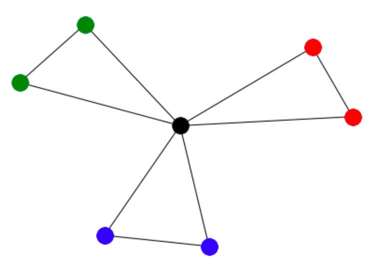
THE FRIENDSHIP THEOREM

MICHELLE HEWSON, MAXWELL PITNEY, DAKOTA THOMPSON

Abstract. The Friendship Theorem poses the situation where in a group of people, given that any pair of people have precisely one common friend, then there must exist somebody who is everybody's friend. We will discuss the background mathematics involved, a detailed analysis of Paul Erdoös' proof, how it directly connects mathematics to real world situations, and the popular extension of the friendship theorem known as Kotzig's conjecture.

1. Introduction

Picture a party where not *everyone* at this gathering is closely acquainted, but every distinct pair of people seem to have one common friend. If we were to visualise this, we would be have something similar to the graph below:



In this graph, the vertices represent distinct people and the connecting edges represent a friendship between two people. For example, the two green vertices are friends, and their common friend is the black vertex. This is because both green vertices are connected by an edge, and they are both individually connected to the black vertex with an edge. Similarly, the two red vertices are friends and the two blue vertices are friends, and the common friend for these pairs is the black vertex.

Who is this black vertex? Since the black vertex shares an edge with every other vertex in the graph, the black vertex is considered the "politician." The politician is friends with every person at the party. The politician also shares a common friend with any given pair of individuals at the party. The common friend between the politician and one red vertex is the other red vertex, and the common friend between the politician and one green vertex is the other green vertex. This pattern continues with every possible pair of distinct people at the party. This scenario is a real life portrayal of a concept known in mathematics as the Friendship Theorem.

Theorem 1 (The Friendship Theorem Summarized)**.** *Suppose in a group of people we have the situation that any pair of persons have precisely one common friend. Then there is always a person (the "politician") who is everybody's friend.* [5]

The goals of this paper include introducing the Friendship Theorem and its functionalities, explaining the proof created by Paul Erdoös, and discussing further conceptualizations of the theorem.

2. Background Information

In this section, the definitions and theorems that are relevant to the Friendship Theorem, its proof, and its further results will be introduced. The proof that we will study intertwines concepts from graph theory and linear algebra, so we will discuss standard principles from both.

2.1. **Graph Theory.** A *vertex* is a point on a graph that connects edges together. An *edge* can be used to define a relationship between vertices, such as "friendship." A *path* is a finite sequence of adjacent vertices and adjacent edges where neither can be repeated. A *cycle of length* n, denoted as C_n, is a path where the only repeated vertices are the first and last. A vertex is *adjacent* (also referred to as a *neighbor*) to another vertex if they share a common edge. The *degree* of a vertex is how many neighbors a vertex has. A graph is considered k*-regular* if every vertex in the graph has degree k. A graph with n vertices is considered *complete* if every vertex is adjacent to every other vertex in the graph, and is denoted by K_n.

2.2. **Linear Algebra.** An *identity matrix* is a square matrix whose entries in the main diagonal are all 1's and all other elements are 0's. An *adjacency matrix* is a square matrix that represents the neighbors of the vertices in a graph. Each row and each column represents a vertex, and each entry represents whether or not the two vertices are adjacent. In the matrix, $a_{ij} = 1$ if the vertices i and j are adjacent where $i \neq j$, and $a_{ij} = 0$ otherwise. A *symmetric matrix* is a matrix where $a_{ij} = a_{ji}$ for all i and j. Eigenvalues (often denoted λ) are scalars which uphold the relationship $A\vec{x} = \lambda\vec{x}$, where A is some $n \times n$ matrix and $\vec{x}$ is the $n \times 1$ nonzero eigenvector. A *diagonal matrix* is a matrix whose diagonal entries are nonzero, but all other entries in the matrix are zeroes. A *diagonalizable* matrix is a matrix which can be transformed into a diagonal matrix. Lastly, the *trace* of a matrix is the sum of its diagonal elements.

