Math for Elementary Teachers

April 29, 2016

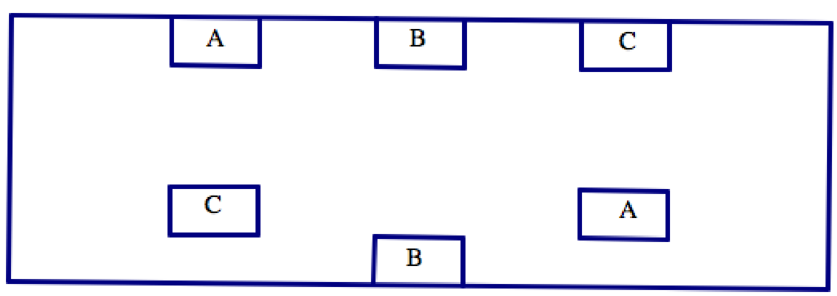
Contents

Chapter 1

Problem Solving

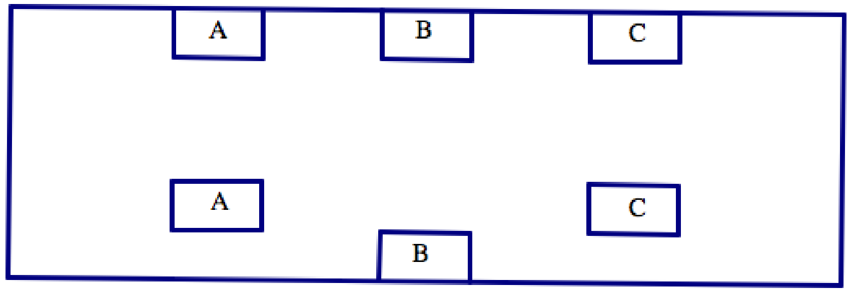
♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

Problem 1 (ABC)**.** Draw curves connecting A to A, B to B, and C to C. Your curves can't cross or even touch each other, and they can't go outside the box.



Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What makes this problem difficult? Can you change the problem slightly so that it would be easier to solve?

Problem Solving Strategy 1 (Wishful Thinking)**.** *Don't you* **wish** *the picture in the problem looked more like this one? Could you solve the problem in that case?*



Can you use a solution to this easier problem to help you solve the original problem? How? Think about moving the boxes around once the lines are already drawn.

♣♣♣ Fellow: [Would be great to have an animation here showing how one solution transforms into the other. Not sure how hard that would be to create...]

The Common Core State Standards for Mathematics (`http://www.corestandards.org/Math/Practice`) identify eight "Mathematical Practices" — the kinds of expertise that all teachers should try to foster in their students, but that go far beyond any particular piece of mathematics content. They describe what mathematics is really about, and why it is so valuable for students to master. The very first Mathematical Practice is:

> "**Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary."

This chapter will help you develop these very important mathematical skills, so that you'll be better prepared to help your future students develop them.

1.1 Problem or Exercise?

The main activity of mathematics is **solving problems**. But what most people experience in most mathematics classrooms is **practice exercises**. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should even look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then *practice* on very similar problems, with the goal of mastering that skill.

Note: What is a **problem** for some people may be an **exercise** for other people who have more background knowledge! For a young student just learning addition, this might be a problem:

Fill in the blank to make a true statement: $\_\_\_\_ + 4 = 7.$

But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice lots of skills — hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court, breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient, keeping control of the ball while making quick turns in soccer, shooting free throws in basketball, catching high fly balls in baseball, and so on — but the whole point of the sport is to *play the game*. You practice the skills so that you're better at playing the game!

The game of math, that's solving problems!

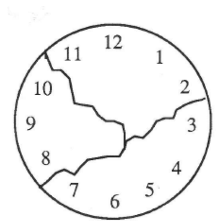
On Your Own

For each question below, decide if it is a **problem** or an **exercise**. (You do not need to solve the problems! Just decide which category it fits for you.)

After you have labeled each one, compare your answers with a partner.

♣♣♣ Fellow: [Add a mix of exercises. Use the problems provided here. Throw in some straightforward computations like adding fractions with unlike denominators, multiplying two-digit numbers, and solving some linear equations. Make a couple of them "word problems" but exercise-y ones like: "What number is 3 more than 20?" You can just flip through the book. Choose about six or seven exercises to go with the problems shows, and intersperse them.]

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?



   Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

   ♣♣♣ Fellow: [The clock picture is scanned & stolen from some long-forgotten source. Any chance we can re-create a version of it? Even take a picture of a real clock and draw some lines on it?]

2. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.

3. Letters stand for digits 0–9. In a given problem: the same letter always represents the same digit, and different letters always represent different digits. There is no relation between problems (so "A" in problem one and "A" in problem 3 might be different).

$$\begin{array}{cccc} & A & B & C \\ + & A & C & B \\ \hline & C & B & A \end{array} \qquad \begin{array}{cccc} & O & N & E \\ + & O & N & E \\ \hline & T & W & O \end{array} \qquad \begin{array}{ccc} & & A \\ & & A \\ + & & A \\ \hline & H & A \end{array}$$

Notes: "O" represents the letter O and not the number zero. Two and three digit numbers never start with 0.

4. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

   ♣♣♣ Fellow: [Can we get a picture of a balance scale? Best if it's one that is in the public sphere or that we create ourselves.]

5. How many squares are on a standard 8×8 chess board?

6. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

1.2 Problem Solving Strategies

Think back to the first problem in this chapter. What did you do to solve it? Even if you didn't figure it out completely by yourself, you probably worked towards a solution and figured out some things that *didn't* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

1.2.1 George Polya

♣♣♣ Fellow: [Get a picture of Polya and write a SHORT bio. You can pull information from the web and from the textbook. But don't overdo it.]

1.2.2 Polya's "How to Solve it"

In 1945, Polya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.

2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually **do** these steps?!?! Steps (1) and (2) are particularly mysterious! How do you "make a plan"? That's where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We'll articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve lots of problems!

We have already seen one problem solving strategy, which we called "Wishful Thinking." Don't be afraid to change the problem! As yourself "what if" questions: What if the picture was different? What if the numbers were simpler? What if I just made up some numbers? You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

Problem Solving Strategy 2 (Try Something!)**.** *If you're really trying to solve a problem, the whole point is that you don't know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what's going on.*

And equally important: If what you tried first doesn't work, try something else! Play around with the problem until you have a feel for what's going on.

1.2.3 Two More Strategies

Problem 2 (Payback)**.** Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had left. Who got the most money from Alex?

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What did you figure out about the problem, even if you haven't solved it completely.

This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before seeing the solution.

Problem Solving Strategy 3 (Draw a Picture)**.** *Some problems are obviously about a geometric situation, and it's clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that isn't geometric, like this one, thinking visually can help! Can you represent something in the situation by a picture?*

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you're totally stuck), you can watch someone else's solution: ♣♣♣ Fellow: [Add an animation of the solution described as shown in Monique's book?]

Problem Solving Strategy 4 (Make Up Numbers)**.** *Part of what makes this problem difficult is that it's about money, but there are no numbers given. That means the numbers must not be important. So just make them up!*

You can work forwards: Assume Alex had some specific amount of money when she showed up at school, say \$100. Then figure out how much he gives to each person. Or you can work backwards: suppose he has some specific amount left at the end, like \$10. Since he gave Chris half of what he had left, that means he had \$20 before running into Chris. Now, work backwards and figure out how much each person got.

