Laboratory activity 1: Position state–space control of a DC servomotor

Riccardo Antonello* Francesco Ticozzi*

March 30, 2021

1 Activity goal

This laboratory activity is articulated in two parts: in the first part, some improvements to the position PID–control system designed in the previous laboratory activity are introduced. The improvements consist in the implementation of an anti–windup scheme to reduce the large overshoot occurring in the step response when the controller output saturates, and a friction plus inertia feedforward compensator, which allows to enhance both the accuracy and speed of response of the conventional feedback controller.

The second part of the activity is devoted to the design of a continuous–time position control system by using state–space techniques. Both *nominal* and *robust* tracking designs are considered. Nominal tracking is performed by exploiting a conventional feedforward scheme; on the other hand, robust tracking is achieved by either exploiting an integral action (for robust tracking of constant set–points, or perfect rejection of constant load disturbances), or by resorting to the internal model principle (for robust tracking and perfect rejection of more general, possibly time–varying signals).

2 Position PID–control improvements

2.1 Feedforward compensation

The benefits of using *feedback* in a control system are well known; they include the ability of reducing the control sensitivity to parameter variations and external disturbances, the possibility of stabilising unstable plants, and the capability of enhancing both the control precision and speed of response. Nevertheless, the performance of a conventional feedback (i.e. closed–loop) control system is often improved by resorting to open-loop or *feedforward* control actions. These additions to the control architecture are typically introduced for:

1. pre–filtering the reference signal in order to match the overall control system transfer function (from the reference input to the controlled output) to that of a desired reference model.

   With reference to Fig. 1a, it holds that

$$Y(s) \;=\; T'(s)\,H_1(s)\,R(s) \qquad \text{with} \qquad T'(s) = \frac{Y(s)}{R'(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} \tag{1}$$

   Therefore, the feedforward compensator (reference pre–filter) $H_1(s)$ can be used to introduce some adjustments to the overall closed–loop system dynamics (from the reference input r to

*Dept. of Information Engineering (DEI), University of Padova; email: {antonello, ticozzi}@dei.unipd.it

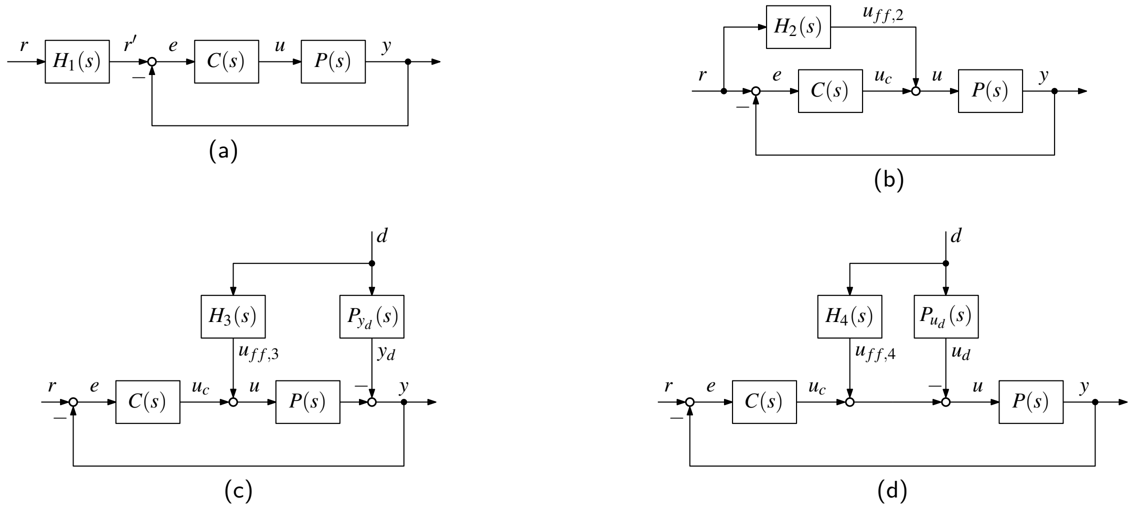


Figure 1: Possible modifications of the control architecture: (a) pre–filtering of the reference signal; (b) feed-forward compensation of the reference signal; (c) cancellation of known output disturbances; (d) cancellation of known input disturbances.

the controlled output y). In particular, it can be used to match the closed–loop response to that of a reference model with transfer function $T_m(s)$ (under certain assumptions for $T_m(s)$).

2. improving the control accuracy and speed of response.

   With reference to Fig. 1b, it holds that

   $$Y(s) \;=\; \frac{C(s)P(s) + H_2(s)P(s)}{1 + C(s)P(s)}\, R(s) \tag{2}$$

   For perfect tracking, it is desired to have $y(t) = r(t)$ for all $t \in \mathbb{R}$ and any reference signal $r(t)$. This specification cannot be guaranteed with a simple feedback controller. Indeed, what can be usually achieved with a simple feedback control is the *perfect asymptotic tracking* of *specific* reference inputs, namely the capability of following certain signals with zero tracking error at steady state. This is done by exploiting the *internal model principle*, namely by embodying a model of the signal to track (or the disturbance to reject) in the controller dynamics. For example, an integral action is typically included in the controller to achieve perfect steady state regulation of constant set–points (i.e. step reference inputs), and perfect steady state rejection of constant load disturbances (i.e. step disturbances entering at the plant input).

   Perfect tracking "in the general sense" can be ideally achieved by employing a feedforward action: in fact, from (2), it follows that the perfect tracking condition $Y(s) = R(s)$ can be satisfied by choosing

   $$H_2(s) \;=\; \frac{1}{P(s)} \tag{3}$$

   Obviously, this choice is not always practicable. In particular, it is requested that

   - $P(s)$ does not contain zeros with positive real part (i.e. *minimum–phase* condition).
   - $P(s)$ is a proper transfer function.
   - the plant dynamics does not contain relevant input–output (I/O) time–delays.

   However, some conditions can be relaxed, on condition of modifing the structure of the feed-forward compensator. In fact

- strictly proper transfer functions can be tolerated, provided that the time derivatives of the reference signal (up to the relative degree of the plant transfer function) are known. In this case, the feedforward compensator is replaced by multiple feedforward actions, one for every requested derivative of the reference input.

  For example, if $P(s) = 1/s^2$, then the single feedforward compensator $H_2(s) = s^2$ that performs the double derivative of the reference signal is not practically implementable (it has a non–proper transfer function). However, if the double derivative of the reference input is known, then it can be directly forwarded, and this avoids the issue of implementing a non–proper feedforward compensator.

- fixed time–delays can be tolerated, provided that the reference input is known in advance over an horizon which is larger than the time delay.

  As for the previous case, the feedforward action is implemented by forwarding the the time–advanced replica of the reference signal. For example, if $P(s) = e^{-sT_d}$, then the feedforward compensation can be performed by directly forwarding the reference input value $r(t + T_d)$, without requiring to implement any feedforward compensator.

