Simple Model of Cardiovascular System

Project Work – Laboratory Exercises in Modeling and Simulating Physiological Processes

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The problem statement and the models in this project work are a summary from

1. *J. J. Bazel, F. Kappel, D. Schneditz, H. T. Tran, 'Cardiovascular and Respiratory Systems – Modeling, Analysis, and Control', ISBN-13: 978-0-898716-17-7, 2007*
2. *J. Keener, J. Sneyd, 'Mathematical Physiology', ISBN: 0-387-98381-3, 1998*

Step 1 – Defining the System

The goal of this project is to describe the overall reaction of the cardiovascular system under a constant workload over a period of time of 15 minutes. We are interested only in the time behavior of the heart rate and the time course of the blood pressure.

Step 2 – Identifying System Components

We are not interested in the blood flow in the arterial tree and do not distinguish between individual vessels in the venous and arterial part of the systemic or pulmonary circuit. Instead, we lump them together into four single compartments: **Venous Systemic Compartment** and **Arterial Systemic Compartment; Venous Pulmonary Compartment** and **Arterial Pulmonary Compartment**. We assume that those compartments are compliant and only characterized by the blood pressure, which determines the blood volume.

The venous systemic and the arterial systemic compartments on one side; the venous pulmonary and arterial pulmonary compartments on the other side; are connected by capillaries, arterioles and venules, which we lump together into two single compartments: **Systemic Peripheral Compartment** and **Pulmonary Peripheral Compartment**. We assume that both peripheral compartments are only characterized by the resistance of their walls to blood flow.

The venous systemic and the arterial pulmonary compartments are connected by the **Right Ventricle Compartment**. We assume that the right atrium is part of the venous systemic compartment. The venous pulmonary and the arterial systemic compartments are connected by the **Left Ventricle Compartment**. We assume that the left atrium is part of the venous pulmonary compartment.

Step 3 – Modeling the System with Equations

Since the venous systemic, arterial pulmonary, venous pulmonary and arterial systemic compartments are compliance compartments, we have the relation

$V = c \,.\, P$ [1] where V is the blood volume, P is the blood pressure and c is the compliance constant of the given compartment.

The left and right ventricle compartments are characterized by the blood flow generated by the ventricle, i.e. the cardiac output, which is given by

$Q_{co} = H \,.\, V_{str}$ [2] where H is the heart rate and V_{str} is the stroke volume, i.e., the volume of blood ejected by one beat of the ventricle.

The blood flow F through the pulmonary and systemic peripheral compartments depends on the blood pressure in the adjacent compartments and the resistance R against blood in the peripheral regions.

$F = \frac{1}{R}\,(P_a - P_v)$ [3] where a stands for arterial and v stands for venous.

If we use the subscripts l and r to characterize the left heart and the right heart, we obtain the following so-called Gordins **equations for the four compliance compartments**: [4]

Arterial Systemic Compartment:	$C_{as} \,.\, \dot{P}_{as} = Q_l - F_s$
Venous Systemic Compartment:	$C_{vs} \,.\, \dot{P}_{vs} = F_s - Q_R$
Arterial Peripheral Compartment:	$C_{ap} \,.\, \dot{P}_{ap} = Q_r - F_p$
Venous Peripheral Compartment:	$C_{vp} \,.\, \dot{P}_{vp} = F_p - Q_l$

For the systemic and pulmonary peripheral compartments, we have: [5]

Systemic Peripheral Compartment:	$F_s = \frac{1}{R_s}\,(P_{as} - P_{vs})$
Pulmonary Peripheral Compartment:	$F_p = \frac{1}{R_p}\,(P_{ap} - P_{vp})$

To model the **filling process**, we consider following equation giving the variation of the ventricle volume at time t, where R is the total resistance to the inflow into the ventrical; $P_v(t)$ is the venous filling pressure at time t; $P(t)$ is the pressure in the ventricle at time t: $\dot{V}(t) = \frac{1}{R}\,(P_v(t) - P(t))$. The initial value is the end-systolic ventricle volume $V(0) = V_{sys}$
Further we assume that $P_v(t)$ is constant during diastole and is equal P_v. We assume that for the relaxed ventricle we have the volume-pressure relation $V(t) = c.P(t)$ where c is the compliance constant. By integrating the differential equation above, we obtain: $V(t) = V_{sys} \,.\, e^{-(cR)^{-1}t} + cP_v(1 - e^{-(cR)^{-1}t})$
Let $t_d = t_d(H) = \frac{1}{H^{\frac{1}{2}}}(\frac{1}{H^{\frac{1}{2}}} - k)$ denote the duration of the filling process after Bazett, where k is a constant in the range of 0.0387-0.0516; then we have $V(t_d) = V_{dias}$, the end-diastolic volume.

