MODELLING AND SIMULATION

Lesson 4- SS 2014 – Michel Kana

What do we do in today's practice?

1. Summary of the previous practice
2. Population model with age structure
3. Epidemiology models
4. Summary

Summary of the previous practice

[Population models]
Two species populations models of predator – prey
Two species populations models with competition
Two species populations models with mutualism

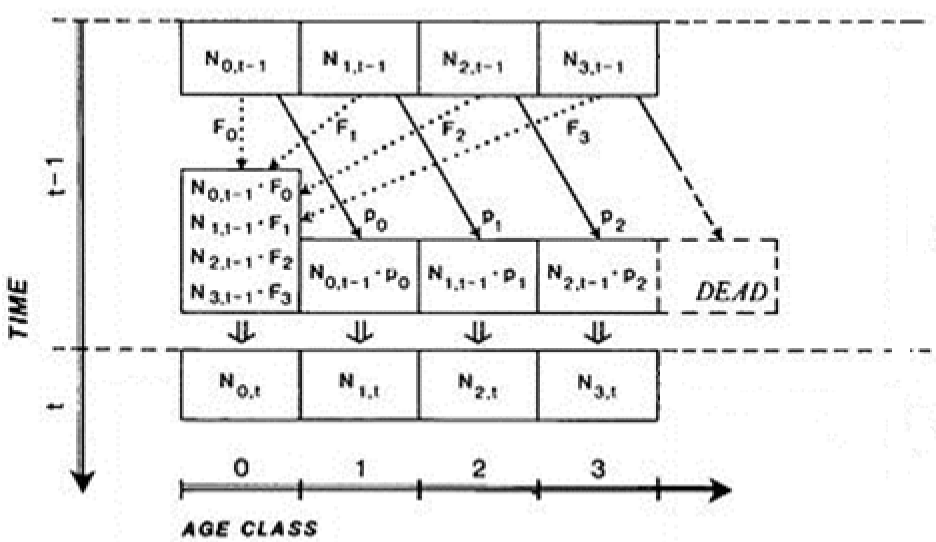
Population models with age structure

- A single species is not only modeled as one population, but instead as a structured population
  - demographic differences among individuals as a function of stages or classed
  - stage or class can be age, size, etc.
- Stages can be discrete or continuous function of time
  - Discrete if experimental data is collected at regular time intervals, e.g. yearly or seasonal
  - Continuous if experimental data is collected continuously over time
- Examples
  - partition an insect population into egg, larva , pupa, and adult stages
  - partition a mammal population into age classes

Population model with discrete age structure

- We assume that
  - the population is a group of individuals of a single species.
  - the population is closed
  - The population is divided in $n + 1$ age groups
  - reproduction and survival depend highly upon age
  - all births happen "at once" in each time step.
- $N_{i,t}$ represents the number of individuals in the age group i at time t.
  - $N_{0,t}$ represents the number of offspring at time t.
  - $N_{n,t}$ represents the number of individuals in the oldest age group at time t
- F_i represents fertility, i.e. the birth rate per capita in age group i
- p_i represents mortality, i.e. the survival rate per capita in age group i.

Population model with discrete age structure



Source: *http://mathbio.colorado.edu*

$$\begin{cases} N_{0,t} = F_0 \bullet N_{0,t-1} + F_1 \bullet N_{1,t-1} + F_2 \bullet N_{2,t-1} + F_3 \bullet N_{3,t-1} \\ N_{1,t} = p_0 . N_{0,t-1} \\ N_{2,t} = p_1 . N_{1,t-1} \\ N_{3,t} = p_2 . N_{2,t-1} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} N_{0,t} = F_0 \bullet N_{0,t-1} + F_1 \bullet N_{1,t-1} + F_2 \bullet N_{2,t-1} + F_3 \bullet N_{3,t-1} \\ N_{1,t} = p_0 \bullet N_{0,t-1} + 0 \bullet N_{1,t-1} + 0 \bullet N_{2,t-1} + 0 \bullet N_{3,t-1} \\ N_{2,t} = 0 \bullet N_{0,t-1} + p_1 . N_{1,t-1} + 0 \bullet N_{2,t-1} + 0 \bullet N_{3,t-1} \\ N_{3,t} = 0 \bullet N_{0,t-1} + 0 . N_{1,t-1} + p_2 . N_{2,t-1} + 0 \bullet N_{3,t-1} \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} N_{0,t} \\ N_{1,t} \\ N_{2,t} \\ N_{3,t} \end{bmatrix} = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 \\ p_0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \end{bmatrix} \bullet \begin{bmatrix} N_{0,t-1} \\ N_{1,t-1} \\ N_{2,t-1} \\ N_{3,t-1} \end{bmatrix}$$

