Modelling and simulation

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Models of two species populations

- Models of predator – prey
  - One population prospers, the other doesn´t prosper
- Models with competition
  - Both population suffers from mutual contact
- Models with cooperation
  - Mutually beneficial interaction of two different populations.

Predator – prey model

relative fertility of prey

probability that predator will kill prey when prey and predator are meeting

$$\Delta x_n = \boxed{k_1} x(t)\Delta t$$
$$\Delta x_m = \boxed{k_2} x(t)y(t)\Delta t$$

$$\Delta x_n - \Delta x_m = [k_1 x(t) - k_2 x(t)y(t)]\Delta t$$

$$\Delta y_n = \boxed{k_3} k_2 x(t)y(t)\Delta t$$
$$\Delta y_m = \boxed{k_4} y(t)\Delta t$$

$$\Delta y_n - \Delta y_m = [k_3 k_2 x(t)y(t) - k_4 y(t)]\Delta t$$

conversion efficiency of the biomass of prey to predator biomass

relative mortality of predators

Predator – prey model

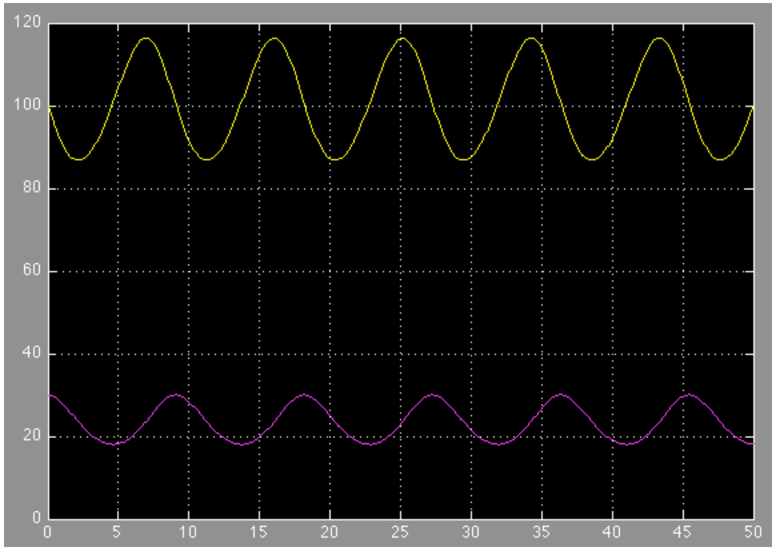
- Equation of Lotka – Volterra

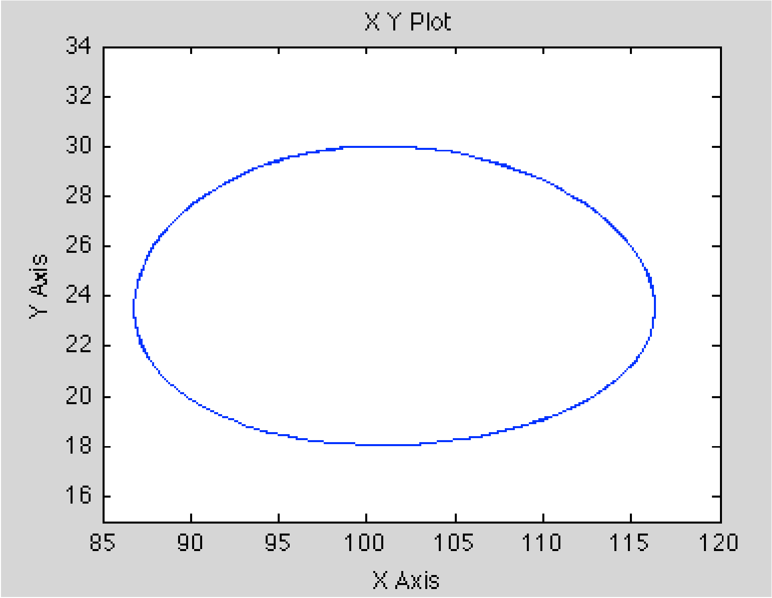
$$x'(t) = R_1 x(t) - k_2 x(t) y(t)$$

$$y'(t) = k_2 k_3 x(t) y(t) - k_4 y(t)$$

Visualization of the dynamics of populations

The development of both populations as a function of time





One state population as a function of the state second population

(block XY Graph)