Watch the solution only after you tried this strategy for yourself: ♣♣♣ Fellow: [Add an animation of the solution described as shown in Monique's book?]

If you use the "Make Up Numbers" strategy, it's really important to remember what the original problem was asking! You don't want to answer something like "Everyone got \$10." That's not true in the original problem; that's an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and **answer what was asked**!

1.2.4 Four More Strategies

Problem 3 (Squares on a Chess Board)**.** How many squares are on a standard 8×8 chess board? (The answer is *not* 64! It's a lot bigger!)

Remember Polya's first step is to understand the problem. If you're not sure what's being asked, or why the answer is not just 64, be sure to ask someone!

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What did you figure out about the problem, even if you haven't solved it completely.

It's pretty clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you've found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be *absolutely positive* that you found the right answer. You should never ask the teacher, "Is this right?" Instead, you should declare, "Here's my answer, and here's why I know it's correct!"

Problem Solving Strategy 5 (Try a Simpler Problem)**.** *Polya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an* 8×8 *checkerboard is pretty big. Can you solve the problem for smaller boards? Like* 1×1*?* 2×2*?* 3×3*?*

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that's Polya's step (2)).

Problem Solving Strategy 6 (Work Systematically)**.** *If you're working on simpler problems, it's useful to keep track of what you've figured out and what changes as the problem gets more complicated.*

For example, in this problem you might keep track of how many 1×1 squares are on each board, how many 2×2 squares on are each board, how many 3×3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	1×1 squares	2×2 squares	3×3 squares	4×4 squares	...
1×1	1	0	0	0	...
2×2	4	1	0	0	...
3×3					...
$\vdots$					

Problem Solving Strategy 7 (Use Manipulatives to Help You Investigate)**.** *Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!*

For example, in this problem it can be difficult to keep track of which squares you've already counted. You might want to cut out 1×1 squares, 2×2 squares, 3×3 squares, and so on. You can actually move the smaller squares across the checkerboard in a systematic way, making sure that you count everything once and don't count anything twice.

♣♣♣ Fellow: [Make a video showing how to do this on a 5×5 board, using cutouts of a 2×2 and / or a 3×3?]

Problem Solving Strategy 8 (Look for and Explain Patterns)**.** *Sometimes the numbers in a problem are so big, there's no way you will actually count everything up by hand. For example, if the problem in this section were about a* 100×100 *chess board, you wouldn't want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a* 100×100 *chess board just with a calculation.*

Think/Pair/Share. If you haven't done so already, extend the Table above all the way to an 8×8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8×8 chess board. Then:

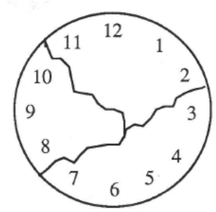
- Describe all of the patterns you see in the table.
- Can you *explain* and *justify* any of the patterns you see? How can you be sure they will continue?

- What calculation would you do to find the total number of squares on a 100×100 chess board?

(We'll come back to this question in Section 1.3. So if you're not sure right now how to explain and justify the patterns you found, that's OK.)

1.2.5 Two More Strategies

Problem 4 (Broken Clock)**.** This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?



Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

♣♣♣ Fellow: [Replace clock picture as before.]

Remember that your first step is to understand the problem. Work out what's going on here. What are the sums of the numbers on each piece? Are they consecutive? (What does that mean?)

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What progress have you made?

Problem Solving Strategy 9 (Find the Math, Remove the Context)**.** *Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.*

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
- How do I know when I'm done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it doesn't translate into solving the clock problem.

Problem Solving Strategy 10 (Check Your Assumptions)**.** *When solving problems, it's easy to limit your thinking by adding extra assumptions that aren't in the problem. Be sure you ask yourself: Am I constraining my thinking too much?*

In the clock problem, because the first solution has the clock broken *radially* (all three pieces meet at the center, so it looks like slicing a pie), many people assume that's how the clock must break. But the problem doesn't require the clock to break radially. It might break into pieces like this:

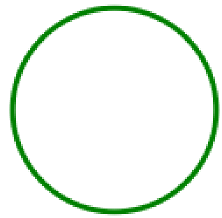
♣♣♣ Fellow: [Add a picture of a clock broken into pieces with the breaks going across, not radial. For example, three pieces: $\{11, 12, 1, 2\}$, $\{10, 9, 3, 4, 5\}$, and $\{6, 7, 8\}$.]

Were you assuming the clock would break in a specific way? Try to solve the problem now, if you haven't already.

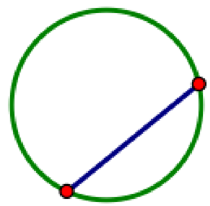
1.3 Beware of Patterns!

The "Look for Patterns" strategy can be particularly appealing, but you have to be careful! Don't forget the **"and Explain"** part of the strategy. Not all patterns are obvious, and not all of them will continue.

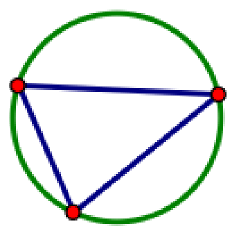
Problem 5 (Dots on a Circle)**.** Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots. How many pieces will you get?

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you haven't already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You'll want to draw a *big* circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It's probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and don't count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it *seems* to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 31! The pattern simply doesn't hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we can't explain *why* a pattern would occur, then we aren't willing to just believe it!

For example, if my number pattern starts out

$$2, 4, 8, \ldots$$

I can find *lots* of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- $2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, \ldots.$
  This is a a repeating pattern, cycling through the numbers $2, 4, 8$ and then starting over with 2.

- $2, 4, 8, 32, 256, 8192, \ldots.$
  To get the next number, multiply the previous two numbers together.

- $2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$

- $2, 4, 8, 14, 22, 32, 44, 58, 74 \ldots.$

Think/Pair/Share. Work on your own and then share your ideas with a partner:

1. For the last two patterns above, describe in words how the number sequence is being created.

2. Find at least two other ways to continue the sequence $2, 4, 8, \ldots$ that looks different from all the ones you've seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- $5 \times 5 = 25$ squares of size 1×1.
- $4 \times 4 = 16$ squares of size 2×2.
- $3 \times 3 = 9$ squares of side 3×3.
- $2 \times 2 = 4$ squares of size 4×4.
- $1 \times 1 = 1$ squares of size 5×5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5×5 chess board.

You can probably guess how to continue the pattern to any size board, but how can you be *absolutely sure* the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it's clear why the pattern *must* continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!)**.** *One of the most powerful tools we have is the use of a* variable. *If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.*

Think/Pair/Share. For now, just work on *describing* the pattern with variables.

- Stick with a 5×5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5×5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ *should* fit on a chess board of size 10×10?

- What if you don't know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ *should* fit on a chess board of size $n \times n$?

Now comes the tough part: *explaining* the pattern. Let's focus on an 8×8 board. Since it measures 8 squares on each side, we can see that we get $8 \times 8 = 64$ squares of size 1×1. And since there's just a single board, we get just one square of size 8×8. But what about all the sizes in-between?

