The binomial coefficient counting the number of different ways we can place k links among the N-1 nodes

# Degree distribution

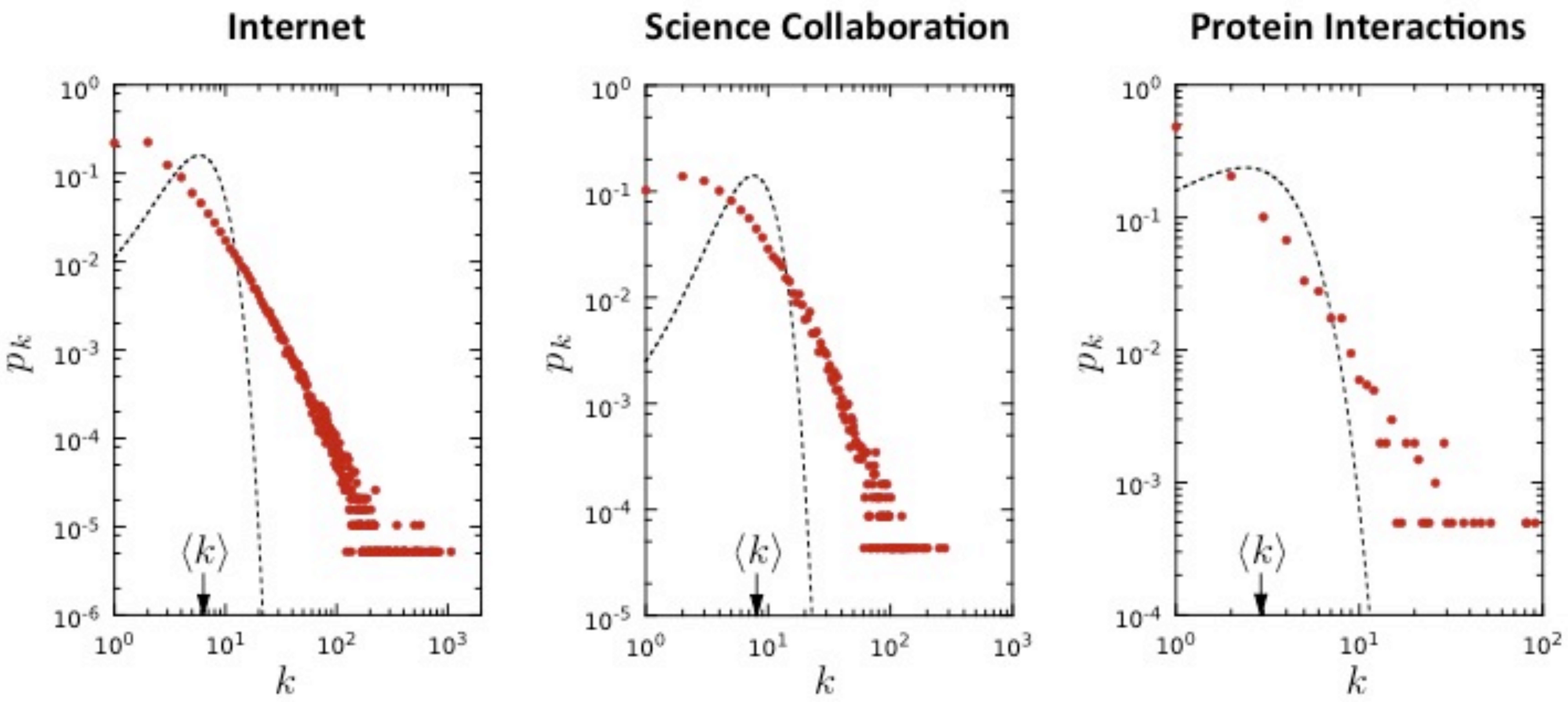
$$p(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

For large N, and $\langle k \rangle \ll N$

$$p(k) = e^{-<k>} \frac{<k>^k}{k!}$$



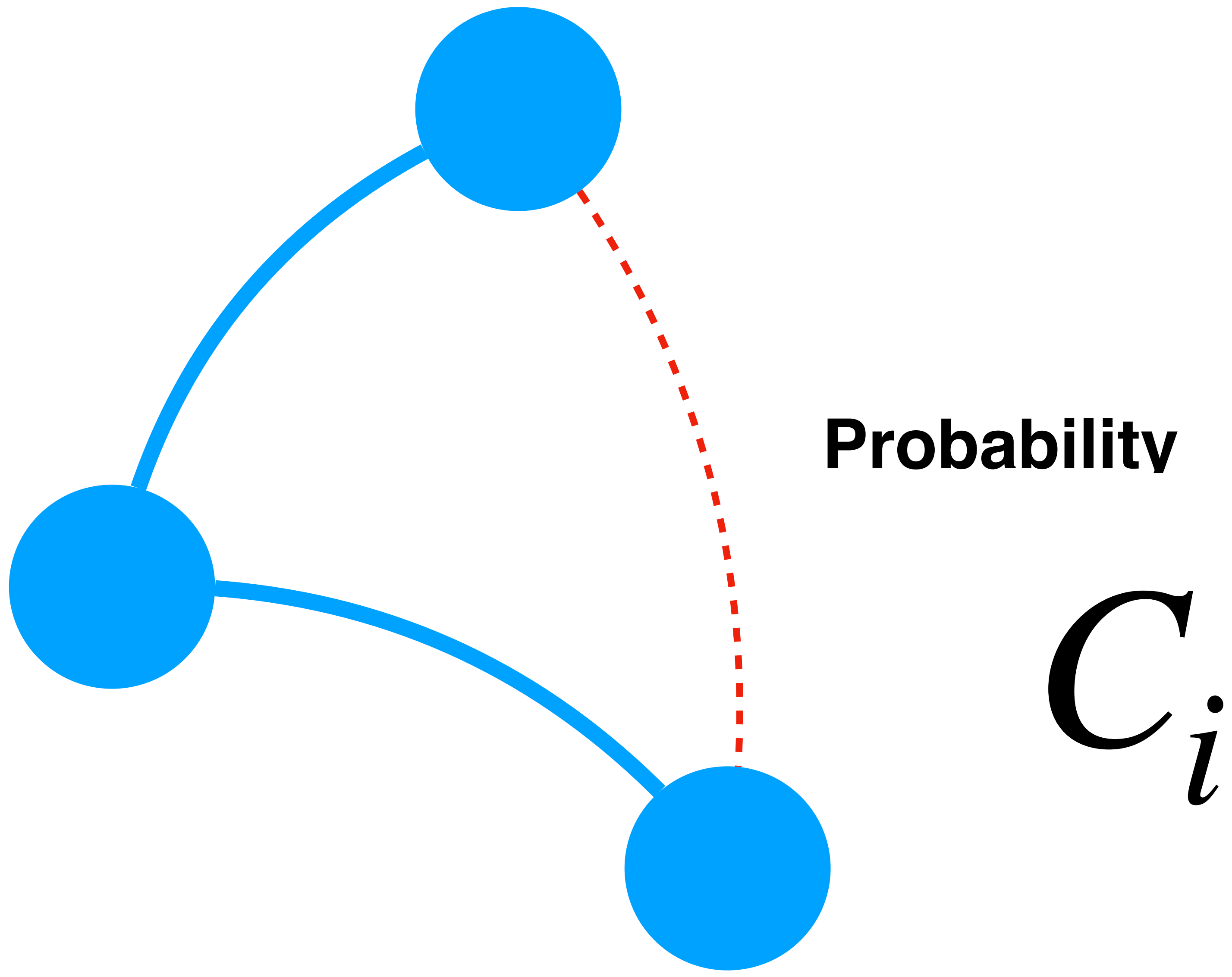
# Degree distribution



# Clustering

What about clustering?

