PHYS 512 Assignment 4

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Problem 1

Refer to code **ps5_Problem1.py.** Use delta function to shift gaussian.

$$f * g = \int f(x)g(\Delta - x)dx \tag{1}$$

$$f_{gauss} * \delta(\Delta - x) = \sum_i f_{gauss}(x_i)\delta(\Delta - x_i) \tag{2}$$

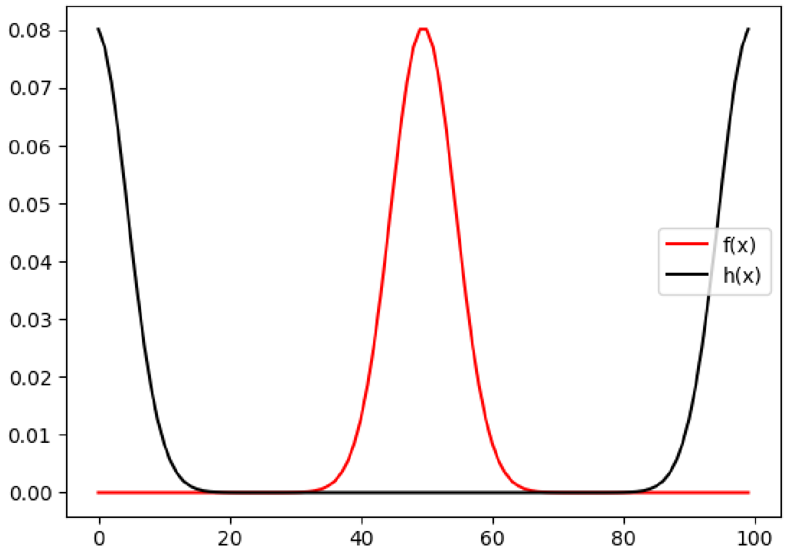

Figure 1: Shifted gaussian by half array

Problem 2

Refer to code ps5_problem2.py

Want: auto-correlation of gaussian

$$h(x) = \int f(x)g(x+y)dx$$

Go into k-space to multiply instead:

$$ift(FT(f)FT^*(g))$$

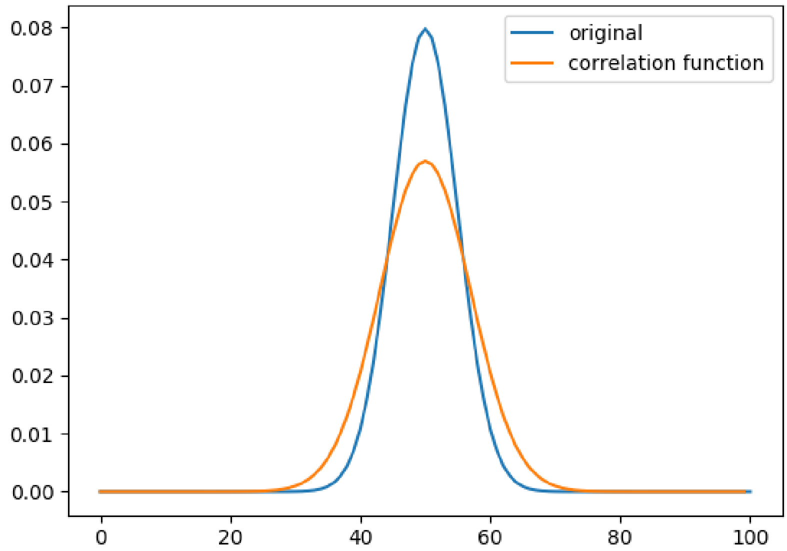

Figure 2: autocorrelation of gaussian

Problem 3

It's interesting that the autocorrelation doesn't shift with the shifted gaussian it's the autocorrelation of. The autocorrelation is independent of the shift, possibly because it's only correlation with itself so from its pov it's not shifted.Refer to code ps5_problem3.py

