MODELLING AND SIMULATION

Lesson 5- SS 2014 – Michel Kana

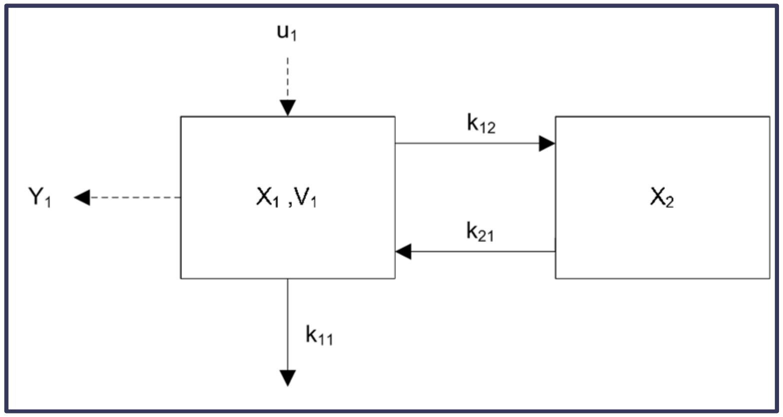
What do we do in today's lesson?

1. Summary of the previous practice
2. State-space notation of multi-compartmental models
3. Compartmental modeling of the cardiovascular system
4. Summary

Summary of the previous practice

Epidemiology models
Compartmental models

Example of compartmental model

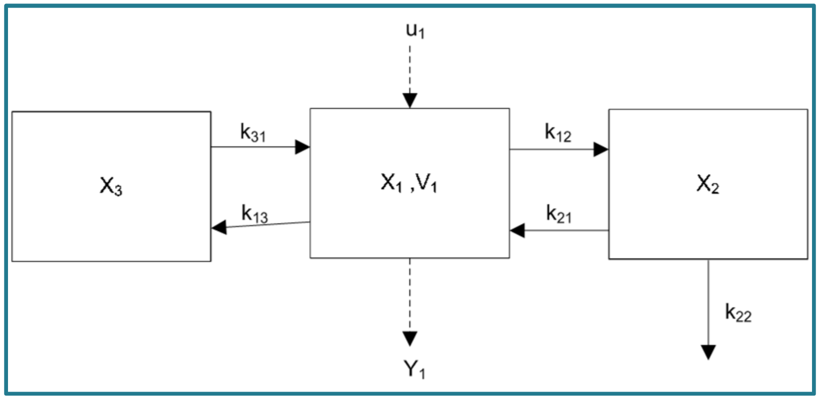


$$\dot{X}_1 = (-k_{11} - k_{12}) \cdot X_1 + k_{21} \cdot X_2 + u_1$$
$$\dot{X}_2 = k_{12} \cdot X_1 + (-k_{21}) \cdot X_2 + 0$$
$$Y_1 = \frac{1}{V_1} \cdot X_1 + 0 \cdot X_2$$
$$Y_2 = 0 \cdot X_1 + 0 \cdot X_2$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$
$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$
$$U = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$$
$$A = \begin{bmatrix} (-k_{11} - k_{12}) & k_{21} \\ k_{12} & -k_{21} \end{bmatrix}$$
$$\mathrm{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} \frac{1}{V_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\mathrm{X}} = \mathrm{A.X} + \mathrm{B.U}$$
$$\mathrm{Y} = \mathrm{C.X}$$

