1. Solutions of linear equations

(a) Consider the following set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z = 0$$

Place these equations into a matrix, and row reduce the matrix.

Solution:

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$
$$R_3 = R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy! (Also, feel free to keep all the numbers as non-fractional values by finding the least common multiple of the two numbers you are trying to cancel out.)

$$R_2 = \frac{1}{-2.5}R_2$$
$$R_3 = -2R_3$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Convert the row reduced matrix back into equation form.

Solution:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 3y + 5z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

(c) Intuitively, what does the last equation from the previous part tell us?

Solution: It tells us that there are infinite solutions to the equations. $0x + 0y + 0z = 0$ is satisfied by **any** x, y, z.

(d) Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

Solution: No. $x = 1, y = 1, z = 1$ doesn't work, for instance.

(e) What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

Solution: z is a free variable. If $z = t$, then

$$y = -3z = -3t$$

$$2x + 3y + 5z = 0 \implies 2x - 9t + 5t = 0 \implies 2x = 4t \implies x = 2t$$

The general solution is then $t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!

2. First Proof

Prove that a subset of a finite linear independent set of vectors is linearly independent.

Solution: **Given:** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. This, by definition of linear independence, means that if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that:

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n = 0$$

then

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

In other words, the only solution to the above αs is that the αs are all 0.
To Prove: $\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \ldots + \beta_k \vec{v}_k = 0 \implies \beta_1 = \beta_2 = \ldots = \beta_k = 0$.
Note that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are a subset of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Assume that $\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \ldots + \beta_k \vec{v}_k = 0$ is true but not $\beta_1 = \beta_2 = \ldots = \beta_k = 0$.

Consider $\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \ldots + \beta_k \vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n$. If $\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \ldots + \beta_k \vec{v}_k = 0$ then

$$\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \ldots + \beta_k \vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n = 0$$

However, since we assumed that not all $\beta_1, \beta_2, \ldots, \beta_k$ are 0, this means that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not linearly independent, which is a contradiction because it is given that the set is linearly independent. Therefore, $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ must have been true.

3. Inverses!

(a) Find the inverse of:

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Solution: We know that for 2×2 matrices, we can simply use the formula for the inverse. The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(1)(5)-(3)(2)} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

(b) Find the inverse of:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}$$

using Gaussian Elimination.

Solution: To find the inverse of a matrix, start with the equation

$$\mathbf{A} = \mathbf{I}\mathbf{A}$$

Do row operations on the left hand side of this equation, to end up with

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}$$

$$R_2 = R_2 - 2R_1$$

$$\textit{and}$$

$$R_3 = R_3 - 3R_1.$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A}$$

$$R_2 = -0.5R_2$$

$$\begin{bmatrix}1 & 3 & 5\\0 & 1 & 2\\0 & -3 & -7\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & \frac{-1}{2} & 0\\-3 & 0 & 1\end{bmatrix}\mathbf{A}$$

$$R_1 = R_1 - 3R_2$$

$$\textit{and}$$

$$R_3 = R_3 + 3R_2.$$

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & \frac{3}{2} & 0\\1 & \frac{-1}{2} & 0\\0 & \frac{-3}{2} & 1\end{bmatrix}\mathbf{A}$$

$$R_3 = -1R_3$$

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & \frac{3}{2} & 0\\1 & \frac{-1}{2} & 0\\0 & \frac{3}{2} & -1\end{bmatrix}\mathbf{A}$$

$$R_1 = R_1 + R_3$$

$$\textit{and}$$

$$R_2 = R_2 - 2R_3$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 3 & -1\\1 & \frac{-7}{2} & 2\\0 & \frac{3}{2} & -1\end{bmatrix}\mathbf{A}$$

Now this equation looks like $\mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$

$$\begin{bmatrix}-2 & 3 & -1\\1 & -3.5 & 2\\0 & 1.5 & -1\end{bmatrix}$$

A rotation matrix is a matrix that takes a vector and rotates it by some number of degrees. That matrix looks like:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$$

for some angle θ. For example, if we had a rotation matrix with $\theta = 45^\circ$, and we multiplied it with the vector [.5, .5], what would you expect?

