MODELLING AND SIMULATION

Lecture 2 - SS 2014 – Michel Kana

Agenda

1. **Corrections of entrance test**
2. **Summary of the previous exercise**
3. **Malthus population model**
4. **Logistic population model**
5. **Summary**

Correction of the entrance test

Summary of the previous practice

[Foundations of Mathematics]
Coefficients of polynomial, order of polynomial, roots of polynomial
Equation with one variable, linear system of equations, non - linear system of equations
Square schema of numbers, row and column
Linear differential equation and system of equations

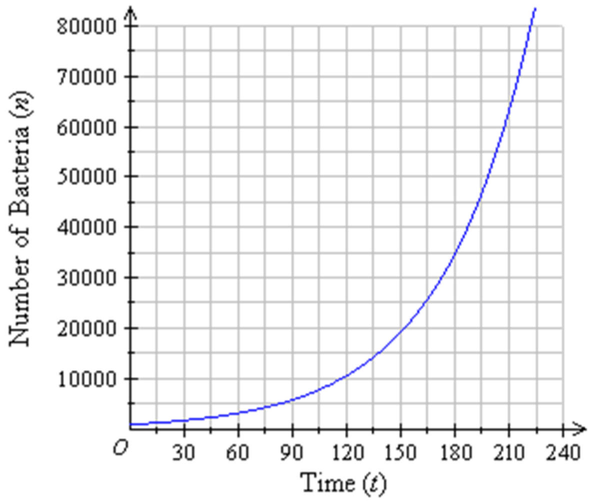
[Modeling guidelines]
Defining the Problem, Designing the Experiment, Describing the Model Mathematically, Performing Identifiability Analysis, Implementing the Model, Performing Parameters Estimation, Performing Sensitivity Analysis, Validating the Model

Population modeling

- **Definitions**
  - ***Population*** is group of individuals of any species that live in a well defined geographical area, share or compete for similar resources
  - ***Population size*** is the total number of individuals in the population at a given time
  - ***Birth Rate*** is the average number of new individuals added per unit population due to births, hatchings and germinations.
  - ***Death Rate*** is the average number of natural deaths per unit population per unit time.
  - A population at any given time is composed of individuals of different age/***age group***
  - Age groups are: pre-reproductive, reproductive and post- reproductive
  - ***Population growth*** depends on factors such as food, weather, predators, competition
  - ***Population Density*** indicates the number of individual or the size of population found in a unit area or space at a given time
- **Basic problem**
  - identify classes of population interactions that could lead to the observed type of dynamics
  - identify unique predictions of each competing hypothesis that can be tested using observational data
  - understanding patterns in cross-species comparisons
  - managing fisheries/hunting/harvesting for the highest possible sustainable yield
  - developing recovery plans for species threatened by extinction
  - trying to contain or prevent the spread of invasive species
- **Approach**
  - Model the change in population size due to the interaction of organisms with the environment, with individuals of their own kind, and with other types of organisms