Model with cooperation

the rate how the 2. population affects a 1. population

$$x_1'(t) = R_1 x_1(t)\left(1 - \frac{x_1(t)}{\boxed{K_1}} + \frac{\boxed{b_{12}}\, x_2(t)}{K_1}\right)$$

capacity of the environment for first population

the rate how the 1. population affects a 2. population

$$x_2'(t) = R_2 x_2(t)\left(1 - \frac{x_2(t)}{\boxed{K_2}} + \frac{\boxed{b_{21}}\, x_1(t)}{K_2}\right)$$

capacity of the environment for second population

Model with competition

the rate how the 2. population affects a 1. population

$$x_1'(t) = R_1 x_1(t)\left(1 - \frac{x_1(t)}{\boxed{K_1}} - \frac{\boxed{b_{12}} x_2(t)}{K_1}\right)$$

capacity of the environment for first population

the rate how the 1. population affects a 2. population

$$x_2'(t) = R_2 x_2(t)\left(1 - \frac{x_2(t)}{\boxed{K_2}} - \frac{\boxed{b_{21}} x_1(t)}{K_2}\right)$$

capacity of the environment for second population

Practice 1 – assignment (model with competition)

- Chemostat, in which was maintained monoculture brewer's yeast, was infected. Both cultures compete for the nutrients provided by chemostat.

- Create a model for the above case in Simulink
- Determine the equilibrium states of the model
- Simulate the dynamics of both populations and displayed based on the time

Practice 1 - assignment

- Use the following values for the simulation :
  - amount of yeast is before infecting foreign cultures 0.42 g
  - amount of infecting cultures (IC) is in start of infection 0.63 g
  - yeast growth rate is 0.26 per hour
  - infecting culture grows 4.5 times slower
  - environment is sufficient for 12 g yeast maximum
  - IC effect on yeast is described by constant 0.22
  - capacity of the environment for IK is 6 g
  - effect of IC on yeast is given by the constant 0.50

Practice 1– solution

- Determination of the equilibrium states

DE of model:

$$x_1'(t) = R_1 x_1(t)\left(1 - \frac{x_1(t)}{K_1} - \frac{b_{12}x_2(t)}{K_1}\right)$$

$$x_2'(t) = R_2 x_2(t)\left(1 - \frac{x_2(t)}{K_2} - \frac{b_{21}x_1(t)}{K_2}\right)$$

By adjusting get:

$$x_1'(t) = \overset{a}{R_1} x_1(t) - \overset{c}{\frac{R_1}{K_1}} x_1{}^2(t) - \overset{b}{\frac{R_1 b_{12}}{K_1}} x_1(t)x_2(t)$$

$$x_2'(t) = \underset{d}{R_2} x_2(t) - \underset{e}{\frac{R_2}{K_2}} x_2{}^2(t) - \underset{f}{\frac{R_2 b_{21}}{K_2}} x_1(t)x_2(t)$$

Practice 1– solution

- Determination of the equilibrium states

after substituting:
$$x_1'(t) = ax_1(t) - bx_1^2(t) - cx_1(t)x_2(t)$$
$$x_2'(t) = dx_2(t) - ex_2^2(t) - fx_1(t)x_2(t)$$

For equilibrium states applies:
$$x_1'(t) = 0$$
$$x_2'(t) = 0$$

Equilibrium states:
$$\boxed{(0,0), \left(0, \frac{d}{e}\right), \left(\frac{a}{b}, 0\right), \left(\frac{ae-cd}{be-cf}, \frac{bd-af}{be-cf}\right)}$$