$$C_i = \frac{2L_i}{k_i(k_i - 1)}$$



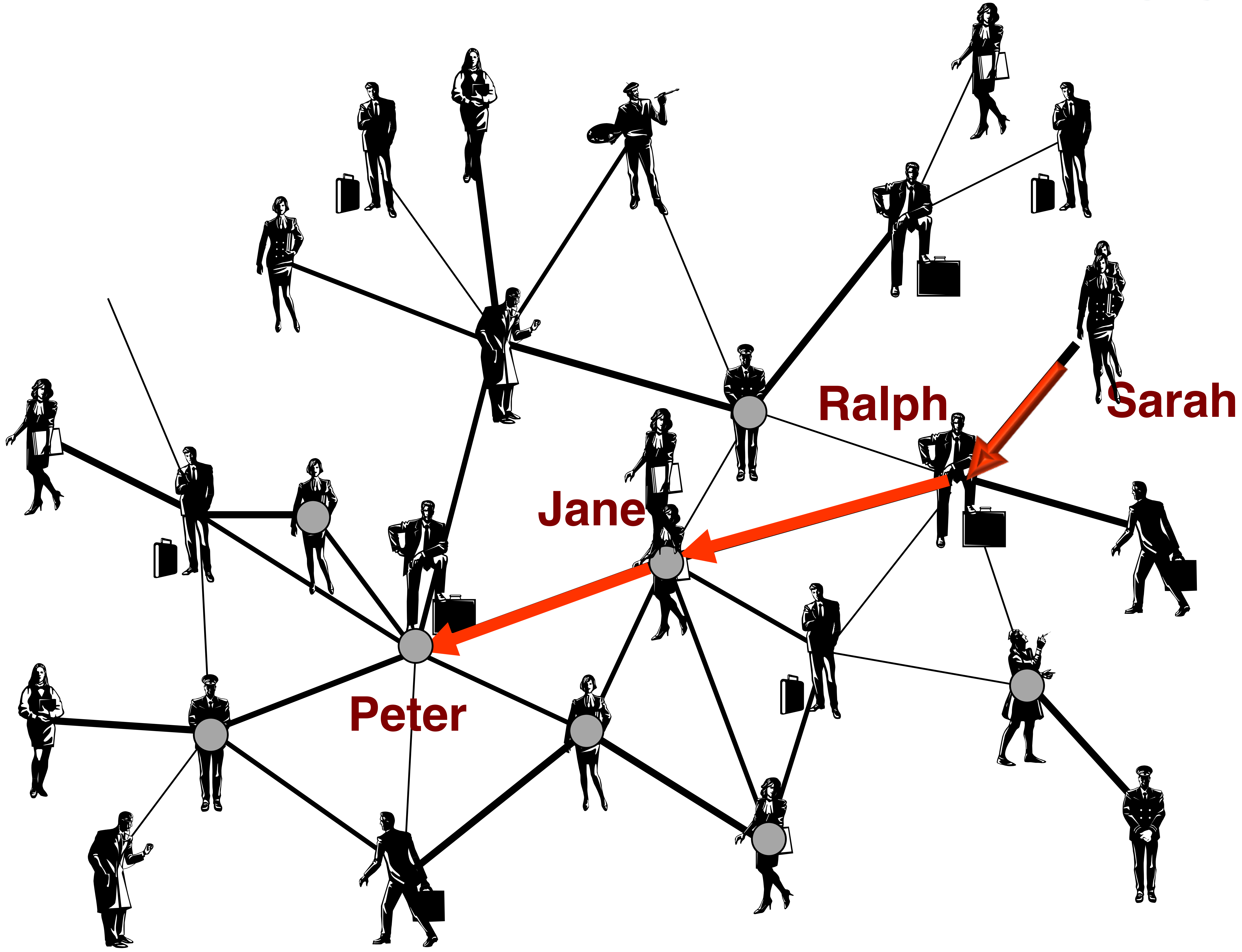

$$C_i = p$$

**We CAN constrain the clustering (but uniform)!**
So if we want high clustering, we need large p!

**We are constraining the average degree!**
So if we want SPARSENESS, we need small p

# Distances

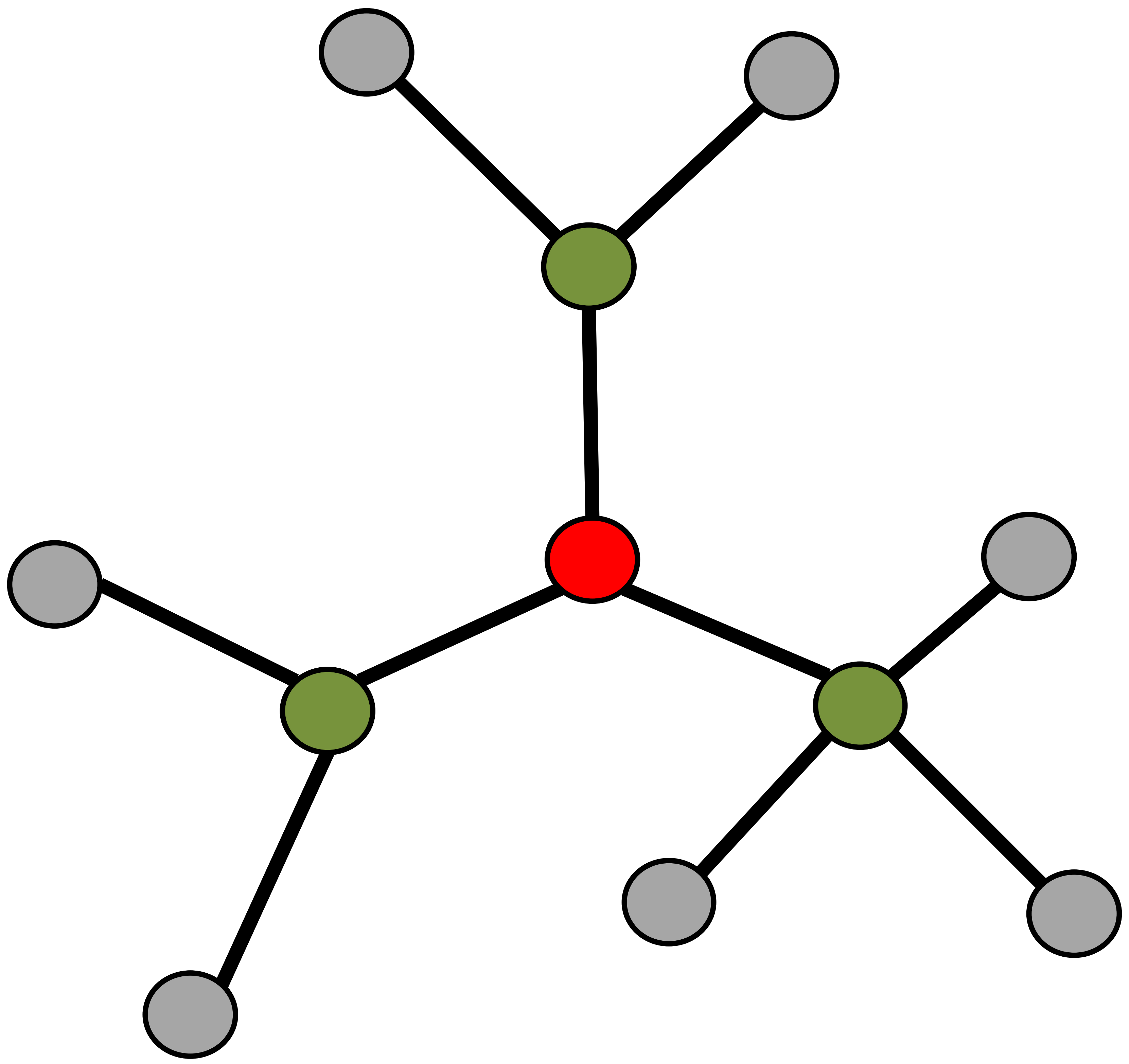
What about distances?




*Frigyes Karinthy, 1929*
*Stanley Milgram, 1967*

# Distances

Let's try an easy case



<k> nodes at distance d=1

<k>$^2$ nodes at distance d=2

<k>$^3$ nodes at distance d=3

...

$$1 + \langle k \rangle + \langle k \rangle^2 + \langle k \rangle^3 + \ldots + \langle k \rangle^{d_{max}} = N$$

$$\frac{\langle k \rangle^{d_{max}+1} - 1}{\langle k \rangle - 1} = N \quad \rightarrow \quad d_{max} \simeq \frac{\log N}{\log \langle k \rangle}$$

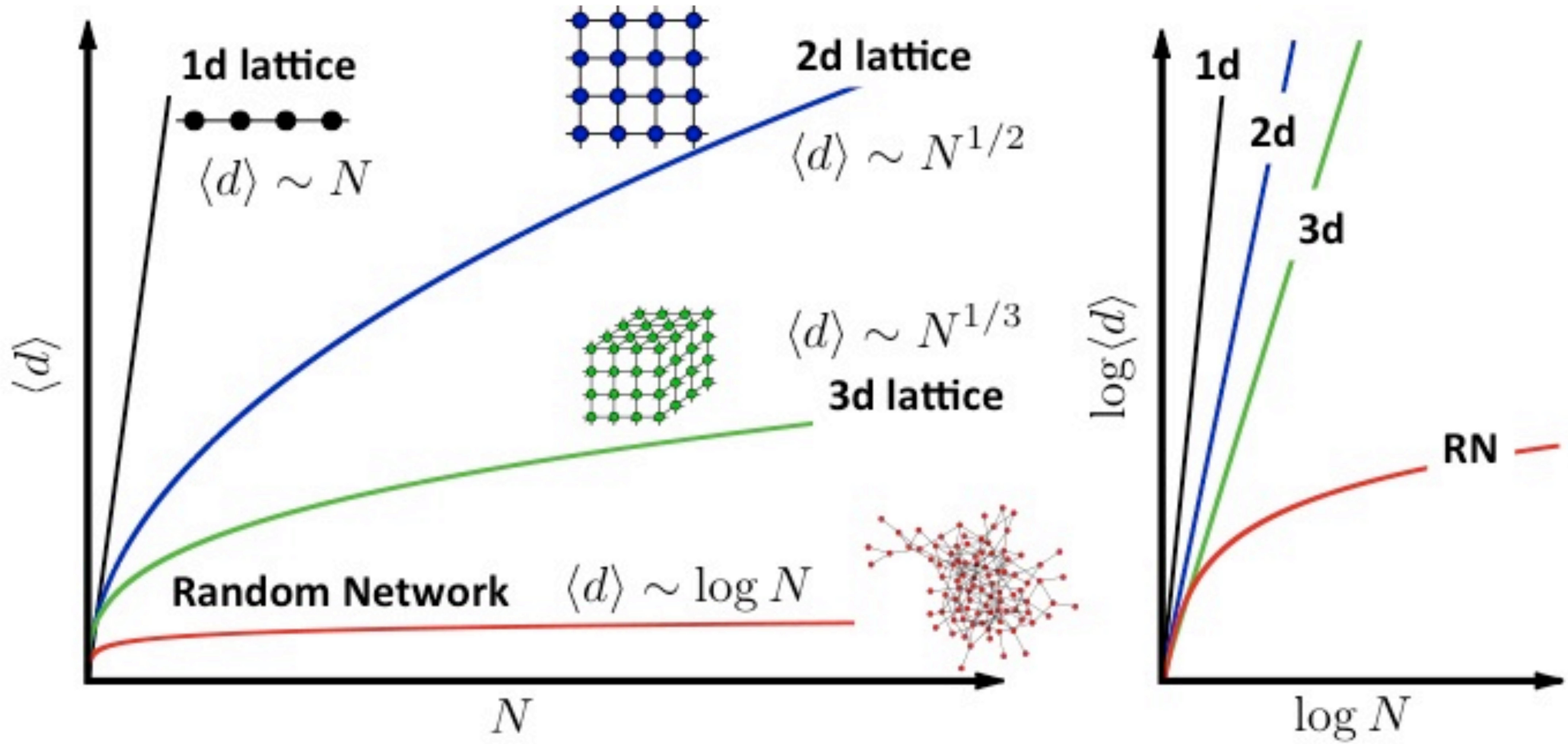
Wrong! This is actually closer to the average distance! $\langle d \rangle \simeq \dfrac{\log N}{\log \langle k \rangle}$

# Erdős-Renyi random model

List of main results:

- We can reproduce sparseness using N and *p*
- Degree distribution is Poissonian and not power-law/broad
- We can reproduce high-clustering but not low density, or viceversa
- Small world property emerges naturally