3. compensating known (measured or estimated) input or output disturbances.

   With reference to Fig. 1c, it holds that

$$Y(s) \;=\; \frac{C(s)P(s)}{1 + C(s)P(s)}\, R(s) \;+\; \frac{H_3(s)P(s) - P_{y_d}(s)}{1 + C(s)P(s)}\, D(s) \tag{4}$$

   The effect of the known (measured or estimated) output disturbance d can be cancelled out by choosing

$$H_3(s) \;=\; \frac{P_{y_d}(s)}{P(s)} \tag{5}$$

   Similar considerations to those done for (3) regarding the inversion of the plant transfer function are also valid for (5). For the cancellation of a disturbance entering at the plant input, the scheme of Fig. 1d can be adopted, with $H_4(s) = P_{u_d}(s)$.

The feedforward compensations illustrated above can be conveniently applied to improve the performance of the position PID–controller designed in the previous laboratory activity. To this aim, reconsider the *simplified* model of the DC gearmotor, whose block diagram is reported in Fig. 2a. After some elementary manipulations, the equivalent block diagram of Fig. 2b can be obtained, where the original torque disturbance τ_d (static friction torque) is represented as an equivalent input voltage disturbance u_d. With reference to Fig. 2b, it holds that

$$\Theta_l(s) \;=\; P(s)\,[\,U(s) \,-\, U_d(s)\,] \;=\; P(s)\,[\,U(s) \,-\, P_{u_d}(s)\,\tau_d(s)\,] \tag{6}$$

where

$$P(s) \;=\; \frac{k_m}{N s\,(T_m s \,+\, 1)}\,, \qquad P_{u_d}(s) \;=\; \frac{R_{eq}}{k_{drv}\, k_t\, N} \tag{7}$$

with

$$k_m \;=\; \frac{k_{drv}\, k_t}{R_{eq}\, B_{eq} + k_t\, k_e}\,, \qquad T_m \;=\; \frac{R_{eq}\, J_{eq}}{R_{eq}\, B_{eq} + k_t\, k_e} \tag{8}$$

Given (6), two feedforward compensations can be readily implemented, namely the feedforward compensation of the reference signal of Fig. 1b, and the feedforward cancellation of the input distur-

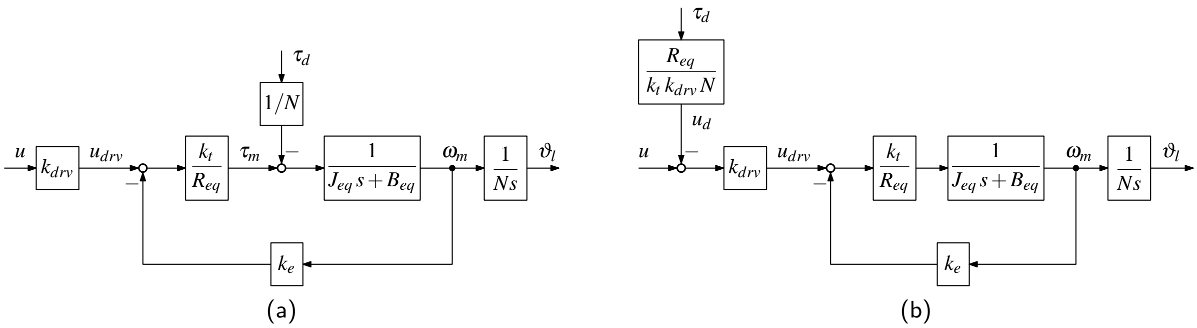


Figure 2: DC gearmotor block diagram: (a) simplified model with torque disturbance; (b) simplified model with equivalent voltage input disturbance.

bance of Fig. 1d. In Fig. 1b and 1d, $C(s)$ denotes the position PID–controller, and r is the position reference signal ϑ_l^*.

Regarding the scheme of Fig. 1b, the feedforward action can be implemented by choosing

$$H_2(s) \;=\; \frac{1}{P(s)} \;=\; \frac{N}{k_m}\Big(T_m s^2 \,+\, s\Big) \tag{9}$$

This feedforward compensator is not implementable in practice, since it requires to compute the first and second derivative of the reference signal. However, when the reference signal is *known in advance*, the two derivatives can be computed offline, and then used to implement the required feedforward action as a simple algebraic expression of the pre–computed derivatives. In fact, from (9) it follows that (by inverse Laplace transformation, reminding that r in Fig. 1b is the position reference signal ϑ_l^*):

$$u_{ff,2} \;=\; \frac{N\,T_m}{k_m}\,\frac{d\omega_l^*}{dt} \,+\, \frac{N}{k_m}\,\omega_l^* \tag{10}$$

$$\;=\; \frac{N\,R_{eq}\,J_{eq}}{k_{drv}\,k_t}\,\frac{d\omega_l^*}{dt} \,+\, \frac{N\,(R_{eq}\,B_{eq} + k_t\,k_e)}{k_{drv}\,k_t}\,\omega_l^* \tag{11}$$

where ω_l^* and $d\omega_l^*/dt$ are the precomputed first and second order derivatives of the position reference ϑ_l^*, provided that this latter quantity is known in advance.

For the cancellation of the input disturbance u_d, assume that τ_d is the static friction torque $\tau_d = \tau_{sf}\,\mathrm{sign}(\omega_l)$. Then, with reference to the scheme of Fig. 1d, the following feedforward compensation signal can be used

$$u_{ff,4} \;=\; \frac{R_{eq}}{k_{drv}\,k_t\,N}\,\tau_{sf}\,\mathrm{sign}(\omega_l^*) \tag{12}$$

By combining (12) with (11), the following total feedforward action results

$$\begin{aligned} u_{ff} \;&=\; u_{ff,2} \,+\, u_{ff,4} \;= \\ &=\; \frac{N\,R_{eq}\,J_{eq}}{k_{drv}\,k_t}\,\frac{d\omega_l^*}{dt} \,+\, \frac{N\,(R_{eq}\,B_{eq} + k_t\,k_e)}{k_{drv}\,k_t}\,\omega_l^* \,+\, \frac{R_{eq}}{k_{drv}\,k_t\,N}\,\tau_{sf}\,\mathrm{sign}(\omega_l^*) \end{aligned} \tag{13}$$

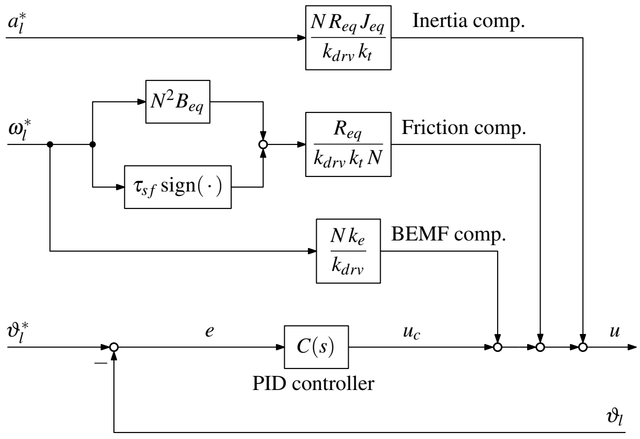


Figure 3: Position PID–control with feedforward compensation.

