Modeling and Simulation - Test – Michel Kana, PhD - 20.05.2014

Name: ____________________________

1. **Write the differential equations for the following models and shortly explain the meanings of variables and parameters**
   - Logistic population model **2**

$$\frac{dX(t)}{dt} = \rho \cdot \left(1 - \frac{X(t)}{K}\right) \cdot X(t)$$

   - Population-prey population model **4**

$$\frac{dX(t)}{dt} = k_1 \cdot X(t) - k_2 \cdot X(t) \cdot Y(t)$$
$$\frac{dY(t)}{dt} = k_3 \cdot k_2 \cdot X(t) \cdot Y(t) - k_4 \cdot Y(t)$$

   - Two species population model with cooperation **4**

$$\frac{dX_1\,(t)}{dt} = \rho_1 \cdot \left(1 - \frac{X_1(t)}{K_1} + b_{12} \cdot \frac{X_2(t)}{K_1}\right) \cdot X_1(t)$$
$$\frac{dX_2\,(t)}{dt} = \rho_2 \cdot \left(1 - \frac{X_2(t)}{K_2} + b_{21} \cdot \frac{X_1(t)}{K_2}\right) \cdot X_2(t)$$

   - Epidemiology model – SIR **5**

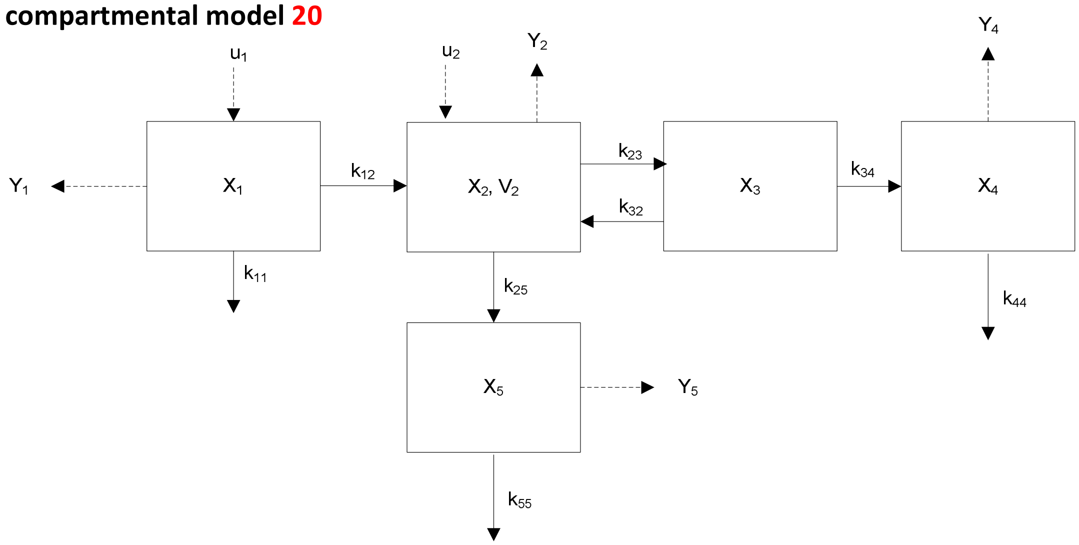
$$\frac{dS\,(t)}{dt} = -r \cdot S(t) \cdot I(t)$$
$$\frac{dI\,(t)}{dt} = r \cdot S(t) \cdot I(t) - a \cdot I(t)$$
$$\frac{dR\,(t)}{dt} = a \cdot I(t)$$
$$S(t) + I(t) + R\,(t) = N$$

2. **Give the Leslie matrix and initial population vector for the following population 5**

6 age groups. 10 Individuals in each age group at time 0. Age group 0 and 1 are not fertile. In age groups 2, 3 and 4 the fertility coefficient is 35%. Fertility is 10% in age group 5. In each age group, except of group 5, the survival rate is 80%.

$$A = \begin{bmatrix} 0 & 0 & 0.35 & 0.35 & 0.35 & 0.10 \\ 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \end{bmatrix} X_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