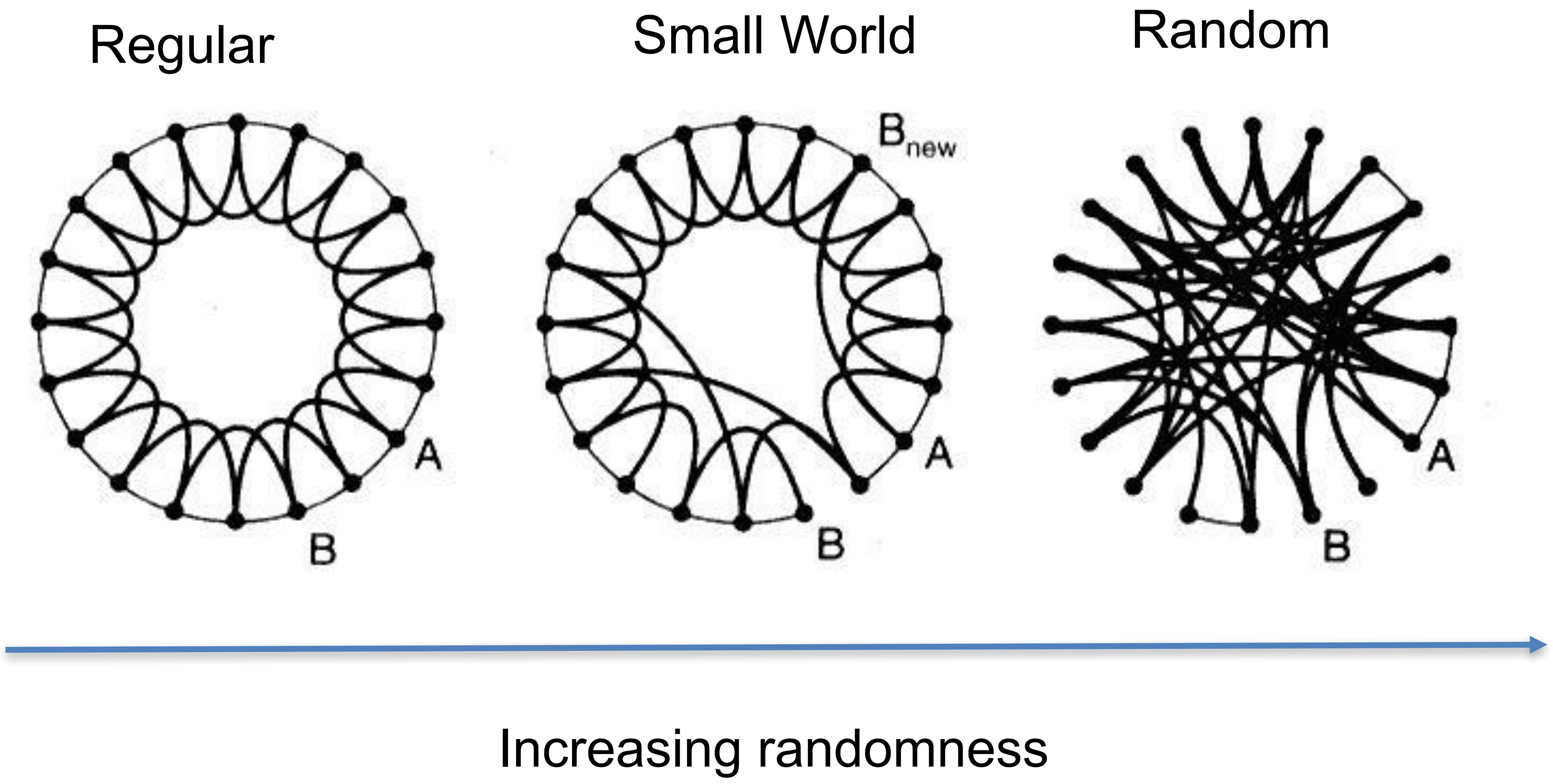
# Distances

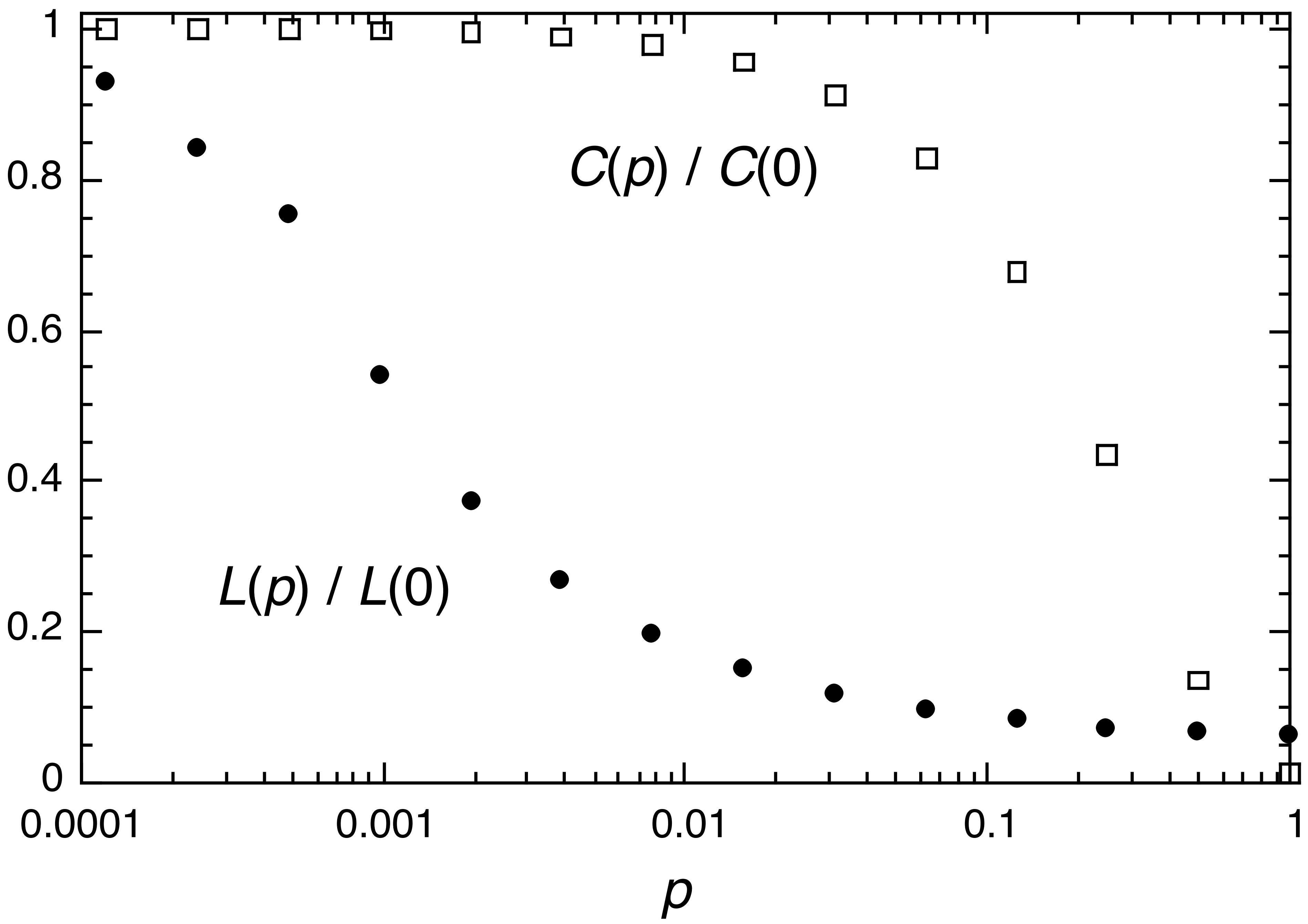


1d lattice
$\langle d \rangle \sim N$
2d lattice
$\langle d \rangle \sim N^{1/2}$
$\langle d \rangle \sim N^{1/3}$
3d lattice
Random Network
$\langle d \rangle \sim \log N$
$\langle d \rangle$
$N$
1d
2d
3d
RN
$\log \langle d \rangle$
$\log N$

# Watts-Strogatz random model

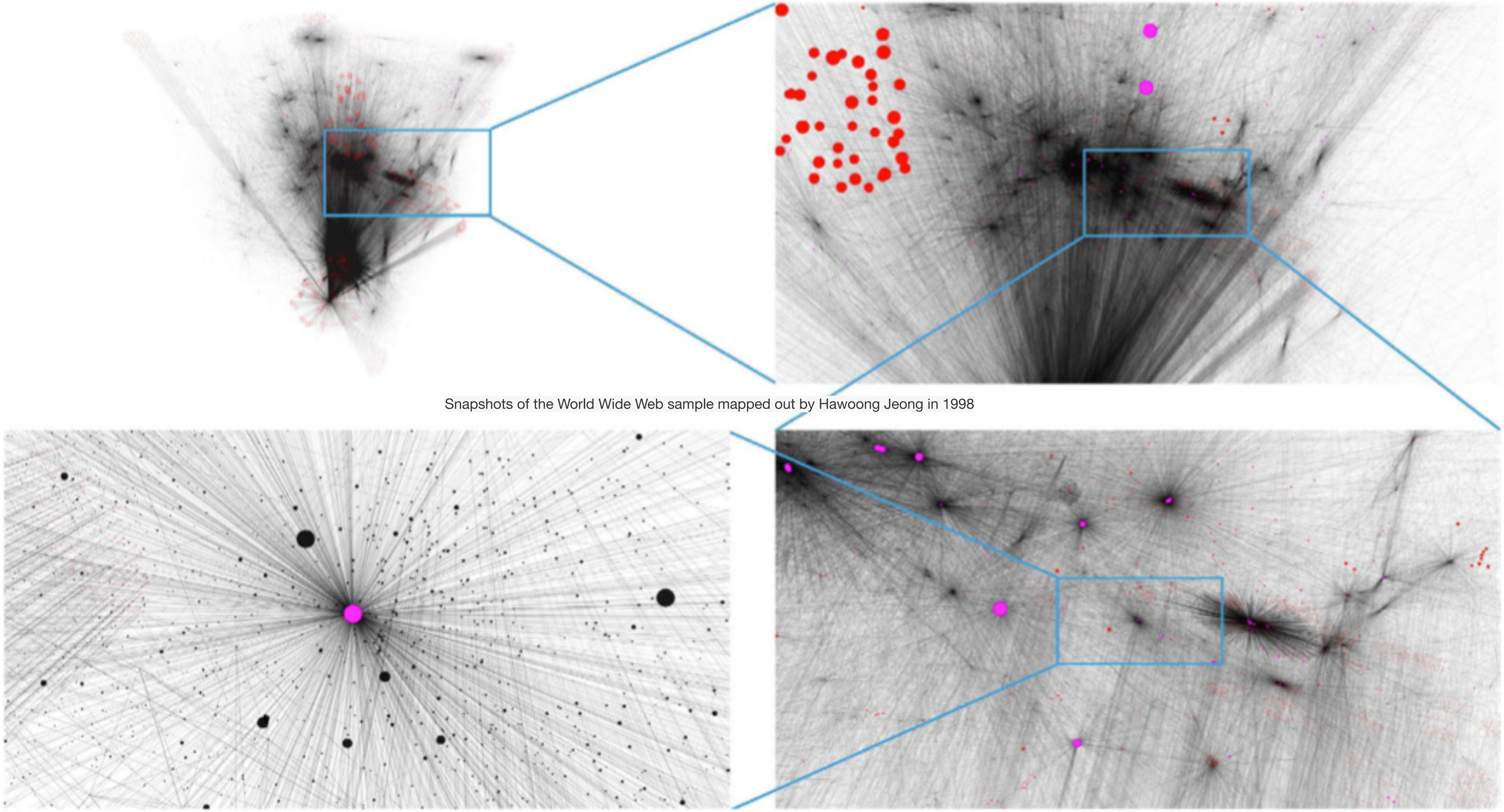
Can we reconcile small world and high clustering in a single model? Yes




*Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of 'small-world' networks." nature 393.6684 (1998): 440-442.*

# The meaning of scale-free



Snapshots of the World Wide Web sample mapped out by Hawoong Jeong in 1998

# The meaning of scale-free

A scale-free network is a network whose degree distribution follows a power law.

**Discrete formalism**

$$p_k = Ck^{-\gamma}$$

$$\sum_{k=1}^{\infty} p_k = 1$$

$$C\sum_{k=1}^{\infty} k^{-\gamma} = 1 \qquad C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

**Continuous formalism**

$$p(k) = Ck^{-\gamma}$$

$$\int_{k_{min}}^{\infty} p(k)dk = 1$$

$$C = \frac{1}{\int_{k_{min}}^{\infty} p(k)dk} = (\gamma - 1)k_{min}^{\gamma-1}$$

$$p(k) = (\gamma - 1)k_{min}^{\gamma-1}k^{-\gamma}$$

# Scale-free networks

- Real world networks can be very large but they are finite.
- It exists a **maximum degree** in each finite network.

$$\int_{k_{max}}^{\infty} p(k)dk \simeq \frac{1}{N}$$

We **assume** there is only one node with degree max or above.