♣♣♣ Fellow: [Insert video showing why the count for 2×2 squares should be $7 \times 7 = 49$. Model it on the general solution below.]

Think/Pair/Share. Using the video as a model, work with a partner to carefully explain why the number of 3×3 squares will be $6 \times 6 = 36$, and why the number of 4×4 squares will be $4 \times 4 = 16$.

Here's what a final justification might look like:

Solution (Chess Board Pattern)**.** Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n-k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n-k)$ times, until it hits the top right corner of the chess board. The square is in $(n-k+1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n-k)$ rows below the square, I can shift it down $(n-k)$ times until it hits the bottom row. This makes $(n-k+1)$ total rows that the square moves across, counting the top row.

So there are $(n-k+1)$ rows with $(n-k+1)$ squares in each row. That makes $(n-k+1)^2$ total squares.

Once we're sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10×10 chess board?

- What calculation would you do to solve that problem for a 100×100 chess board?

There *is* a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why* it's the right pattern for this problem, and not just another pattern that seems to work but might not continue.

1.4 Problem Bank

You have several problem solving strategies to work with. Here are the ones we've described in this Section (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help You Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you're stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

♣♣♣ Fellow: [If you have any favorite problems that don't have a lot of mathematical prerequisites — this is the first chapter! — throw them in! The more the merrier!]

Problem 6. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

♣♣♣ Fellow: [Balance scale picture?]

Problem 7. You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8. You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9. Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10. There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and can't see a thing, but you can move the coins around, and you can flip them over. However, you can't tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11. The digital root of a number is the number obtained by adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number. For example, the digital root of 98 is 8, since

$$9 + 8 = 17 \quad \text{and} \quad 1 + 7 = 8.$$

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns? Can you predict the digital root of a number without computing it?

Problem 12. If this lattice were continued, what number would be directly to the right of 98?

	3		6		9		12		...
1	2	4	5	7	8	10	11	13	...

Problem 13. Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

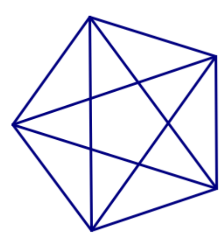
Problem 14. There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high-fives were there?

Problem 15. In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

Problem 17. Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18. How many triangles are in this picture?

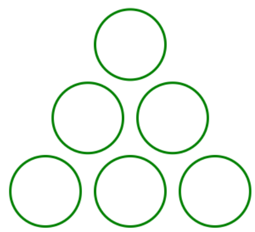


Problem 19. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.
Example: This is a difference triangle, but it doesn't work because it uses 1 twice and doesn't have a 6:

$$\begin{array}{ccccc} 4 & & 5 & & 3 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

Problem 20. Certain pipes are sold in lengths of 6 inch, 8 inch, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21. Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22. Find a way to cut a circular pizza into 11 pieces using just four straight cuts.

1.5 Careful use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it's not. Mathematics is a social endeavor. We don't just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves lots of scratch paper and careful thinking.

2. Convincing *yourself* that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.

3. Convincing *someone else* that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you're not able to do that last step, then you haven't really solved the problem. We'll talk more about how to write up a solution in the next section. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

1.5.1 Mathematical Statements

Definition 1.5.1. A **mathematical statement** is a complete sentence that is either true or false.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It's important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.

Think/Pair/Share. For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not, in what way does it fail?

1. I like the color blue.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. Roses are red.
7. UH Manoa is the best college in the world.
8. $1/2 = 2/4$.
9. Go to bed.
10. There are a total of 204 squares on an 8×8 chess board.

Now, work with a partner. Write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "$1/2 = 2/4$" is a perfectly good mathematical statement. It doesn't look like an English sentence, but read it out loud. The subject is "$1/2$." The verb is "equals." And the object is "$2/4$." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a *paradox*: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements.

1.5.2 Precision

When we use words in an everyday situation, we often rely on context and shared understanding. The American Academy of Dermatology has this sentence on their web page:

"One American dies of melanoma every hour."

Taken literally (as a mathematician would), this statement makes an absurd claim: There is one person in America who keeps dying over and over. In fact, he dies *every single hour*.

A more precise statement would be this: "Every hour, someone in America dies of melanoma."

Think/Pair/Share. Compare the two sentences:

- "One American dies of melanoma every hour."
- "Every hour, someone in America dies of melanoma."

What is the (subtle) difference? Why does that small difference change the meaning so dramatically?

If we're working on mathematical problem, we need to work with clear and correct statements. We cannot make assumptions about context or shared understanding. We have to say exactly what we mean.

On Your Own

Work on the exercises below to reinforce the idea of using precise language.

1. Consider this ambiguous sentence:

   "The man saw the woman with a telescope."

   Find two unambiguous (but natural sounding) sentences equivalent to the sentence above, one in which the man has the telescope, and one in which the woman has the telescope.

2. Here are three ambiguous newspaper headlines. For each one, rewrite it in a way that avoids the unintended second meaning. But keep it short and pithy, like a newspaper headline should be.

   (a) Sisters reunited after 10 years in checkout line of Longs Drugs.

   (b) Large hole appears on H-1. County authorities are looking into it.

   (c) Governor Abercrombie says bus passengers should be belted.

3. This hospital notice says exactly the opposite of what it means to say.

   "No head injury is too trivial to ignore."

   Rewrite the sentence so it would still fit on the sign, but would convey its intended meaning.

1.5.3 And / or

Consider this sentence

"After work, I will go to the beach or I will do my grocery shopping."

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means "either one or the other or both," or whether he means "either one or the other but not both." (Some people use the awkward phrase "and/or" to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition 1.5.2. In mathematics, the word "or" *always means* "either one or the other or both."

Think/Pair/Share. For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.

- Choose a different value of x that makes the statement true (or say why that's not possible).
- Choose a different value of x that makes the statement false (or say why that's not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is negative.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.

1.5.4 Quantifiers

Problem 23 (All About the Benjamins)**.** You are handed an envelope filled with money, and you are told "Every bill in this envelope is a \$100 bill."

- What would convince you *beyond any doubt* that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you *beyond any doubt* that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: "There is a \$100 bill in this envelope."

- What would convince you *beyond any doubt* that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you *beyond any doubt* that the sentence is false? How could you convince someone else that the sentence is false?

Think/Pair/Share. After you have thought about the problem on your own for a while, talk through your ideas with a partner. What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- "Every time…"
- "For all numbers…"
- "For every choice…"
- "It's always true that…"

These are *universal statements*. Such statements claim that something is always true, no matter what.

- To prove a *universal statement* is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a *universal statement* is true, you must either check every single case, or you must find a **logical reason** why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes…"
- "There is some number…"
- "For some choice…"
- "At least once…"

These are *existential statements*. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an *existential statement* is true, you may just find the example where it works.

- To prove an *existential statement* is false, you must either show it fails in every single case, or you must find a **logical reason** why it can't be true. (Sometimes the first option is impossible!)

Think/Pair/Share. For each statement below, do the following:

- Decide if it is a *universal statement* or an *existential statement*. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)

- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.

2. Every prime number is odd.

3. For all positive numbers x, $x^3 > x$.

4. There is some number x such that $x^3 = x$.

5. The points $(-1, 1)$, $(2, 1)$, and $(3, 0)$ all lie on the same line.

6. Addition (of real numbers) is commutative.

7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That's okay for now! Divide your answers into four categories:

(a) I am confident that the justification I gave is good.