(c) find the inverse of this matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Note that you don't have to actually do the normal math for finding inverses (and are probably pretty sad if you did). Instead think of what the inverse of a rotation matrix should probably do. What is the inverse of rotating 30 degrees? Rotating -30 (or 330) degrees! So by plugging -30 or 330 into the general form of the rotation matrix you can get the correct answer

(d) i. Will a rotation matrix always have an inverse? Why or why not?

ii. Consider a matrix that mirrors a vector across the x-axis. Will it always have an inverse?

iii. Consider a matrix flattens a vector on to the x-axis (so for example $[3, 5]^\top$ becomes $[3, 0]^\top$). Will it have an inverse?

Solution: i) Yes, you can always rotate in the opposite direction.
ii) Yes, you can always invert back (In fact, the inverse would be itself)
iii) No, since the transformation loses information, there probably is NOT an inverse.

4. Invertibility and equations

(a) Consider the following system of equations

$$2x - 2y = -6$$
$$x - y + z = 1$$
$$3y - 2z = -5$$

Write these equations in matrix form. Then, write an expression for the solution to the equations using inverses, but don't compute the inverse.

Solution:

$$\begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}$$

This can be rewritten as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}$$

(b) Let the system of equations be $\mathbf{A}\vec{x} = \vec{y}$. What does it mean if $\mathbf{A}$ is not invertible?
Hint: The solution to the previous part.

Solution: If $\mathbf{A}$ is not invertible, then the system cannot be solved uniquely. We may have infinite solutions or no solutions. **Make sure you read through the entire problem. We talk more in depth about what invertibility means in later parts!**

(c) Consider the matrix

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix}$$

Is it invertible?

Solution: We don't actually need to do gaussian elimination on this matrix to check whether it is invertible. It is not. An R^{NxM} matrix, where $N \neq M$ is **never** invertible. Why? Because when we do gaussian elimination, for invertible matrix, we must get 3 pivots since we have 3 rows. But we cannot get 3 pivots because we have only 2 columns.

(d) Does the system of equations that is represented by the following have any solutions?

$$\underbrace{\begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{B}\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 9 \\ -7 \end{bmatrix}}_{\vec{y}}$$

Solution: From part (b), we want to say that the system doesn't have any solutions. We saw in the previous part that $\mathbf{B}$ is not invertible. But... let's convert this system to actual equations.

$$-1x_1 + x_2 = 5$$
$$-2x_1 + x_2 = 9$$
$$1x_1 - 3x_2 = -7$$

These are 3 equations in 2 variables. Surely, they could have a solution. If we solve them, we get $(x_1 = -4, x_2 = 1)$ as a solution. How could this happen? $\mathbf{B}$ was not invertible!

The invertibility test actually only holds well for the case when $\mathbf{B}_{NxN}\vec{x}_{Nx1} = \vec{y}_{Nx1}$ are the dimensions of the matrices and vectors in question. In this case, $\mathbf{B}$ being non-invertible, means that gaussian elimination gives you a row of zeros.

- If you have infinite solutions, then in your augmented matrix, you will have a row of 0s and the corresponding element from $\vec{y}$ will also be 0. (Why does this mean infinite solutions?)
- If you have no solutions, then in your augmented matrix, you will have a row of 0s, but the corresponding element from $\vec{y}$ will be non-zero. (Why does this mean zero solutions?)

However, if you have an equation of the form $\mathbf{B}_{MxN}\vec{x}_{Nx1} = \vec{y}_{Mx1}$, then you have M equations in N variables.

- If $M > N$ (like in this example), then you have more equations than variables.
  - This could certainly have a solution if the equations are linearly dependent. For instance, $x + y = 1, 2x + 2y = 2, x - y = 4$ definitely has a solution. The first 2 equations are linearly dependent, so we can remove one of them. Notice how Gaussian Elimination would help you realize this and find the one solution!