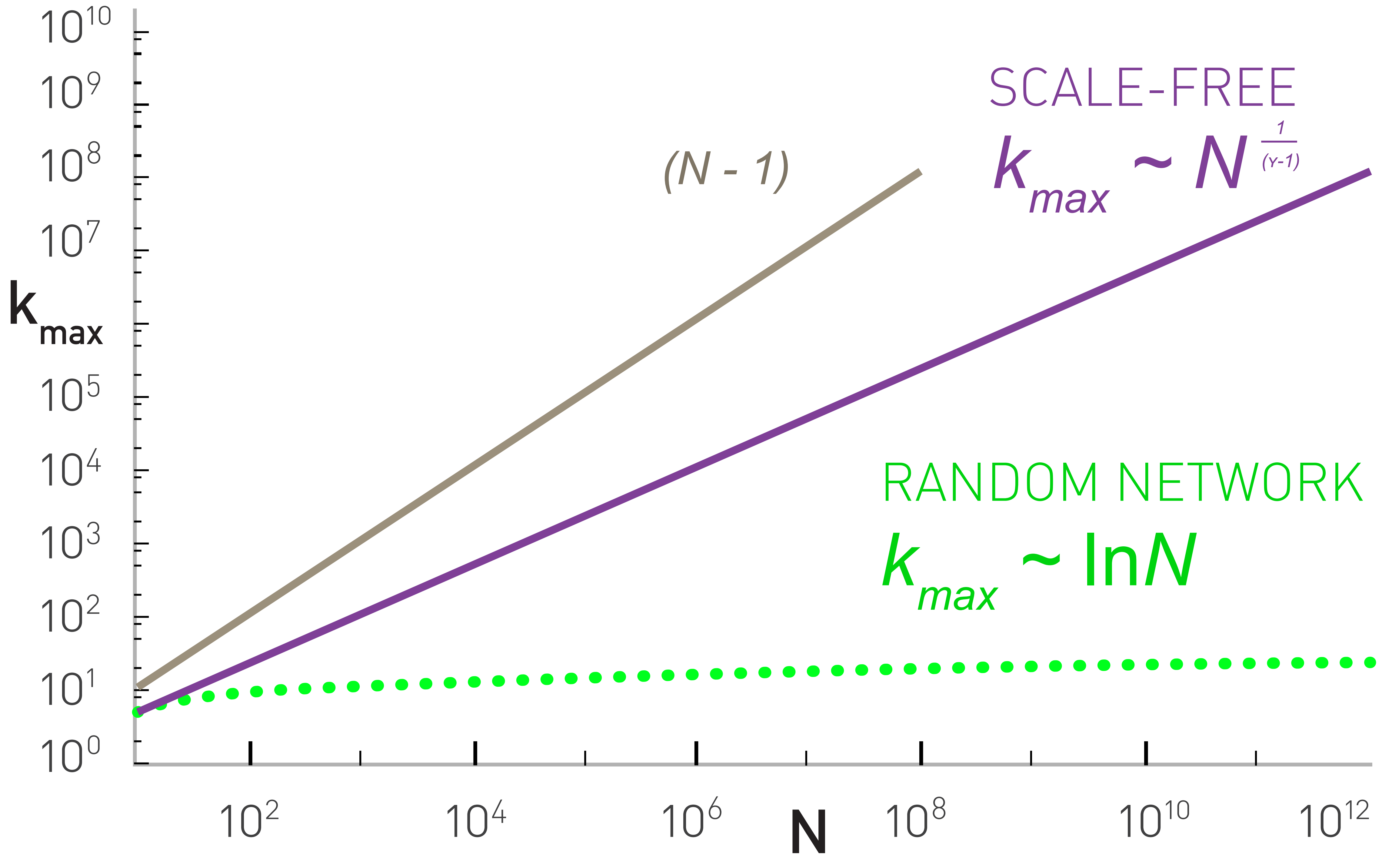
$$\int_{k_{max}}^{\infty} p(k)dk = (\gamma-1)k_{min}^{\gamma-1}\int_{k_{max}}^{\infty} k^{-\gamma}dk = \frac{\gamma-1}{-\gamma+1}k_{min}^{\gamma-1}[k^{-\gamma+1}]_{k_{max}}^{\infty} = \frac{k_{min}^{\gamma-1}}{k_{max}^{\gamma-1}} \simeq \frac{1}{N}$$

$$\boxed{k_{max} = k_{min}N^{\frac{1}{\gamma-1}}}$$

# Scale-free networks

$$\boxed{k_{max} = k_{min} N^{\frac{1}{\gamma - 1}}}$$

- $k_{max}$ increases with the size of the network ==> bigger system, bigger hub
- For γ>2, $k_{max}$ increases slower than N ==> decreasing fraction of links as N increases.
- For γ=2, $k_{max}$~N ==> The size of the biggest hub is O(N)
- For γ<2 $k_{max}$ increases faster than N: condensation phenomena ==> the largest hub will grab an increasing fraction of links.



# More divergences

$$< k^m >= \int_{k_{min}}^{\infty} k^m p(k) dk \qquad p(k) = (\gamma - 1) k_{min}^{\gamma-1} k^{-\gamma}$$

$$< k^m >= (\gamma - 1) k_{min}^{\gamma-1} \int_{k_{min}}^{\infty} k^{m-\gamma} dk = \frac{\gamma - 1}{m - \gamma + 1} k_{min}^{\gamma-1} \left[ k^{m-\gamma+1} \right]_{k_{min}}^{\infty}$$

if $m - \gamma + 1 < 0$: $\qquad < k^m >= \dfrac{\gamma - 1}{m - \gamma + 1} k_{min}^m$

if $m - \gamma + 1 > 0$: $\qquad < k^m > \to \infty$

This implies:

For $\gamma < 3, \quad < k^2 > \to \infty$

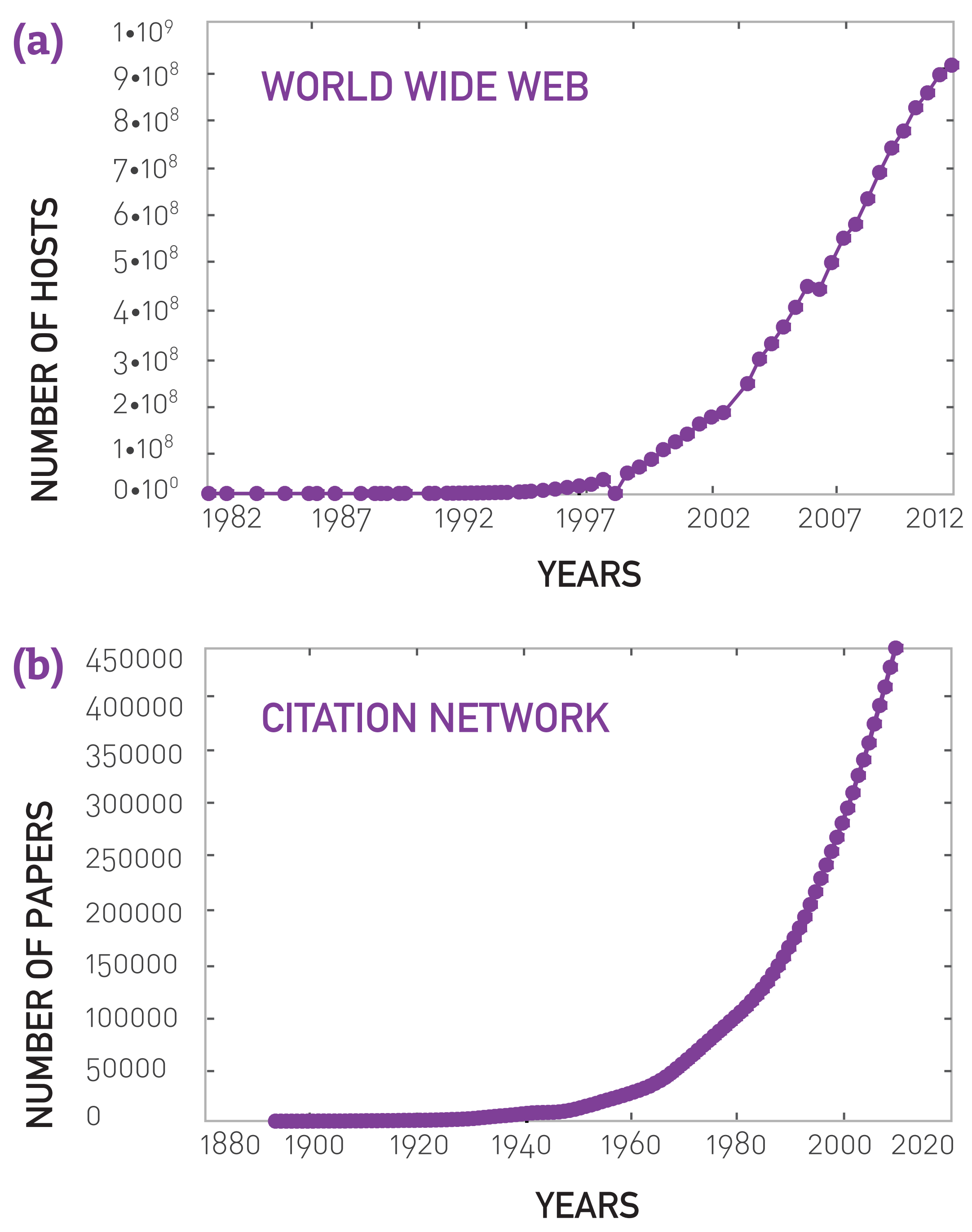
# Barabási-Albert model

- In static ensemble models (last lecture) we defined network by constraints.