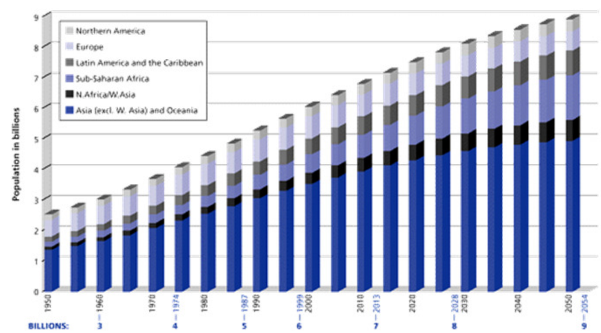
Malthus model

- We assume the population to be ***unstructured***, i.e. we ignore differences between individuals, and assume that the population size – irrespective of age, sex, diseases, etc. – provides all the necessary information for predicting future population changes
- ***Fundamental Balance Law*** for total population size $X(t)$
  - $X(t+h) = X(t) + Births + Immigration - Deaths - Emigration$
  - $X(t+h) = X(t) + B + I - D - E$
- We assume ***constant birth and death rates***
  - # Births = (# parents) × (births/parent/time) × (length of time interval)
    - $B = X(t) \cdot b \cdot h$
  - # Deaths= (# parents) × (deaths/parent/time) × (length of time interval)
    - $D = X(t) \cdot d \cdot h$
- We assume a ***closed*** population, i.e. there is no immigration and no emigration
  - # Immigration = # Emigration = 0
    - $I = E = 0$
    - $X(t+h) = X(t) + X(t) \cdot b \cdot h - X(t) \cdot d \cdot h$
    - $\frac{X(t+h)-X(t)}{h} = (b-d) \cdot X(t)$
    - $\frac{X(t+h)-X(t)}{h} = r \cdot X(t)$
- Assuming $h \to 0$, we obtain the ***Malthus population model***
  - $\frac{dX}{dt} = r \cdot X(t)$
  - $X(t) = X(0) \cdot e^{r \cdot t}$



Malthus model - Analysis

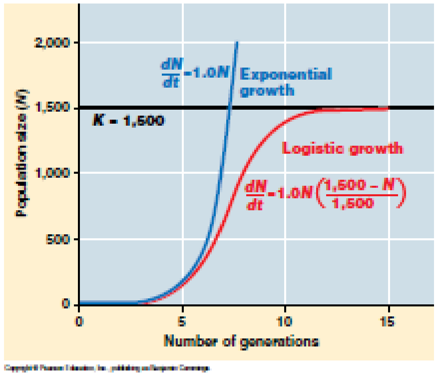
- ***Malthus population model*** $X(t) = X(0) \cdot e^{r \cdot t} = X(0) \cdot e^{(b-d) \cdot t}$
  - if $r > 0$ (i.e. if $b > d$) the population grows exponentially without limit
    - The ***doubling time*** is defined as the time it takes to double the population size
      - $2 \cdot X(0) = X(0) \cdot e^{r \cdot t}$
      - $t = \frac{\ln 2}{r}$
  - if $r < 0$ (i.e. if $b < d$) the population decreases to 0.
    - The ***half life*** is defined as the time it takes to loose half of the population size
      - $\frac{X(0)}{2} = X(0) \cdot e^{-r \cdot t}$
      - $t = \frac{\ln 2}{r}$
  - The ***expected life-span*** is $\frac{1}{d}$ time units.
  - The ***fitness*** $R_0 = \frac{b}{d}$ is the expected number of offspring of an individual over his entire life-span



- Malthus (1798) found that the local population had a doubling time of 30 years
  - $30 = \frac{\ln 2}{r}$
  - $r = 0.0231$ per year, i.e. 2.31% human growth rate per year

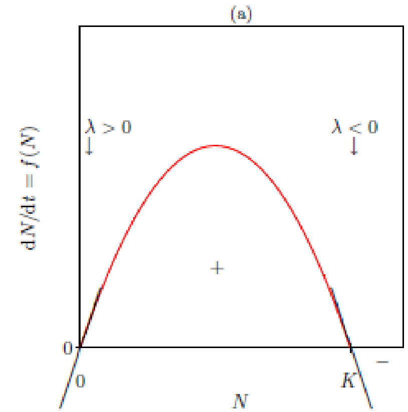
Logistic model

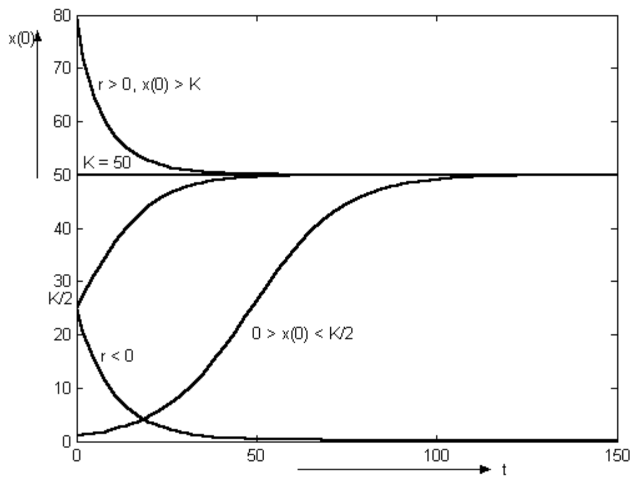
- Malthus model is not density-dependent
  - Birth and death rates are typically not fixed because the processes of birth and death often depend on the population size.
  - Due to competition at high population densities birth rates may become lower and death rates higher when the population size increases
- Density-dependent death
  - Assuming a simple linear increase of the per capita death rate with the population size: $f(X) = 1 + \frac{X}{K}$
    - $\frac{dX}{dt} = \left[b - d \cdot (1 + \frac{X}{K})\right] \cdot X(t)$
    - The death rate doubles when $X = K$
- Density-dependent birth
  - Assuming a simple linear decrease of the per capita birth rate with the population size: $f(X) = 1 + \frac{X}{K}$
    - $\frac{dX}{dt} = \left[b \cdot (1 - \frac{X}{K}) - d\right] \cdot X(t)$
    - The birth rate becomes 0 when $X = K$
- The ***logistic population model***
  - $\frac{dX}{dt} = \left[b \cdot (1 - \frac{X}{K}) - d \cdot (1 + \frac{X}{K})\right] \cdot X(t)$
  - $\frac{dX}{dt} = (b - d) \cdot (1 - \frac{X}{K}) \cdot X(t)$
  - $X(t) = \frac{K \cdot X(0)}{X(0) + e^{r \cdot t} \cdot (K - X(0))}$
  - K is the ***carrying capacity*** of the population