3. The Friendship Theorem

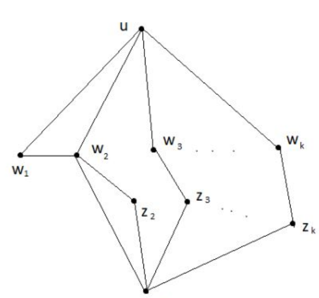
The Friendship Theorem can be translated to graph-theoretic terms and proven using concepts from both graph theory and linear algebra.

Theorem 2 (The Friendship Theorem)**.** *Suppose that G is a finite graph in which any two vertices have precisely one common neighbor. Then there is a vertex which is adjacent to all other vertices.* [5]

Proof. We will prove this using a contradiction. Suppose the theorem is false, and that the graph G is a counterexample. This means that G has no specific vertex that is adjacent to all other vertices, thus no "politician" vertex exists. The lack of a politician vertex is the main condition our graph G must satisfy. Note that the assumption that the Friendship Theorem is false still requires any two vertices to have precisely one common neighbor. It follows that cycles of length 4 cannot exist in G because vertices opposite from one another share 2 common neighbors. This will be referred to as the C_4-condition, and this will be a key point further in our proof.

(1) Initially, we need to make the claim that G is a regular graph, or that the degree of all the vertices in G are equal. Two cases must be considered: non-adjacent vertices having the same degree and adjacent vertices having the same degree.

First, we will prove that any two non-adjacent vertices have equal degree. Suppose some vertex, u, has a degree of k. Therefore, it must have k neighbors, we can call them $w_1, \ldots, w_k$. Without loss of generality, we know that exactly one w_i must be adjacent to some other vertex v because any given pair of vertices will have one common neighbor. Let w_2 be this common neighbor. We also know that there must be a common neighbor between u and w_2, so we can let that specific vertex be w_1. Lastly, we also know that all of the w_i's must also have exactly one common neighbor with v. The vertex v has a common neighbor w_2 with w_1, leaving $w_2, \ldots, w_k$. We will let $z_2, \ldots, z_k$ be the common neighbor between $w_2, \ldots, w_k$ and v. We know $z_2, \ldots, z_k$ must be distinct because if $z_i = z_j$ where $i \neq j$, we would form a cycle of length 4. This would contradict the C_4-condition we referenced earlier. This situation results in the diagram pictured below, which represents the relationship between two non-adjacent vertices in our graph G:



As this is only a part of our graph G, v must have at least k neighbors $\{w_2, z_2, \ldots, z_k\}$, so $d(v) \geq k = d(u)$. Since u and v are entirely arbitrary, we can find that $d(u) \geq k = d(v)$, and so by symmetry, we know $d(u) = d(v) = k$.

Next, we must show that all adjacent vertices have the same degree. This can easily be shown when considering any vertex different from w_2. We know that any other vertex must only be adjacent to either u or v, otherwise u and v would have more than one common neighbor. By using what we just proved about any two non-adjacent vertices, we can conclude that all vertices aside from w_2 must have degree k. Since we know G is a counterexample and has no politician, there must exist some vertex, x, that is non-adjacent to w_2. Thus, the vertices w_2 and x must be a pair of non-adjacent vertices. So, we know $d(x) = d(w_2) = k$. Therefore, we can safely claim that G is a k-regular graph.

We want to relate the number of vertices to the degree, and this can be accomplished with simple combinatorics. If we sum the degrees of the k neighbors of u we get k^2. Since every vertex (except u) has exactly one common neighbor, we have effectively counted every vertex once, except for u, which was counted k times. Thus, the equation for the number of vertices can be written as

$$n = k^2 - k + 1$$

completing the first part of the proof.