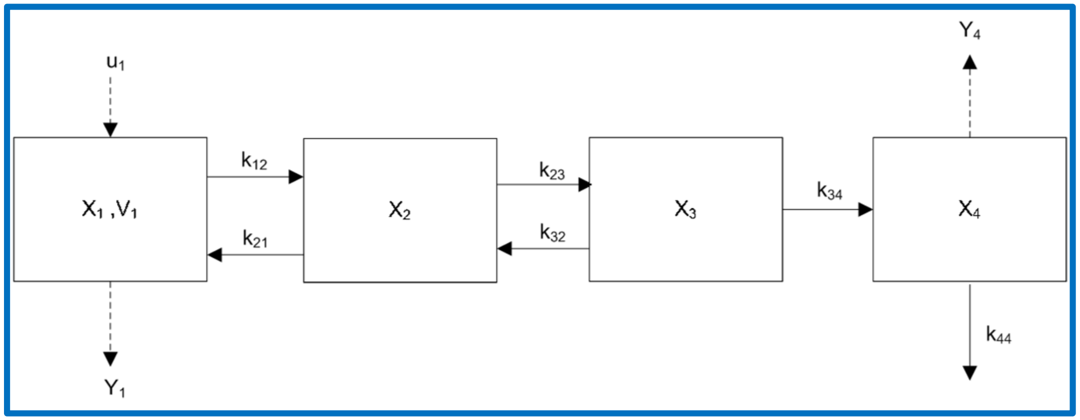
Example of compartmental model



$$\dot{X}_1 = (-k_{12} - k_{13}) \cdot X_1 + k_{21} \cdot X_2 + k_{31} \cdot X_3 + u_1$$
$$\dot{X}_2 = k_{12} \cdot X_1 + (-k_{21} - k_{22}) \cdot X_2 + 0.X_3 + 0$$
$$\dot{X}_3 = k_{13} \cdot X_1 + 0.X_2 + (-k_{31}) \cdot X_3 + 0$$
$$Y_1 = \frac{1}{V_1} \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3$$
$$Y_2 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3$$
$$Y_3 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$
$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$
$$U = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}$$
$$A = \begin{bmatrix} -k_{12} - k_{13} & k_{21} & k_{31} \\ k_{12} & -k_{21} - k_{22} & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix}$$
$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} \frac{1}{V_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example of compartmental model



$$\begin{aligned}
\dot{X}_1 &= (-k_{12})\cdot X_1 + k_{21}\cdot X_2 + 0\cdot X_3 + 0\cdot X_4 + u_1\\
\dot{X}_2 &= k_{12}\cdot X_1 + (-k_{21}-k_{23})\cdot X_2 + k_{32}\cdot X_3 + 0\cdot X_4 + 0\\
\dot{X}_3 &= 0\cdot X_1 + k_{23}\cdot X_2 + (-k_{32}-k_{34})\cdot X_3 + 0\cdot X_4 + 0\\
\dot{X}_4 &= 0\cdot X_1 + 0\cdot X_2 + k_{34}\cdot X_3 + (-k_{44})\cdot X_4 + 0
\end{aligned}$$

$$\begin{aligned}
Y_1 &= \frac{1}{V_1}\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 + 0\cdot X_4\\
Y_2 &= 0\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 + 0\cdot X_4\\
Y_3 &= 0\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 + 0\cdot X_4\\
Y_4 &= 0\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 + 1\cdot X_4
\end{aligned}$$

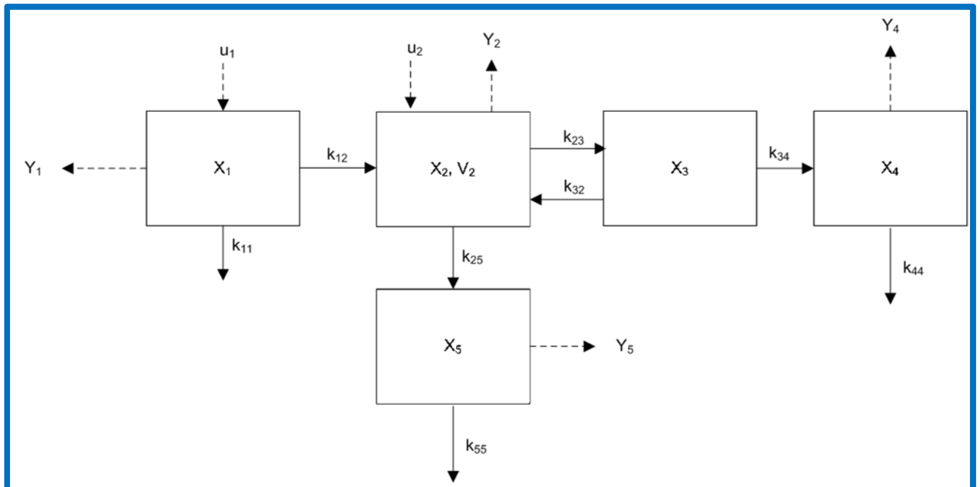
$$X = \begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4 \end{bmatrix} \quad Y = \begin{bmatrix} Y_1\\ Y_2\\ Y_3\\ Y_4 \end{bmatrix} \quad U = \begin{bmatrix} u_1\\ 0\\ 0\\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -k_{12} & k_{21} & 0 & 0\\ k_{12} & (-k_{21}-k_{23}) & k_{32} & 0\\ 0 & k_{23} & (-k_{32}-k_{34}) & 0\\ 0 & 0 & k_{34} & -k_{44} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{V_1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example of compartmental model