By rearranging terms, it can be noticed that

$$u_{ff} = \underbrace{\frac{N\,R_{eq}\,J_{eq}}{k_{drv}\,k_t}\frac{d\omega_l^*}{dt}}_{\text{Inertia compensation}} + \underbrace{\frac{R_{eq}}{k_{drv}\,k_t\,N}\left[N^2 B_{eq}\,\omega_l^* + \tau_{sf}\,\mathrm{sign}(\omega_l^*)\right]}_{\text{Friction compensation}} + \underbrace{\frac{N\,k_e}{k_{drv}}\,\omega_l^*}_{\text{BEMF compensation}} \tag{14}$$

In fact, in the friction compensation term, the torque

$$\tau_f = N^2 B_{eq}\,\omega_l^* + \tau_{sf}\,\mathrm{sign}(\omega_l^*) \tag{15}$$

represents an estimate of the total friction torque acting at *load side*, while the voltage

$$u_e = N\,k_e\,\omega_l^* \tag{16}$$

in the BEMF compensation term is an estimate of the actual BEMF generated by the DC motor at the rotor speed $N\omega_l^*$. On a conventional feedback–based position PID–control scheme, the feedforward compensation (14) can be introduced as illustrated in Fig. 3.

2.2 Integrator anti–windup mechanism

In any real control system the actuator output has a limited dynamic range. The block diagram of Fig. 4a takes into account this limitation by including a saturation block between the controller output and the plant input. Whenever the actuator saturates, the command u provided to the plant is constant, regardless of the tracking error e, so that the feedback loop turns out to be effectively disabled (opened), and both the plant and the PI-controller operate in open–loop.

Suppose that the reference set–point is large enough to cause the actuator to saturate at its upper bound $\bar{u} = \bar{u}_{\max}$. While the saturation is active, the integrator in the PID controller still continues to integrate the (positive) tracking error e (*integrator windup*), and hence the controller output $\bar{u}$ keeps growing; however, the increasingly growing control command is ineffective for the plant, since it exceeds the actuator limits. When the plant output rises enough to bring the tracking error to zero, the actuator is still saturated because the integrator has (unecessarily) accumulated a large amount of tracking error in the previous phase. To bring the actuator out of saturation, it is necessary that the tracking error remains negative for enough time to allow the integrator output to

come back within the linear operating region of the actuator (*integrator unwinding*). The effect of the integrator windup on the closed–loop step response is an excessively large overshoot, which is however required to produce the necessary unwinding error for the integrator.
A possible solution to mitigate the integrator windup effects consists of implementing an *integrator anti-windup circuit* in the PID controller, as shown in Fig. 4b.

A *saturation detection mechanism* is used within the controller to detect when the actuator is saturated. If the controller is implemented digitally, this mechanism can be implemented as a logical test, such as: "if the PID controller output $\bar{u}$ exceeds the actuator saturation levels $\pm\bar{u}_{\max}$, i.e. $\bar{u} \geq \bar{u}_{\max}$, then the actuator is saturated". In analog controllers, the easiest way to detect the actuator saturation consists of employing a "simulated saturation" within the controller, and then computing the difference between its input $\bar{u}$ and output u. If the difference is not zero, a saturation in the controller command is occurring. Obviously, for an effective detection of the actuator saturation, the saturation levels in the 'simulated saturation" block must be matched with those of the real actuator.

When an actuator saturation is detected, the integration has to be either stopped or compensated to avoid the integrator windup issue. In the anti–windup mechanism depicted in Fig. 4b, the integrator input is compensated by a signal proportional to the amount of saturation occurring in the actuator. When a saturation occurs, the local feedback around the integrator becomes active, and the integrator response is modified into that of a first order low–pass filter of the type

$$H(s) = \frac{1}{s + K_W} \tag{17}$$

Since the filter is BIBO stable, its output remains bounded (provided that the tracking error e is bounded), thus preventing the integrator windup phenomenon. In other words, the purpose of the

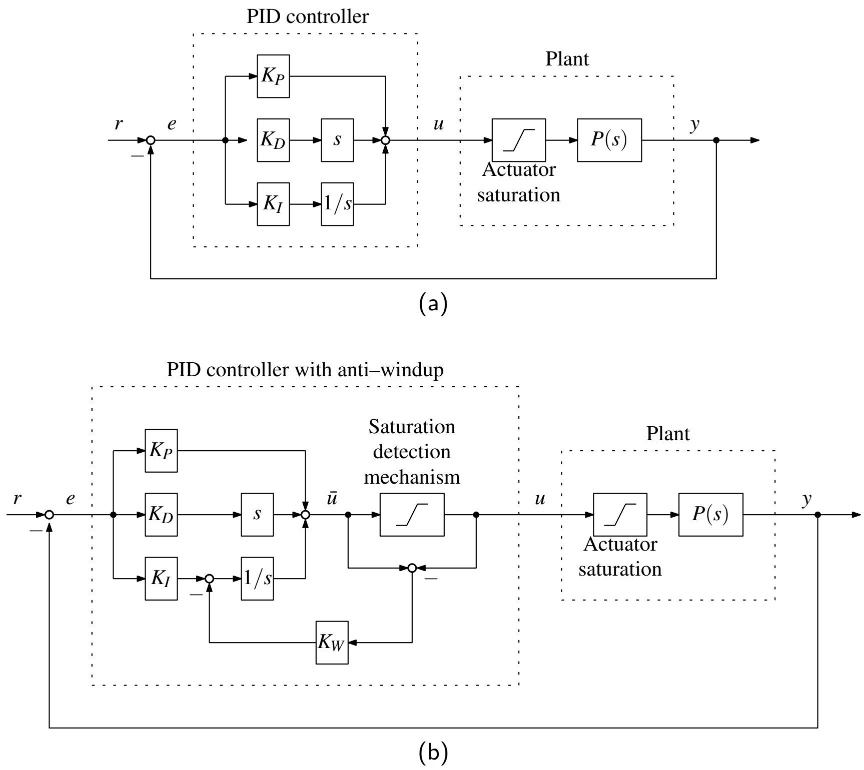


Figure 4: PID control in presence of actuator saturation: (a) conventional PID controller; (b) PID controller with anti–windup mechanism.

anti–windup mechanism is to provide a local feedback around the integrator in order to make the controller stable alone when the main control loop is inactive (because of the saturation occurring in the actuator). Regarding the choice of the anti–windup gain $K_W = 1/T_W$, it should be chosen large enough to keep the integrator input small under all error condition. A possible tentative value for the gain is obtained by setting $T_W = t_{s,5\%}/5$, where $t_{s,5\%}$ is the settling time of the control system to a step reference input. However, the initial tentative value may require manual adjustments to get a satisfactory response.

