Stay in Your Assigned Seat

Michelle Hewson

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Math 479 Project #3

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- What happens if a stickler enters the CatTran and sees his seat is taken?

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- There are two cases:
  - The stickler's assigned seat is free and he sits down. The CatTran departs!
  - The stickler's seat is taken. Here's where the commotion begins.

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- What is the probability that poor Adam will be forced to move from his randomly-chosen seat?

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- This means the probability that Adam gets booted is $1/2$.

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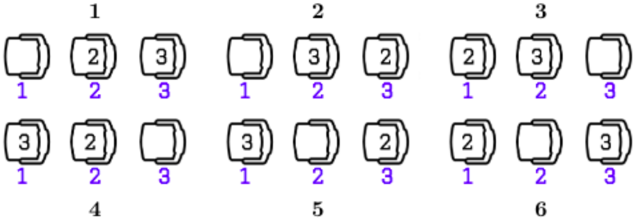

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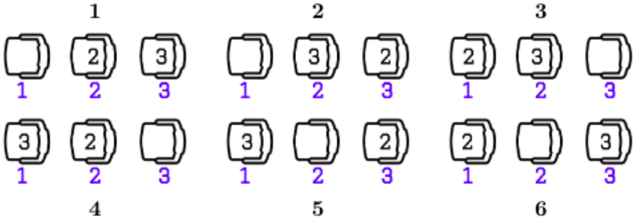

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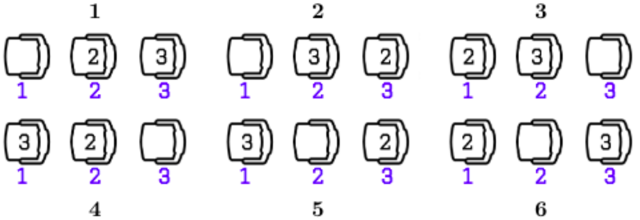

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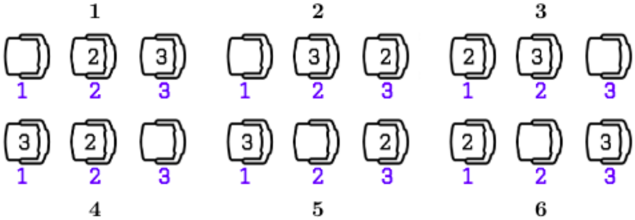

- In 3 of the arrangements, Adam is forced to move.
- Is the probability that Adam will be forced to switch seats always $1/2$?
- We will investigate this question by representing the disturbance on the CatTran with permutations.

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- A *transformation* is a function that maps a set to itself, or $f : A \mapsto A$.

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    - There are $n!$ ways to permute a set with n elements.

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- These permutations will represent the seating arrangements on the CatTran.
- The passengers, not including Adam and the stickler, will be represented by the values 3 through n.
- Assume for each $k \in [n]$, passenger k is officially assigned seat k.

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    - If $\sigma(1) = 1$, then the stickler is happily seated in his assigned seat and the CatTran takes off.
    - If $\sigma(1) \neq 1$, then the stickler will refuse to sit until his assigned seat is open and chaos ensues.

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- The takeaway is that persons $3, 4$, and 6 were forced to move from their randomly selected seats, and these values are also in the first cycle of the one-line representation of σ.

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Theorem

In a set of $n!$ random permutations of $[n]$, there will be $\frac{n!}{2}$ permutations that contain 1 and 2 in the same cycle.

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- Let k be the size of a cycle in a permutation of $[n]$ that contains the values 1 and 2, where $2 \leq k \leq n$.
- There are $\binom{n-2}{k-2}$ ways to select the other elements in the cycle, not including 1 and 2.

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- Since this cycle will follow the k-length cycle, we have defined an injective mapping from $[n] \mapsto [n]$.
- This means every value of n has been utilized in the permutation.

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- With our previous observations, we get the sum:

$$
\begin{aligned}
Q &= \sum_{k=2}^{n} \binom{n-2}{k-2} (k-1)!(n-k)! \\
&= \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!(k-2)!}(k-1)!(n-k)! \\
&= (n-2)! \sum_{k=2}^{n} (k-1) \\
&= (n-2)! \sum_{k=1}^{n-1} k \\
&= (n-2)! \frac{(n-1)n}{2} \\
&= \frac{n!}{2}
\end{aligned}
$$

Creating a Bijection

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$$\mathcal{A}_{(12)} = \{\sigma \in \mathcal{S}_n : (1 \ldots 2 \ldots) \ldots\}$$
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$$\sigma = (136)(24)(75) \mapsto U(\sigma) = (13624)(75) = \tau.$$

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- Define the inverse transformation $U^{-1} : \mathcal{A}_{(12)} \mapsto \mathcal{A}_{(1)(2)}$

$$\tau = (13624)(75) \mapsto U^{-1}(\tau) = (136)(24)(75) = \sigma.$$

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- $\mathcal{A}_{(1)(2)} \sqcup \mathcal{A}_{(12)} = \mathcal{S}_n$
- $n!$ possible permutations can be created from $\mathcal{S}_n$, so $|\mathcal{A}_{(1)(2)}| + |\mathcal{A}_{(12)}| = n! \rightarrow |\mathcal{A}_{(1)(2)}| = |\mathcal{A}_{(12)}| = \frac{n!}{2}$

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- This means that for any value of n, Adam has $1/2$ probability of being forced from his seat.

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- $S^{-1} : \mathcal{S}_n \mapsto \mathcal{S}_n$ where
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Forcing Every Passenger from their Seat

- We want to find the probability that values $\{2, 3, \ldots, k\}$ are in the first cycle of a random permutation of $[n]$ with $2 \leq k \leq n$.

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- Let $\tau = [a_1, a_2, \ldots, a_k]$ be the one-line representation of a random ordering of $[n]$.
- Since we know that S^{-1} is a bijection, τ has a unique permutation σ whose first cycle is $(a_1, a_2, \ldots, 1)$.

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- The probability that $\{2, 3, \ldots, k\}$ precede 1 in τ is the same as the probability that 1 is the last number in a random ordering of $[k]$.
- Every placement of 1 is equally likely, so the probability that 1 comes last is $1/k$.

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- Remember that we found that this probability is $1/2$ with $k = 2$ (the probability that Adam got bumped from his seat).
- When $k = n$, we find that the probability that every passenger gets bumped is $1/n$.

Forcing Every Passenger from their Seat

- The probability that the values $\{2, 3, \ldots, k\}$ are in the first cycle of a random permutation of $[n]$ must be $1/k$ with $2 \leq k \leq n$.
- Remember that we found that this probability is $1/2$ with $k = 2$ (the probability that Adam got bumped from his seat).
- When $k = n$, we find that the probability that every passenger gets bumped is $1/n$.
- This probability reflects all of the random permutations of $[n]$ that have 1 as the last position.

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  - The probability of everyone being bumped from their seat is $1/n$.
- More questions can be asked and answers can be arrived at with our bijections.

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- Spitzer's identity is a result in combinatorial probability where the main idea is that there is a bijection between $\mathcal{S}_n$ and $\mathcal{S}_n$, just like we had created.
- The Robinson–Knuth–Schensted, or RKS, correspondence is a combinatorial bijection created between permutations and pairs of Young tableaux, which are combinatorial objects that can describe the general symmetric groups.

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- Understanding the combinatorial structure of the passengers on the CatTran and how bijections can explain the phenomena is a great first step in realizing how important bijections are in several realms of mathematics.

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