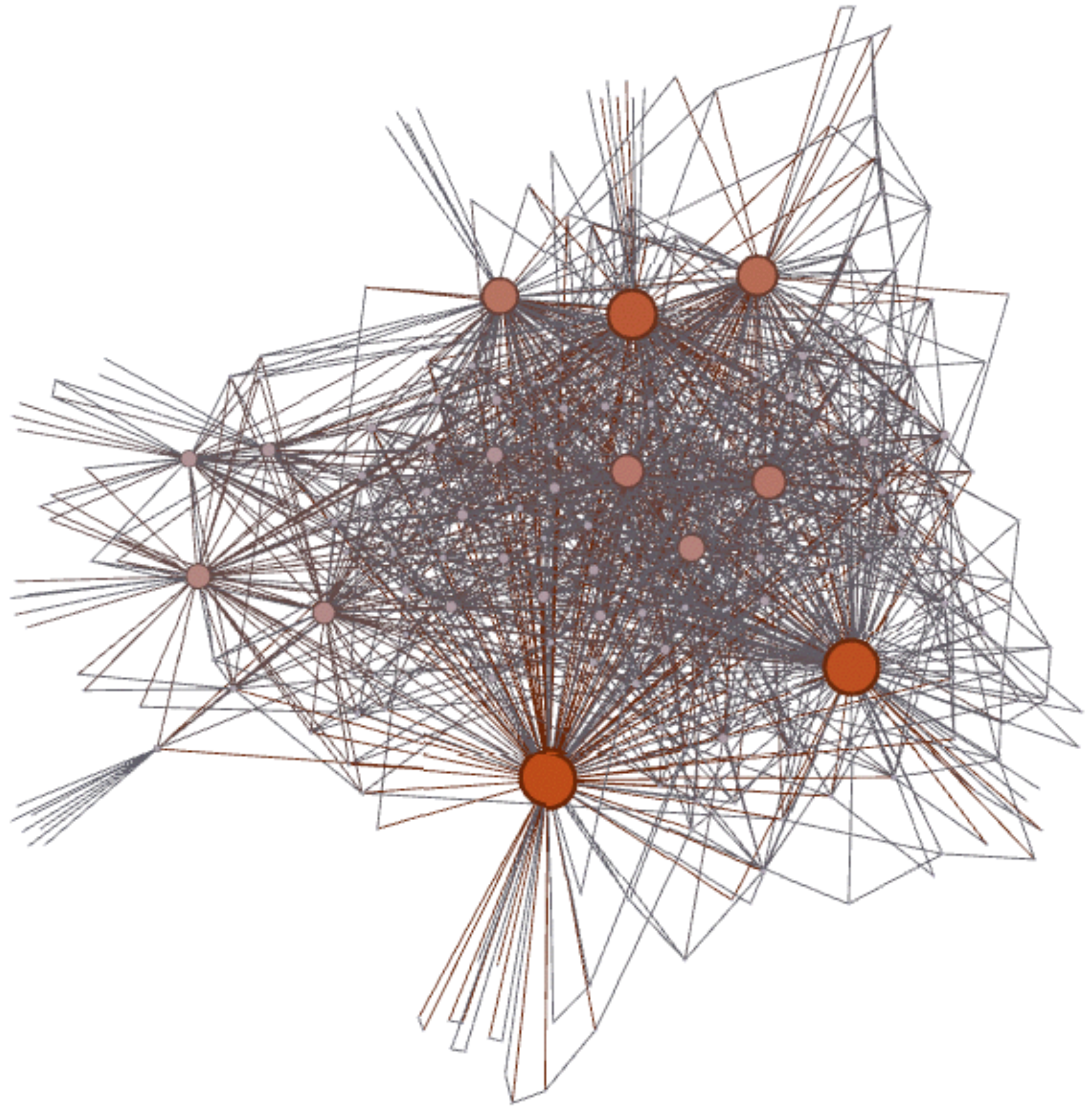
- In evolving/growing network, we **define growth rules** and look for asymptotic stationary behaviour

# Barabási-Albert model

First ingredient:
Growth/time




Second ingredient:
Not all links are equally likely!



**GROWTH:**

At each timestep we add a new node with *m* ($\leq m_0$) links that connect the new node to *m* nodes already in the network.

**PREFERENTIAL ATTACHMENT:**

the probability that a node connects to a node with *k* links is proportional to *k*.

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

# BA degree dynamics

$$\frac{dk_i}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j} \qquad \sum_{j=1}^{N-1} k_j = 2mt - m \overset{t \gg 1}{\approx} 2mt$$

$$\frac{\partial k_i}{k_i} = \frac{\partial t}{2t} \qquad \int_m^k \frac{\partial k_1}{k_i} = \int_{t_i}^{t} \frac{\partial t}{2t} \qquad \ln\left(\frac{k}{m}\right) = \frac{1}{2}\ln\left(\frac{t}{t_i}\right) = \left[\left(\frac{t}{t_i}\right)^{\frac{1}{2}}\right]$$

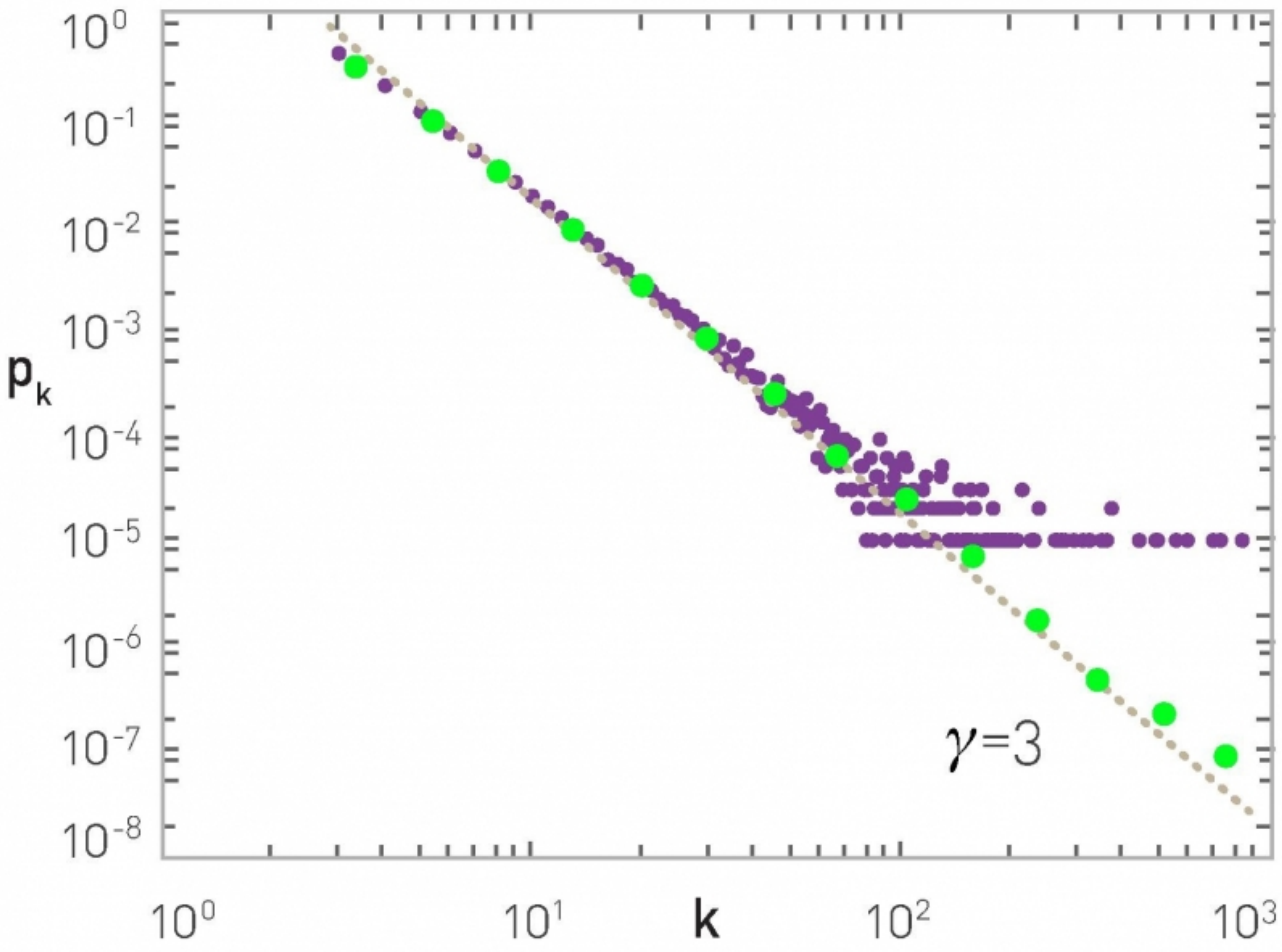
$$\boxed{k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \qquad \beta = \frac{1}{2}}$$

First mover advantage!

# BA degree distribution

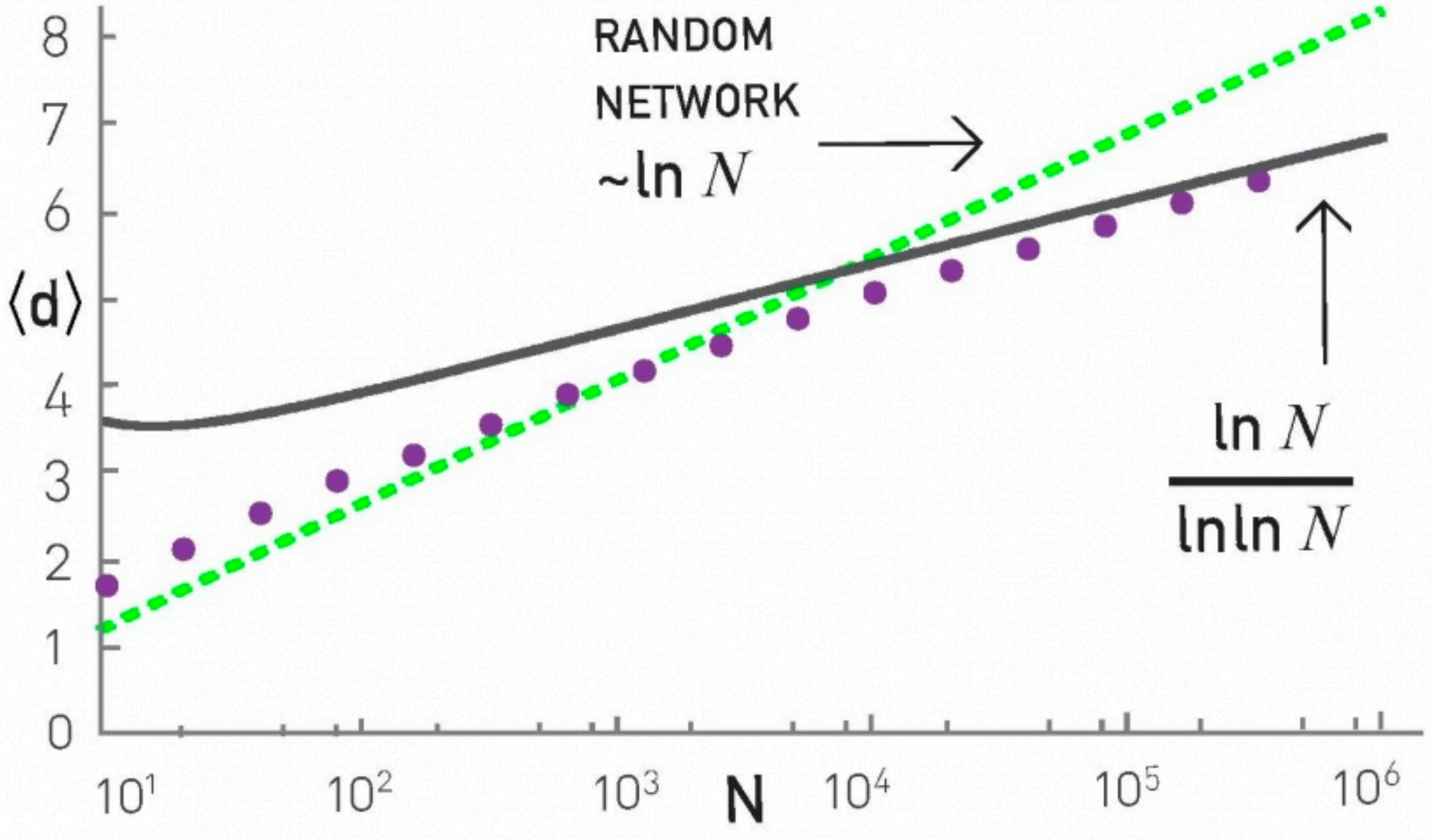
$$P(k) = \frac{2m(2m+1)}{k(k+1)(k+2)} \sim k^{-3}$$

