STAY IN YOUR ASSIGNED SEAT

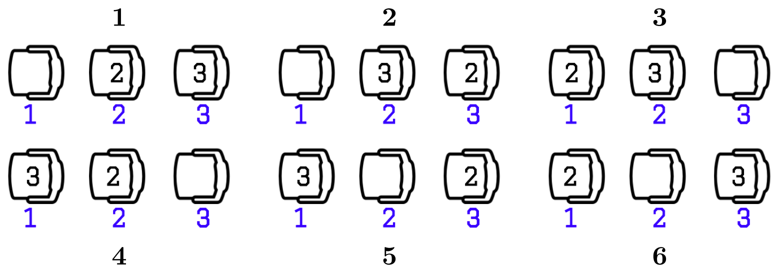
Abstract. Suppose that n passengers of a CatTran are assigned to n different seats and are required to sit there. Many passengers are lax and ignore this rule, all but one passenger— the stickler. If the stickler sees anyone sitting in his assigned seat, he will cause a commotion on the CatTran and force passengers back to their assigned seats dependent on where they were originally sitting. By utilizing permutations to represent the seating on the bus, investigating the cycles within, and building bijections, we explore and answer several questions about the fate of the passengers and their seats.

1. Introduction

Breaking news: Western Carolina University has just implemented a new policy for every CatTran passenger. Each passenger is assigned a seat number on the CatTran and is encouraged to abide by the rule and sit in their assigned seat. Despite the new rule, most of the students at WCU are very casual and will sit in any seat available. However, not every student is this easygoing regarding the new seating policy, and we will refer to these students as *sticklers*. Sticklers want to sit in their assigned seat and have no problem causing a huge commotion to make that happen. What happens if a stickler enters the CatTran and sees his seat is taken?

Imagine Adam, a freshman at WCU who is eager to get to class (and is unaware of the new rule), and is always the first person to board the n-seat CatTran. After he boards, $n-2$ more students board the CatTran and sit in random seats. The last student to board is a stickler, and he will not accept someone else sitting in his seat. If someone is sitting in the stickler's seat, he will insist that person goes to their personal assigned seat and displace whoever is sitting in it. The stickler makes sure that every successively displaced passenger goes to their specific assigned seat until no one else is displaced. The commotion that arises on this CatTran leads to an interesting question about Adam's CatTran fate: what is the probability that Adam, who boarded the CatTran first, will be forced to move from the seat he randomly chose?

It is clear that if Adam and the stickler are the only two passengers on the CatTran, the probability that Adam is booted from his randomly selected seat is $1/2$. This is because his only options are either sitting in his assigned seat or the stickler's seat. When the number of seats increases to $n=3$, there are more seating arrangement possibilities. Consider the following visualization of seating arrangements where the stickler boards last and has yet to take a seat. In the figure, the black numbers in the seat represent the passengers on the CatTran and the blue numbers represent the seat numbers. For ease, throughout this paper we will label 1 as the stickler, 2 as Adam, and $3, \ldots, n$ being the other passengers on the CatTran.



Notice that in cases 1 and 2, the stickler's only option is to sit in his own seat when he boards, therefore he is happy and no one else is displaced from their seat. In case 4, person 3 is seated in the stickler's seat so person 3 is booted to his own seat, and since seat 3 is open, only person 3 is displaced. Since Adam was in his own assigned seat the whole time so he never had to move. In cases 3 and 6, Adam is seated in the stickler's seat, and when he boards he forces Adam to move his seat. In case 5, passenger 3 is seated in the stickler's seat and is forced to move to seat 3, where Adam is sitting. In turn, this forces Adam to move to his assigned seat. In 3 out of the 6 possible seating arrangements, the stickler presses Adam to follow the rules and sit in his own assigned seat.

It is now clear that when $n = 3$, the probability that the stickler forces Adam to move is also $1/2$, just like when $n = 2$. This realization leads to a more generalized question– is the probability that Adam will be thrown from his seat always $1/2$, regardless of how many seats there are on the CatTran? This paper focuses on investigating this question through several different approaches by representing the disturbance on the CatTran with permutations.

2. Background Information

In this section, the definitions and necessary background information that are relevant to the CatTran seating fiasco will be introduced.