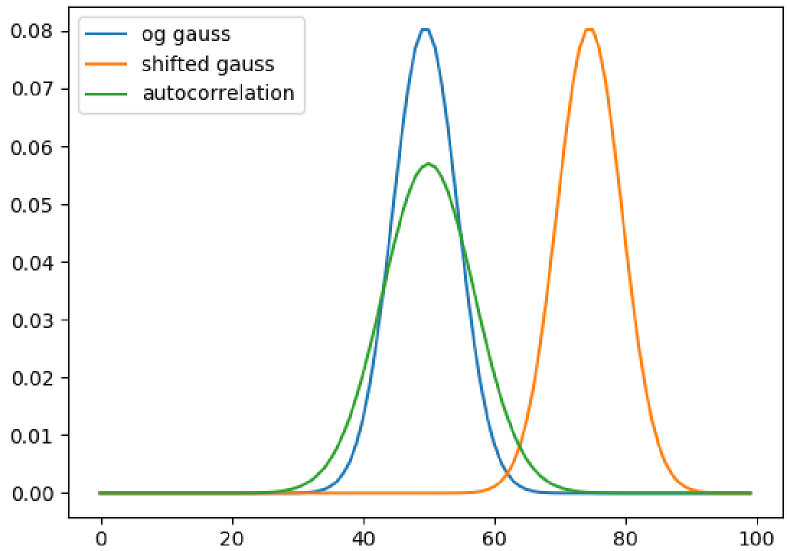

Figure 3: Autocorrelation of shifted gaussian

Problem 4

f,g aren't the same length but padded with zeros to be the same length to convolve. The length of the output is the length of the input array + the extra zeros added. Refer to code ps5_problem4.py

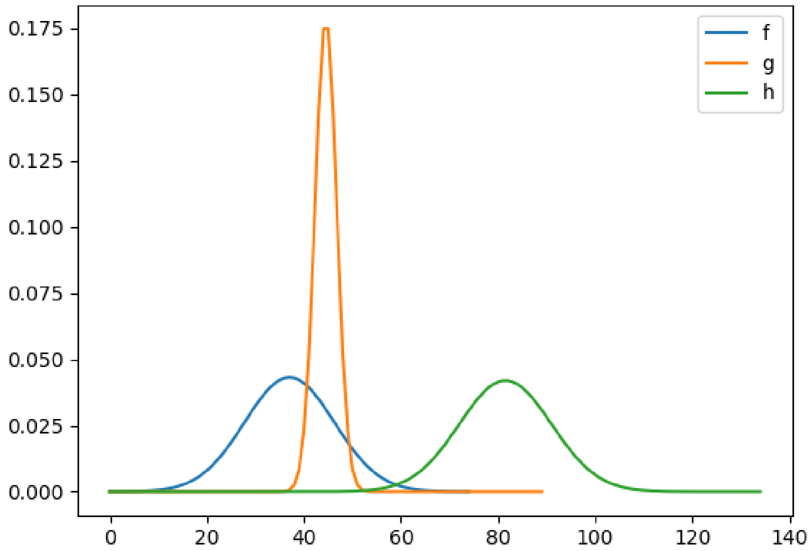

Figure 4: convolution no wrap-around

Problem 5

Part A

Geometric series:

$$\sum_{x=0}^{N} \alpha^x = 1 + \alpha + \alpha^2 + ... + \alpha^N$$
$$= \frac{1-\alpha^{N+1}}{1-\alpha} \tag{3}$$

Plug in $\alpha = e^{-\frac{2\pi ik}{N}}$
Therefore:

$$\sum_{x=0}^{N-1} \alpha^x = \frac{1-\alpha^N}{1-\alpha}$$
$$= \frac{1-e^{-2\pi ik}}{1-e^{-\frac{2\pi ik}{N}}} \tag{4}$$

Part B

As k approaches 0:

$$\begin{aligned}\sum_{x=0}^{N-1} \alpha^x &= \lim_{k\longrightarrow 0} \frac{1-e^{-2\pi ik}}{1-e^{-\frac{2\pi ik}{N}}} \\ &= \lim_{k\longrightarrow 0} \frac{\frac{d}{dk}(1-e^{-2\pi ik})}{\frac{d}{dk}(1-e^{-\frac{2\pi ik}{N}})} \\ &= \lim_{k\longrightarrow 0} \frac{2\pi i e^{-2\pi ik}}{\frac{2\pi i}{N} e^{-\frac{2\pi ik}{N}}} \\ &= \frac{2\pi i}{\frac{2\pi i}{N}} \\ &= N\end{aligned} \tag{5}$$

Show it's 0 for any integer k, not multiple of N .. need to satisfy this to be 0, integer k.

$$
\begin{aligned}
\frac{1-e^{-2\pi ik}}{1-e^{-\frac{2\pi ik}{N}}} &= \frac{1-(cos(2\pi k)-isin(2\pi k))}{1-(cos\frac{2\pi k}{N}-isin\frac{2\pi k}{N})} \\
1-(cos(2\pi k)-isin(2\pi k)) &= 0 \\
1 &= cos(2\pi k) \\
0 &= sin(2\pi k)
\end{aligned} \tag{6}
$$