$$\Downarrow$$

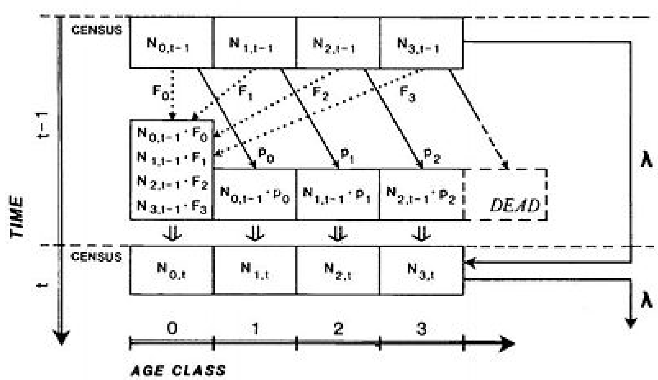
$$N_t = A \cdot N_{t-1}$$

Population models with discrete age structure

- The **Leslie population model**
- A is the Leslie matrix
- Eigenvalues of A, noted λ represents the asymptotic growth rate of the population in the a stable age distribution v
  - $\mathrm{A} \cdot v = \lambda \cdot v$
- As soon as a stable age distribution is reached, the population will undergo exponential growth with rate λ.

$$\begin{bmatrix} N_{0,t+1} = \sum F_i.x_{i,t} \\ N_{1,t+1} = p_0.x_{0,t} \\ N_{2,t+1} = p_1.x_{1,t} \\ N_{3,t+1} = p_2.x_{2,t} \\ \vdots \\ N_{n,t+1} = p_{n-1}.x_{n-1,t} \end{bmatrix} = \begin{bmatrix} F_0 & F_1 & F_2 & \cdots & F_{n-1} & F_n \\ p_0 & 0 & 0 & & 0 & 0 \\ 0 & p_1 & 0 & & 0 & 0 \\ 0 & 0 & p_2 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix} . \begin{bmatrix} N_{0,t} \\ N_{1,t} \\ N_{2,t} \\ N_{3,t} \\ \vdots \\ N_{n,t} \end{bmatrix}$$

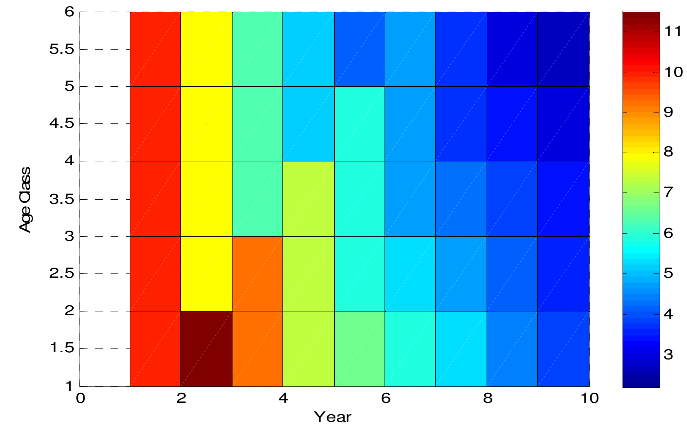
$$N_t = A^t \cdot N_0$$

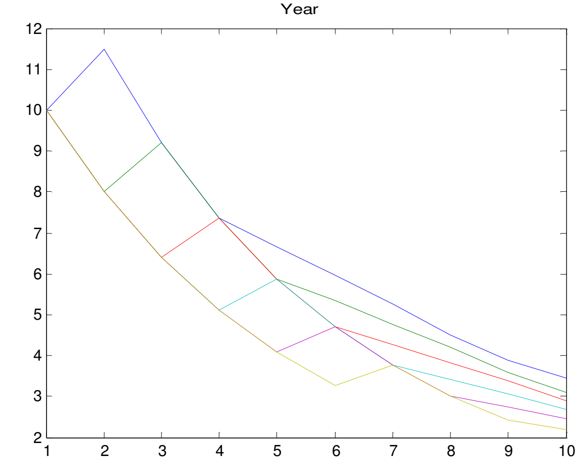


Population models with discrete age structure

- Consider a structured population with 6 age groups.
- 10 individuals in each age group at time 0.
- The age group 0 and 1 are not fertile.
- In the age group 2-4, there are 0.35% of offspring per individual at each time step.
- In the age group 5, there are 0.10% of offspring per individual at each time step.
- In all age groups (except of group 5), 80% of individuals survive with each time step.