Consider a function f that maps elements from set A to set B. This is represented with the notation $f : A \mapsto B$. A *transformation* is a function that maps a set to itself, or $f : A \mapsto A$. The function f is considered *surjective* if there exists an $a \in A$ for every $b \in B$, with $f(a) = b$. The function f is considered *injective* if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$, for $a_1, a_2 \in A$. A *bijective* function is a function that is both surjective and injective. If f is bijective, then the set A is the *domain* of f and the set B is the *image* of f. The following theorem will be useful when we build bijections:

Theorem 1. *The inverse of a bijection is a bijection.*

Proof. Assume $f : A \mapsto B$ is a bijection. This means that f is injective and surjective.

First we will show f^{-1} is injective. Assume that $f^{-1}(b_1) = f^{-1}(b_2)$ for some $b_1, b_2 \in B$. Since f is surjective, there exist $a_1, a_2 \in A$ such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Since we assumed $f^{-1}(b_1) = f^{-1}(b_2)$, we know $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$. Since f^{-1} is the inverse of f, this means

$a_1 = a_2$. It follows that $f(a_1) = f(a_2)$ because f is a bijective function, and therefore $b_1 = b_2$. Thus, f^{-1} is injective.

We will now show f^{-1} is surjective. Since $f : A \mapsto B$, for every $a \in A$ there exists an element $b \in B$ such that $b = f(a)$. Applying the inverse of f, we get $f^{-1}(b) = f^{-1}(f(a)) = a$. Thus, for every $a \in A$, there exists an element $b \in B$ such that $f^{-1}(b) = a$. Hence, f^{-1} is surjective.

Since f^{-1} is both injective and surjective, f^{-1} is bijective.[2] □

The seating arrangements on the bus can be viewed as a random permutation on the set $[n]$, where a *permutation* is a bijection between all of the elements in the set $[n]$. For example, a random permutation of $[6] = \{1, 2, 3, 4, 5, 6\}$ can be represented in the following form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 6 & 2 & 1 \end{pmatrix}$$

which is referred to as the two-line notation. This notation means that 1 maps to 3, 3 maps to 4, 4 maps to 6, 6 maps to 1, and so on. It can also be represented in one-line notation

$$\sigma = (1346)(25)$$

with both the one-line and two-line representations equivalent. In this paper, the one-line representation will be referred to as the *first-cycle* representation of σ. The *symmetric group*, denoted $\mathcal{S}_n$, is the group of permutations on a set with n elements. There are $n!$ ways to permute a set with n elements.

Two more theorems that will greatly assist our exploration in the probability that Adam is booted from his seat are *The Addition Principle* and *The Multiplication Principle.*

Theorem 2 (The Addition Principle)**.** *Let $S_1, S_2, \ldots S_n$ be a finite collection of pairwise disjoint sets. Then the total number of elements in $S_1 \cup S_2 \cup \ldots \cup S_n$ is given by the sum of the cardinalities of the sets: $|S_1 \cup S_2 \cup \ldots \cup S_n| = |S_1| + |S_2| + \ldots + |S_n|$.*

Theorem 3 (The Multiplication Principle)**.** *Let $S_1, S_2, \ldots S_n$ be a finite collection of sets. Then the number of elements in $S_1 \times S_2 \times \ldots \times S_n$ is given by $|S_1 \times S_2 \times \ldots \times S_n| = |S_1| \times |S_2| \times |S_n|$.*

3. Main Results

3.1. **Relating the Seating Arrangement to Permutations.** To visualize the CatTran situation mathematically, we will translate the seating arrangement of the passengers on the bus to a permutation. First, we will label all passengers (other than the stickler and Adam) as unique numbers 3 through n, and we will assume that for each $k \in [n]$, passenger k is officially assigned seat k. Consider a random permutation $\sigma : [n] \mapsto [n]$ where $\sigma(k)$ represents the seat that passenger k is sitting in. When the stickler enters the CatTran, the only empty seat remaining will be $\sigma(1)$. If $\sigma(1) = 1$, then the stickler is happily seated in his assigned seat and the CatTran takes off. If $\sigma(1) \neq 1$, then the stickler will refuse to sit until his assigned seat is open and chaos ensues.

The two-line representation of the random permutation of $[6] = \{1, 2, 3, 4, 5, 6\}$ that was introduced earlier can be interpreted as where each passenger on a CatTran with $n = 6$ seats sat before the stickler boarded.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 6 & 2 & 1 \end{pmatrix}$$

The first row of the two-line representation is the number assigned to the person and the second row is the seat that the person from the first row has occupied. In this example, the only available empty seat when the stickler enters the CatTran is seat 3. The stickler forces person 6 from his assigned seat and makes person 6 take their assigned seat, too. This forces person 4 from their seat, who then forces person 3 from seat 4. Since seat 3 is open, person 3 retreats to seat 3 and the stickler can sit down in his assigned seat smugly.