Power-law degree distribution with fixed decay exponent: -3



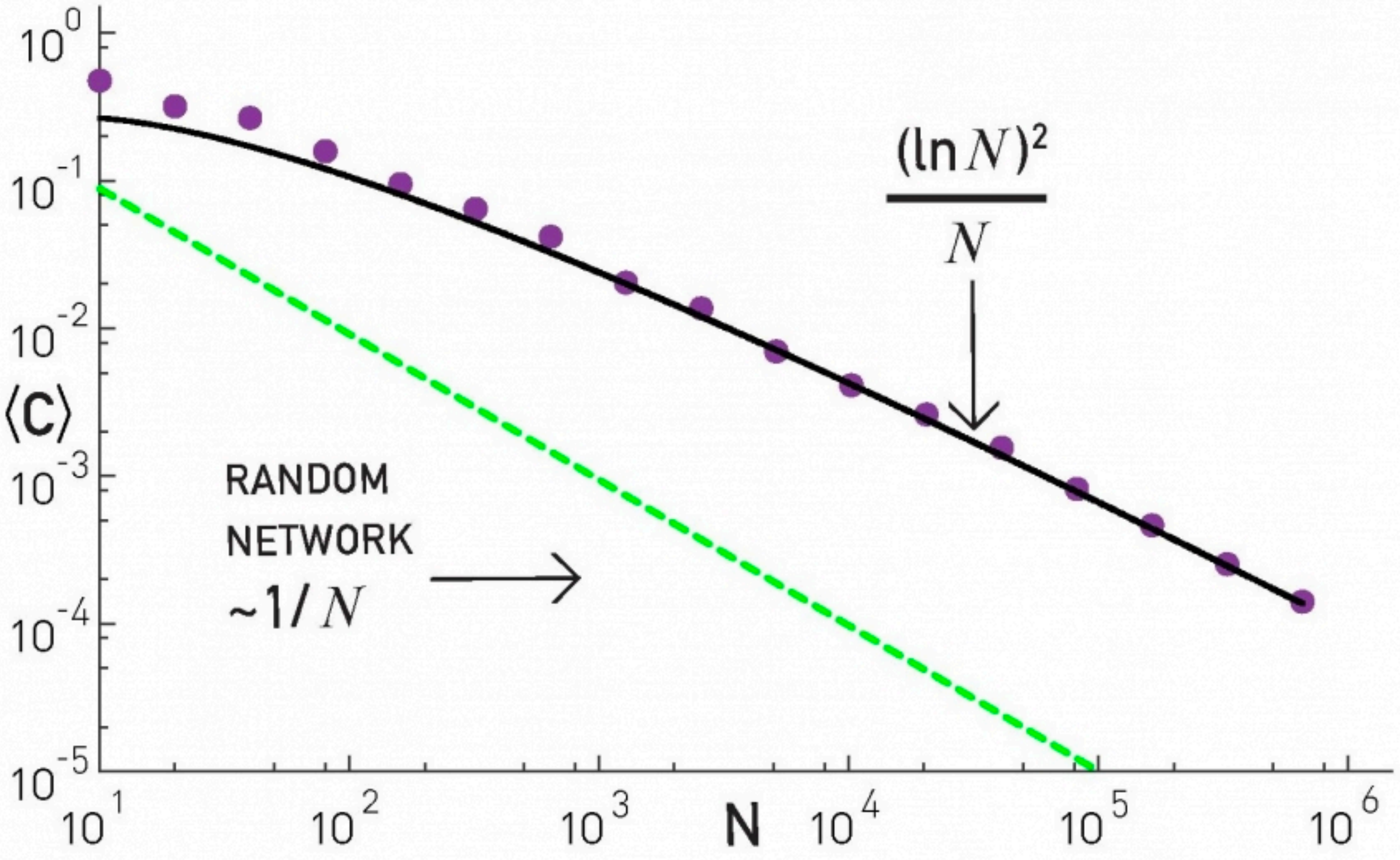
# BA diameter

$$\langle d \rangle \sim \frac{\ln N}{\ln \ln N}$$



# BA clustering

$$\langle C \rangle \simeq \frac{m}{4}\frac{(\ln N)^2}{N}$$



# Barabási-Albert summary

- Power law with fixed exponent, equal to -3
- Ultrasmall world
- Undirected
- Vanishing clustering

- Does not capture:
  - variations in the shape of the degree distribution
  - variations in the degree exponent
  - size-independent clustering coefficient

**Number of Nodes**

$N = t$

**Number of Links**

$N = mt$

**Average Degree**

$\langle k \rangle = 2m$

**Degree Dynamics**

$k_i(t) = m\,(t/t_i)^{\beta}$

**Dynamical Exponent**

$\beta = 1/2$

**Degree Distribution**

$p_k \sim k^{-\gamma}$

**Degree Exponent**

$\gamma = 3$

**Average Distance**

$\langle d \rangle \sim \log N / \log \log N$

**Clustering Coefficient**

$\langle C \rangle \sim (\ln N)^2 / N$

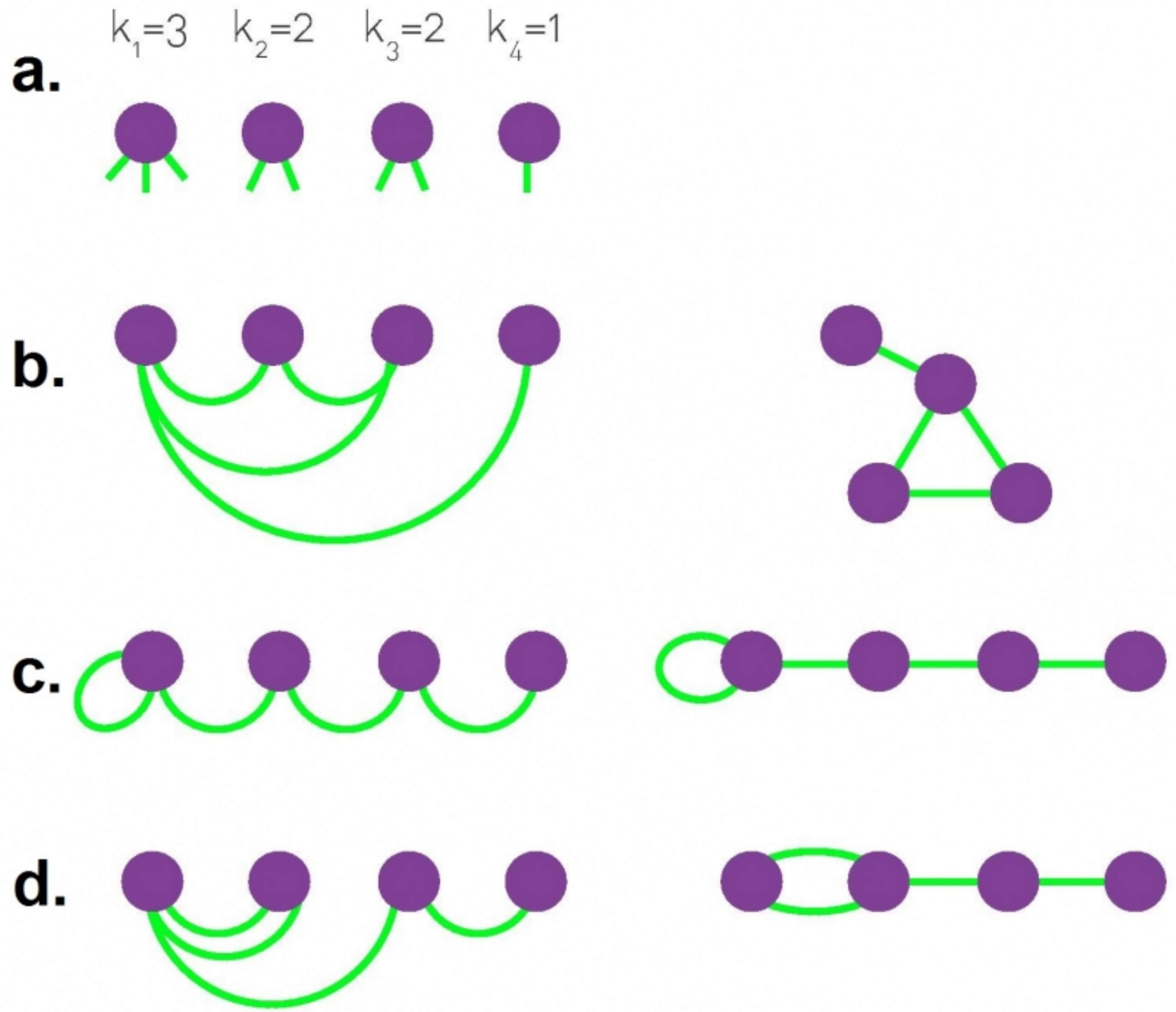
# Configuration model

- Can we constrain a random model of a static network to be scale-free?

- We can do it by **fixing the degree sequence** and then wire nodes together preserving the degree of each node

# Configuration model

1. Given a degree sequence $\boldsymbol{k} = \{k_1, k_2, k_3, \ldots, k_n\}$
2. Assign to each node $k_i$ stubs
3. Select random pairs of unmatched stubs and connect them
4. Repeat 3 while there are unmatched stubs