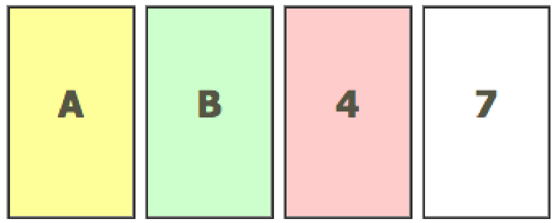
(b) I am not confident in the justification I gave.

(c) I am confident that the justification I gave is *not* good, or I could not give a justification.

(d) I could not decide if the statement was true or false.

1.5.5 Conditional statements

Problem 24 (Card Logic)**.** These cards are on a table.



Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side." Which cards *must* you flip over to be certain that your friend is telling the truth?

♣♣♣ Fellow: [Picture is stolen from a long-forgotten source. Can we re-create it so it's not any kind of copyright violation?]

Think/Pair/Share. After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Problem 25 (IDs at a Party)**.** You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a beer, another has a Coke, but their IDs happen to be face down so you cannot see their ages.

- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. Unfortunately, you are not sure if they are drinking Seven-up or vodka and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

Think/Pair/Share. After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition 1.5.3. A **conditional statement** can be written in the form

$$\text{If } \underbrace{\textit{some statement}}_{\text{hypothesis}} \text{ then } \underbrace{\textit{some statement}}_{\text{conclusion}}.$$

Think/Pair/Share. These are each conditional statements, though they are not all stated in "if/then" form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent "if/then" statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If $2^n - 1$ is prime, then n is prime.
   n is prime. / $2^n - 1$ is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It's either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the *hypothesis* and the *conclusion* are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 1.5.4. Here is a conditional statement:

$$\text{“If } \underbrace{\text{I win the lottery}}_{\text{hypothesis}}\text{, then } \underbrace{\text{I'll give each of my students \$1,000}}_{\text{conclusion}}\text{.”}$$

There are four things that can happen:

True hypothesis, true conclusion I do win the lottery, and I do give everyone in class \$1,000. I kept my promise, so the conditional statement is TRUE.

True hypothesis, false conclusion I do win the lottery, but I decide **not to** give everyone in class \$1,000. I broke my promise, so the conditional statement is FALSE.

False hypothesis, true conclusion I don't win the lottery, but I'm exceedingly generous, so I go ahead and give everyone in class \$1,000. I didn't break my promise! (Do you see why?) So the conditional statement is TRUE.

False hypothesis, false conclusion I don't win the lottery, so I don't give everyone in class \$1,000. I didn't break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If the statement is not false, it must be true.

Example 1.5.5. Here's another conditional statement:

"If you live in Honolulu, then you live in Hawaii."

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let's think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.

- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it doesn't matter! The statement is true either way. The right way to understand such a statement is as a *universal statement*: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here's how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since

Honolulu is in Hawaii), so the statement is true for that person. I don't need to consider people who don't live in Honolulu. The statement is *automatically* true for those people, because the hypothesis is false!"

Example 1.5.6. How do we show a (universal) conditional statement is false? You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

"If you are a good swimmer, then you are a good surfer."

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think/Pair/Share. For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If the units digit of a number is 4, then the number is even.
6. If a number is even, then the units digit of that number is 4.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.
9. If you are tall, then you have long hair.

Think/Pair/Share (Two truths and a lie). On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but don't say which are true and which is false. See if your partner can figure it out!

1.6 Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you haven't really *solved* the problem until you've explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, and grant proposals. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Polya's step (5).

Writing in mathematics is different from writing poetry or an English paper. The goal of mathematical writing is not florid description, but clarity. If your reader doesn't understand, you haven't done a good job. Here are some tips for good mathematical writing.

Don't Turn in Scratch Work When you are solving *problems* and not *exercises*, you're going to have lots of false starts. You're going to try lots of things that don't work. You're going to make lots of mistakes. You're going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is *not* what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)

(Re)state the Problem Don't assume your reader (even if it's the teacher who assigned the problem!) knows what problem you're solving. If the problem has a very long description, you can summarize it. You don't have rewrite it word-for-word or give all of the details. But make sure the question is clear.

Clearly Give the Answer It's not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you're trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be *clearly* stated somewhere in the writeup, and it should be easy to find.

Be Correct Of course, everyone makes mistakes as they're working on a problem. But we're talking about after you've solved the problem, when you are writing up your solution to share with someone else. The best writing in the world can't save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.

Justify Your Answer You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you *know* your answer is right?"

Be Concise There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.

Use Variables and Equations Often an equation is much easier to read and understand (and is more concise!) than a long paragraph of text describing a calculation. Mathematical writing often has way fewer words (and way more equations) than other kinds of writing.

Define your Variables If you use variables or equations in the solution of your problem, always say what the variable stands for *before* you use it. If you use an equation, say where it comes from and why it applies to this situation. Don't make your reader guess!

Use Pictures If pictures helped you solve the problem, include those pictures in your final solution. Even if you didn't draw a picture to solve the problem, it still might help your reader understand the solution. And that's your goal!

Use Correct Spelling and Grammar Proofread your work. A good test is to read your work aloud. There should be complete, natural-sounding sentences. This includes reading the equations and calculations aloud. They should read naturally and make sense. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.

Format Clearly Don't write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Don't write too small. Don't make your reader struggle to read and understand your work.

Acknowledge Collaborators If you worked with someone else on solving the problem, give them credit!

Here is a problem you've already seen:

Problem. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

Think/Pair/Share. Below you will find several solutions that were turned in by students. Using the criteria above, how would you score these solutions on a scale of 1 to 5? Give reasons for your answers.

Solution (Solution 1)**.** 41312432

This is the largest eight-digit b/c the #s 1, 2, 3, 4 & all separated by the given amount of spaces.

Solution (Solution 2)**.** 41312432

You have to have the 4 in the highest place and work down from there. However unable to follow the rules the 2 and the 1 in the 10k and 100k place must switch.

Solution (Solution 3)**.** 41312432

First, I had to start with the #4 because that is the largest digit I could start with to get the largest #.

Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4s.

Next, I place 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be.

I then placed the 3 because opening spaces showed me that I could fit three digits in between the two 3s.

Lastly, I had to input the final 2s, which worked out because there were two digits separating them.

Solution (Solution 4)**.**

1x1

2xx2

3xxx3

4xxxx4
Answer: 41312432

Solution (Solution 5)**.**

$$\underline{4}\ \underline{3}\ \_\ \underline{2}\ \underline{4}\ \underline{3}\ \underline{2}\ \_$$

$$\underline{4}\ \underline{2}\ \_\ \_\ \underline{2}\ \underline{4}\ \_\ \_$$

$$\underline{4}\ \_\ \underline{1}\ \underline{3}\ \underline{1}\ \underline{4}\ \_\ \underline{3}$$

$$\star\underline{4}\ \underline{1}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{4}\ \underline{3}\ \underline{2}$$

4 needs to be the first # to make it the biggest. Then check going down from next largest to smallest. Ex:

$$\underline{4}\ \underline{3}\ \underline{\hspace{4cm}}\times$$

$$\underline{4}\ \underline{2}\ \underline{\hspace{4cm}}\times$$

$$\underline{4}\ \underline{1}\ \underline{\hspace{4cm}}\checkmark$$

Solution (Solution 6)**.** 41312432

I put 4 at the 10,000,000 place because the largest # should be placed at the highest value.

