Chaos and Fractals

Research Skills Mini Project

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1 Chaos theory

Chaos theory describes the behavior of dynamical systems that appear random but in fact often follow deterministic laws and patterns, and whose complex behavior arises from sensitivity to initial conditions. It is possible for chaotic systems which are free of random elements to still be so sensitive to initial perturbations that the outcomes are divergent in nature, making them unpredictable.

This seemingly erratic behavior can be meaningfully measured by the Lyapunov exponent, which describes the rate of separation of infinitesimally close trajectories (and thus is useful for computing their propensity to diverge). The Lyapunov exponent is useful for determining how sensitive a given system is to perturbations of its initial conditions.

Suppose there is some initial separation $\delta\mathbf{Z}_0$, the rate of divergence is then expressed as $|\delta\mathbf{Z}(t)| \approx e^{\lambda t}|\delta\mathbf{Z}_0|$, and λ is the Lyapunov exponent. The set of such exponents has cardinality dimension of the phase space and its maximal element (MLE) is particularly useful for determining whether the dynamical system is chaotic. Define the MLE as

$$\lambda_{MLE} = \lim_{t\to\infty} \lim_{|\delta\mathbf{Z}_0|\to 0} \frac{1}{t} \ln \frac{|\delta\mathbf{Z}(t)|}{|\delta\mathbf{Z}_0|}$$

A positive λ_{MLE} corresponds to a divergent system, and in the case of the Lorenz attractor below (where $\lambda_{MLE} = 0.9$) the system in fact diverges rapidly).

1.1 Lorenz attractor

Known popularly as the "butterfly effect," the Lorenz attractor models fluid flow relevant to atmospheric physics. This example of a strange attractor arises from a system of ordinary differential equations where certain initial conditions and parameter values give rise to chaotic solutions. Just as the analogy of a butterfly flapping its wings in the US can cause a hurricane in China suggests, the imperfect constraint of even infinitesimally small perturbations in the initial conditions will eventually lead to unpredictable behavior.

The Lorenz attractor arises from the simplified system of ordinary differential equations describing two-dimensional fluid flow:

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

where the constants σ, ρ, and β describe diffusivity, buoyancy, viscosity, and other physical properties of the fluid. Lorenz used $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$ and initial conditions $x = z = 0$, $y = 1$. As such, these are the values used for solving the system in the code. A range of two minutes is chosen for t to draw the Lorenz attractor in Figure 1 because it is sufficient to demonstrate the behavior of the system and generate the familiar butterfly image without being computationally intensive.

One of the contributions of this ODE system historically was that it gave one of the earliest examples of the butterfly effect. To underscore this phenomena, the code overlays two runs of the solver with the same initial values of x and z and y differing by a minuscule amount - on the order of 10^{-5}. It is clear from Figure 2 that though the two systems' behavior is well-defined and similar, the slightly perturbed y evaluates to a much different result. The plot is rotated to a view from directly above (i.e. azimuth = 0° and elevation = 90°) to highlight the behavior of the attractor when it is plotted as a trajectory in phase space.

It is crucial to note that though there appear to be repeated crossings between the two circuits, this is a result of projecting the three-dimensional image and is clearly not the case when rotating Figure 1.

1.2 Julia set

Julia sets are the fractal sets formed by polynomials in complex space. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial map and $\{p_1, .., p_n\}$ an orbit under f. We can define the filled Julia set for f as the set

$$\{p_1 \in \mathbb{C} \mid \exists R > 0 \text{ such that } |p_n| \leq R \; \forall n \in \mathbb{N}\}.$$

Now consider $f(z) = z^2 = f(re^{i\theta}) = r^2 e^{i2\theta}$. Then for $|p_1| \leq 1$ we have the orbit is bounded and thus converges, meaning the filled Julia set for $f(z)$ is precisely the closed unit disk. In general, we can define the Julia set recursively as the set of all parameters $c \in \mathbb{C}$ such that the sequence $z_{n+1} = z_n^2 + c$ is bounded.

In Part II of the code, we consider the example of the Douady rabbit, where $f(z) = z^2 - 0.123 + 0.754i$. The majority of orbits in this case escape to infinity, but it is possible to generate the filled set for those orbits which are bounded. The result is plotted on the axes of its real and imaginary parts. The function *julia_set* takes as its sole argument a value for a parameter c which determines where the orbits are bounded.

Though the given example used to generate Figure 3 is $c\_douady = -0.123 + 0.745i$, it is easy to pass through a different value of c (such as $c = 0$ which gives the original unit disk solution) to the function and generate the resulting fractal image.

1.3 The Mandelbrot set

One interesting example of the Julia set is determined when the Mandelbrot criterion, which exploits the connectendess of the polynomials in complex space, is met. In this case, the filled Julia set for some map f is connected $\iff$ its orbit under $z = 0$ is bounded.

Because of its close connection to the Julia set, it should be possible to reuse code from the previous section. However, because one of the key features of the Mandelbrot set is its self-similarity, the visualization of which requires more computational intensity than the general case, the code is left in a non modular state.

In particular, the *plot_mandelbrot_set* function uses twice as many iterations as the *julia_set* and higher resolution in the *meshgrid*. This results in a longer code run time but also makes it possible, when zooming in on Figure 4, to appreciate the unique geometric structure of the object.

2 References

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