The stroke volume is given by $V_{str} = V_{dias} - V_{sys}$. The stroke volume can be also modeled by the formula of Frank-Starling: $V_{str} = \frac{S}{P_a} V_{dias}$ where S is the contractility of the ventricle and P_a is the arterial pressure against which the ventricle has to eject. By combining all linear equations obtained, we have: [6]

$$V_{str} = \frac{c\, a(H)\, P_v\, S}{a(H)\, P_a + k(H)S}$$

$$V_{dias} = c.P_v - \frac{c\, k(H)\, P_v\, S}{a(H)\, P_a + k(H)S}$$

$$V_{sys} = c\, P_v - \frac{c\, P_v\, S}{a(H)P_a + k(H)S}$$

Where $k(H) = e^{-(cR)^{-1}t_d(H)}$ and $a(H) = 1 - k(H)$

Equations [2], [4], [5], [6] together give a nonlinear system of ordinary differential equations, usually called **Grodins' model of the mechanical part of the cardiovascular system**: [7]

Arterial Systemic Compartment:	$C_{as}\ \dot{P}_{as} = H\frac{c_l\ P_{vp}\ a_l(H)S_l}{a_l(H)P_{as}+k_l(H)S_l} - \frac{1}{R_s}(P_{as} - P_{vs})$
Venous Systemic Compartment:	$C_{vs}\ \dot{P}_{vs} = \frac{1}{R_s}(P_{as} - P_{vs}) - H\frac{c_r\ P_{vs}\ a_r(H)S_r}{a_r(H)P_{ap}+k_r(H)S_r}$
Arterial Pulmonary Compartment:	$C_{ap}\ \dot{P}_{ap} = H\frac{c_r\ P_{vs}\ a_r(H)S_r}{a_r(H)P_{ap}+k_r(H)S_r} - \frac{1}{R_p}(P_{ap} - P_{vp})$
Venous Pulmonary Compartment:	$C_{vp}\ \dot{P}_{vp} = \frac{1}{R_p}(P_{ap} - P_{vp}) - H\frac{c_l\ P_{vp}\ a_l(H)S_l}{a_l(H)P_{as}+k_l(H)S_l}$
Total Blood Volume is constant:	$V_{tot} = C_{as}\,P_{as} + C_{vs}\,P_{vs} + C_{ap}\,P_{ap} + C_{vp}\,P_{vp}$

In order to model the reaction of the cardiovascular system under an ergonomic workload, we have to include the regulation of the heart rate. We assume that the contractilities of the ventricles vary according to the changes in heart rate in the same direction. Thus we model **the variation of ventricle contractilities** by the second-order differential equations: [8] where $\gamma_l\ \alpha_l\ \beta_l\ \gamma_r\ \alpha_r\ \beta_r$ are positive constants.

$$\ddot{S}_l + \gamma_l\dot{S}_l + \alpha_l S_l = \beta_l H$$
$$\ddot{S}_r + \gamma_r\dot{S}_r + \alpha_r S_r = \beta_r H$$

The **local metabolic control** in tissues is achieved in increasing or decreasing the diameter of arterioles, which results in a change of vessel resistance to blood supply with oxygen. Thus we use Peskin model, where R_s is the peripheral resistance, A_{pesk} is a positive constant and C_{v,O_2} is the concentration of oxygen in the venous blood in the capillary region: $R_s = A_{pesk}C_{v,O_2}$

The metabolic rate M_T in the tissue is satisfied both by oxygen supply and anaerobic energy flow M_b provided by anaerobic biochemical reactions as described by the equation following, where C_{a,O_2} is the oxygen concentration in the arterial blood, which is assumed to be constant: $M_T = F_s\left(C_{a,O_2} - C_{v,O_2}\right) + M_b$

The biochemical energy flow in assumed to depend on the rate of change of venous oxygen level, thus: $M_b = -K\frac{d}{dt}C_{v,O_2}$ where K is a positive constant. By differentiating the equations in this section, we obtain the following differential equation: [9]

$$\dot{R}_s = \frac{1}{K}\left(A_{pesk}\left(\frac{P_{as} - P_{vs}}{R_s}C_{a,O_2} - M_T\right) - (P_{as} - P_{vs})\right)$$