3 Position state–space control design

In a conventional course on linear system theory, state–space control methods are typically introduced to solve a *regulation problem*, which consists of designing a stabilising control law capable of steering any initial perturbed state to an equilibrium point, with prescribed convergence rate. This section describes how to extend the state–space methods to solve a *tracking problem*, namely how to design a control law capable of forcing the controlled output y to follow a desired reference signal r at steady state (i.e. asymptotically). Both the *nominal* and *robust* tracking problems will be addressed in this activity. Remind that a property or control specification is satisfied in a *nominal sense* when it is guaranteed only for a nominal condition; instead, it is satisfied in a *robust sense* when it is guaranteed in presence of model uncertainties and external disturbances (within a certain class to be specified in the design). A controller designed with state–space methods can be either based on a *static state feedback* or a *dynamic output feedback*. In the former case, the control law depends only on the current plant state, which is assumed to be fully accessible; instead, in the later case the state is considered either unmeasurable or only partially accessible, and therefore the state–feedback control law is implemented by resorting to a state estimate provided by either a full or reduced order state observer. This activity will be limited to the design of tracking controllers based on full–state feedback; the dynamic output feedback case will be considered in the next laboratory activity. Moreover, the attention will be restricted to single–input single–output (SISO) strictly proper plants with state–space models $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0)$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times 1}$ and $\boldsymbol{C} \in \mathbb{R}^{1\times n}$. Nevertheless, the results presented below are still valid for multi–input multiple–output (MIMO) systems, but obviously with an increased notational complexity.

3.1 State–space model of the DC gearmotor

A possible state–space realisation $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, D)$ of the DC gearmotor transfer function (7) is the reachable canonical form

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{T_m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{R_{eq}B_{eq} + k_t k_e}{R_{eq}\, J_{eq}} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ \dfrac{k_m}{N\, T_m} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{k_{drv}\, k_t}{N\, R_{eq}\, J_{eq}} \end{bmatrix} \tag{18}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$

with state vector $\boldsymbol{x} = [\vartheta_l,\, \omega_l]^T$. It can be easily verified that (18) is both reachable and observable.

3.2 Nominal tracking design

Consider the problem of asymptotically tracking a step reference input $r(t) = r_\infty\,\delta_{-1}(t)$ in the ideal situation of perfect knowledge of the plant dynamics and lack of any external disturbance (*nominal tracking* problem). Denote with $\boldsymbol{x}_\infty$, u_∞ and y_∞ the values reached at steady state by, respectively, the plant state, input and output variables, i.e.

$$\boldsymbol{x}_\infty = \lim_{t\to+\infty} \boldsymbol{x}(t)\,, \qquad u_\infty = \lim_{t\to+\infty} u(t)\,, \qquad y_\infty = \lim_{t\to+\infty} y(t) \tag{19}$$

The perfect steady state tracking condition $y_\infty = r_\infty$ imposes that $\boldsymbol{x}_\infty$ and u_∞ satisfy the conditions

$$\begin{cases} \boldsymbol{A}\,\boldsymbol{x}_\infty + \boldsymbol{B}\,u_\infty = \dot{\boldsymbol{x}}_\infty = \boldsymbol{0} & (20)\\ \boldsymbol{C}\,\boldsymbol{x}_\infty = y_\infty = r_\infty & (21) \end{cases}$$

where (20) is obtained by noting that the steady state $\boldsymbol{x}_\infty$ is an equilibrium state for the model $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0)$ with constant input u_∞. From (20) and (21) it is evident that both $\boldsymbol{x}_\infty$ and u_∞ depend linearly on the value r_∞ of the reference input, namely

$$\boldsymbol{x}_\infty = \boldsymbol{N}_x\,r_\infty\,, \qquad u_\infty = N_u\,r_\infty \tag{22}$$

where the two gains $\boldsymbol{N}_x \in \mathbb{R}^{n\times 1}$ and $N_u \in \mathbb{R}$ are determined by solving the system of linear equations

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix} \tag{23}$$

which is immediately obtained by replacing (22) within (20) and (21). With the steady state quantities introduced above, define the following control law:

$$\begin{aligned} u &= u_\infty - \boldsymbol{K}\,(\boldsymbol{x} - \boldsymbol{x}_\infty) && (24)\\ &= N_u\,r_\infty - \boldsymbol{K}\,(\boldsymbol{x} - \boldsymbol{N}_x\,r_\infty) && (25)\\ &= -\boldsymbol{K}\boldsymbol{x} + \underbrace{(N_u + \boldsymbol{K}\boldsymbol{N}_x)}_{\triangleq\, N_r}\,r_\infty && (26) \end{aligned}$$

where $\boldsymbol{K} \in \mathbb{R}^{1\times n}$ is a state feedback gain that asymptotically stabilises the closed–loop system, i.e that places all the eigenvalues of the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ in the open left–half plane (LHP). In (25), the control law is written as a state feedback plus two feedforward actions, one for the state and the other for the control variable (see Fig. 6a); instead, in (26) the control action contains only a state feedback plus a single feedforward reference compensation (see Fig. 6b). Obviously, the two configurations are equivalent. It is immediate to verify that the control law (24)–(26) guarantees nominal perfect tracking of the constant reference input at steady state. In fact, the state of the closed–loop system evolves according to the following dynamical equation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,u && (27)\\ &= \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,[u_\infty - \boldsymbol{K}\,(\boldsymbol{x} - \boldsymbol{x}_\infty)] && (28)\\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,(\boldsymbol{x} - \boldsymbol{x}_\infty) + \underbrace{\boldsymbol{A}\,\boldsymbol{x}_\infty + \boldsymbol{B}\,u_\infty}_{\dot{\boldsymbol{x}}_\infty = \boldsymbol{0}} && (29) \end{aligned}$$

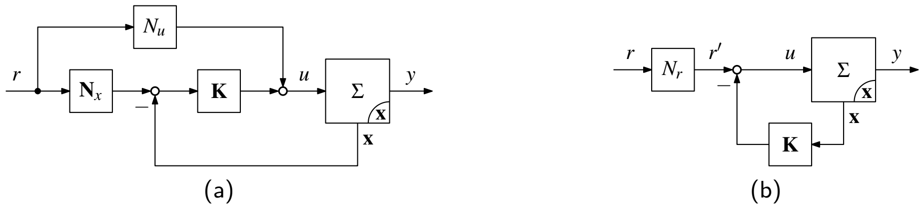


Figure 5: Nominal tracking by static state feedback: (a) with feedforward action on state and control variables; (b) with feedforward compensation of the reference signal.

which implies that $\boldsymbol{x} \to \boldsymbol{x}_\infty$ at steady state, since $\boldsymbol{A} - \boldsymbol{BK}$ is stable. Then, because of (21), it also follows that $y \to y_\infty$, which is the desired perfect tracking condition.