Practice 1– solution

- Analysis of the stability of equilibrium points
  - We look for eigenvalues λ of the matrix A.
  - The matrix is called the Jacobian matrix

$$A = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{vmatrix}$$

$$f_1 = x_1'(t) = ax_1(t) - bx_1^2(t) - cx_1(t)x_2(t)$$

$$f_2 = x_2'(t) = dx_2(t) - ex_2^2(t) - fx_1(t)x_2(t)$$

Practice 1– solution

- Analysis of the stability of equilibrium points

$$A = \begin{vmatrix} a - 2 \cdot b \cdot x_1 - c \cdot x_2 & -c \cdot x_1 \\ -f \cdot x_2 & d - 2 \cdot e \cdot x_2 - f \cdot x_1 \end{vmatrix}$$

- Eigenvalues λ matrix A are determined by solution of the characteristic equation

$$\det \left| A - \lambda \cdot I \right| = 0$$

- Where I is diagonal matrix: $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

Practice 1– solution

- Analysis of the stability of equilibrium points
  - Simplify with putting into the matrix and removal of substitution.

$$\det\begin{vmatrix} a-2\cdot b\cdot x_1-c\cdot x_2-\lambda & -c\cdot x_1 \\ -f\cdot x_2 & d-2\cdot e\cdot x_2-f\cdot x_1-\lambda \end{vmatrix}=0$$

$$\boxed{(x_1,x_2)=(0,0)}$$

$$\det\begin{vmatrix} a-\lambda & 0 \\ 0 & d-\lambda \end{vmatrix}=0 \qquad \det\begin{vmatrix} R_1-\lambda & 0 \\ 0 & R_2-\lambda \end{vmatrix}=0$$

$$\lambda^2-\lambda(R_1+R_2)+R_1\cdot R_2=0$$

$$R_1=0{,}26hour^{-1} \qquad \lambda_1=0{,}26$$

$$R_2=0{,}0578hour^{-1} \qquad \lambda_2=0{,}0578$$

Practice 1– solution

- Analysis of the stability of equilibrium points
  - If the real value is at least one eigenvalue is positive, then the equilibrium is unstable, if the real value of all eigenvalues is negative, then the equilibrium is stable. If the eigenvalues are purely imaginary, we are talking about neutral stability.

$$\boxed{(x_1, x_2) = (0{,}0)}$$

$$\lambda_1 = 0{,}26$$

$$\lambda_2 = 0{,}2167$$

Both eigenvalues λ are positive.
Therefore the equilibrium points are unstable.

The same procedure can investigate the stability of other equilibrium points

$$(x_1, x_2) = (0{,}0); \left(0, \frac{d}{e}\right); \left(\frac{a}{b}, 0\right); \left(\frac{ae - cd}{be - cf}, \frac{bd - af}{be - cf}\right)$$

Practice 2 - assignment (predator-prey model)

- Estimating the number of pigeons was 140 000 in Prague in 2006 . It was given 10 birds of prey into the city.
  - Success of pigeons nesting is 50%, ie only 50% of the pigeons are capable of nesting. Each pair nests annually and has 3 cubs.
  - Pigeon has chance of survival 99% during meeting with predators.
  - The relative mortality of birds of prey is 1.8 per year
  - Constant of conversion of pigeons biomass to predators biomass is 0.02

Practice 2 - assignment

- Create a model in Simulink based on the equations of Lotka - Volterra
- Perform the simulation the next 40 years
- Briefly describe how they developed pigeon populations in the absence of predators
- On the basis of general differential equations of the model, decide whether it is possible exterminate predators or prey with nonzero initial condition of the population. If yes, please give an example.

Practice 2 - assignment

- Display the phase diagram of the model
- Shape thus obtained state trajectory briefly describe and explain its shape

- Find the equilibrium points of the model and determine their stability

Practice 2– desired output

- Model file *. mdl with correctly described blocks
- Short paper in *. pdf containing
  - Definition equation model
  - Table of all the parameters of the original model with columns: symbol, importance, value, unit
  - Table of all state variables of the model with columns: symbol, meaning the initial value, unit
  - Graphical representation of the simulation results
  - Short description of the behavior of the model in the absence of predators
  - Determine whether it can occur extinction of predators or prey (optionally give an example)

Practice 2– desired output

- Model file *. mdl with correctly described blocks
- Short paper in *. pdf containing (continuation)
  - Phase diagram with description
  - Calculation of the equilibrium states and their stability