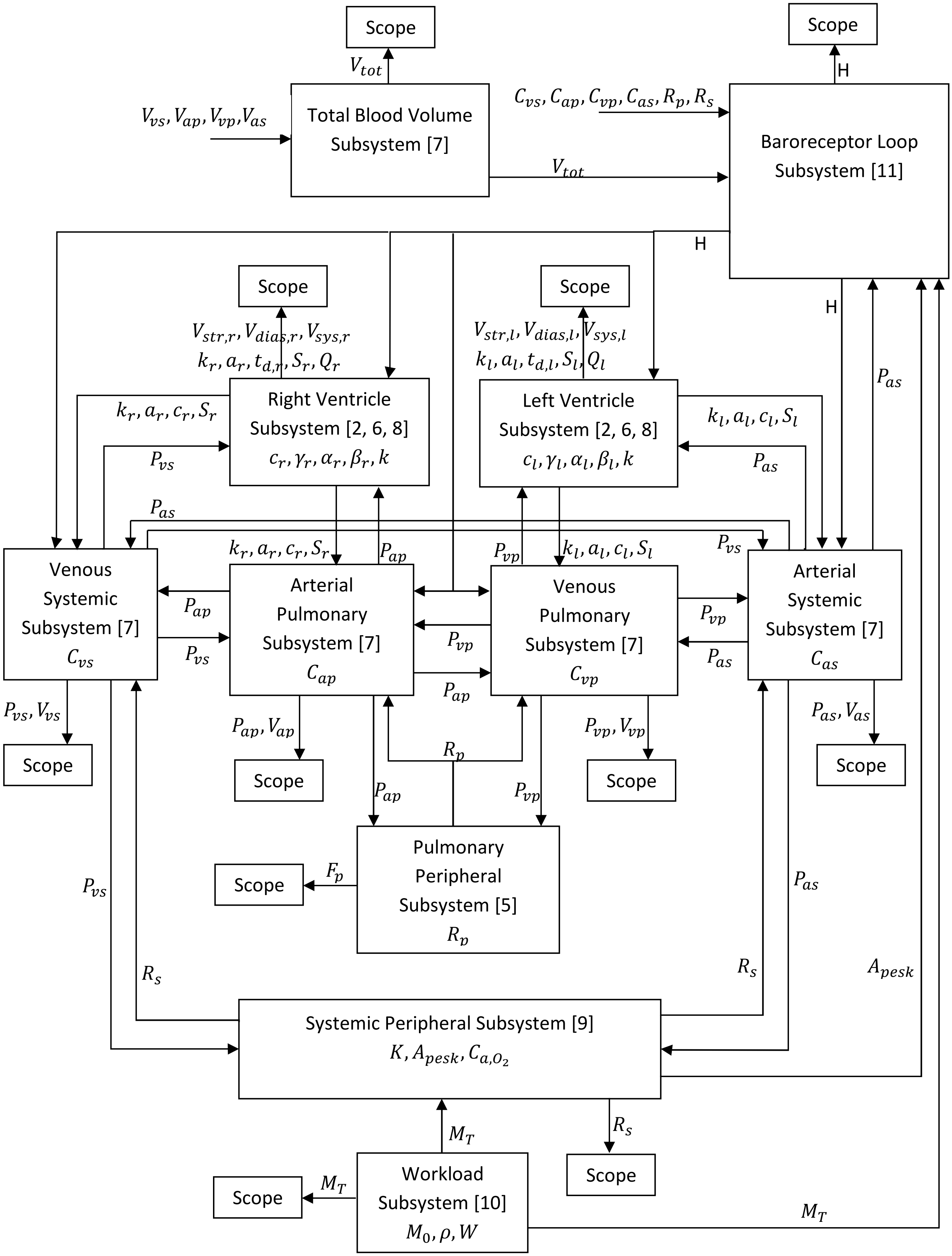
Now we include a constant ergonomic workload W imposed on a test person and rewrite the resulting metabolic rate as: [10]

$M_T = M_0 + \rho W$ where M_0 is the metabolic rate in the systemic tissue region corresponding to zero workload and ρ is a positive constant.

We assume that the **baroreceptor loop** is modeled by a feedback law $u(t)$, which controls the heart rate: [11]

$$H = u(t) = \frac{\left(\frac{1}{C_r}(2C_{vs} + C_{as}) + \frac{2}{C_l}C_{vp}\right)\left(A_{pesq}M_T + P_{as}\right) + \frac{1}{C_r}(P_{as}C_{as} - 2V_{tot})}{(2V_{tot} - P_{as}C_{as})R_s - \left(A_{pesq}M_T + P_{as}\right)(C_{vs}R_s + C_{ap}R_p + C_{as}R_s)}$$

Step 4 – Building the Simulink Block Diagrams



As homework students should model the subsystems illustrated above using Simulink block diagrams, with respect of inputs /outputs from/to other subsystems. The scopes are part of subsystems and should multiplex and display the required signals. We count max 3 students per group.