It is worth to notice that the role of feedforward gain N_r in (26) is to make the DC gain of the closed–loop system from the reference input r to the controlled output y equal to one (in the nominal case), so that the perfect tracking condition $y_\infty = r_\infty$ can be achieved at steady state. In fact, the closed–loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{BK}\right)^{-1}\boldsymbol{B}N_r \tag{30}$$

and hence the DC gain is

$$\begin{aligned}
T(0) &= \boldsymbol{C}\left(-\boldsymbol{A} + \boldsymbol{BK}\right)^{-1}\boldsymbol{B}N_r && (31)\\
&= \boldsymbol{C}\left(-\boldsymbol{A} + \boldsymbol{BK}\right)^{-1}\left(\boldsymbol{B}N_u + \boldsymbol{BKN}_x\right) \qquad \text{(using} \quad N_r = N_u + \boldsymbol{KN}_x\text{)} && (32)\\
&= \boldsymbol{C}\left(-\boldsymbol{A} + \boldsymbol{BK}\right)^{-1}\left(-\boldsymbol{AN}_x + \boldsymbol{BKN}_x\right) \qquad \text{(using} \quad \boldsymbol{AN}_x + \boldsymbol{B}N_u = \boldsymbol{0}\text{)} && (33)\\
&= \boldsymbol{C}\left(-\boldsymbol{A} + \boldsymbol{BK}\right)^{-1}\left(-\boldsymbol{A} + \boldsymbol{BK}\right)\boldsymbol{N}_x && (34)\\
&= \boldsymbol{CN}_x = 1 && (35)
\end{aligned}$$

In practice, this condition implies that $N_r = 1/T'(0)$, where $T'(s)$ is the transfer function from the uncompensated reference input r' to the output y.

3.3 Robust tracking design with integral action

One way to guarantee perfect tracking of a step reference input $r(t) = r_\infty\,\delta_{-1}(t)$ in presence of model uncertainties or constant disturbances entering at the plant input (*robust tracking* problem) consists of introducing an integral action in the control law. With reference to the control law (24),

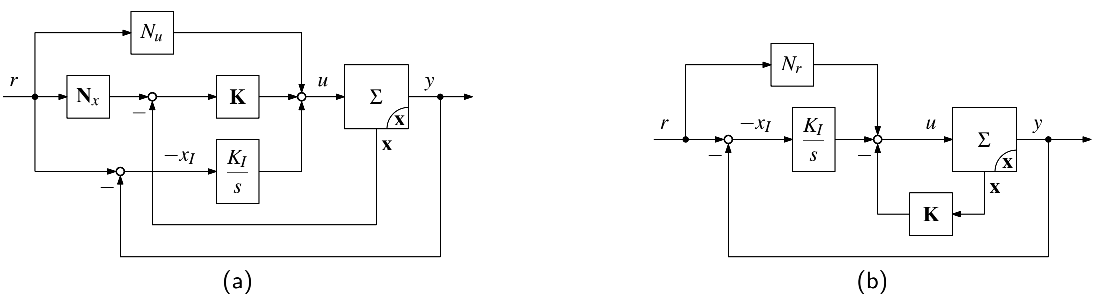


Figure 6: Robust tracking with integral action: (a) with feedforward action on state and control variables; (b) with feedforward compensation of the reference signal.

the integral action can be included as follows

$$u \;=\; u_\infty \;-\; \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{x}_\infty) \;-\; K_I \int_0^t [y(\tau)-r(\tau)]\, d\tau \tag{36}$$

which extends the control schemes of Fig. 5 as shown in Fig. 6. By introducing the integrator state variable $x_I = \int_0^t [y(\tau)-r(\tau)]d\tau$, the control law (36) can be rewritten as follows

$$\begin{cases} \dot{x}_I \;=\; y \;-\; r & (37)\\ u \;=\; -\boldsymbol{K}_e\,\boldsymbol{x}_e \;+\; (u_\infty + \boldsymbol{K}\boldsymbol{x}_\infty) & (38)\end{cases}$$

where $\boldsymbol{x}_e = [x_I,\ \boldsymbol{x}]^T$ is the state of the augmented state system

$$\Sigma_e \;:\; \begin{cases} \begin{bmatrix}\dot{x}_I(t)\\ \dot{\boldsymbol{x}}(t)\end{bmatrix} \;=\; \underbrace{\begin{bmatrix}0 & \boldsymbol{C}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}}_{\triangleq\,\boldsymbol{A}_e} \underbrace{\begin{bmatrix}x_I(t)\\ \boldsymbol{x}(t)\end{bmatrix}}_{\triangleq\,\boldsymbol{x}_e} + \underbrace{\begin{bmatrix}0\\ \boldsymbol{B}\end{bmatrix}}_{\triangleq\,\boldsymbol{B}_e} u(t) \;-\; \begin{bmatrix}1\\ \boldsymbol{0}\end{bmatrix} r(t) & (39)\\[2ex] y(t) \;=\; \underbrace{\begin{bmatrix}0 & \boldsymbol{C}\end{bmatrix}}_{\triangleq\,\boldsymbol{C}_e} \begin{bmatrix}x_I(t)\\ \boldsymbol{x}(t)\end{bmatrix} & (40)\end{cases}$$

while $\boldsymbol{K}_e = [K_I,\ \boldsymbol{K}]$ is a state–feedback matrix for the augmented state. It can be proved that if the pair $(\boldsymbol{A},\boldsymbol{B})$ is reachable and the open loop transfer function $H(s) = \boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$ has no zeros at $s=0$, then Σ_e is reachable and therefore the eigenvalues of the closed–loop system, namely the eigenvalues of

$$\boldsymbol{A}_{cl} \;\triangleq\; \boldsymbol{A}_e - \boldsymbol{B}_e\boldsymbol{K}_e \;=\; \begin{bmatrix} 0 & \boldsymbol{C}\\ -\boldsymbol{B}K_I & \boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}\end{bmatrix} \tag{41}$$

can be placed at arbitrary position by properly selecting the feedback gain matrix $\boldsymbol{K}_e$.

3.4 Robust tracking design with the internal model principle

The concept of integral action can be extended to achieve the robust steady state reference tracking and perfect disturbance rejection of signals that are more general than a simple constant value. Two methods are available for the purpose: the *error–space* approach and the design based on the *extended estimator*. Although different in their implementation, both methods rely on the same *internal model principle* to achieve the robust tracking property.

