The Friendship Theorem

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Math 479 Project #2

Introduction

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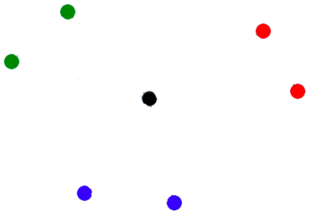

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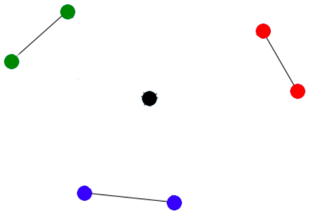

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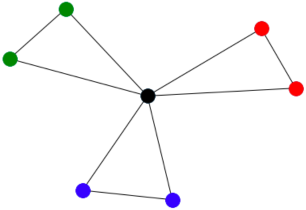

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- Although the problem's creator is unknown, there are several famous proof methods.
- We'll tackle this proof using graph theory and linear algebra.

Graph Theory

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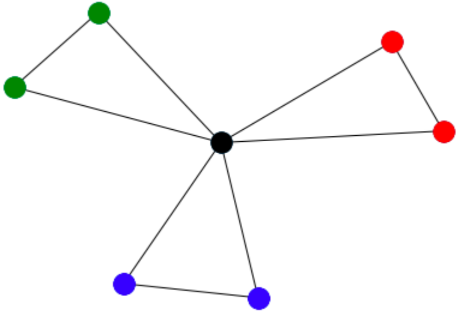
- An **edge** can be used to define a relationship between vertices.
- A **path** is a finite sequence of adjacent vertices and adjacent edges where neither can be repeated.

Graph Theory

- An example of a graph:



Linear Algebra

- An identity matrix:

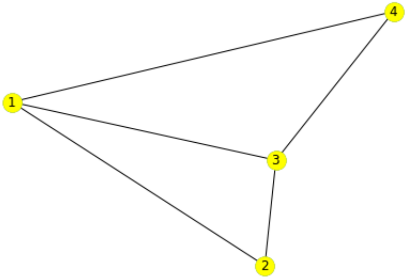
$$I_1 = \begin{bmatrix} 1 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- An adjacency matrix:



$$A = \begin{array}{c c} & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Linear Algebra

- *Eigenvalues* (λ) are special scalars that have the property,

$$Ax = \lambda x,$$

if x is the eigenvector where $x \in \mathbb{R}^n \neq \mathbf{0}$ and A is a matrix.

Linear Algebra

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  if x is the eigenvector where $x \in \mathbb{R}^n \neq \mathbf{0}$ and A is a matrix.
- The *trace* of a square matrix is the sum of the elements in the main diagonal.

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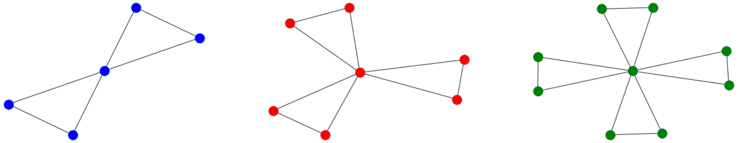
- The *vertices* represent the people.
- Any given two people that are considered friends will be represented with a connecting *edge*.

Visualization of The Friendship Theorem

- Graphs that fulfill the properties of the Friendship Theorem are **windmill graphs**.



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- To do this, we will create a counterexample graph.

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- Note that G still must adhere to the condition of being a friendship graph.
- **Claim:** The graph G is a regular graph, or that $d(u) = d(v)$ for all vertices u and v.

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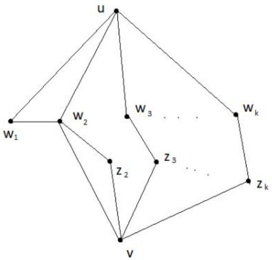
- Suppose $d(u) = k$, where $w_1, w_2, \ldots, w_k$ are the neighbors of u.
- Without loss of generality, let w_2 be adjacent to v and exactly one other w_i be adjacent to w_2, we'll say w_1.
- Then we know that w_i must also have precisely one common neighbor, z_i, where $i \geq 2$, with v.

Proof of the Friendship Theorem

- This gives us this diagram of G.



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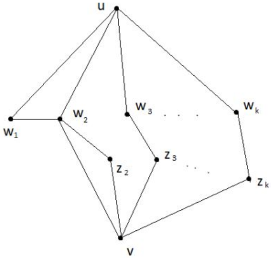

- We can now conclude that $d(v) \geq k = d(u)$, and by symmetry, $d(u) = d(v) = k$.

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- Recall that G is a counterexample, so there is no politician. This means there must exist some non-neighbor of w_2, therefore w_2 must also have degree k.

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  - k^2 is given by summing the degrees of the k neighbors of u.
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- This concludes the first part of this proof.

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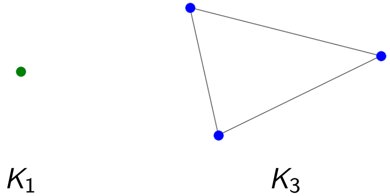
- If $k = 2$, we get $k^2 - k + 1 = 3 = n$.
- These k and n values result in K_1 and K_3. Both graphs have a "politician" vertex.



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- A must be symmetric because $a_{ij} = a_{ji}$.

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- Therefore, A^2 has eigenvalues $k - 1 + n = k^2$ (of multiplicity 1) and $k - 1$ (of multiplicity n - 1).

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- Note that $r+s=n-1$.

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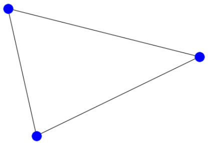

- **Contradiction!** $\square$

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- Proved that all vertices are of degree k.

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- Showed that the adjacency matrix A is symmetric and diagonalizable.
- Used the eigenvalues of A to produce a contradiction.

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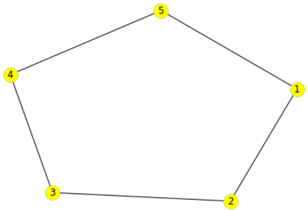

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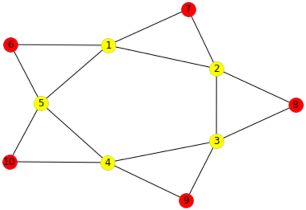

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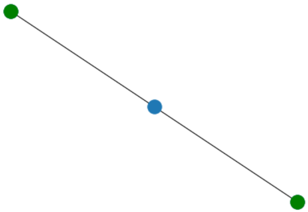

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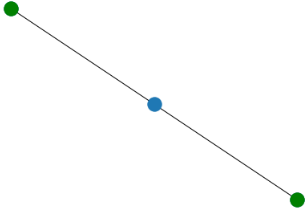

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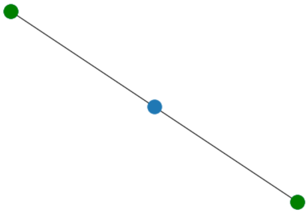

- This reconstruction is another way to present the friendship condition.
- What if the path length is greater than 2?

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- A general proof of this conjecture has yet to be formulated.

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- It uses applications of both graph theory and linear algebra.
- The Friendship Theorem is a fascinating example of a real life problem that can be translated to and solved with mathematics.

References

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