Part C

Here we analytically compute dft for non-integer sine wave, and show that it's close to machine precision. Refer to code ps5_problem5c.py

$$sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

$$
\begin{aligned}
FT(sin\frac{2\pi k'x}{N}) &= \sum_{x=0}^{N-1} e^{-\frac{2\pi ikx}{N}} \frac{e^{\frac{i2\pi k'x}{N}}-e^{-\frac{i2\pi k'x}{N}}}{2i} \\
&= \frac{1}{2i}\left[\sum_{x=0}^{N-1} e^{-\frac{2\pi i(k-k')x}{N}} - \sum_{x=0}^{N-1} e^{-\frac{2\pi i(k+k')x}{N}}\right] \\
&= \frac{1}{2i}\left[\frac{1-e^{-2\pi i(k-k')}}{1-e^{-\frac{2\pi i(k-k')}{N}}} - \frac{1-e^{-2\pi i(k+k')}}{1-e^{-\frac{2\pi i(k+k')}{N}}}\right]
\end{aligned} \tag{7}
$$

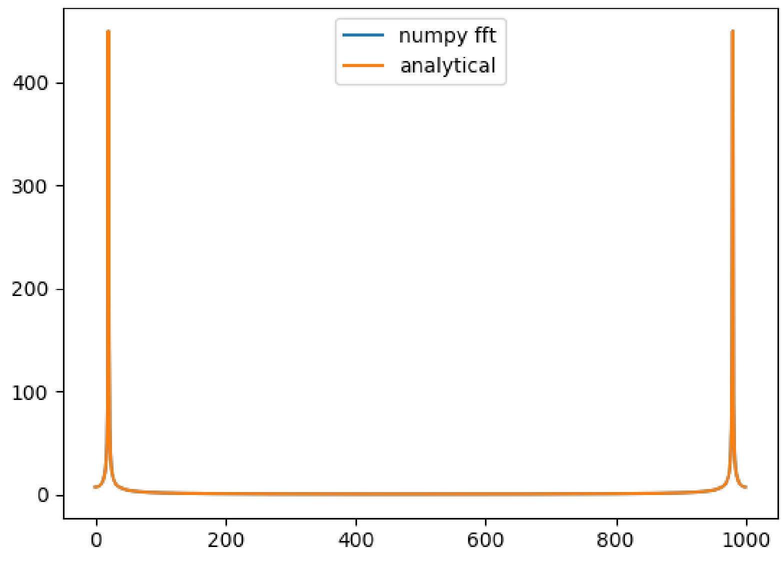

Figure 5: fourier transform analytical vs numpy

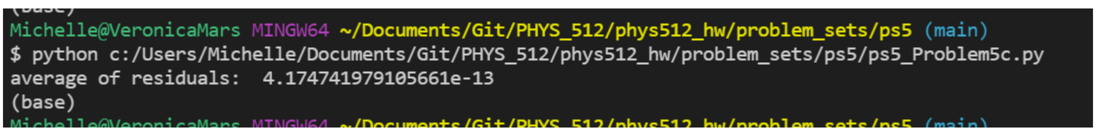

Figure 6: Residuals: analytical vs numpy

Part D

Comparing windowing with no windowing, refer to code ps5_problem5d.py

We can see that the spectral leakage is smaller with the windowing function and our delta functions are narrower.

Window function used was 0.5 - 0.5cos(2πx/N)

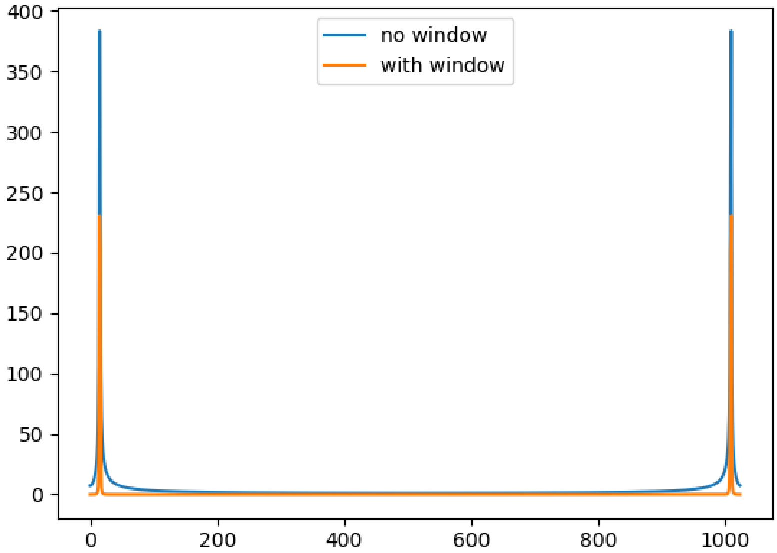

Figure 7: windowing vs no windowing

Part E

Window function used was 0.5 - 0.5cos(2πx/N), refer to code ps5_problem5e.py

Note: the number of points used were N = 1024 points, so this follows the trend N/2 and N/4 , etc.