3.4.1 Signal models

For the application of the internal model principle, it is necessary to assume that both the reference input r and the load disturbance w are signals that are generated according to certain "models", which are typically specified in terms of homogeneous linear ordinary differential equations (ODEs) of the type

$$\begin{aligned} r^{(q)}(t) \;+\; \beta_{q-1}\, r^{(q-1)}(t) \;+\; \ldots \;+\; \beta_1\, r^{(1)}(t) + \beta_0 r(t) \;&=\; 0\\ w^{(v)}(t) \;+\; \gamma_{v-1}\, w^{(v-1)}(t) \;+\; \ldots \;+\; \gamma_1\, w^{(1)}(t) + \gamma_0 w(t) \;&=\; 0 \end{aligned} \tag{42}$$

where $r^{(i)}$ and $w^{(i)}$ denote the time derivatives of order i. Being r and w the solutions of the ODEs (42) for some non–zero initial conditions, it follows that the two signals are linear combinations of elementary functions (system modes) of the type

$$\begin{array}{c} \frac{t^l}{l!}\, e^{\lambda_i t} \qquad \lambda_i \in \mathbb{R} \\ \frac{t^l}{l!}\, e^{\sigma_i t} \cos \omega_i t \qquad \frac{t^l}{l!}\, e^{\sigma_i t} \sin \omega_i t \qquad \sigma_i,\, \omega_i \in \mathbb{R} \end{array} \tag{43}$$

where λ_i and $\sigma_i \pm j\omega_i$ are the generic real and complex-conjugate roots of the *characteristic polynomials* associated to the ODEs, i.e.

$$\begin{aligned} p_r(s) &= s^q + \beta_{q-1}\, s^{q-1} + \ldots + \beta_1\, s + \beta_0 \\ p_w(s) &= s^v + \gamma_{v-1}\, s^{v-1} + \ldots + \gamma_1\, s + \gamma_0 \end{aligned} \tag{44}$$

It is obvious that the only system modes that are relevant for the reference tracking and disturbance rejection problems are those that do not converge to zero for $t \to +\infty$: in fact, if both r and w fades out at steady state, then the two problems can be solved as a simpler regulation problem, namely by stabilising the system state to the origin with a proper state feedback. Therefore, in the following it will be assumed that all the roots of (44) have nonnegative real parts, i.e. they are located in the closed right half plane (RHP).

The two high order ODEs (42) can be alternatively represented as two sets of first order ODEs, namely two autonomous state–space models of the type

$$\Sigma_r \;:\; \begin{cases} \dot{\boldsymbol{x}}_r = \boldsymbol{A}_r\, \boldsymbol{x}_r \\ r = \boldsymbol{C}_r\, \boldsymbol{x}_r \end{cases} \qquad \Sigma_w \;:\; \begin{cases} \dot{\boldsymbol{x}}_w = \boldsymbol{A}_w\, \boldsymbol{x}_w \\ w = \boldsymbol{C}_w\, \boldsymbol{x}_w \end{cases} \tag{45}$$

The two state–space models (45) are usually called *exo–systems*. It is worth to notice here that even though the models for generating r and w have been defined as different ODEs/exo–systems, it is not restrictive to assume that the two signals are generated according to a same model. In fact, let

$$p(s) = s^m + \alpha_{m-1}\, s^{m-1} + \ldots + \alpha_1\, s + \alpha_0 \tag{46}$$

be the least common multiple of the two polynomials in (44). Then, both r and w satisfying (42) are also solutions of the homogeneous ODE with characteristic polynomial (46) (with proper choices of the initial conditions), which can therefore be assumed as the common model for the generation of the two signals. For convenience, the most typical signal models used in control systems are reported below (with reference to a generic signal z, which can be thought as either the reference r or the disturbance w):

- constant signal: $z(t) = z_0\, \delta_{-1}(t)$
  - homogenous ODE:

    $$\dot{z} = 0 \tag{47}$$

    with characteristic polynomial

    $$p_z(s) = s \tag{48}$$

– exo–system: $\Sigma_z = (\boldsymbol{A}_z, \boldsymbol{C}_z)$ with

$$\boldsymbol{A}_z = 0, \qquad \boldsymbol{C}_z = 1 \tag{49}$$

- constant + linear ramp signal: $z(t) = (z_1\, t + z_0)\, \delta_{-1}(t)$
  – homogenous ODE:
  $$\ddot{z} = 0 \tag{50}$$
  with characteristic polynomial
  $$p_z(s) = s^2 \tag{51}$$
  – exo–system: $\Sigma_z = (\boldsymbol{A}_z, \boldsymbol{C}_z)$ with
  $$\boldsymbol{A}_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{52}$$
- sinusoidal signal: $z(t) = z_0 \sin(\omega_0\, t + \varphi_0)\, \delta_{-1}(t)$
  – homogenous ODE:
  $$\ddot{z} + \omega_0^2\, z = 0 \tag{53}$$
  with characteristic polynomial
  $$p_z(s) = s^2 + \omega_0^2 \tag{54}$$
  – exo–system: $\Sigma_z = (\boldsymbol{A}_z, \boldsymbol{C}_z)$ with
  $$\boldsymbol{A}_z = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{55}$$
  Note: an alternative exo–system can be
  $$\boldsymbol{A}_z = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{56}$$
  which consists of a state–space representation of (53) in Jordan canonical form.
- constant + sinusoidal signal: $z(t) = [\, z_1 + z_0 \sin(\omega_0\, t + \varphi_0)\,]\, \delta_{-1}(t)$
  – homogenous ODE:
  $$z^{(3)} + \omega_0^2\, z^{(1)} = 0 \tag{57}$$
  with characteristic polynomial
  $$p_z(s) = s^3 + \omega_0^2\, s \tag{58}$$
  – exo–system: $\Sigma_z = (\boldsymbol{A}_z, \boldsymbol{C}_z)$ with
  $$\boldsymbol{A}_z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_0^2 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{59}$$

Possible alternatives are:

$$\boldsymbol{A}_z \;=\; \begin{bmatrix} 0 & 1 & 0 \\ -\omega_0^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z \;=\; \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{60}$$

$$\boldsymbol{A}_z \;=\; \begin{bmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_z \;=\; \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{61}$$

3.4.2 Robust tracking with the error–space approach

The problem of tracking the reference r and rejecting the disturbance w can be reformulated as a problem of regulating the tracking error e to zero in a robust sense, i.e. even in presence of modelling errors and exogenous perturbations that do not vanish at steady state.
Assume that both r and w satisfy a same homogenous ODE, i.e.

$$\begin{aligned} r^{(m)} + \alpha_{m-1} r^{(m-1)} + \ldots + \alpha_1 r^{(1)} + \alpha_0 r \;&=\; \sum_{i=0}^{m} \alpha_i\, r^{(i)} \;=\; 0 \\ w^{(m)} + \alpha_{m-1} w^{(m-1)} + \ldots + \alpha_1 w^{(1)} + \alpha_0 w \;&=\; \sum_{i=0}^{m} \alpha_i\, w^{(i)} \;=\; 0 \end{aligned} \tag{62}$$

