MODELLING AND SIMULATION

Lesson 3- SS 2014 – Michel Kana

What do we do in today's practice?

1. Summary of the previous practice
2. Two species populations models of predator - prey
3. Two species populations models with competition
4. Two species populations models with cooperation
5. Epidemiology models
6. Summary

Summary of the previous practice

[Population models]
Population models with age structure
Two species populations models of predator - prey: *Lotka – Volterra*

Interacting populations

- Interspecific interactions are those interactions of population of two different species
- Predation
  - Individuals of one species called Predator kill and consume individuals of the other species called Prey
- Parasitism
  - Individuals of one species depend and benefit on individuals of the other species (Host) for food and shelter, while damaging them
- Competition
  - individuals of different species compete for the same resources that are limited
- Commensalism
  - Individuals of one species depend and benefit on individuals of the other species who are neither harmed nor benefitted
- Mutualism
  - Both species are benefitted

Two species populations models with competition

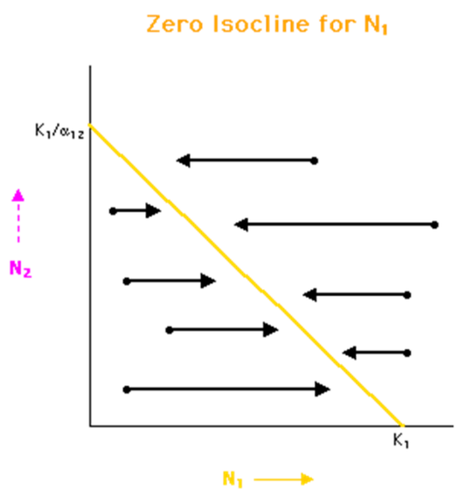
- $X_1(t)$ is the total population size for species 1 with growth rate ρ_1 and carrying capacity K_1
  - Assuming species 1 to be a closed population
    - $\frac{dX_1(t)}{dt} = \rho_1 \cdot \left(1 - \frac{X_1(t)}{K_1}\right) \cdot X_1(t)$
- $X_2(t)$ is the total population size for species 1 wit growth rate ρ_2 and carrying capacity K_2
  - Assuming species 2 to be a closed population
    - $\frac{dX_2(t)}{dt} = \rho_2 \cdot \left(1 - \frac{X_2(t)}{K_2} - b_{21} \cdot \frac{X_1(t)}{K_2}\right) \cdot X_2(t)$
- Assuming that both populations suffer from mutual contact: **Lotka-Volterra competition model**
  - the effect of species 2 on the growth rate of species 1 is proportional to the density of the species 2
    - $\frac{dX_1(t)}{dt} = \rho_1 \cdot \left(1 - \frac{X_1(t)}{K_1} - b_{12} \cdot \frac{X_2(t)}{K_1}\right) \cdot X_1(t)$
    - b_{12} represents the mutual competitive impact of one individual of species 2 on the growth rate of species 1.
  - the effect of species 1 on the growth rate of species 2 is proportional to the density of the species 1
    - $\frac{dX_2(t)}{dt} = \rho_2 \cdot \left(1 - \frac{X_2(t)}{K_2} - b_{21} \cdot \frac{X_1(t)}{K_2}\right) \cdot X_2(t)$
    - b_{21} represents the mutual competitive impact of one individual of species 1 on the growth rate of species 2.

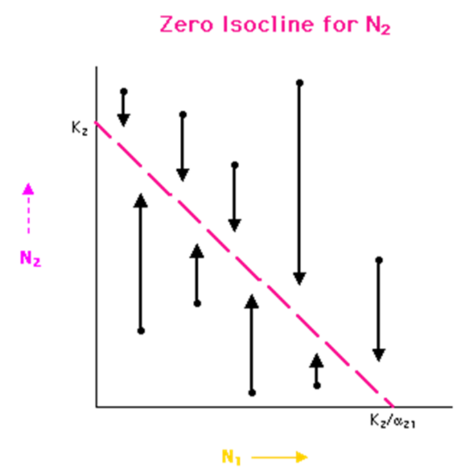
Lotka-Volterra competition model - analysis

