The Division Algorithm

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Math 479 Project #1

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- The division of integers is conceptualized by the Division Algorithm.
  - This algorithm falls under the scope of Euclidean division.
- Its proof and applications are diverse and important in theoretical and applied mathematics.

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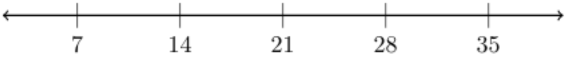

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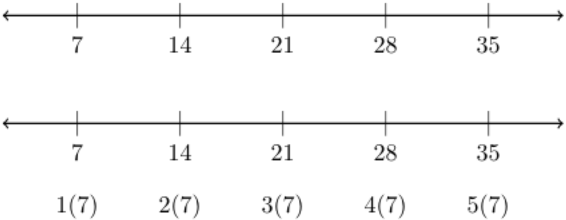

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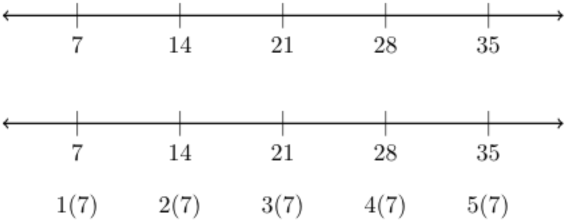

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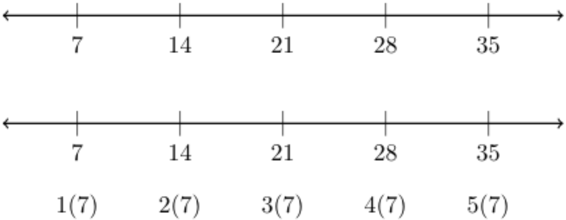

$$26 = 7 + 7 + 7 + r$$

$$26 = 3(7) + 5$$

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$$\text{dividend} \div \text{divisor} = \text{quotient} + \text{remainder}$$

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Every nonempty set of non-negative integers has a smallest element.

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- We will use the Well Ordering Principle to prove the Division Algorithm.

The Division Algorithm

Theorem

If a and b are integers, with $b > 0$, there exist unique integers q and r such that

$$a = qb + r \quad \textit{with} \quad 0 \leq r < b.$$

The integer q represents the quotient and r represents the remainder of the division of a by b.

The Division Algorithm Visualization

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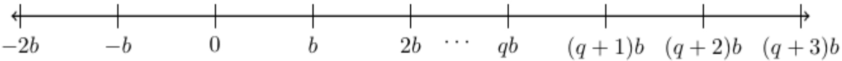

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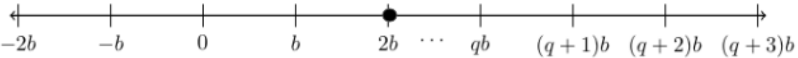

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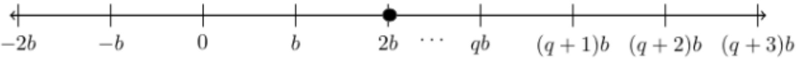

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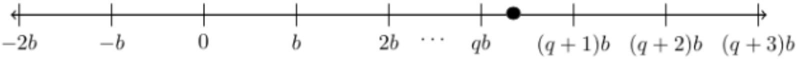

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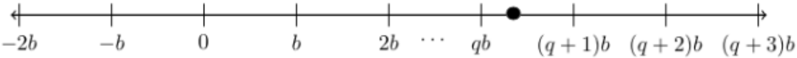

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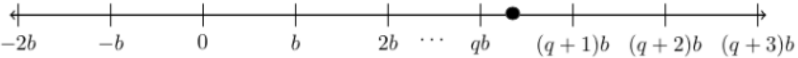

$$r \neq 0$$

$$0 < r < b$$

Idea of the Proof of the Division Algorithm

Idea of the Proof:

- Show that q and r both exist and are unique integer values.

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  - Introduce two new integers, r' and q'.
  - Show $r' = r$ and $q' = q$.

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x	$26 - 7x$
-2	40
-1	33
0	26
1	19
2	12
3	5

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- Notice that $r = 5$ is the smallest element in S and $x = q = 3$.

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    $a - bx = a - b(a) = a(1 - b) \geq 0 \Rightarrow a(1 - b) \in S$.
- Both cases result in S being nonempty.

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$$\begin{aligned} r &= a - bq & &\geq b \\ r - b &= a - bq - b & & \\ r - b &= a - b(q-1) & & \\ r - b &= a - b(q-1) & &\geq b - b = 0 \end{aligned}$$

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- So, $r - b \in S$ but $r - b < r$. Contradiction!
- Thus, r and q both exist.

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- Suppose there exist integers r, r', q, q' such that $a = bq + r$, $a = bq' + r'$, $0 \leq r < b$, and $0 \leq r' < b$.

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- Notice the left hand side is a multiple of b and $0 \leq r' - r < b$ on the right hand side.
- Thus, $b(q - q') = r' - r = 0$ so $r = r'$ and since $b > 0, q = q'$. $\square$

Application of the Division Algorithm

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- **The Euclidean Algorithm** is an algorithm that allows one to find the greatest common divisor of two integers by repeatedly performing the Division Algorithm.
  - The GCD of two integers a and b is the greatest positive integer d that is a divisor of both a and b, denoted $d = \gcd(a, b)$.

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- The Euclidean Algorithm spells out the following steps to compute $d = \gcd(a, b)$:

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a &= bq_1 + r_1 \\
b &= r_2 q_2 + r_2 \\
r_1 &= r_2 q_3 + r_3 \\
&\vdots \\
r_{n-2} &= r_{n-1} q_n + r_n \\
r_{n-1} &= r_n q_{n+1}
\end{aligned}
$$

where $r_n = d$.

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- The Euclidean Algorithm, combined with the Extended's integers r, s, both aid in creating and deciphering keys in RSA cryptography.
    - RSA cryptography is used to enhance the security of data encryption for emails, transactions, etc.

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  - It's the basis of several number theory concepts, including the Euclidean Algorithm, which was created around 300 BC.
  - Its uses can also extend to significant real-world problems, like online security.

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