3. Write the differential equations, linear equations and matrices X, Y, U, A, B, C for the following compartmental model 20



$$\dot{X}_1 = (-k_{11} - k_{12}) \cdot X_1 + \quad 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + u_1$$
$$\dot{X}_2 = k_{12} \cdot X_1 + (-k_{23} - k_{25}) \cdot X_2 + k_{32} \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + u_2$$
$$\dot{X}_3 = 0 \cdot X_1 + k_{23} X_2 + (-k_{32} - k_{34}) \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + 0$$
$$\dot{X}_4 = 0 \cdot X_1 + 0 \cdot X_2 + k_{34} \cdot X_3 + (-k_{44}) \cdot X_4 + 0 \cdot X_5 + 0$$
$$\dot{X}_5 = 0 \cdot X_1 + k_{25} X_2 + 0 \cdot X_3 + 0 \cdot X_4 + (-k_{55}) \cdot X_5 + 0$$
$$Y_1 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_2 = 0 \cdot X_1 + \frac{1}{V_2} \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_3 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_4 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 1 \cdot X_4 + 0 \cdot X_5$$
$$Y_5 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 1 \cdot X_5$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \qquad U = \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -k_{11} - k_{12} & 0 & 0 & 0 & 0 \\ k_{12} & -k_{23} - k_{25} & k_{32} & 0 & 0 \\ 0 & k_{23} & -k_{32} - k_{34} & 0 & 0 \\ 0 & 0 & k_{34} & -k_{44} & 0 \\ 0 & k_{25} & 0 & 0 & -k_{55} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{V_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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4. Design a model, describe it mathematically and implement it in Simulink/Matlab for solving the following problems. Send your .mdl or .m files per email to michel.kana@gmail.com before the end of the test.

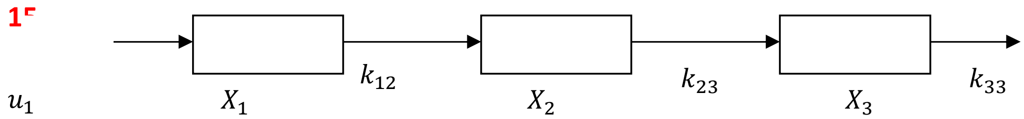
Red blood cells are produced in the bone marrow at a rate of 200 x 10^9 cells per day. They have a density independent death rate of 0.2% per day. Assuming that the body contain 30 x 10^{12} cells at time t = 0, how many cells will we have after 365 days? **15**

$$\frac{dN}{dt} = m - d \cdot N$$

$$m = 200 \cdot 10^9, d = 0.002, N(0) = 30 \cdot 10^{12}$$

$$\boldsymbol{N(365) = \textbf{cells count at time t } = 365}$$

Before patients undergo a minor operation doctors inject a certain amount of anesthesia in the muscle of your upper arm. From there it slowly flows into the blood where it exerts its sedating effect. From the blood it is picked up by the liver, where it is ultimately degraded. For chosen parameter values and initial conditions, estimate how long does it take before half of the injected amount has flown from the muscle to the blood? **1**



$$\frac{dX_1}{dt} = -k_{12} \cdot X_1$$

$$\frac{dX_2}{dt} = k_{12} \cdot X_1 - k_{23} \cdot X_2$$

$$\frac{dX_3}{dt} = k_{23} \cdot X_2 - k_{33} \cdot X_3$$

$$X_1(0) = u_1, X_2(0) = 0, X_3(0) = 0$$

$$\boldsymbol{X_2\left(t_{1/2}\right) = \frac{u_1}{2},} \qquad \boldsymbol{t_{1/2}} \textbf{ is the time when half of the injected amount has flown from the muscle to the blood}$$