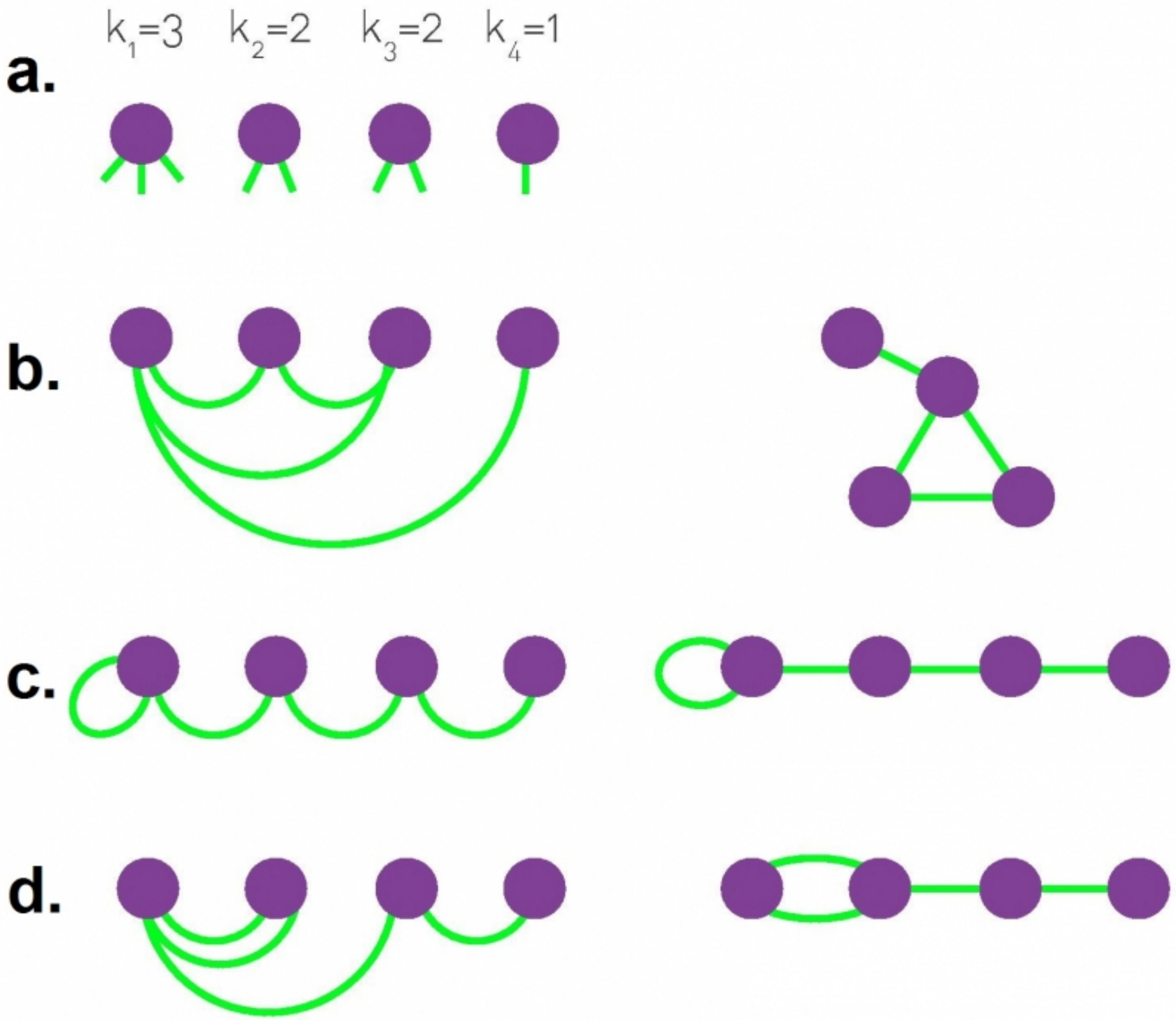
# Configuration model

- The probability of having a link between two nodes of degree $k_i$ and $k_j$ is

$$p_{ij} = \frac{k_i k_j}{2L - 1}$$

- Often, this quantity is approximated as

$$p_{ij} = \frac{k_i k_j}{2L}$$



# Correlations

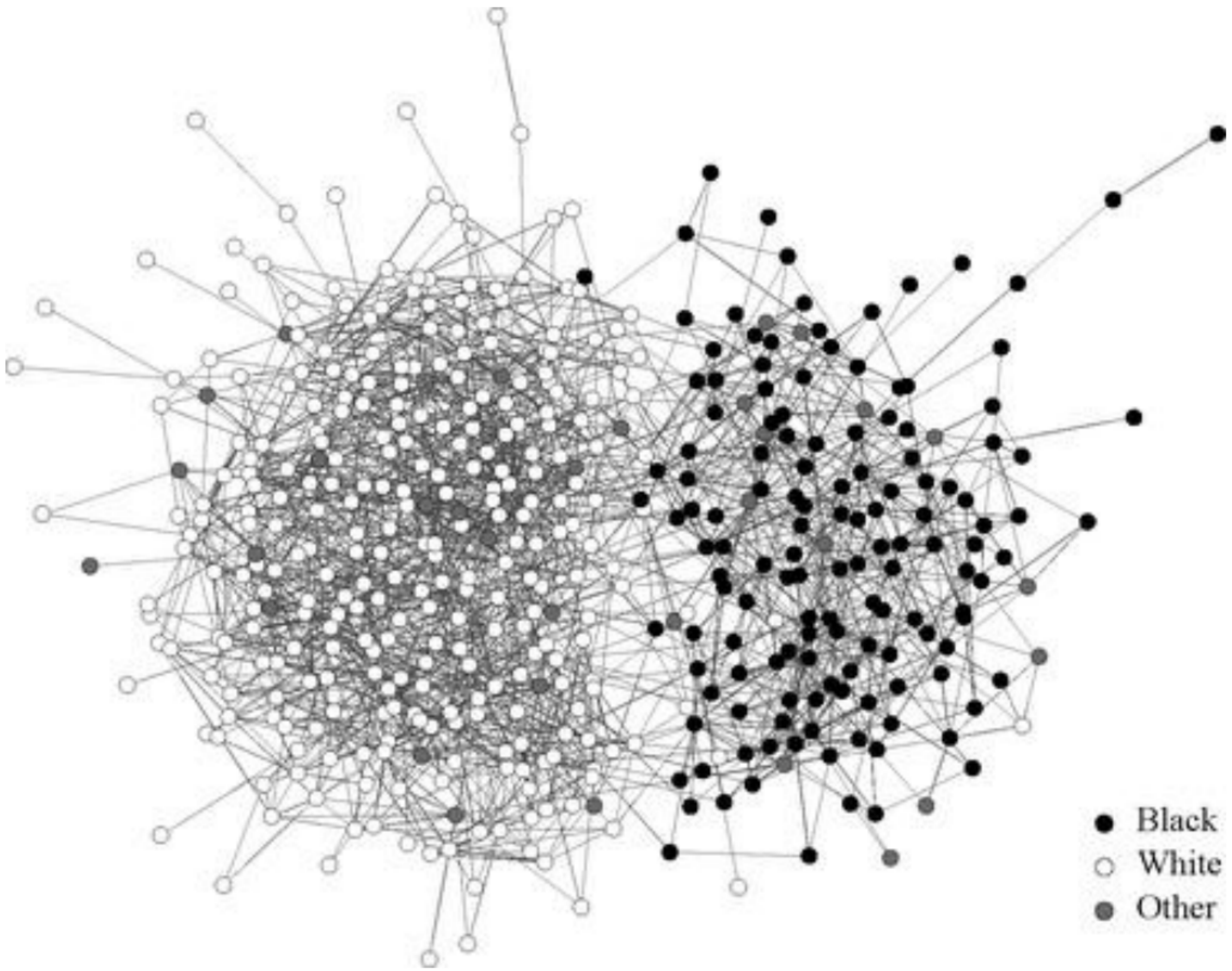
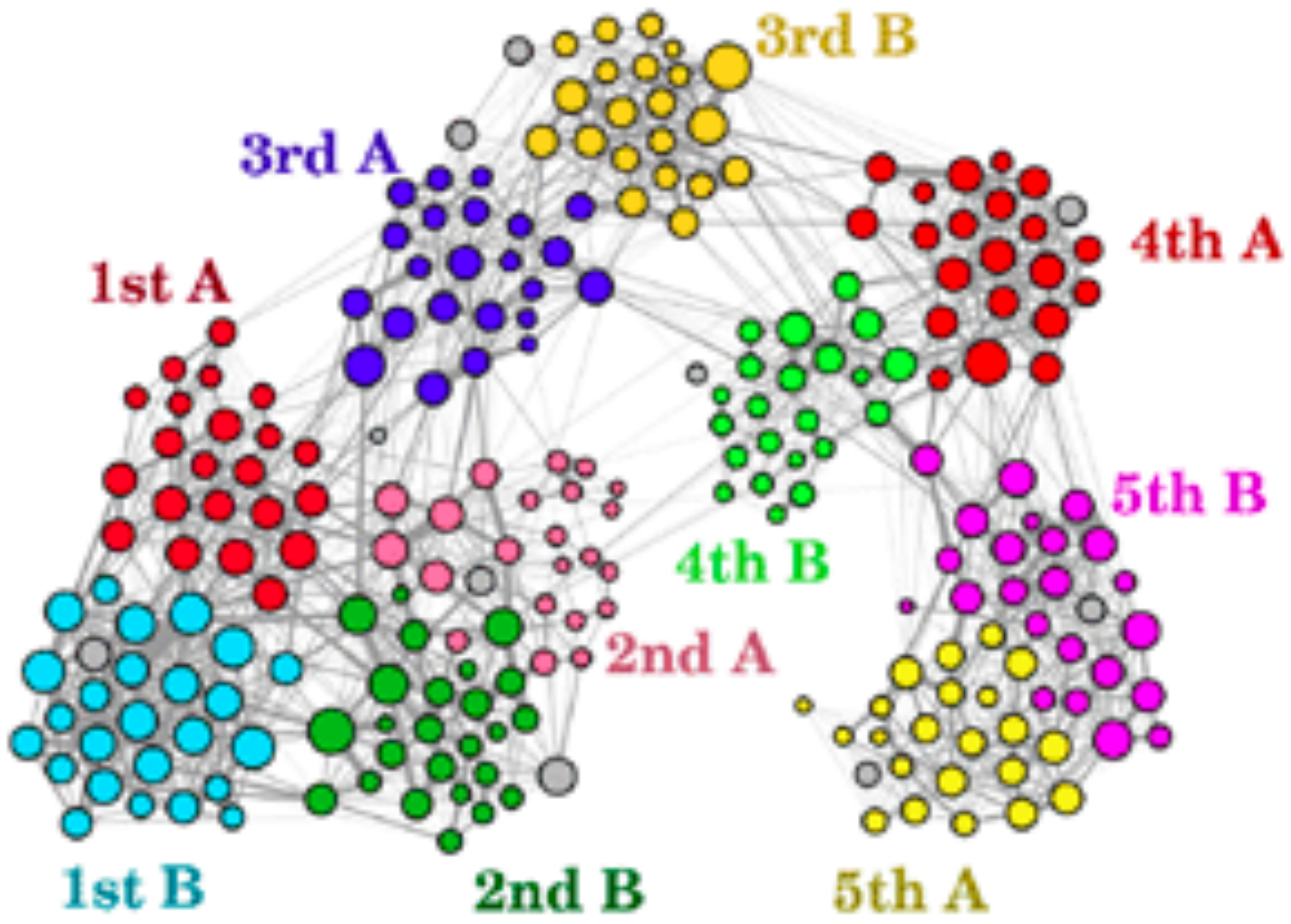



**Figure 7.10: Friendship network at a US high school.** The vertices in this network represent 470 students at a US high school (ages 14 to 18 years). The vertices are color coded by race as indicated in the key. Data from the National Longitudinal Study of Adolescent Health [34, 314].