- The Lotka-Volterra model
  - $\frac{dX_1(t)}{dt} = \rho_1 \cdot \left(1 - \frac{X_1(t)}{K_1} - b_{12} \cdot \frac{X_2(t)}{K_1}\right) \cdot X_1(t)$
  - $\frac{dX_2(t)}{dt} = \rho_2 \cdot \left(1 - \frac{X_2(t)}{K_2} - b_{21} \cdot \frac{X_1(t)}{K_2}\right) \cdot X_2(t)$
- Phase plane is a visual display of certain characteristics of differential equations
  - The X_1-nullclines is a set of points in the (X_1, X_2) plane where $\frac{dX_1(t)}{dt} = 0$
    - These are $(0, \frac{K_1}{b_{12}})$ and $(K_1, 0)$
  - The X_2-nullclines is a set of points in the (X_1, X_2) plane where $\frac{dX_2(t)}{dt} = 0$
    - These are $(0, K_2)$ and $(\frac{K_2}{b_{21}}, 0)$

Lotka-Volterra competition model - analysis

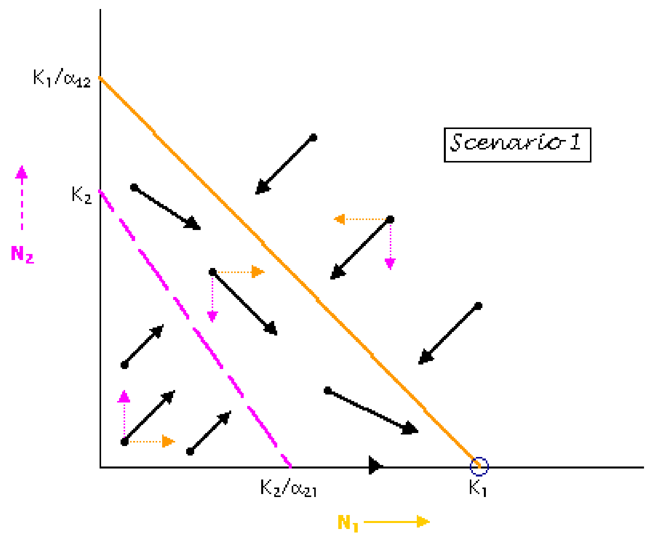
- Below or to the left of the nullcline, the population increases because its size is lower than the carrying capacity
- Above and to the right of the nullcline, the population decreases because its size is greater than the carrying capacity
- For species 1, the nullcline intersects the x-axis when X_1 reaches its carrying capacity K_1 and no individuals of species 2 are present
- For species 1, the nullcline intersects the y-axis when when the carrying capacity of species 1 is filled by the equivalent number of individuals of species 2 and no individuals of species 1 are present





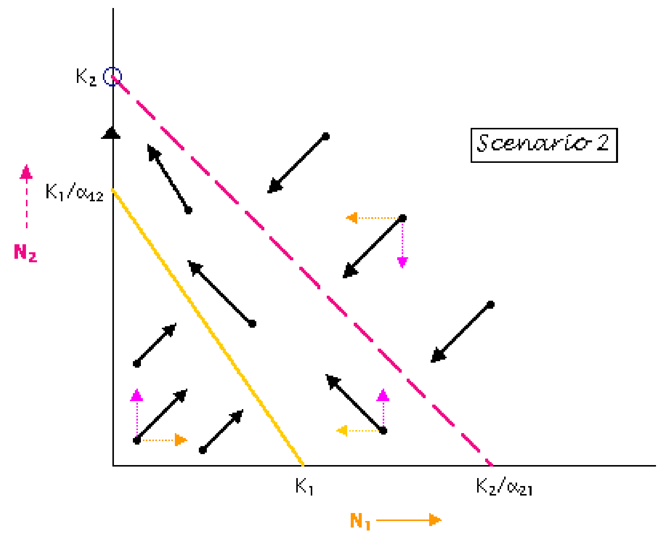
Source: http://www.tiem.utk.edu/~gross/bioed/bealsmodules/competition.html

Lotka-Volterra competition model - analysis



species 2 becomes extinct and species 1 increases until it reaches carrying capacity K_1

$\frac{K_2}{b_{21}} < K_1$ and $\frac{K_1}{b_{12}} > K_2$

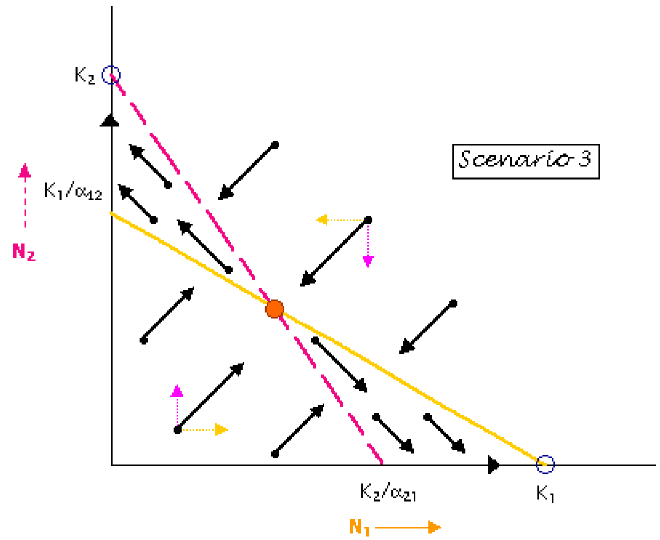


species 1 becomes extinct and species 2 increases until it reaches carrying capacity $K_1 K_2$