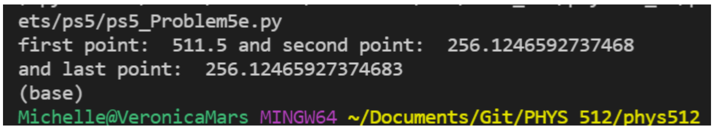

Figure 8: Numerically comparing first and last points of FT of window function

1 Problem 6

Part A

From class, we saw the correlation function between 2 points separated by dx, is proportional to N - $|dx|$ or c- $|\delta|$, which takes the shape of a triangular function.

Let $\delta = x, c = N$: Plotting our function y = N - $|x|$, bound goes from -N to N, and y-int/amplitude at N. Using integration by parts, we can evaluate fourier transform .

$$
\begin{aligned}
FT(c-|\delta|) &= \int_{-N}^{0}(x+N)e^{-\frac{2\pi ikx}{N}}dx + \int_{0}^{N}(-x+N)e^{-\frac{2\pi ikx}{N}}dx \\
&= \left[\frac{1+2\pi ik}{4\pi^2(k/N)^2} - \frac{e^{2\pi ik}}{4\pi^2(k/N)^2}\right] - \left[\frac{2\pi ik-1}{4\pi^2(k/N)^2} + \frac{e^{-2\pi ik}}{4\pi^2(k/N)^2}\right] \\
&= \frac{2-e^{2\pi ik}-e^{-2\pi ik}}{4\pi^2(k/N)^2} \\
&= -\frac{e^{-2\pi ik}(e^{2\pi ik}-1)^2}{4\pi^2(k/N)^2} \\
&= -\frac{e^{-2\pi ik}e^{2\pi ik}(2i)^2 sin^2(\pi k)}{4\pi^2(k/N)^2} \\
&= N^2\frac{(sin(\pi k))^2}{(\pi k)^2} = N^2 sinc^2(k)
\end{aligned}
\tag{8}
$$

From Fig. 9, the amplitudes of our sinc^2 function has a $\frac{1}{k^2}$ behaviour. Therefore, the fourier transform of the correlation function of the random walk has the behaviour $\frac{N^2}{k^2}$, which is proportional to k^{-2}

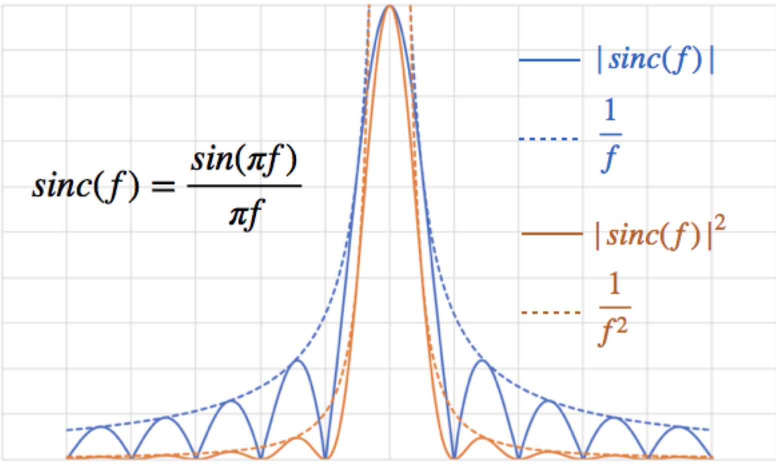

Figure 9: Sinc graph. Taken from: https://medium.com/tangibit-studios/2d-spectrum-characterization-e288f255cc59

Part B

refer to code ps5_Problem6b.py PSF was fitted to a curve proportional to k^{-2}

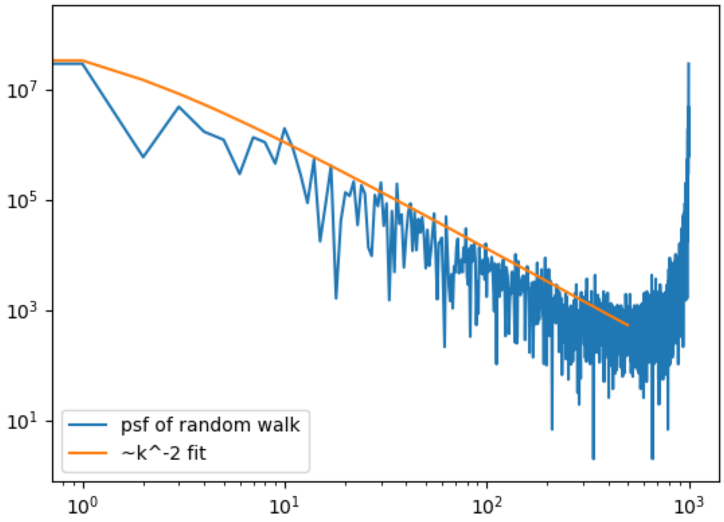

Figure 10: PSF of random walk