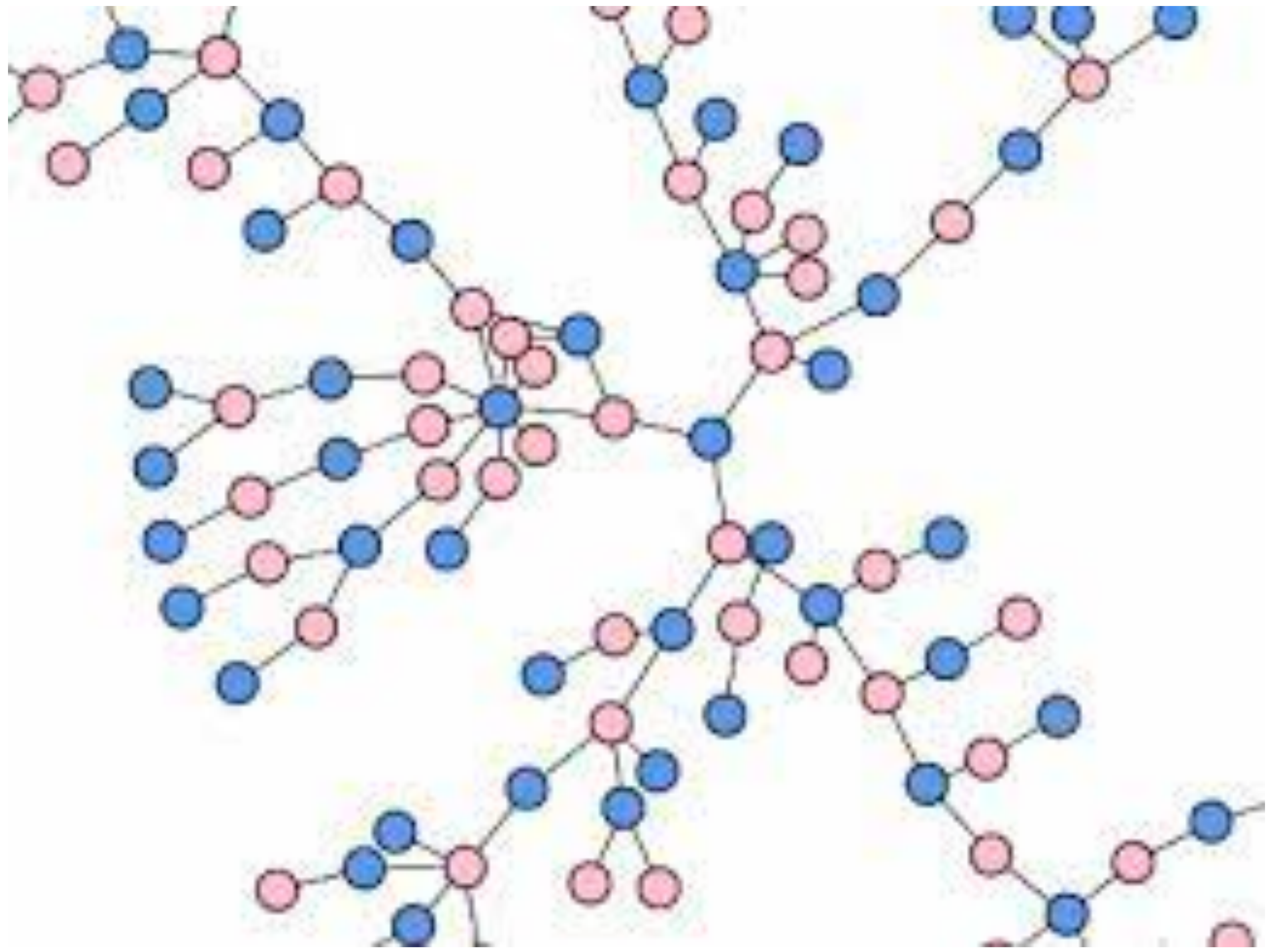
# Correlations

**Homophily:** This is not news to sociologists, who have long observed and discussed such divisions.

Assortative: like is associated with like




Disassortative: like is associated with not-like



# Correlations

Let's consider a network of **N nodes and m links**.

Given $c_i$ class or type of vertex i (1,...,$n_C$ = total number of classes), then the total number of edges that run between vertices of the same type is:

$$\sum_{edges(i,j)} \delta(c_i, c_j) = \frac{1}{2}\sum_{ij} A_{ij}\delta(c_i, c_j)$$

However, we want to control for the random expectation of the mixing: $\frac{1}{2}\sum_{ij}\frac{k_i k_j}{2m}\delta(c_i, c_j)$

$$Q = \frac{1}{2m}\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\delta(c_i, c_j)$$

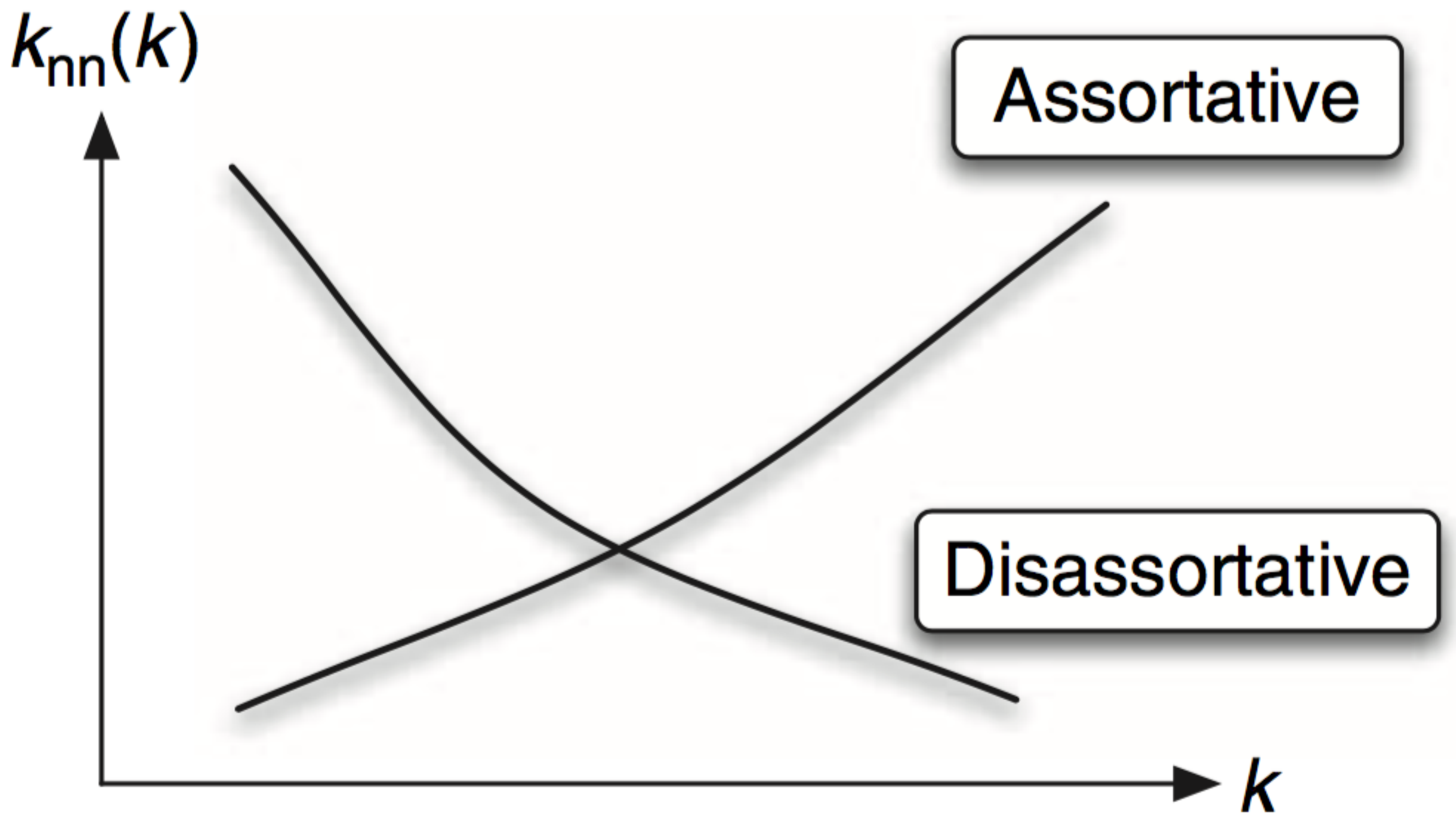
**Modularity**: It is strictly less than 1, takes positive values if there are more edges between vertices of the same type than we would expect by chance, and negative ones if there are less.

# Degree correlations

**Average nearest neighbours degree**

$$k_{nn,i} = \frac{1}{k_i} \sum_{j \in \nu(i)} k_j$$

$$k_{nn} = \frac{1}{N_k} \sum_{i, k_j = k} k_{nn,i} = \sum_{k'} k' P(k'|k)$$



# Random baseline

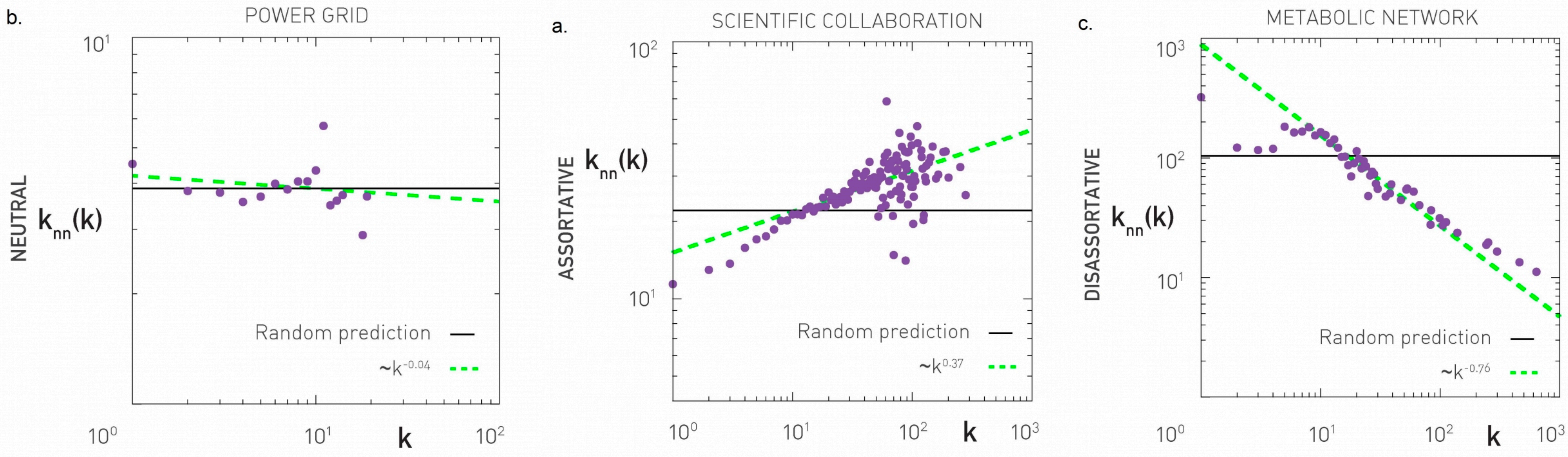
$$k_{nn}^{\mathbf{unc}} = \sum_{k'} k' P^{\mathbf{unc}}(k' \,|\, k) = \sum_{k'} k' \frac{k' P(k')}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

$$\boxed{P^{\mathbf{unc}}(k' \,|\, k) = \frac{k'}{2L} P(k') N = \frac{k' P(k')}{\langle k \rangle}}$$

This observation will be extensively used to simplify analytical calculations when dealing with dynamical processes (SIR, SIS) on networks.

# Random baseline

$$k_{nn}^{\text{unc}} = \sum_{k'} k' P^{\text{unc}}(k' \mid k) = \sum_{k'} k' \frac{k' P(k')}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$



# Next.. Epidemics on networks