$\frac{K_2}{b_{21}} > K_1$ and $\frac{K_1}{b_{12}} < K_2$

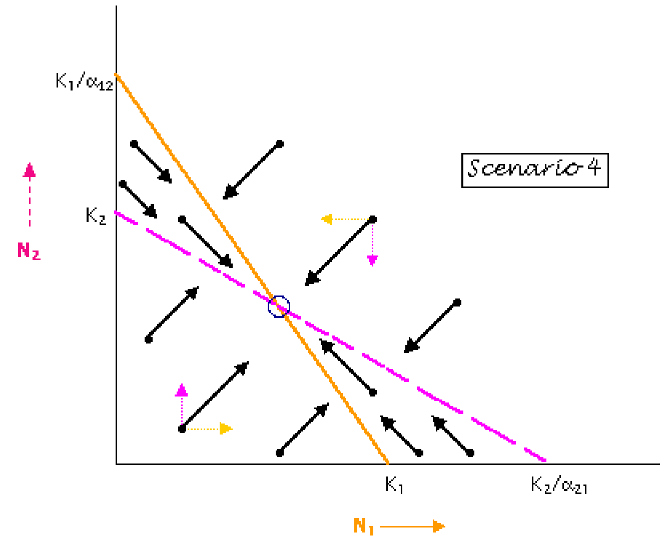
Source: http://www.tiem.utk.edu/~gross/bioed/bealsmodules/competition.html

Lotka-Volterra competition model - analysis



the outcome depends on the initial number of individuals of the two species

$\frac{K_2}{b_{21}} < K_1$ and $\frac{K_1}{b_{12}} < K_2$



Coexist of both species

$\frac{K_2}{b_{21}} > K_1$ and $\frac{K_1}{b_{12}} > K_2$

Source: http://www.tiem.utk.edu/~gross/bioed/bealsmodules/competition.html

Two species populations models with mutualism

- Mutually beneficial interaction of two different populations..
- $X_1(t)$ represents the number of individuals in the first population.
- $X_2(t)$ represents the number of individuals in the second population.
- ρ_1 represents the relative fertility of the first population.
- ρ_2 represents the relative fertility of the second population.
- K_1 is the capacity of the environment first population.
- K_2 is the capacity of the environment second population.
- b_{12} represent the mutually beneficial effect of the first population on the second.
- b_{21} represent the mutually beneficial effect of the second population on the first.

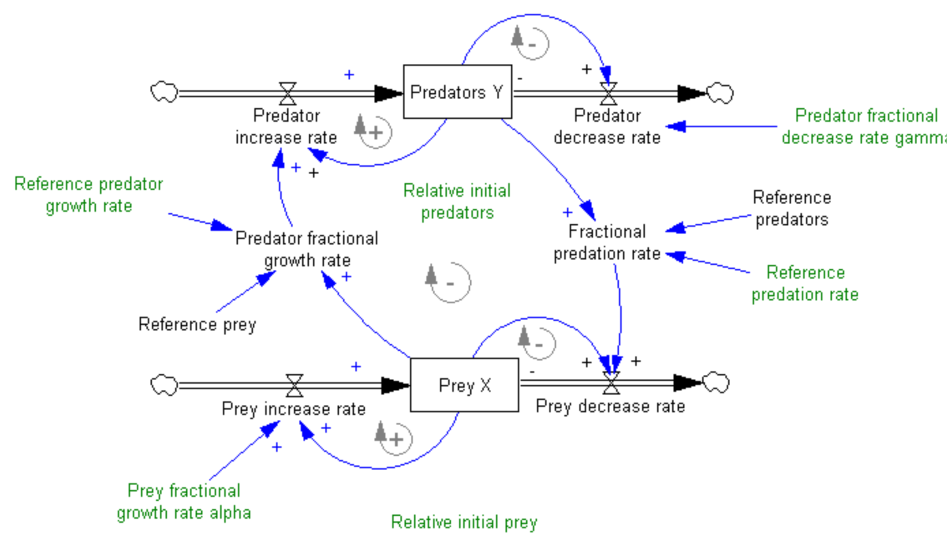
$$\frac{dX_1\ (t)}{dt} = \rho_1 \cdot \left(1 - \frac{X_1(t)}{K_1} + b_{12} \cdot \frac{X_2(t)}{K_1}\right) \cdot X_1(t)$$

$$\frac{dX_2\ (t)}{dt} = \rho_2 \cdot \left(1 - \frac{X_2(t)}{K_2} + b_{21} \cdot \frac{X_1(t)}{K_2}\right) \cdot X_2(t)$$