Recall that the the one-line representation of σ is $\sigma = (1346)(25)$. Here, the stickler is always the first value in the first cycle of the one-line representation because he boards the CatTran last and begins the switching of seats. If his seat were to be empty in a random permutation, the one-line representation would have the form $\sigma = (1)(\ldots)(\ldots)\ldots$ where the first two ellipses can represent any disjoint subsets of $[n]\backslash 1$. In this case, the stickler is still the first value in the first cycle.

The key takeaway from the commotion that occurred on the CatTran is that persons $3, 4, 6$ were forced to move from their randomly selected seats, and these values are the same values that are in the first cycle with 1 in the one-line representation of σ.

Theorem 4. *Adam has to move seats if and only if Adam is in the first cycle of the one-line representation of σ.*

Proof. We will first prove the forward direction that if Adam is forced to move, then he is in the first cycle. Suppose Adam is forced to move from his seat after the stickler boards the CatTran. This means that Adam must be in the same cycle as the stickler in σ, otherwise he would not have had to switch seats. If Adam and the stickler were not in the same cycle, this would mean that Adam and the persons in his cycle are in each others' respective seats and do not mind. Since Adam and the stickler must be in the same cycle, we also know that this cycle must be the first in the one-line representation, because the stickler boards the CatTran last and that is when people are forced to move from their seats if the stickler's seat is taken.

Now we will prove that if Adam is in the first cycle of the one-line representation of σ, then he will have to move. Assume that Adam is in the first cycle. We know that the stickler is always the first value in the first cycle of the one-line representation of the permutation. Thus, Adam and the stickler are in the same cycle. Since they are in the same cycle, the stickler will eventually force Adam to move from his seat as he forces everyone in the cycle to return to their assigned seats. So, regardless of Adam's position in the cycle, as long as he is in the first cycle, he will be forced to move by the stickler. □

This result shows that regardless of what value n is, Adam will be forced to move from his seat if and only if he is in the first cycle of the one-line representation of σ. This result leads to the question: how many permutations of a set $[n]$ have 1 and 2 in the same cycle?

3.2. **Counting the Specific Permutations.** There are two interesting methods to reach the solution to how many permutations of $[n]$ have both Adam and the stickler in the same cycle. The first method includes utilizing combinatorial counting calculations.

Theorem 5. *In a set of n random permutations of $[n]$, there will be $\frac{n!}{2}$ permutations that contain 1 and 2 in the same cycle.*

Proof. We will denote Q as the number of permutations with 1 and 2 in the same cycle, and we will count how many there are using a combinatorial counting procedure.

Since we are permuting n objects from the set $[n]$, there can be cycles of lengths $2, 3, \ldots, n$. Note that no permutation that has 1 and 2 in the same cycle of length i can have 1 and 2 in a cycle of length j with $i \neq j$. This reasoning is because 1 and 2 can only occur once in a permutation. We will partition the permutations of $[n]$ that we want to count into disjoint subsets and then add the cardinalities of the subsets together using the addition principle in a sum. It follows that we will sum these from $k = 2$ to n.

To begin, for any value $2 \leq k \leq n$, we can count the number of ways we can create a $k-$element cycle that contains 1 and 2. The combination $\binom{n-2}{k-2}$ is the number of ways we can select the other $k-2$ elements of $[n]$ for the $k-$element cycle. Now that we have the elements for the cycle, we must order them inside the $k-$length cycle. For a cycle of size k, there are $(k-1)!$ possible ways to order the cycle where no two orderings are equivalent. This is because we will list 1 as the first element of the cycle, and then there are $(k-1)$ choices for the next element, $(k-2)$ choices for the following element, and so on until we reach the end of the cycle. Note the ordering of these elements is independent of the elements we chose to be in the cycle itself, meaning they do not rely on each other. Since we have utilized k elements to create a cycle, $n-k$ elements remain. The number of ways to permute these elements is $(n-k)!$. This value is also independent of the choices we previously made regarding the values in the $k-$length cycle and their ordering[1]. Now that we have three ordered steps that are independent, we can use the multiplication principle to reach the total number of permutations with 1 and 2 in the same cycle in the following sum:

$$\begin{aligned}
Q &= \sum_{k=2}^{n} \binom{n-2}{k-2}(k-1)!(n-k)! \\
&= \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!(k-2)!}(k-1)!(n-k)! \\
&= (n-2)!\sum_{k=2}^{n}(k-1) \\
&= (n-2)!\sum_{k=1}^{n-1} k \\
&= (n-2)!\frac{(n-1)n}{2} \\
&= \frac{n!}{2}
\end{aligned}$$

Thus, $\frac{n!}{2}$ of permutations of $[n]$ have 1 and 2, or Adam and the stickler, in the same cycle.