Logistic model - analysis

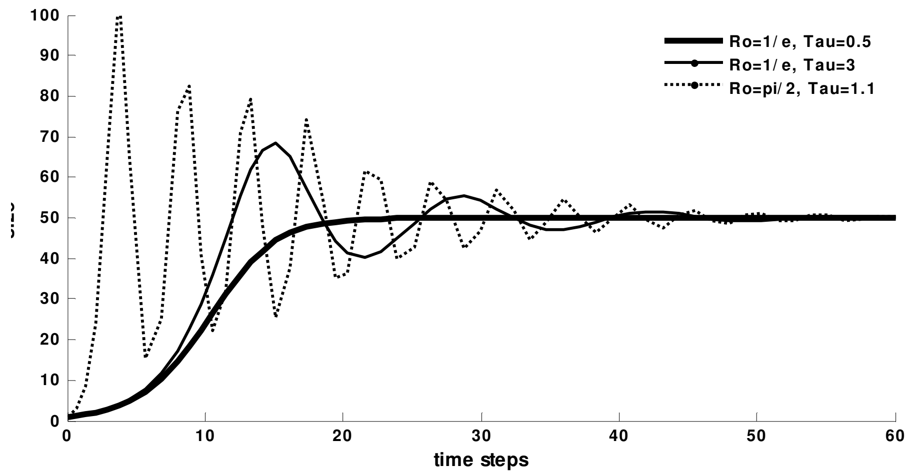
- The steady state of the logistic model is obtained at any X when $\frac{dX}{dt} = 0$
  - $\mathrm{X} = 0$
  - $\mathrm{X} = \mathrm{K}$
- Effect of the initial population size
  - $r > 0$ and $0 < X(0) < \frac{K}{2}$
    - The graph of $X(\mathrm{t})$ is sigmoid with inflection at $\frac{K}{2}$
  - $r > 0$ and $\frac{K}{2} < X(0) < K$
    - The graph of $X(\mathrm{t})$ is a concave curve increasing to K
  - $\rho > 0$ and $X(0) > K$
    - The graph of $X(\mathrm{t})$ is a concave curve decreasing to K





Logistic model with delay

- Oscillations of the population size is observed in some populations
- Oscillations are explained by a delay in fertility of individuals
  - $\frac{dX(t)}{dt} = r \cdot \left(1 - \frac{X(t-\tau)}{K}\right) \cdot X(t)$
  - τ is the mean time to achieve fertility.
- Analysis
  - For $\tau \cdot r < \frac{1}{e}$ the solution is similar to the normal logistic model
  - For $\frac{1}{e} < \tau \cdot r < \frac{\pi}{2}$ the solution is oscillatory and approaches the equilibrium
  - For $\tau \cdot r > \frac{\pi}{2}$ the solution is unstable



Summary of today's practice

[Population models]
Study the dynamics of unstructured populations due to the interactions with the environment, and with individuals of own kind.
Malthus model: growth is unlimited
Logistic model: growth is limited by the capacity of the studied population
Logistic model with delay: growth is delayed by the mean time to reach fertility

[What is next?]
Next week we will continue with model of two species populations.