(2) Now that we have found the equation to find the number of vertices in the graph G, we can exclude some k values. Letting $k = 0$ results in

$$\begin{aligned} n &= k^2 - k + 1 \\ n &= (0)^2 - 0 + 1 \\ n &= 1 \end{aligned}$$

and letting $k = 1$ results in

$$\begin{aligned} n &= k^2 - k + 1 \\ n &= (1)^2 - 1 + 1 \\ n &= 1. \end{aligned}$$

When $k = 2$, we get

$$n = k^2 - k + 1$$
$$n = (2)^2 - 2 + 1$$
$$n = 3.$$

These two n values, $n = 1$ and $n = 3$, result in the complete graphs K_1 and K_3, respectively.

Figure 1. K_1 Graph

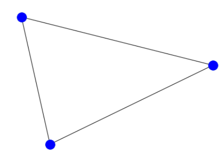


Figure 2. K_3 Graph

Note that in order to make sure our initial assumptions hold true, k must be greater than 2. This is because when $k \leq 2$, we find that n must either be 1 or 3, which means $G = K_1$ or $G = K_3$, which both have a politician vertex. Now that we have a direct relation between the number of vertices, n, and the degree of said vertices, we can apply linear algebra to achieve a contradiction.

Consider the adjacency matrix, A, of the graph G. We know that the main diagonal of A is composed of all zeros because a vertex cannot be adjacent to itself. The rest of the entries are composed of either ones or zeros. This is unhelpful because we do not explicitly know what G is. However, we do know that A must be symmetric because of the nature of having exactly one common neighbor. This can be seen by $a_{ij} = a_{ji}$, which we know is only either one or zero. Multiplying our A by itself will prove to be helpful because it takes any ambiguity out of the problem. No matter what A is, we know that A^2 fits the following form by simple matrix multiplication.

Note that A^2 can be rewritten as $(k-1)I + J$, where J is the $n \times n$ matrix composed of all ones, and the I is the $n \times n$ identity matrix. We can easily find the eigenvalues of J because it is an $n \times n$ matrix of all ones. This means that the determinant of J is 0. Thus, one of J's eigenvalues must be 0 because the determinant is a product of its eigenvalues. The nullity of J, equal to the total number of columns minus the matrix's rank, is $n - 1$. This implies that the multiplicity of the eigenvalue 0 is $n - 1$. The final eigenvalue has a multiplicity of 1. We find that it is n because $n \times \mathbf{x}$, where $\mathbf{x}$ is the only linearly independent vector, gives us J. This means $A\mathbf{x} = n\mathbf{x}$.

Then the eigenvalues of A^2 can be found using the following conjecture:

Lemma. *If B is an $n \times n$ matrix and p is an eigenvalue of B, then the matrix $W = B + c(I)$ has an eigenvalue of $\lambda = p + c$.*

Proof. Assume B is an $n \times n$ matrix and p is an eigenvalue of B. Then by definition, $B\vec{x} = p\vec{x}$ where $\vec{x}$ is the eigenvector for B associated with p.

Let $W = B + c(I)$, then

$$\begin{aligned} W &= B + c(I) \\ W\vec{x} &= (B + c(I))\vec{x} \\ &= B\vec{x} + c\vec{x} \\ &= p\vec{x} + c\vec{x} \quad \text{(recall that } B\vec{x} = p\vec{x}) \\ &= (p + c)\vec{x} \end{aligned}$$

Therefore, $p + c$ is an eigenvalue for W. □

Recall that $A^2 = J + (k-1)I$. Using the above conjecture, we find that p is an eigenvalue from our J matrix and c is $k - 1$. This leaves us with the eigenvalues $k - 1 + n$ (with multiplicity 1) and $k - 1$ (with multiplicity $n-1$). Notice that $k-1+n = k^2$ from part **(1)**. To get the eigenvalues of A, we will take the square root of the eigenvalues of A^2 because A is diagonalizable. Thus, A has the eigenvalues k (with multiplicity one) and $\pm\sqrt{k-1}$ (with multiplicity $n-1$).