□

Since there are $n!$ random ways to form a permutation of $[n]$ and we now know that $\frac{n!}{2}$ of these random permutations contain 1 and 2 in the same cycle, we know that Adam has to switch seats with 1/2 probability.

3.3. **Forming a Bijection between Permutations.** To arrive at the same result in the previous subsection, we could instead create a bijection between two sets of permutations and interpret the cycles within. First, let $\mathcal{S}_n$ be the set of permutations on $[n]$, and then partition S_n into two different subsets: $\mathcal{A}_{(12)}$ being the subset of $\mathcal{S}_n$ whose elements have 1 and 2 in the same cycle, and $\mathcal{A}_{(1)(2)}$ being the subset of $\mathcal{S}_n$ whose elements do not have 1 and 2 in the same cycle. The cycles in these two subsets will have the form:

$$\mathcal{A}_{(12)} = \{\sigma \in \mathcal{S}_n : (1 \ldots 2 \ldots) \ldots\}$$
$$\mathcal{A}_{(1)(2)} = \{\sigma \in \mathcal{S}_n : (1 \ldots)(2 \ldots) \ldots\}$$

We will define the transformation $U : \mathcal{A}_{(1)(2)} \mapsto \mathcal{A}_{(12)}$ by erasing the back-to-back parenthesis ")(" that comes before the 2 in any permutation $\sigma \in \mathcal{A}_{(1)(2)}$. A visualization of the transformation U with $n = 7$ can be

$$\sigma = (136)(24)(75) \mapsto U(\sigma) = (13624)(75) = \tau.$$

We will define the inverse transformation $U^{-1} : \mathcal{A}_{(12)} \mapsto \mathcal{A}_{(1)(2)}$ by placing a pair of back-to-back parenthesis ")(" in front of the 2 in a $\tau \in \mathcal{A}_{(12)}$. A visualization of the transformation U^{-1} with $n = 7$ can be

$$\tau = (13624)(75) \mapsto U^{-1}(\tau) = (136)(24)(75) = \sigma.$$

It is clear that these these two transformations are both bijective because they map every unique element from the domain to a unique element in the image. Since there is a bijection defined between $\mathcal{A}_{(1)(2)}$ and $\mathcal{A}_{(12)}$, it must mean that the two sets are of the same size. This is because in the transformation U^{-1}, every element in $\mathcal{A}_{(1)(2)}$ is mapped to exactly once. The same logic holds for the transformation U and the image $\mathcal{A}_{(12)}$. Since n is finite, this means that both sets have the same number of elements. Furthermore, the two sets are disjoint and share no elements, therefore their disjoint union is the set $\mathcal{S}_n$. Since the number of permutations that can be randomly created from the set $\mathcal{S}_n$ is $n!$, it follows that the sum of the cardinalities of $\mathcal{A}_{(1)(2)}$ and $\mathcal{A}_{(12)}$ is $n!$. This means that $|\mathcal{A}_{(1)(2)}| = |\mathcal{A}_{(1)(2)}| = \frac{1}{2}n!$.

Once again, we have arrived at the result that Adam and the stickler will be in the same cycle with probability 1/2. This method of creating a bijection between the two sets, the possibilities where Adam is seated in the same cycle as the stickler and the possibilities where he is not, helps us develop a better understanding of why this probability holds.

4. Further Discussion and Extensions

We have mainly focused on the fate of Adam and his seat when the stickler boards the CatTran, but what about the other passengers who risk being forced from their seat by the stickler? We can extend the original CatTran seating problem by taking into account what the other passengers are going to experience.

4.1. Forming Another Bijection that Answers New Questions.

The simple bijection that we previously built helped us answer questions about Adam specifically, but we would like to focus on each and every passenger. In order to do so, a new bijection must be built.