- You could also have no solution, if the equations are all linearly independent. $x+y=3, x-y=-1, x+2y=0$ does not have any solutions. Notice how Gaussian Elimination would help you realize there are no solutions! *If you don't see it, try it out. Do you get a row of 0s on the left part of the augmented matrix, but not a corresponding zero element?*
- You could also have infinite solutions. Consider $x+y=1, 2x+2y=2, 3x+3y=3$. Gaussian elimination helps here too!

- If $M < N$, then you have more variables than equations.
  - This could have many solutions. Consider 1 equation: $x+y=3$.
  - This could have no solutions. Consider $x+y+z=1$ and $x+y+z=2$
  - This cannot possibly have just one solution. You have more variables than you have equations!

  Notice though that Gaussian Elimination would help you realize all 3 of these cases.

Big take away: Do not pattern match. If you hear it once "no inverse means no solutions", don't pattern match. That holds in a *particular* case. Remember that at the end, these are all equations, and use your intuition about equations.

5. Are you linear?

(a) Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Note that a, b, c can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

Solution: To prove whether a transformation is linear, we must check whether it preserves scalar multiplication, addition and the zero vector.

Scalar multiplication

Let $\alpha \in \mathbb{R}$. Is $\mathbf{S}(\alpha\vec{x}) = \alpha\vec{y}$?

$\mathbf{S}\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix} = \alpha \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Try it!

Addition

Is $\mathbf{S}(\vec{x}_1 + \vec{x}_1) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$?

Let $\vec{x}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$. Then $\mathbf{S}(\vec{x}_1 + \vec{x}_2) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$. Try it out!

Zero vector

Is $\mathbf{S} \cdot \vec{0} = \vec{0}$? Yes.

This proves that $\mathbf{S}$ is indeed a linear transformation.

(b) Now let's consider another matrix $\mathbf{Q}$ which takes a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\begin{bmatrix} a+5 \\ b \\ c \end{bmatrix}$. Is this matrix a linear operator?

Solution: Let's try the preservation of the zero vector first. Is $\mathbf{Q} \cdot \vec{0} = \vec{0}$? Nope, it is $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$. This matrix is not a linear operator! Notice that even though matrix-vector multiplication is generally linear

(c) Let's dive deeper. Write out the matrix $\mathbf{S}$ and $\mathbf{Q}$. Are they invertible?

Solution: $\mathbf{S} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. This matrix is not invertible, but it was still linear!

Writing out the matrix for $\mathbf{Q}$ is actually a trick question. There is no easy way to do this. In fact, you cannot write it just using numbers. Try it out. Let the first row of $\mathbf{S}$ be some $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}$. Consider the first equation in the matrix vector multiplication: $\alpha_1 a + \alpha_2 b + \alpha_3 c = a + 5$. Using α_is from $\mathbb{R}$, there are no α_is that satisfy this equation. Since it isn't possible to write such a matrix, the invertibility question is invalid.

Solution: **Disclaimer: Read the following with caution. It abuses notation, and matrices in EE16A are not typically seen in the way that they are presented below (with their rows being functions of vectors).** You cannot simply write $\mathbf{Q}$ without considering the context in which it is being used. Say that the context is multiplication with a vector.

$$\mathbf{Q}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+5 \\ b \\ c \end{bmatrix}$$

In this case,

$$\mathbf{Q} = \begin{bmatrix} 1 & \frac{5}{b} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how it is impossible to write the matrix out entirely using just numbers and that we need to use either a, b or c inside the matrix itself. Finally, note that the first row of this matrix could be written in other ways too. It could be $\begin{bmatrix} 1+\frac{5}{a} & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & \frac{3}{b} & \frac{2}{c} \end{bmatrix}$ too. But the essential idea is that a **non-linear transformation matrix cannot be expressed using just scalars**.
So it is invertible? We can't say.