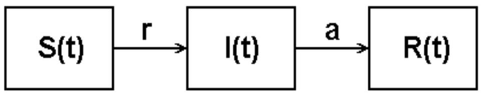
$$A = \begin{bmatrix} 0 & 0 & 0.35 & 0.35 & 0.35 & 0.10 \\ 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \end{bmatrix} \qquad N_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$





Epidemiology models - SIR

- We assume to have an epidemic with the following characteristics
  - The disease spreads through contact or close proximity between infected and healthy individuals
  - The probability for two individuals to come to contact is the same within the population
  - There is no incubation period and the disease becomes effective immediately after contact
  - The population is closed with constant size (no births neither deaths)
- SIR is a simple model for many infectious diseases, including measles , mumps and rubella
  - $S(t)$ represents the number of individuals susceptible to infection
  - $I(t)$ represents the number of infected individuals, i.e. those who show signs of illness and spreads disease further.
  - $R(t)$ represents the number of removed individuals, i.e. those in a period of isolation or resistant individuals who were previously infected and have recovered with immunity.
  - r represents the average spreading rate of infection, i.e. the adequate number of contacts sufficient for the transmission of infection between individuals.
  - a represents the removal rate, the speed of isolation or treatment of infected individuals.

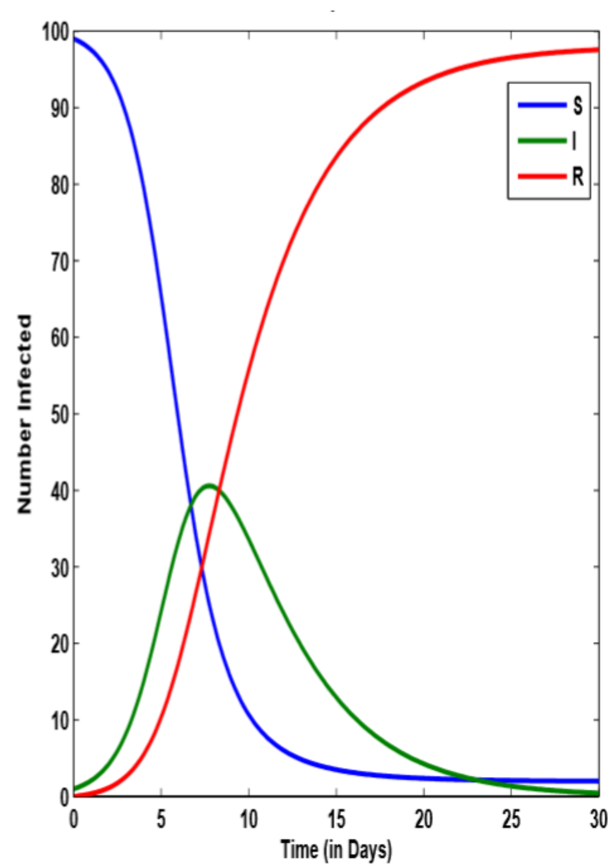


Epidemiology models - SIR

- N is the total number of individuals in the population.
- $\frac{I(t)}{N}$ represents the proportion of infected individuals in the population.
- $\frac{r \cdot I(t)}{N}$ represents the rate infected individual gives rise to new infections.
- $\frac{r \cdot I(t)}{N} \cdot S(t)$ represents the rate at which susceptible individuals encounter infected individuals and become infected.
- $a \cdot I(t)$ is the rate at which infected individuals are removed from the infective class

$$\boxed{S(t)} \xrightarrow{r} \boxed{I(t)} \xrightarrow{a} \boxed{R(t)}$$

$$\frac{dS\,(t)}{dt} = -r \cdot S(t) \cdot I(t)$$
$$\frac{dI\,(t)}{dt} = r \cdot S(t) \cdot I(t) - a \cdot I(t)$$
$$\frac{dR\,(t)}{dt} = a \cdot I(t)$$
$$S(t) + I(t) + R\,(t) = N$$



Epidemiology models - SIR

$$\frac{dS(t)}{dt} = -r \cdot S(t) \cdot I(t)$$
$$\frac{dI(t)}{dt} = r \cdot S(t) \cdot I(t) - a \cdot I(t)$$
$$\frac{dR(t)}{dt} = a \cdot I(t)$$
$$S(t) + I(t) + R(t) = N$$

- Equilibrium occurs
  - Before disease begins spreading $S(0) = N$ and $R(0) = 0$
  - Disease begins to spread if $\frac{dI(t)}{dt} > 0$
    - $r \cdot S(0) \cdot I(t) - a \cdot I(t) > 0$
    - $(r \cdot S(0) \cdot -a) \cdot I(t) > 0$
    - $r \cdot S(0) \cdot -a > 0$
    - $\frac{r}{a} \cdot S(0) > 1$
      - $\frac{r}{a} \cdot S(0)$ is the basic contact number
      - $\frac{r}{a} \cdot S(0) > 1$: infection will be established in the population. Infection peaks and then disappears.
      - $\frac{r}{a} \cdot S(0) < 1$: the infection dies out and there is no epidemic.
  - After disease has moved through the entire population $S = 0$ and $R = N$

Summary of today's lesson

[Population models]
Models of structured populations
Epidemiology models

[What is next?]
1-compartment model.