Street-Fighting Mathematics

- ✓ Most material is from Sanjoy Mahajan's edX course 6.SFMx and book *Street-Fighting Mathematics*

Estimating Quantities

- ✓ Break it down into parts you can estimate
- ✓ Don't be afraid to lump – errors will cancel out
- ✓ Make sure the dimensions work out

Multiplying large numbers [5.1]

- ✓ Convert numbers to scientific notation $k \cdot 10^n$
- ✓ Lump k into 1 (0 to 1.8), *few* (1.8 to 5.6), 10 (5.6 to 10)
- ✓ $\mathit{few}^2 = 10$
- ✓ Add powers of multiplicands $n_1, n_2, ...$

Fractional changes [5.2]

$$(m + \Delta m\%)(n + \Delta n\%) \approx m \cdot n + (\Delta m + \Delta n)\%$$

$$\frac{\Delta x^n}{x^n} \approx n \cdot \frac{\Delta x}{x}$$

✓

Ex: m'	Big part (m)	Δm
5.3	5 +	$0.3/5 \approx 6\%$
3.04	3 +	$0.04/3 \approx 1.3\%$
	15 +	$7.3\% \cdot 15 \approx 1.1 = 16.1$

- ✓ Ex: 4% increase in sides of rectangle → 8% increase in area
- ✓ Ex: inflation by 10% + discount of 15% → 5% decrease
- ✓ Ex: division, $n = -1$. $1/13 \approx 1/10 - 30\% = 0.07$
- ✓ Ex: square roots, $n = 1/2$. $\sqrt{10} \approx \sqrt{9} + (1/2)(1/9) \approx 3.17$

Improving fractional estimations

- ✓ Increase the accuracy of the big part, e.g. by multiplying both numerator and denominator by a convenient number.
- ✓ Ex: $1/13 \cdot 8/8 = 8/104 \approx 8/100 - 4\% = 0.0768$
- ✓ In the case of division, results in a quadratic improvement: $\frac{\Delta x^{-1}}{x^{-1}} \approx -1 \cdot \frac{\Delta x}{x} + (-1)^2 \cdot \frac{\Delta x}{x}^2$
- ✓ Ex: $\sqrt{10} = \sqrt{360}/6 \approx \sqrt{361}/6 = 19/6 \approx 3.167$
- ✓ Polya's method: $\sqrt{2} = \sqrt{4/3}/\sqrt{2/3} \approx (1 + (1/6))/(1 - (1/6)) = 7/5 \approx 1.4$
- ✓ Another ex: $\ln 2 = \ln \frac{4}{3} - \ln \frac{2}{3} \approx \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

Useful approximations

- ✓ $(1 + z)^n \approx 1 + nz$, $z \ll 1$ and $nz \ll 1$
- ✓ $(1 + z)^n \approx e^{nz}$, $z \ll 1$ and nz unrestricted
- ✓ $\ln(1 + z) \approx z$, small z
- ✓ $\ln(2) \approx 0.7$
- ✓ $\ln(10) \approx 2.3$
- ✓ Ex: 5% bacteria mutated per round, how many unmutated after 140 rounds? $0.95^{140} = (1 - 1/20)^{140} \approx e^{-140/20} \approx e^{-6.9} \approx 10^{-3} = 0.001$
- ✓ $\sin\theta \approx \theta$ at small angles
- ✓ $\cos\theta \approx 1 - \frac{\theta^2}{2}$ at small angles
- ✓ Rule of 72: if a quantity increases $x\%$ per time unit (e.g. year), it doubles in $72/x$ time units (years).

Estimating from bounds

- ✓ Estimate a lower bound l and an upper bound u for the desired quantity
- ✓ Your estimate = GeometricMean$(l, u) = \sqrt{l \cdot u}$

Guessing a Formula

Dimensional analysis [1]

- ✓ You can only make meaningful comparisons with items of the same dimension.
- ✓ Gather together the elements you think play a role in the formula.
- ✓ Compare dimensions to guess at formula.
- ✓ Ex: Impact speed v: dim LT^{-1}, g: dim LT^{-2}, height h: dim L. $v \sim \sqrt{g \cdot h}$
- ✓ Special cases:
  - Exponents must be dimensionless.
  - d/dt = 'a little bit of t': dim T
  - d^2t = 'a little bit of a little bit of t': dim T
  - dt^2 = dim T^2
  - Summation and integration signs do not affect dimensions.
- ✓ Warning: there will sometimes be > 1 solution!
- ✓ Take advantage of symmetries. Consider solutions such as $a^2 + ab + b^2$.
- ✓ Additive terms must all have the same dimensions.
- ✓ To judge between solutions find the constant term, try easy cases.

Easy cases [2]

- ✓ Try $x = 0, 1, y, \infty$
- ✓ Reduce dimensions
- ✓ When n has arbitrary range, try a small number of terms

Solving formulae

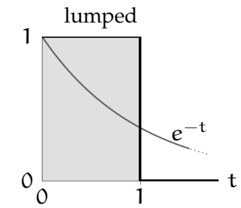
Quadratic equations [5.HW]

$$ax^2 + bx + c = 0$$

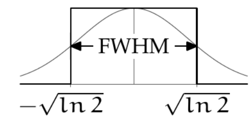
- ✓ $x \approx 0$. Remove small ax^2 term, $x \approx -c/b$
- ✓ $x \gg 1$. Remove small c, $x \approx -b/a$
- ✓ Substitute existing estimates of x into eqn to iterate better solutions

Evaluating integrals

- ✓ Can you use one of the approximations above to simplify the integrand? (ideally to exponential)
- ✓ Don't be afraid to extend the bounds to infinity
- ✓ $\int_{-\infty}^{\infty} e^{-\alpha t^2} \mathrm{d}t = \sqrt{\pi/\alpha}$
- ✓ Archimedes' theorem: an inverted parabola encloses 2/3 of a rectangle's area
- ✓ $1/e$ technique: Find the width Δx that produces a change of e, use the resulting rectangle area as an estimate.



- ✓ Full-Width Half-Maximum (FWHM) technique: find the width where the y-value is half the maximum.



Evaluating derivatives

- ✓ Significant change approximation:

$$\frac{df}{dx} \sim \frac{\text{significant } \Delta\text{f near x}}{\Delta\text{x}}$$

- ✓ $\frac{d^2x}{dt^2} \sim \frac{\text{significant } \Delta x}{(\Delta t \text{that produces a significant} \Delta x)^2}$

Find the convergence point of a recurrence relation

- ✓ Substitute x for x_n and x_{n-1} etc, and solve.
- ✓ Ex: $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$, $n \geq 0$
- ✓ Solve with: $x = \frac{1}{2}(x + \frac{2}{x}) \rightarrow x = \sqrt{2}$

Finding a closed formula for a summation

- ✓ Use the Euler-MacLaurin summation formula:

$$\sum_a^b f(k) = \int_a^b f(k)dk + \frac{f(b) + f(a)}{2} + \frac{f'(b) - f'(a)}{12} - \frac{f^{(3)}(b) - f^{(3)}(a)}{720} + ...$$

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