$$\dot{X}_1 = (-k_{11} - k_{12}) \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + u_1$$
$$\dot{X}_2 = k_{12} \cdot X_1 + (-k_{23} - k_{25}) \cdot X_2 + k_{32} \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + u_2$$
$$\dot{X}_3 = 0 \cdot X_1 + k_{23} X_2 + (-k_{32} - k_{34}) \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5 + 0$$
$$\dot{X}_4 = 0 \cdot X_1 + 0 \cdot X_2 + k_{34} \cdot X_3 + (-k_{44}) \cdot X_4 + 0 \cdot X_5 + 0$$
$$\dot{X}_5 = 0 \cdot X_1 + k_{25} X_2 + 0 \cdot X_3 + 0 \cdot X_4 + (-k_{55}) \cdot X_5 + 0$$
$$Y_1 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_2 = 0 \cdot X_1 + \frac{1}{V_2} \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_3 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 0 \cdot X_5$$
$$Y_4 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 1 \cdot X_4 + 0 \cdot X_5$$
$$Y_5 = 0 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 + 0 \cdot X_4 + 1 \cdot X_5$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \qquad U = \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -k_{11} - k_{12} & 0 & 0 & 0 & 0 \\ k_{12} & -k_{23} - k_{25} & k_{32} & 0 & 0 \\ 0 & k_{23} & -k_{32} - k_{34} & 0 & 0 \\ 0 & 0 & k_{34} & -k_{44} & 0 \\ 0 & k_{25} & 0 & 0 & -k_{55} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

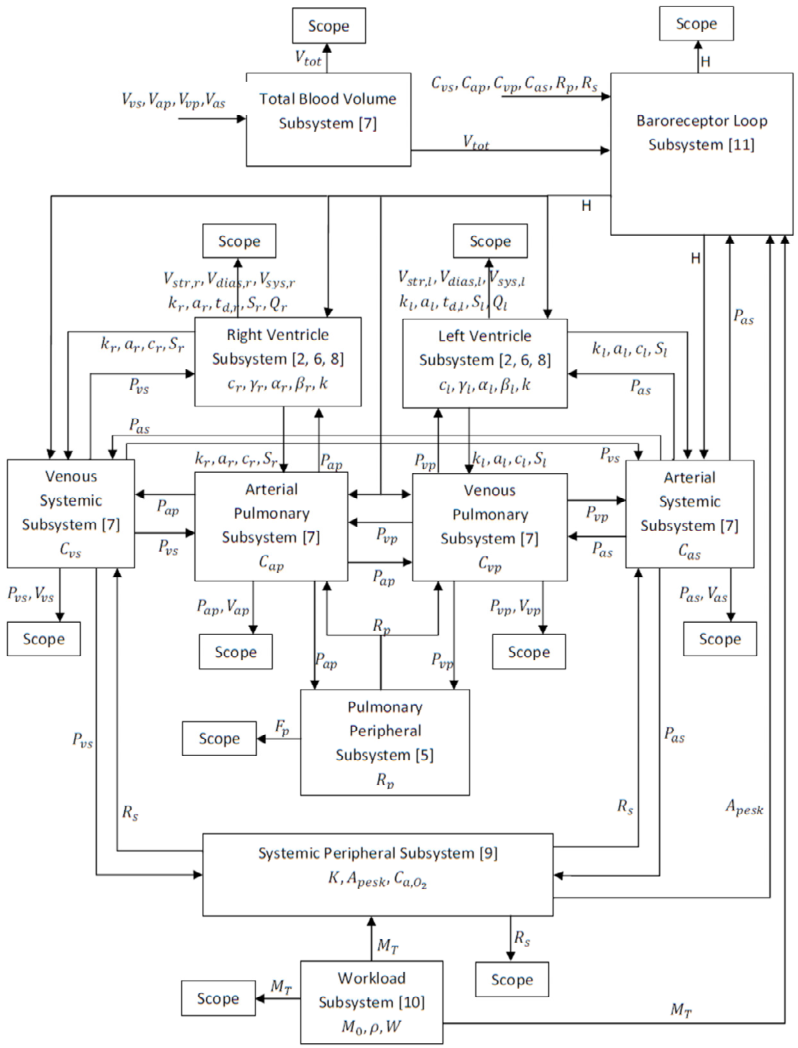
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{V_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Compartmental model of cardiovascular system