Numbers 2 & 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly.

So I ended up placing the #1 there. In the 100,000 place I put the #3 because it was the second higehst number.

Solution (Solution 7)**.** 41312432

Since the problem asks you for the largest 8 digit #, I knew 4 had to be the first # since it's the greatest # of the set.

To solve the rest of the problem, I used the guess and test method. I tried many different combinations. First using the #3 as the second digit in the sequence, but came to no answer. Then the # 2, but no combination I found correctly finished the sequence.

I then finished with the #1 in the second digit in the sequence and was able to successfully fill out the entire #.

Solution (Solution 8)**.**

$\underline{4}\ \_\ \_\ \_\ \_\ \underline{4}\ \_\ \_$

4 has to be the first digit, for the number to be the largest possible. That means the other 4 has to be the 6th digit in the number, because 4s have to be separated by four digits.

$\underline{4}\ \_\ \underline{3}\ \_\ \_\ \underline{4}\ \underline{3}\ \_$

3 must be the third digit, in order for the number to be largest possible. 3 cannot be the second digit because the other 3 would have to be the 6th digit in the number, but 4 is already there.

$\underline{4}\ \underline{1}\ \underline{3}\ \underline{1}\ \_\ \underline{4}\ \ \underline{3}\ \_$

1s must be separated by one digit, so the 1s can only be the 2nd and 4th digit in the number.

$\underline{4}\ \underline{1}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{4}\ \ \underline{3}\ \underline{2}$

This leaves the 2s to be the 5th and 8th digits.

Solution (Solution 9)**.** With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4 then 1 then 3, since I'm going for the highest # possible.

My answer: 41312432

1.7 The Last Step

A lot of people — from Polya to the writers of the Common Core State Standards and lots of people in between — talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example 1.7.1. Recall Problem 3, "Squares on a Chess Board":

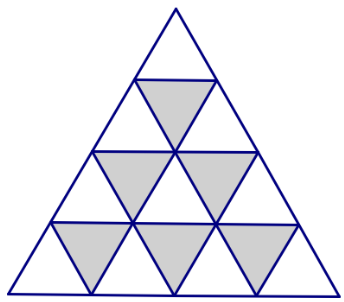
> How many squares are on a standard 8×8 chess board? (The answer is *not* 64! It's a lot bigger!)

We've already talked about some obvious follow-up questions like "What about a 10×10 chess board? Or 100×100? Or $n \times n$?"

But there are lots of interesting (and less obvious . . . and harder) questions you might ask:

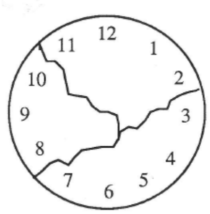
- How many *rectangles* can you find on a standard 8×8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1×2 and 5×3. How could you possibly count them all?)

- How many triangles can you find in this picture?



Example 1.7.2. Recall Problem 4, "Squares on a Chess Board":

> This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?



The original problem only asks if you can find *one* other way. The obvious follow-up question: "Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify that you have found every possibility."

Think/Pair/Share. Choose a problem from the Problem Bank in Section 1.4 (preferably a problem you have worked on, but that's not strictly necessary). What follow-up or similar questions could you ask?

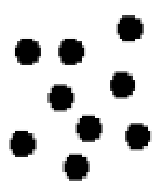
Chapter 2

Place Value

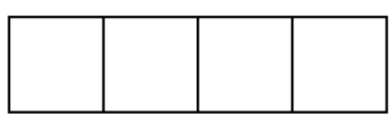
♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

2.1 Dots and Boxes

Here are some dots; in fact there's nine of them.



Here are some boxes:

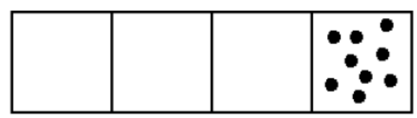


We're going to a play a game in which boxes explode dots and move them around. Here's our first rule:

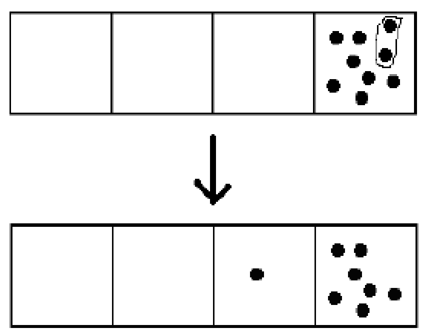
♣♣♣ Fellow: [Have the rules set off in a box somehow?]

The $1 \leftarrow 2$ Rule:
Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

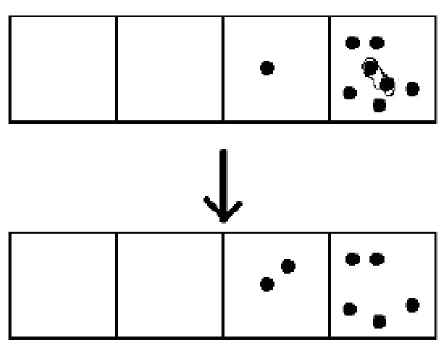
Example 2.1.1 (Nine dots in the $1 \leftarrow 2$ system). We start by placing nine dots in the rightmost box.



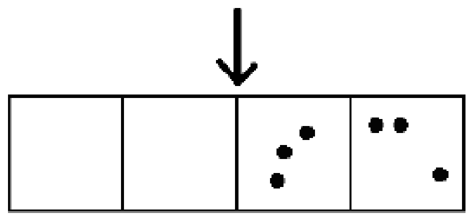
Two dots in that box explode and become one dot in the box to the left.



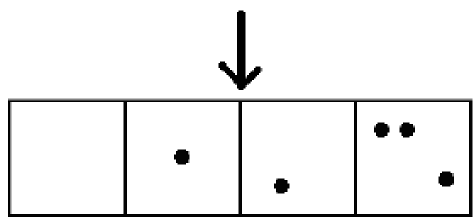
Since there are more than two dots in the rightmost box, it can happen again.



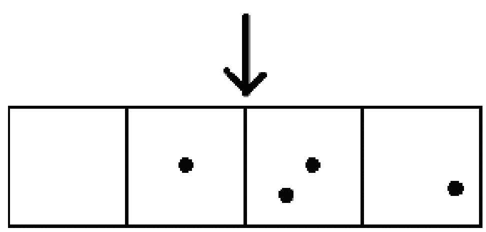
And again!



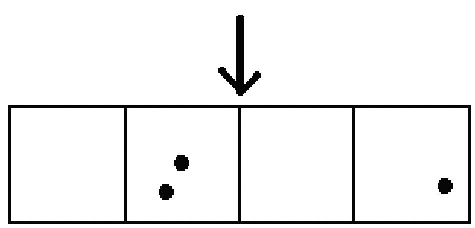
Hey, now we have more than two dots in the second box, so those can explode and move!

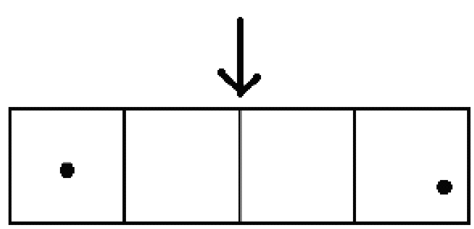


And the rightmost box still has more than two dots.