Consider an insect population consisting of larvae (X_2) and adults (X_1). Assume that adults give birth to larvae (asexual), and that these larvae become adults. Adults have a density independent mortality. Larvae have a mortality that is dependent on the density of adults (and use a simple term for this). Make a model for the growth of such a population, using two ODEs. Draw nullclines and determine the stability of all steady states. Implement the model in Simulink. **15**

$$\frac{dX_2}{dt} = r \cdot X_1 - m \cdot X_2 - d_1 \cdot X_2 \cdot X_1$$

$$\frac{dX_1}{dt} = m \cdot X_2 - d_2 \cdot X_1$$

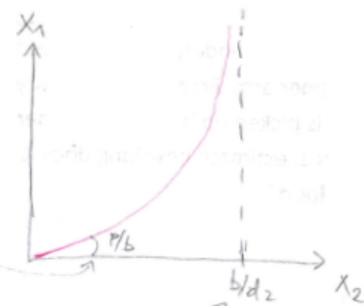
c) Nullcline of x_2

$$\frac{dx_2}{dt}=0 \Rightarrow bx_1 - px_2 - d_2x_2x_1 = 0$$

$$\Rightarrow \boxed{x_1 = \frac{px_2}{b - d_2x_2}}$$

slope: $\left(\frac{dx_1}{dx_2}\right)(0) = p/b$

asymptote: $b - d_2x_2 = 0 \Rightarrow x_2 = b/d_2$

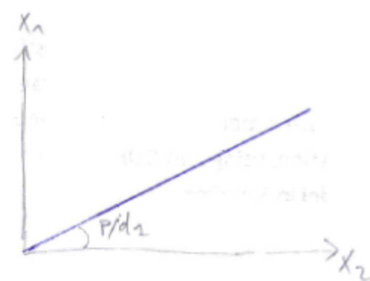


Nullcline of x_1

$$\frac{dx_1}{dt}=0 \Rightarrow px_2 - d_1x_1 = 0$$

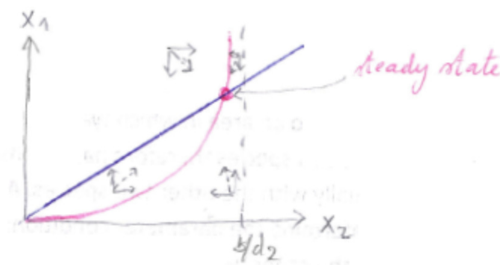
$$\Rightarrow \boxed{x_1 = \frac{p}{d_1}x_2}$$

slope: p/d_1

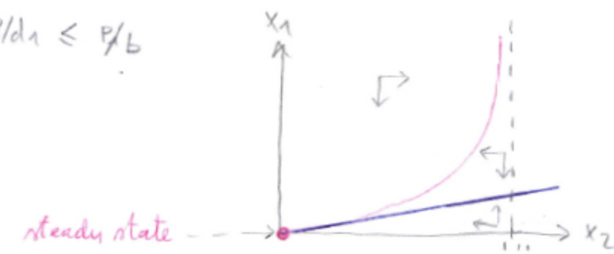


Combining nullclines of x_1 and x_2

(*) if $p/d_1 > p/b$



(*) if $p/d_1 \leq p/b$



Consider a species immigrating into an area in which two other species are present that do not compete with each other. Each of these two species therefore has a density equal to the carrying capacity, and let the new species compete equally with the other two species. Assume that the carrying capacities are the same for all three species. Determine the parameter conditions for successful invasion of the third species in the steady state of the other two. **15**

$$\frac{dN_1}{dt} = r \cdot N_1 \cdot (1 - N_1 - \alpha \cdot N_2)$$
$$\frac{dN_2}{dt} = r \cdot N_2 \cdot (1 - N_2 - \alpha \cdot N_1 - \alpha \cdot N_3)$$

$$\frac{dN_3}{dt} = r \cdot N_3 \cdot (1 - N_3 - \alpha \cdot N_2)$$

úspěšna invaze třetího druhu $\boldsymbol{\Longrightarrow \frac{dN_1}{dt} = 0, \frac{dN_3}{dt} = 0, \frac{dN_2}{dt} > 0 \Longrightarrow 1 - 2\alpha > 0 \Longrightarrow \alpha < {}^{1}\!/_{2}}$