- The goal of this project is to describe the overall reaction of the cardiovascular system under a constant workload over a period of time of 15 minutes
- We are interested only in the time behavior of the heart rate and the time course of the blood pressure

Compartmental model of cardiovascular system

- Compartments of the cardiovascular system
  - **Venous Systemic Compartment**
  - **Arterial Systemic Compartment**
  - **Venous Pulmonary Compartment**
  - **Arterial Pulmonary Compartment**
  - **Systemic Peripheral Compartment**
  - **Pulmonary Peripheral Compartment**
  - **Right Ventricle Compartment**
  - **Left Ventricle Compartment**



J. J. Bazel et. al., 'Cardiovascular and Respiratory Systems – Modeling, Analysis, and Control', ISBN-13: 978-0-898716-17-7, 2007

Basic equations

- Relationship between blood pressure and volume
  - $V = c\,.\,P$ where V is the blood volume, P is the blood pressure and c is the compliance constant of the given compartment.
- Blood flow generated by a ventricle
  - $Q_{co} = H\,.\,V_{str}$ where H is the heart rate and V_{str} is the stroke volume, i.e., the volume of blood ejected by one beat of the ventricle.
- Blood flow through peripheral compartments
  - $F = \frac{1}{R}\,(P_a - P_v)$ where a stands for arterial and v stands for venous.

Equations for blood pressure and flow

- Change of blood pressure over time
  - Arterial Systemic Compartment: $C_{as}.\dot{P}_{as} = Q_l - F_s$
  - Venous Systemic Compartment: $C_{vs}.\dot{P}_{vs} = F_s - Q_R$
  - Arterial Peripheral Compartment: $C_{ap}.\dot{P}_{ap} = Q_r - F_p$
  - Venous Peripheral Compartment: $C_{vp}.\dot{P}_{vp} = F_p - Q_l$
- Change of blood volume over time
  - Volume in the ventricle at time t: $\dot{V}(t) = \frac{1}{R}\left(P_v(t) - P(t)\right)$
    - The initial value is the end-systolic ventricle volume $V(0) = V_{sys}$
    - The final value is the end-diastolic ventricle volume $V(t_d) = V_{dias}$
    - The filling time of a ventricle $t_d = t_d(H) = \frac{1}{H^{\frac{1}{2}}}\left(\frac{1}{H^{\frac{1}{2}}} - k\right)$
    - The stroke volume $V_{str} = V_{dias} - V_{sys}$

Equations for workload

- Local metabolic control
  - metabolic rate in the tissue $M_T = F_s\left(C_{a,O_2} - C_{v,O_2}\right) - K\frac{d}{dt}C_{v,O_2}$
    - C_{v,O_2} is the concentration of oxygen in the venous blood in the capillary region
    - C_{a,O_2} is the concentration of oxygen in the arterial blood in the capillary region
  - the peripheral resistance: $R_s = A_{pesk}C_{v,O_2}$
    - $\dot{R}_s = \frac{1}{K}(A_{pesk}\left(\frac{P_{as}-P_{vs}}{R_s}C_{a,O_2} - M_T\right) - (P_{as} - P_{vs}))$

Summary of today's lesson

Compartmental models

[What is next?]
Pharmacokinetic models.