Keep going, until no box has two dots.



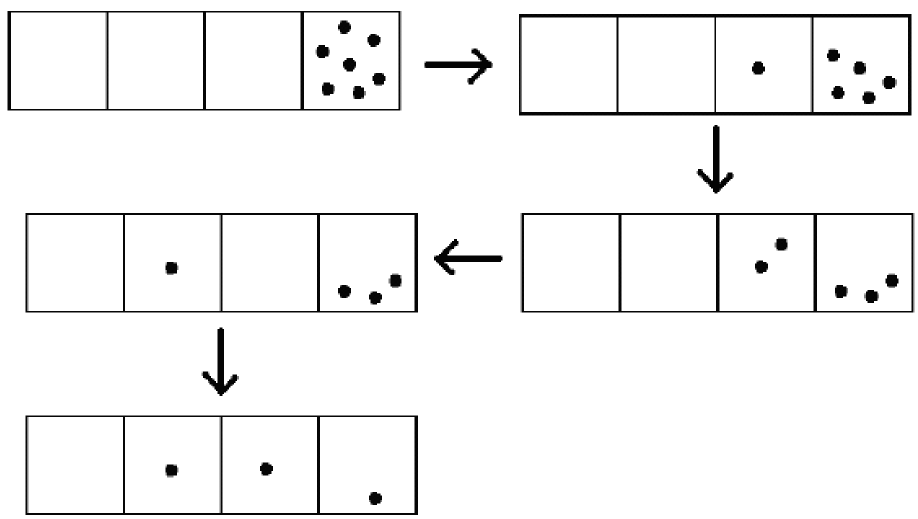


After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

The $1 \leftarrow 2$ code for nine dots is: 1001

On Your Own

Here's a diagram showing what happens for seven dots in a $1 \leftarrow 2$ box. Trace through the diagram, and circle the pairs of dots that "exploded" at each step.



The $1 \leftarrow 2$ code for seven dots is: 111

Problem 26. Note: In solving this problem, you don't need to draw on paper; that can get tedious! Maybe you could use buttons or pennies for dots and do this by hand. What could you use for the boxes?

(a) Draw 10 dots in the right-most box and perform the explosions. What is the $1 \leftarrow 2$ code for ten dots?

(b) Find the $1 \leftarrow 2$ code for thirteen dots.

(c) Find the $1 \leftarrow 2$ code for six dots.

(d) What number of dots has $1 \leftarrow 2$ code 101?

Think/Pair/Share. After you worked on the problem, compare your answer with a partner. Did you both get the same code? Did you have the same process?

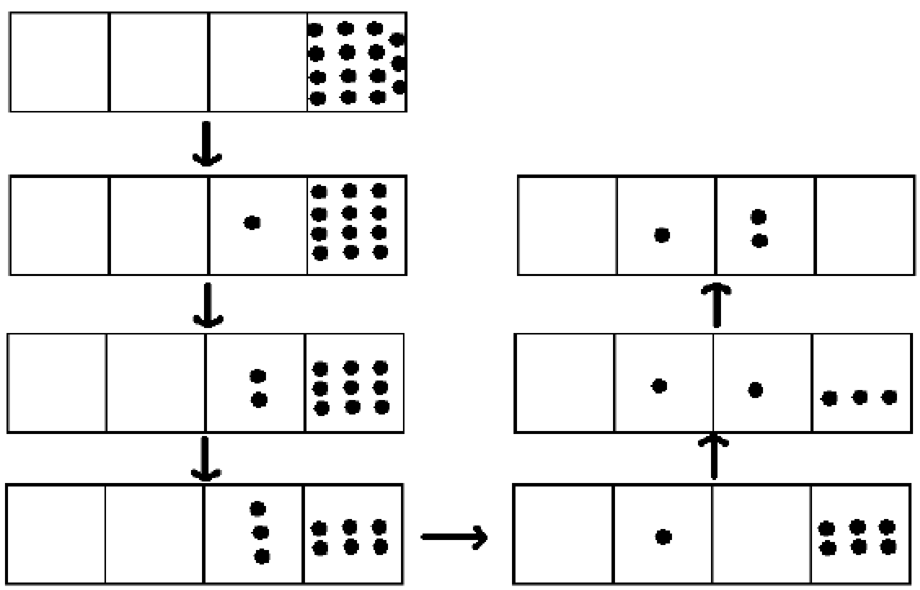
2.2 Other Rules

Let's play the dots and boxes game, but change the rule.

♣♣♣ Fellow: [Have the rules set off in a box somehow?]

> **The $1 \leftarrow 3$ Rule:**
> **Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.**

Example 2.2.1 (Fifteen dots in the $1 \leftarrow 3$ system). Here's what happens with fifteen dots:



The $1 \leftarrow 3$ code for seven dots is: 120

Problem 27.

(a) Show that the $1 \leftarrow 3$ code for twenty dots is 202.

(b) Show that the $1 \leftarrow 3$ code for four dots is 11.

(c) What is the $1 \leftarrow 3$ code for thirteen dots?

(d) What is the $1 \leftarrow 3$ code for twenty-five dots?

(e) What number of dots has $1 \leftarrow 3$ code 1022?

(f) Is it possible for a collection of dots to have $1 \leftarrow 3$ code 2031? Explain your answer.

Problem 28.

(a) Describe how the $1 \leftarrow 4$ rule would work.

(b) What is the $1 \leftarrow 4$ code for the number thirteen?

Problem 29.

(a) What is the $1 \leftarrow 5$ code for the number thirteen?

(b) What is the $1 \leftarrow 5$ code for the number five?

Problem 30.

(a) What is the $1 \leftarrow 9$ code for the number thirteen?

(b) What is the $1 \leftarrow 9$ code for the number thirty?

Problem 31.

(a) What is the $1 \leftarrow 10$ code for the number thirteen?

(b) What is the $1 \leftarrow 10$ code for the number thirty-seven?

(c) What is the $1 \leftarrow 10$ code for the number two hundred thirty-eight?

(d) What is the $1 \leftarrow 10$ code for the number five thousand eight hundred and thirty-three?

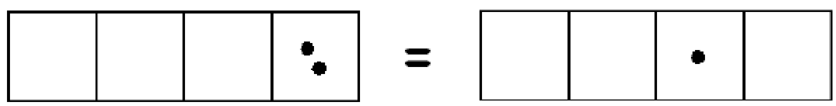
Think/Pair/Share. After you have worked on the problems on your own, compare your ideas with a partner. Can you describe what's going on in Problem 31 and why?