We can then let r be the number of eigenvalues equal to $\sqrt{k-1}$ and s be the number of eigenvalues equal to $-\sqrt{k-1}$. It must be that $r + s = n - 1$ because we know there are $n - 1$ eigenvalues corresponding to $\pm\sqrt{k-1}$. Since we know the Trace$(A) = 0$ and that A is symmetric, we know the sum of the eigenvalues of A is equal to the trace. We can now write

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0$$

Since we previously showed that k must be greater than 2, it must be that $k \neq 0$. Therefore, it must follow that $r \neq s$. When we factor and square the above equation, we arrive at

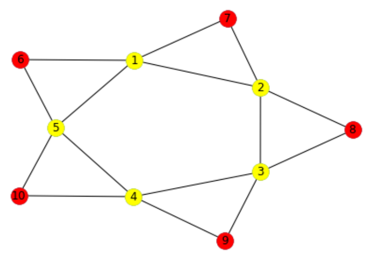
$$(r-s)^2(k-1) = k^2$$

This equation shows that $k - 1$ divides k^2, and the only integer k value that satisfies this condition is $k = 2$. However, when $k = 2$, we previously showed that $n = 3$. This set of n and k values results in the K_3 complete graph, which is a contradiction to our assumption that G has no politician vertex.

□

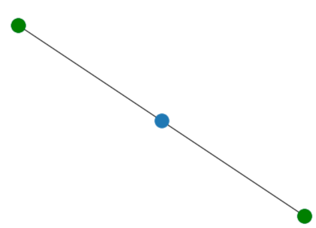
4. Further Discussion

It is important to note that the Friendship Theorem only applies to finite graphs. To show this, we could begin with a C_5 graph. Then a common neighbor could be added to the graph for each pair of vertices currently without a common friend. This process could be repeated using induction until an countable yet infinite graph is presented. However, this friendship graph will not have a "politician".[5] Below is a graph which illustrates the process mentioned. In this graph, the yellow vertices represent the C_5 graph, and the red vertices are the common friends which are added in the first step of induction.



Furthermore, the windmill graphs we presented earlier in this paper cannot exist without a vertex that acts as a common neighbor to all other vertices. Without it, the graph is not a windmill graph by definition. This further illustrates that our countably-infinite graph is not a windmill graph, and therefore does not fulfill the requirements of a Friendship Graph.

Another way to interpret the Friendship Theorem is to translate the concept of a common friend between two vertices into paths of length 2. Any two vertices that share a common friend have a path of length 2 between them. This is illustrated in the figure below, where the blue vertex is the common friend between the two green vertices.



What if we change the restrictions on which vertices have a common friend by increasing the path length? Does the Friendship Theorem still hold? Anton Kotzig wrote a conjecture that deems this question impossible.

Kotzig's Conjecture. *Let $\ell > 2$, then there are no finite graphs with the property that between any two vertices there is precisely one path of length ℓ.* [5]

The above conjecture has been verified for any $\ell \leq 33$, with Kotzig himself verifying $\ell \leq 8$, and Alexandr Kostochka verifying the remainder. [5]

5. Conclusion

The original person to pose the question is unknown, however, there have been various attempts at solving the puzzle. The most accomplished proof was done by Paul Erdös, Alfred Rényi, and Vera Sós. The proof discussed here closely follows their proof, with expansion in a few areas when needed. The proof connects two branches of mathematics, linear algebra and graph theory, in a quite elegant manner. It uses the relationships between people at a party and standard results involving eigenvalues to arrive at a clever contradiction.

References

1. Amulya Bhattaram, *The Friendship Theorem,* Aug. 2020, http://simonrs.com/eulercircle/pftb2020/amulya-friendship.pdf.
2. Elizabeth Lambers, *The Friendship Theorem,* https://math.mit.edu/∼ apost/courses/18.204-2016/18.204_Elizabeth_Walker _final_paper.pdf.
3. Jacob Fox, *Lecture 20: Friends and Policitians 1 ... - MIT MATHEMATICS,* http://math.mit.edu/∼fox/MAT307-lecture20.pdf.
4. Katie Leonard, *The Friendship Theorem and Projective Planes,* 7 Dec. 2005, http://web.pdx.edu/∼caughman/Katie.pdf.
5. Martin Aigner, and Günter M . Zieglar, "Of Friends and Politicians", *Proofs from the Book,* Springer, New York, 2004, pp. 223–225