Consider the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 6 & 7 & 1 & 2 \end{pmatrix}$$

where we would interpret the one-line representation of σ as $\sigma = (1346)(257)$. We will now use an equivalent interpretation of the one-line representation of σ, $\sigma = (3461)(572)$. Note that all of the elements are mapped to the same elements as the other representation, but the order in the cycle is just slightly off. The head of each cycle, or the first value in the cycle, is sent to the back of the cycle instead. Similar to how we erased parentheses in the transformations U and U^{-1}, we will now erase every pair of back-to-back parentheses in σ. We will refer to this transformation as $S : \mathcal{S}_n \mapsto \mathcal{S}_n$. For example, $S(\sigma) = (3461572)$.

Theorem 6. *The transformation $S : \mathcal{S}_n \mapsto \mathcal{S}_n$ is a bijection.*

Proof. To prove that S is a bijection, we must show that S is surjective. This is because S is defined as the mapping of $S : \mathcal{S}_n \mapsto \mathcal{S}_n$, which are both finite sets. Thus, if it is surjective, it is implied that it is injective, too. In order to prove its surjectiveness, we will define an algorithm that shows for every $\tau \in \mathcal{S}_n$, there exists a $\sigma \in \mathcal{S}_n$ where $S(\sigma) = \tau$.

Let τ be a permutation in the one line form $\tau = [a_1, a_2, \dots, a_n]$. A four step algorithm will be used to find a unique σ that fulfills $S(\sigma) = \tau$.

Step 1: Scan through τ until reaching the value 1. Insert back-to-back parentheses after the 1. If 1 is the last value in the cycle, stop.
Step 2: Determine the next smallest value x that has not been scanned over yet.
Step 3: Scan through τ until reaching x. Insert back-to-back parentheses after the x. If x is the last value in the cycle, stop.
Step 4: Repeat steps 2 and 3 until forced to stop.

Once this algorithm comes to an end, one will be left with a unique σ that fulfills $S(\sigma) = \tau$. This proves that S is surjective, therefore implying S is injective. Thus, S is bijective. $\square$

To visualize the algorithm in this proof, let $\tau = (6421735)$. After step 1, $\sigma = (6421)(735)$. Step 2 then has us determine that the next smallest value that has yet to be scanned over is $x = 3$. Step 3 leads to $\sigma = (6421)(73)(5)$. We then repeat steps 2 and 3 with $x = 5$, but since 5 is the last value in the cycle, we stop the algorithm and we are left with the result $\sigma = (6421)(73)(5)$. Note that $S(\sigma) = (6421735) = \tau$, which shows that the algorithm produced a unique σ for the given τ.

With the evidence that $S : \mathcal{S}_n \mapsto \mathcal{S}_n$ is bijective, we also know that the inverse of this bijection, S^{-1}, is also bijective by the theorem and proof discussed earlier in the paper. With the help of both of these bijections, we can now answer questions about the other CatTran passengers. One logical question that one who is concerned about the other passengers might have is: "What is the probability that the stickler will force every passenger on the CatTran out of their seat?"

4.1.1. *Forcing Every Passenger from their Seat.* In terms of permutations, we want to find the probability that values $\{2, 3, \dots, k\}$ are in the first cycle of a random permutation of $[n]$ with $2 \leq k \leq n$. First, let $\tau = [a_1, a_2, \dots, a_k]$ be the one-line representation of a random ordering of $[n]$. This permutation's random generation can be created by sampling the set $[n]$ n times without replacement. Since we know that S^{-1} is a bijection, τ has a unique permutation σ whose first cycle is $(a_1, a_2, \dots, a_j)$ where $a_j = 1$.

At this point, we can tell that the probability that the values $\{2, 3, \dots, k\}$ are in the same cycle as 1 is the same probability that the values $\{2, 3, \dots, k\}$ will appear before 1 in the same cycle. This is because in the random ordering that we generated, the values $\{2, 3, \dots, k\}$ appear randomly, too. Thus, the probability that the $\{2, 3, \dots, k\}$ appear before 1 is the same probability that 1 is the last value in the permutation τ. In this random ordering, 1 can be placed in any spot in the permutation with the same likelihood, therefore the probability that 1 is the last value is $1/k$.

It follows that the probability that the values $\{2, 3, \dots, k\}$ are in the first cycle of a random permutation of $[n]$ must be $1/k$ with $2 \leq k \leq n$. In the first bijection we created, we found that this probability is $1/2$ with $k = 2$ (the probability that Adam got bumped from his seat). When $k = n$, which refers to every passenger of the CatTran, we find that the probability that every passenger gets bumped is $1/n$. This probability reflects all of the random permutations of $[n]$ that have 1 as the last position. Every other random permutation that is not accounted for here can be accounted for with the bijection S^{-1}.