Let $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0)$ be the plant model [1]

$$\Sigma \;:\; \begin{cases} \dot{\boldsymbol{x}} \;=\; \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{B}w \\ y \;=\; \boldsymbol{C}\boldsymbol{x} \end{cases} \tag{63}$$

and define the tracking error as[2]

$$e \;=\; y - r \;=\; \boldsymbol{C}\boldsymbol{x} - r \tag{64}$$

By replacing $r = y - e$ within (62) it follows that

$$\sum_{i=0}^{m} \alpha_i\, r^{(i)} \;=\; -\sum_{i=0}^{m} \alpha_i\, e^{(i)} + \sum_{i=0}^{m} \alpha_i\, \boldsymbol{C}\, \boldsymbol{x}^{(i)} \;=\; -\sum_{i=0}^{m} \alpha_i\, e^{(i)} + \boldsymbol{C} \underbrace{\left(\sum_{i=0}^{m} \alpha_i\, \boldsymbol{x}^{(i)} \right)}_{\triangleq\, \boldsymbol{\xi}} \;=\; 0 \tag{65}$$

In (65), the tracking error dynamics has been reformulated in terms of the new state vector $\boldsymbol{\xi}$, whose evolution is governed by the state–space model

$$\begin{aligned} \dot{\boldsymbol{\xi}} \;=\; \sum_{i=0}^{m} \alpha_i\, \boldsymbol{x}^{(i+1)} \;&=\; \sum_{i=0}^{m} \alpha_i \left(\boldsymbol{A}\boldsymbol{x}^{(i)} + \boldsymbol{B}\boldsymbol{u}^{(i)} + \boldsymbol{B}\boldsymbol{w}^{(i)} \right) \\ &=\; \boldsymbol{A} \underbrace{\left(\sum_{i=0}^{m} \alpha_i\, \boldsymbol{x}^{(i)} \right)}_{=\,\boldsymbol{\xi}} + \boldsymbol{B} \underbrace{\left(\sum_{i=0}^{m} \alpha_i\, u^{(i)} \right)}_{\triangleq\, u_\xi} + \boldsymbol{B} \underbrace{\left(\sum_{i=0}^{m} \alpha_i\, w^{(i)} \right)}_{=\,0} \\ &=\; \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{B}u_\xi \end{aligned} \tag{66}$$

[1] It is assumed here that the disturbance w enters at the plant input; however, the design presented in this section is still valid when the disturbance w adds up in (62) with a matrix $\boldsymbol{B}_w \neq \boldsymbol{B}$.

[2] Note that the tracking error is defined here as $e = y - r$, which is the opposite of the usual definition $e = r - y$.

The combination of (66) with (65) yields the following model $\Sigma_z = (\boldsymbol{A}_z, \boldsymbol{B}_z)$ in *error–space*

$$\Sigma_z \; : \; \left[\begin{array}{c} e^{(1)} \\ e^{(2)} \\ e^{(3)} \\ \vdots \\ e^{(m)} \\ \hline \dot{\boldsymbol{\xi}} \end{array}\right] = \underbrace{\left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & \boldsymbol{0} \\ 0 & 0 & 1 & \cdots & 0 & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \boldsymbol{0} \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{m-1} & \boldsymbol{C} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{A} \end{array}\right]}_{\triangleq \boldsymbol{A}_z} \underbrace{\left[\begin{array}{c} e \\ e^{(1)} \\ e^{(2)} \\ \vdots \\ e^{(m-1)} \\ \hline \boldsymbol{\xi} \end{array}\right]}_{\triangleq \boldsymbol{z}} + \underbrace{\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline \boldsymbol{B} \end{array}\right]}_{\triangleq \boldsymbol{B}_z} u_\xi \tag{67}$$

Note that (67) is independent on both r and w. Therefore, if Σ_z is asymptotically stable, then the state $\boldsymbol{z}$, and hence the tracking error e, converges to zero for $t \to +\infty$, regardless of the initial conditions used in (62) to generate both r and w. If Σ_z is reachable, then it is possible to design a state feedback control law (from the state $\boldsymbol{z}$) that places the closed–loop eigenvalues to any desired location in the complex plane, and in particular on the open left half–plane, so that the closed–loop system becomes asymptotically stable. It is possible to show that Σ_z is reachable if and only if the original plant model Σ is reachable, and the zeros of the characteristic polynomial associated with the ODEs (62) are not zeros of the transfer function of Σ. Let

$$u_\xi \; = \; -\boldsymbol{K}_z \, \boldsymbol{z} \; = \; - \left[\begin{array}{cccc|c} k_0 & k_1 & \cdots & k_{m-1} & \boldsymbol{K}_\xi \end{array}\right] \left[\begin{array}{c} e \\ e^{(1)} \\ \vdots \\ e^{(m-1)} \\ \hline \boldsymbol{\xi} \end{array}\right] \tag{68}$$

be a stabilising control law for Σ_z. By using the definitions of $\boldsymbol{\xi}$ and u_ξ, the control law (68) can be rewritten in terms of the original plant state $\boldsymbol{x}$ and control input u:

$$\begin{aligned} \sum_{i=0}^{m} \alpha_i u^{(i)} \; &= \; - \sum_{i=0}^{m-1} k_i \, e^{(i)} \; - \; \sum_{i=0}^{m} \alpha_i \, \boldsymbol{K}_\xi \, \boldsymbol{x}^{(i)} \\ \Rightarrow \qquad \sum_{i=0}^{m} \alpha_i \, \underbrace{(u + \boldsymbol{K}_\xi \, \boldsymbol{x})}_{\triangleq \tilde{u}}{}^{(i)} \; &= \; - \sum_{i=0}^{m-1} k_i \, e^{(i)} \end{aligned} \tag{69}$$

The equation (69) defines a compensator with input e and output $\tilde{u}$, whose transfer function is

$$\tilde{U}(s) \; = \; - \frac{\sum_{i=0}^{m-1} k_i \, s^i}{\sum_{i=0}^{m} \alpha_i \, s^i} E(s) \; = \; - \underbrace{\frac{k_{m-1} \, s^{m-1} \, + \, \cdots \, + \, k_1 \, s \, + \, k_0}{s^m \, + \, \alpha_{m-1} \, s^{m-1} \, + \, \cdots \, + \, \alpha_1 \, s \, + \, \alpha_0}}_{H(s)} E(s) \tag{70}$$

With the compensator defined above, the control law in the original variables $\boldsymbol{x}$ and u can be written as follows (with a slightly abuse of notation)

$$u \; = \; -\boldsymbol{K}_\xi \, \boldsymbol{x} \; - \; H(s) \, e \tag{71}$$

The final control scheme based on (71) is shown in Fig. 7. Note that $H(s)$ embeds in its denominator the models of the signal to track and the disturbance to reject (internal model principle).