Two species populations models of predator - prey

- One population prospers, the other doesn´t prosper.
- $X(t)$ represents the number of prey in time t.
- $Y(t)$ represents the number of predators in time t.
- k_1 represents the relative fertility prey.
- $k_1 \cdot X(t)$ represents the number of prey that were born during the time interval $\langle t-1 \cdots t\rangle$.
- k_2 represents the probability that predator will kill prey when prey and predator are meeting.
- $k_2 \cdot X(t) \cdot Y(t)$ represents the number of prey caught by predators during the time interval $\langle t-1 \cdots t\rangle$.
- k_3 represents the conversion efficiency of the biomass of prey to predator biomass.
- $k_3 \cdot k_2 \cdot X(t) \cdot Y(t)$ represents the number of births of predators during the time interval $\langle t-1 \cdots t\rangle$.
- k_4 represents the relative mortality of predators.
- $k_4 \cdot Y(t)$ represents the decrease in the population of predators during the time interval $\langle t-1 \cdots t\rangle$.

$$\frac{dX(t)}{dt} = k_1 \cdot X(t) - k_2 \cdot X(t) \cdot Y(t)$$
$$\frac{dY(t)}{dt} = k_3 \cdot k_2 \cdot X(t) \cdot Y(t) - k_4 \cdot Y(t)$$



Two species populations models of predator - prey with delay

- The population of the prey evolves according to the logistic equation
  - ρ_1 represents fertility prey
  - K_1 represents the capacity of the environment prey
  - τ_1 represents the mean time to achieve fertility for prey
  - $\rho_1 \cdot \tau_1 > \frac{\pi}{2}$ enables the creation oscillations
- The increase in the population of predators is defined by $\frac{\rho_2}{K_1} \cdot X(t)$
  - $\frac{\rho_2}{K_1}$ represents the effect of the interaction and conversion of biomass
  - τ_2 represents the mean time to reach reproductive predators
  - $\rho_2 \cdot \tau_2 > \frac{\pi}{2}$ enables the creation oscillations
- The decrease of the population of predators is defined by $\frac{\rho_2}{K_2} \cdot Y(t - \tau_2)$.
  - K_2 represents the capacity of environmental predators

$$\frac{dX(t)}{dt} = \rho_1 \cdot \left(1 - \frac{X(t-\tau_1)}{K_1}\right) \cdot X(t)$$

$$\frac{dY(t)}{dt} = \rho_2 \cdot \left(\frac{X(t)}{K_1} - \frac{Y(t-\tau_2)}{K_2}\right) \cdot Y(t)$$

Kolmogorov models of predator - prey

- Model Lotka - Volterra is not realistic.
  - populations of predators and prey cycles endlessly without stabilization.
  - population of prey in the absence of predator grow exponentially.
- Function A represents the relative rate of reproduction prey population by logistic equation..
  - ρ is the birth rate of the population of prey
  - K_1 is the capacity of the environment prey population
- Function V determines the amount of prey that predator catch per unit of time depending on the condition of the prey population.
  - p*is the maximum increase predator.*
  - a is the amount of prey that is needed to ability of reproduction the predator with speed $\frac{p}{2}$.
  - c is the coefficient of conversion of biomass $\in (0; 1)$.
- Function K indicates the total population growth of predators, which is negative for low levels of prey that is not enough predators to feed.
  - e and m are positive constant

$$\frac{dX(t)}{dt} = A \cdot X(t) - \mathrm{V} \cdot Y(t)$$
$$\frac{dY(t)}{dt} = K \cdot Y(t)$$
$$A = \rho \cdot \left(1 - \frac{X(t)}{K_1}\right)$$
$$V = \frac{p \cdot X(t)}{c \cdot \left(a + X(t)\right)}$$
$$K = e \cdot V - m$$

Equation of Kolmogorov model

Summary of today's practice

[Population models]
Two species populations models of predator – prey: *Lotka– Volterra* with delay, Kolmogorov model
Two species populations models with competition
Two species populations models with cooperation
Epidemiology models.

[What is next?]
Next week we will continue with **compartment models.**