4.1.2. *How Many Passengers Get Bumped?* A similar question to the one just investigated is: "What is the total number of passengers that get bumped from their seat?" In a similar fashion, the bijections S and S^{-1} can assist answering this question. First, consider the value

$$N = |\{j \in \{2, 3, \ldots, k\} \text{ such that } j \text{ get bumped}\}|$$

which represents the size of the set of permutations where j people are forced from their seats. The bounds of N are integer values m with $0 \leq m \leq k - 1$ because N is a random variable that depends on the random ordering of a permutation τ of $[n]$. To answer the raised question, we want to find the probability of the event $\{N = m\}$ for each possible m. Our argument is similar to one before, that the following are equivalent:

$$N = m \Leftrightarrow \text{ in } \tau \text{ there are } m \text{ elements of } \{2, 3, \ldots, k\} \text{ that precede } 1.$$

If we randomly order the values of $[k]$, we know that 1 has the same probability of occurring at each of the k possible places. It follows that the probability that 1 is in the space $m + 1$ is the same for all $0 \leq m < k$. This means that N is normally distributed on the set $\{0, 1, \ldots, k - 1\}$, thus

$$P(N = m) = \frac{1}{k} \text{ for all } 0 \leq m < k.$$

This result is not shocking, as it is the same result we concluded in the previous subsection. However, what is shocking is how the values of m and n do not affect $P(N = m)$. We can further investigate this topic by calculating the expected value of the random variable N.

$$E[N] = \sum_{m=0}^{k-1} m \times \frac{1}{k} = \frac{k(k-1)}{2} \times \frac{1}{k} = (k-1)/2$$

tells us that the expected number of passengers on the CatTran to be bumped is $(k - 1)/2$. This result could have also been reached from earlier results, such as the fact that there are $k - 1$ passengers in the set $\{2, 3, \ldots, k\}$, and each passenger has probability $1/2$ of being forced from their seat. Linearly, it makes sense that half of the passengers on the bus, not including the stickler, are expected to be involved in the stickler's commotion.

5. Conclusion

The reason why studying the CatTran rule, which is a seemingly realistic real-world problem, is so fascinating in mathematics is because we were defining bijections on the symmetric group $\mathcal{S}_n$[3]. Studying the seating arrangements on the CatTran and what can come from them is not the first time the permutations of $\mathcal{S}_n$, bijections, probability, and combinatorics have been related. Spitzer's identity is a result in combinatorial probability which defines the distribution between partial sums and maximal partial sums for different random variables. Although seemingly unrelated to the CatTran problem, the main idea behind this identity is that there is a bijection between $\mathcal{S}_n$ and $\mathcal{S}_n$, just like we had created. Another important bijection that is related to permutations is the Robinson–Knuth–Schensted, or RKS, correspondence. RKS is a combinatorial bijection created between permutations and pairs of Young tableaux, which are combinatorial objects that can describe the general symmetric groups.

The CatTran rule and the stickler's infamous actions can lead to further investigation. For example, one can continue to study and ask new questions about the special subsets of the randomly generated permutations of the passengers and their seats. This may include finding the probability that in the set of derangements D_n of $[n]$, permutations without cycles of size 1, the stickler and Adam will be found in the same cycle. Similar to our explorations before, questions like this can be tackled with creating new bijections and relating generic combinatorial and probability rules.

Even if Western Carolina University realizes the commotion that is occurring on the CatTran and decides to abolish the new rule, we have gained great insight in relating permutations, their cycles, and bijections. Understanding the combinatorial structure of the passengers on the CatTran and how bijections can explain the phenomena is a great first step in realizing how important bijections are in several realms of mathematics.

References

1. M. Budden, "Course Notes for Math 310 Discrete Structures," Western Carolina University, 2021.
2. T. Judson, "Abstract Algebra: Theory and Applications," 1^{st} edition, Virginia Commonwealth University Mathematics **7**, 2009.
3. V. Pozdnyakov, *Buses, Bullies, and Bijections,* Mathematics Magazine **89(3)** (2016).
4. F. Spitzer, *A Combinatorial Lemma and its Application to Probability Theory,* Transactions of the American Mathematical Society **82** (1956).
5. E. Weisstein, *Symmetric Group: Technical From MathWorld,* https://mathworld.wolfram.com/SymmetricGroup.html, (2021).