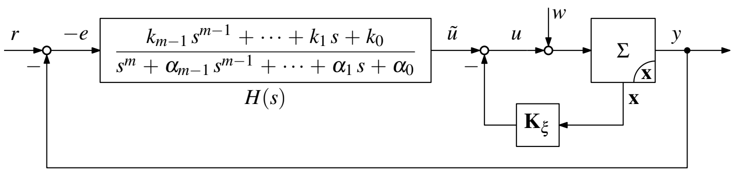


Figure 7: Robust tracking with the error space approach.

3.4.3 Robust tracking with the extended estimator approach

The *extended estimator* approach basically consists of estimating both the reference input and the load disturbance (i.e. disturbance entering at the plant input) with a state estimator, and then using the estimates to perform a reference plus disturbance feedforward compensation. The residual tracking error is finally driven to zero by designing a conventional state regulator.

Consider the connection of Fig. 8a: as far as the steady–state output is concerned, and the eigenvalues of $\boldsymbol{A}_r$ are not zeros of the transfer function of the plant model $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0)$, then it is possible to move the reference signal r to the plant input and combined with the disturbance w to yield an "input–equivalent" exogenous disturbance ρ, generated by the new exo–system

$$\Sigma_\rho \ : \ \begin{cases} \dot{\boldsymbol{x}}_\rho \ = \ \boldsymbol{A}_\rho \, \boldsymbol{x}_\rho \\ \ \rho \ = \ \boldsymbol{C}_\rho \, \boldsymbol{x}_\rho \end{cases} \tag{72}$$

The resulting connection is shown in Fig. 8b. Note that the system Σ_ρ contains all the system modes (without repetitions) of both Σ_r and Σ_w. By combining the original plant model Σ with the new exo–system Σ_ρ, the following extended plant model results

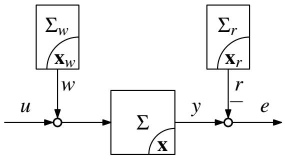
$$\Sigma_e \ : \ \begin{cases} \begin{bmatrix} \dot{\boldsymbol{x}}_\rho \\ \dot{\boldsymbol{x}}' \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}_\rho & \boldsymbol{0} \\ \boldsymbol{B}\boldsymbol{C}_\rho & \boldsymbol{A} \end{bmatrix}}_{\triangleq \boldsymbol{A}_e} \underbrace{\begin{bmatrix} \boldsymbol{x}_\rho \\ \boldsymbol{x}' \end{bmatrix}}_{\triangleq \boldsymbol{x}_e} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B} \end{bmatrix}}_{\triangleq \boldsymbol{B}_e} u & (73) \\[2ex] e = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{C} \end{bmatrix}}_{\triangleq \boldsymbol{C}_e} \begin{bmatrix} \boldsymbol{x}_\rho \\ \boldsymbol{x}' \end{bmatrix} & (74) \end{cases}$$

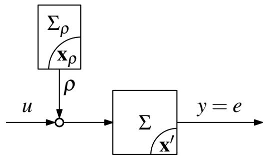
It can be proved[3] that if the two pairs if the two pairs $(\boldsymbol{A}_\rho, \boldsymbol{C}_\rho)$ and $(\boldsymbol{A}, \boldsymbol{C})$ are observable and the polynomials:

$$\boldsymbol{C}\,\mathrm{adj}\,(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} \qquad \text{and} \qquad \det(s\boldsymbol{I} - \boldsymbol{A}_\rho)$$

have no common roots (i.e. the eigenvalues of Σ_ρ are not zeros of the plant transfer function), then the extended plant $\Sigma_e = (\boldsymbol{A}_e, \boldsymbol{B}_e, \boldsymbol{C}_e, \boldsymbol{0})$ is observable. If so, it is possible to design an extended state estimator $\hat{\Sigma}_e$ for the extended plant state $\boldsymbol{x}_e = [\boldsymbol{x}_\rho, \, \boldsymbol{x}']^T$. A *full–order* state estimator is required, since both the state components $\boldsymbol{x}'$ and $\boldsymbol{x}_\rho$ are inaccessible. In fact, it is worth to notice here that the evolution of the state $\boldsymbol{x}'$ in the model Σ of Fig. 8b is usually different from that of state $\boldsymbol{x}$ in the model Σ of Fig. 8a, so that the full knowledge of the state $\boldsymbol{x}$ is irrelevant for the design of the extended estimator (and, in particular, it does not allow to implement a reduced–order

[3]Sufficient condition for the observability of the series connection of two SISO state models (e.g. cascade connection of Σ after Σ_ρ)





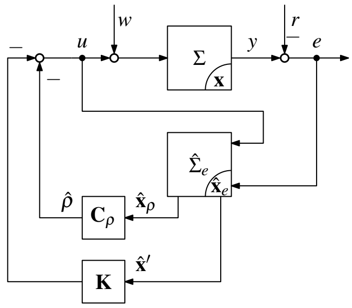


Figure 8: Robust tracking with the extended estimator approach: (a) original plant model with exo–systems for the generation of the reference and disturbance inputs; (b) modified plant model with equivalent input disturbance; (c) robust tracking controller implementation, based on the extended estimator approach.

observer). The full–order state estimator is

$$\hat{\Sigma}_e : \quad \dot{\hat{\boldsymbol{x}}}_e \;=\; \boldsymbol{A}_e \, \hat{\boldsymbol{x}}_e + \boldsymbol{B}_e \, u + \boldsymbol{L}_e \, (e - \boldsymbol{C}_e \, \hat{\boldsymbol{x}}_e) \tag{75}$$

$$= \; (\boldsymbol{A}_e - \boldsymbol{L}_e \boldsymbol{C}_e) \, \hat{\boldsymbol{x}}_e + \boldsymbol{B}_e \, u + \boldsymbol{L}_e \, e \tag{76}$$

where $\boldsymbol{L}_e$ is the estimator gain matrix. With the extended state estimate $\hat{\boldsymbol{x}}_e = [\hat{\boldsymbol{x}}_\rho, \, \hat{\boldsymbol{x}}']^T$ it is possible to implement the following control law

$$u \;=\; -\boldsymbol{C}_\rho \hat{\boldsymbol{x}}_\rho - \boldsymbol{K} \hat{\boldsymbol{x}}' \tag{77}$$

The first term on the right hand side of (77) performs a feedforward cancellation of the the equivalent input disturbance $\rho = \boldsymbol{C}_\rho \boldsymbol{x}_\rho$; once the disturbance has been cancelled, the second term in (77) regulates the plant state $\boldsymbol{x}'$ to zero. In practice, the robust tracking problem is solved by resorting to a more conventional state regulation problem, provided that the equivalent input disturbance is perfectly cancelled with a proper feedforward compensation. The final tracking control scheme is shown in Fig. 8c.