Work Group 1 – Arterial Systemic Subsystem
Work Group 2 – Venous Systemic Subsystem
Work Group 3 – Arterial Pulmonary Subsystem
Work Group 4 – Venous Pulmonary Subsystem
Work Group 5 – Pulmonary and Systemic Peripheral Subsystems
Work Group 6 – Right Ventricle Subsystem
Work Group 7 – Left Ventricle Subsystem
Work Group 8 – Baroreceptor Loop Subsystem
Work Group 9 - Workload, Total Blood Volume Subsystems and Model Integration

Step 5 – Running the Simulation

The following table gives a summary of the parameters (constants) of our model and their values.

Parameter	Meaning	Value	Unit
C_{as}	Compliance of the arterial systemic compartment	0.0101	Liter/mmHg
C_{vs}	Compliance of the venous systemic compartment	0.6425	Liter/mmHg
C_{ap}	Compliance of the arterial pulmonary compartment	0.0357	Liter/mmHg
C_{vp}	Compliance of the venous pulmonary compartment	0.1391	Liter/mmHg
C_l	Compliance of the relaxed left ventricle	0.0220	Liter/mmHg
C_r	Compliance of the relaxed right ventricle	0.0443	Liter/mmHg
R_p	Resistance in the pulmonary peripheral region	0.32	mmHg min/Liter
k	Coefficient in Bazett's formula	0.0516	1
C_{a,O_2}	Oxygen concentration in arterial systemic blood	0.2	1
C_{v,O_2}	Oxygen concentration in venous systemic blood	0.0001	1
K	Constant in the formula for biochemical energy flow	15.959	Liter
W	Constant exercise workload	75	Watt
A_{pesk}	Constant in the formula for peripheral resistance	254	mmHg min/Liter
M_0	Metabolic rate in the systemic peripheral region at zero workload	0.35	Liter/min
ρ	Coefficient in the formula for metabolic rate	0.0011	Liter/(min Watt)
α_l	Coefficient for contractilities of the left ventricle	31.592	min^{-2}
α_r	Coefficient for contractilities of the right ventricle	28.342	min^{-2}
β_l	Coefficient for contractilities of the left ventricle	25.065	mmHg/min
β_r	Coefficient for contractilities of the right ventricle	1.416	mmHg/min
γ_l	Coefficient for contractilities of the left ventricle	-1.332	min^{-1}
γ_r	Coefficient for contractilities of the right ventricle	-2.045	min^{-1}

Step 5 – Validating the Model

The simulation results should be validated against following estimated values for the state variables of our model:

Variable	Meaning	Initial Value	Expected Simulation Value Range	Unit
P_{as}	Pressure in the arterial systemic compartment	105.5	[80 .. 130]	mmHg
P_{vs}	Pressure in the venous systemic compartment	4	[1 .. 20]	mmHg
P_{ap}	Pressure in the arterial pulmonary compartment	15		mmHg
P_{vp}	Pressure in the venous pulmonary compartment	6		mmHg
S_l	Contractility of the left ventricle	61	Around 61	mmHg
S_r	Contractility of the right ventricle	4	Around 4	mmHg
R_s	Resistance in the systemic peripheral region	9.5	Can reach 18	mmHg min/Liter
H	Heart rate		[60 .. 180]	min^{-1}
$V_{str,l}$	Stroke volume of the left ventricle			Liter
$V_{str,r}$	Stroke volume of the right ventricle			Liter
$V_{dias,l}$	End-diastolic volume of the left ventricle		[0.102 .. 0.135]	Liter
$V_{dias,r}$	End-diastolic volume of the right ventricle			Liter
$V_{sys,l}$	End-systolic volume of the left ventricle		[0.031 .. 0.065]	Liter
$V_{sys,r}$	End-systolic volume of the right ventricle			Liter
V_{tot}	Total Blood Volume		[4.5 .. 5.5]	Liter
Q_l	Cardiac Output of the left ventricle		Approx. 5	Liter/min
Q_r	Cardiac Output of the right ventricle			Liter/min

Students should make own research for usual physiological values for healthy adults, in order to validate their models.