2.3 Binary Numbers

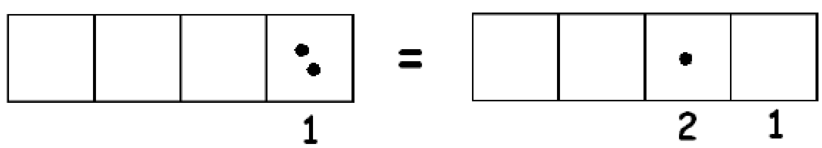
Let's go back to the $1 \leftarrow 2$ rule for a moment:

> **The** $1 \leftarrow 2$ **Rule:**
> **Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.**

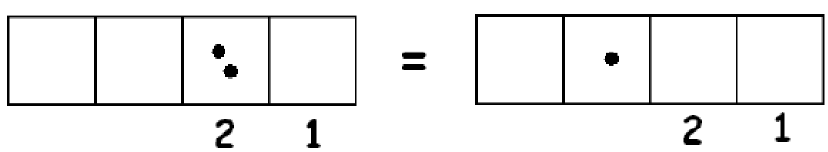
Two dots in the right-most box is worth one dot in the next box to the left.



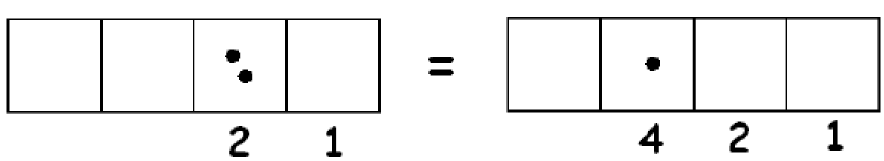
If each of the original dots is worth "one," then the single dot on the left must be worth two.



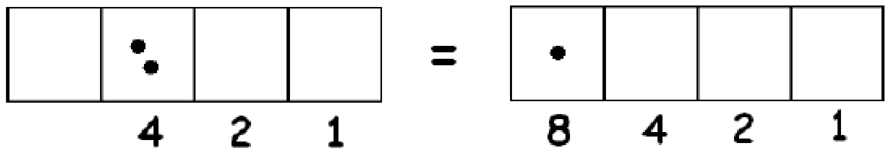
But we also have two dots in the box of value 2 is worth 1 dot in the box just to the left...



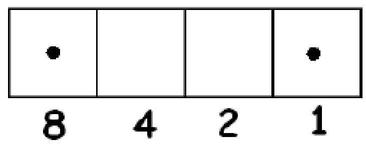
So that next box must be worth two 2s, which is four!



And two of these fours make eight.

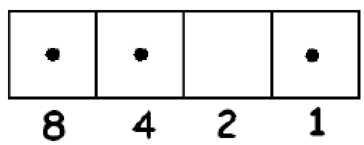


Example 2.3.1. We said earlier that the $1 \leftarrow 2$ code for nine dots was 1001. Lets check:



$$8 + 1 = 9, \text{ so this works!}$$

We also said that thirteen has $1 \leftarrow 2$ code 1101. This is correct.



$$\text{Yep!} \quad 8 + 4 + 1 = 13.$$

Problem 32.

(a) If there were a box to the left of the 8 box, what would the value of that box be?

(b) What would be the value of a box *two* spots to the left of the 8 box? Three spots to the left?

(c) What number has $1 \leftarrow 2$ code 100101?

(d) What is the $1 \leftarrow 2$ code for the number two hundred?

Definition 2.3.2. Numbers written in the $1 \leftarrow 2$ code are called *binary numbers* or *base two* numbers. (The prefix "bi" means "two.") From now on, when we want to indicate that a number is written in base two, we will write a subscript "two" on the number. So 1001_{two} means "the number of dots that has $1 \leftarrow 2$ code 1001," which we already saw was nine.

Important! When we read 1001_{two} we say "one zero zero one base two." We don't say "one thousand and one," because "thousand" is not a binary number.

Think/Pair/Share. Compare you work on problem 32 with a partner.

- Your first goal: come up with a *general method* to find the number of dots represented by any binary number. Clearly describe your method. Test your method out on these numbers, and check your work by actually "unexploding" the dots.

  $$1_{\text{two}} \qquad 101_{\text{two}} \qquad 1011_{\text{two}} \qquad 1111_{\text{two}} \qquad 1101101_{\text{two}}$$

- Explain why binary numbers only contain the digits 0 and 1.

- Here is a new (harder) goal: come up with a *general method* to find the binary number related to any number of dots *without actually going through the "exploding dot" process.* Clearly describe your method. Test your method out on these numbers, and find a way to check your work.

  two dots $= ???_{\text{two}}$ seventeen dots $= ???_{\text{two}}$ sixty-four dots $= ???_{\text{two}}$

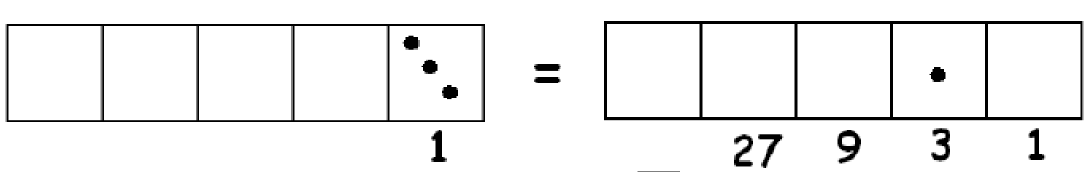
  sixty-three dots $= ???_{\text{two}}$ one thousand dots $= ???_{\text{two}}$

2.3.1 Binary Numbers and Computers

♣♣♣ Fellow: [Can you write a *short* description of the use of binary numbers in computers? Just a paragraph or two getting across the main ideas.]

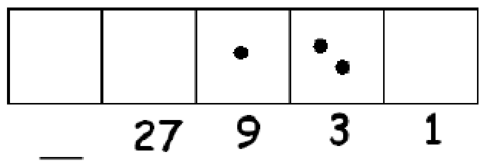
2.4 Other Bases

In the $1 \leftarrow 3$ system, three dots in one box is worth one dot in the box one spot to the left. This gives a new picture:



Each dot in the second box from the left is worth three ones. Each dot in the third box is worth three 3s, which is nine, and so on.

Example 2.4.1. We said that the $1 \leftarrow 3$ code for fifteen is 120. We see that this is correct because



$$9 + 2 \cdot 3 = 9 + 6 = 15.$$

Problem 33. Answer these questions about the $1 \leftarrow 3$ system.

(a) What label should go on the box to the left of the 27 box?

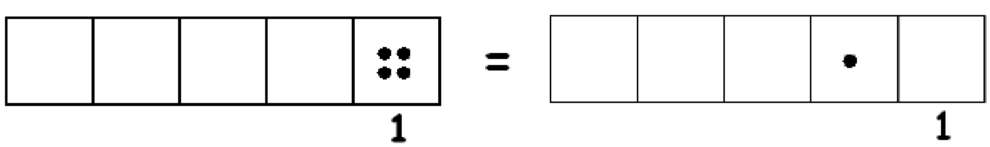
(b) What would be the value of a box *two* spots to the left of the 27 box?

(c) What number has $1 \leftarrow 3$ code 21002?

(d) What is the $1 \leftarrow 3$ code for the number two hundred?

Problem 34. In the $1 \leftarrow 4$ system, four dots in one box are worth one dot in the box one place to the left.

(a) What is the value of each box?

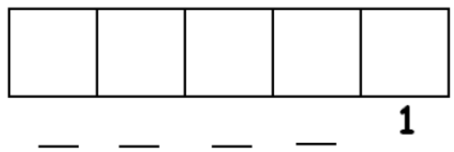


(b) What is the $1 \leftarrow 4$ code for twenty-nine?

(c) What number has $1 \leftarrow 4$ code 132?

Problem 35. In the $1 \leftarrow 10$ system, ten dots in one box are worth one dot in the box one place to the left.

(a) What is the value of each box?



(b) What is the $1 \leftarrow 10$ code for eight thousand four hundred and twenty-two?

(c) What number has $1 \leftarrow 10$ code 95753?

(d) When we write the number 7842 the "7" is represents what quantity? The "4" is four groups of what value? The "8" is eight groups of what value? The "2" is two groups of what value?

(e) Why do human beings like the $1 \leftarrow 10$ system for writing numbers?

Definition 2.4.2. Numbers written in the $1 \leftarrow 3$ system are called *base three numbers*. Numbers written in the $1 \leftarrow 4$ system are called *base four numbers*. Numbers written in the $1 \leftarrow 10$ system are called *base ten numbers*. In general, numbers written in the $1 \leftarrow b$ system are called *base b numbers*.

In a base b number system, each place represents a *power of* b, which means b^k for some positive number k. Remember this means b multiplied by itself k times:

$$b^k = \underbrace{b \cdot b \cdots b}_{k \text{ times}}.$$

- The right-most place is the *units* or *ones* place. (Why is this a *power of b?*)
- The second spot is the "b" place. (In base 10, it's the tens place.)
- The third spot is the "b^2" place. (In base 10, that's the hundreds place, and $100 = 10^2$.)
- The fourth spot is the "b^3" place. (In base 10, that's the thousands place, and $1000 = 10^3$.)
- And so on...the n^{th} spot is the b^{n-1} place.

Notation: Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So 102_{three} is a base three number, 222_{four} is a base four number, and 54321_{ten} is a base ten number. If the base is not written, we assume the number is written in base ten.

Think/Pair/Share.

(a) Find the number of dots represented by:

$$102_{\text{three}}, \quad 222_{\text{four}}, \quad 54321_{\text{ten}}.$$

(b) Represent nine dots in each base:

three, four, five, six, seven, eight, nine, and ten.

(c) Which digits are used in the base two system? The base three system? The base four system? The base five system? The base six system? The base ten system?

(d) What does the *base* tell you about the number system? (Think of as many answers as you can!)

2.4.1 Base b to Base Ten

In Section 2.3, you were asked to come up with *general methods* to translate numbers from base two (binary) to base ten (our standard system). We're now going to describe some general methods for converting from base b to base ten, where b can represent any whole number bigger than one.

If the base is b, that means we're in a $1 \leftarrow b$ system. A dot in the right-most box is worth 1. A dot in the second box is worth b. A dot in the third box is worth $b \times b = b^2$, and so on.

♣♣♣ Fellow: [Can you make a picture of base b boxes with the appropriate labels, say up to b^4?]

So, for example, the number 10123_b represents

$$1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 2 \cdot b + 3 \cdot 1 \text{ dots},$$

because we imagine three dots in the right-most box (each worth one), two dots in the second box (each representing b dots), one dot in the third box (representing b^2 dots), and so on. That means we can just do a short calculation to find the total number of dots, without going through all the trouble of drawing the picture and "unexploding" the dots.

Example 2.4.3. Consider the number 123_{five}. This represents

$$1 \cdot 5^2 + 2 \cdot 5 + 3 = 25 + 10 + 3 = 38 \text{ dots.}$$

On the other hand, the number 123_{seven} represents

$$1 \cdot 7^2 + 2 \cdot 7 + 3 = 49 + 14 + 3 = 66 \text{ dots.}$$

Think/Pair/Share.

- Convert each number to base ten. Compare your answers with a partner to be sure you agree.

$$18_{\text{nine}}, \qquad 547_{\text{eight}}, \qquad 3033_{\text{five}}, \qquad 11011_{\text{three}}.$$

- Which number represents a greater amount of total dots:

$$23,455,443_{\text{six}} \quad \text{or} \quad 23,455,443_{\text{eight}}?$$

  Justify your answer.

2.4.2 Base Ten to Base b

In Section 2.3, you were also asked to come up with *general methods* to translate numbers from base ten to base two. We're now going to describe some general methods for converting from base ten to base b, where b can represent any whole number bigger than one.

We'll work out an example, and then describe the general method.

Example 2.4.4. To convert 321 to a base five number (without actually going through the tedious process of exploding 321 dots in groups of five):

Find the largest power of five that is smaller than 321. We'll just list powers of five:

$$5^1 = 5, \quad 5^2 = 25, \quad 5^3 = 125, \quad 5^4 = 625.$$

So we know that the left-most box we'll use is the 5^3 box. ♣♣♣ Fellow: [add a picture of the appropriately labeled boxes for base 5, up to 5^3?]

How many dots will be in that left-most box? That's the same as asking how many 125s are in 321. Since

$$2 \cdot 125 = 250 \quad \text{and} \quad 3 \cdot 125 = 375,$$

we have two dots in the 5^3 box, representing a total of 250 dots, but rather than drawing dots we'll start writing the digits to represent them.

♣♣♣ Fellow: [picture of the base 5 boxes with a "2" in the 5^3 box?]

How many dots are left unaccounted for? $321 - 250 = 71$ dots are left.

Now just repeat the process: We can put a "2" in the 5^2 box, and that takes care of 50 dots. So so far we have two in the 5^3 box and two in the 5^2 box, so that's a total of

$$2 \cdot 125 + 2 \cdot 25 = 300 \text{ dots}.$$

♣♣♣ Fellow: [picture of the base 5 boxes with 2 in 5^3 box and 2 in the 5^2 box?]

We have 21 dots left to account for. The biggest power of 5 that's less than 21 is just 5. So we can put a "4" in the 5 box, and we have one left over in the one box.

♣♣♣ Fellow: [picture of the base 5 boxes with 2 in 5^3 box and 2 in the 5^2 box 4 in the 5 box and 1 in the 1 box?]

$$2 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 1 = 250 + 50 + 20 + 1 = 321 \text{ dots}.$$

$$\text{So } 321 = 2241_{\text{five}}.$$

The general algorithm to convert from base ten to base b:

1. Start with your base ten number n. Find the largest *power of* b that's less than your number n, say that power is b^k.

2. Figure out how many dots can go in the b^k box without going over the number n. Say that number is a. Put the digit a in the b^k box, and then subtract $n - a \cdot b^k$ to figure out how many dots are left.

3. If your number is now zero, you accounted for all the dots. Put zeros in any boxes that remain, and you have the number. Otherwise, start over at step (1) with the number of dots you have left.

The method seems a little tricky to describe in complete generality. It's probably better to try a few examples on your own to get the hang of it.

Think/Pair/Share. Use the method above to convert 99_{ten} to base three, to base four, and to base five.

The first method we described fills in the boxes from left to right. Here's another method to convert base ten numbers to another base, and this method fills in the digits from right to left. Again, we'll start with an example and then describe the general method:

Example 2.4.5. To convert 712 to a base seven number:

Divide 712 by seven and find the quotient and remainder:

$$712 \div 7 = 101 \text{ R}5.$$

Put the remainder in the ones place:

$$712 = \underline{???}5_{\text{seven}}.$$

Now take the quotient and divide by seven to find the quotient and remainder:

$$101 \div 7 = 14 \text{ R}3.$$

Put the remainder in the sevens place:

$$712 = \underline{???}35_{\text{seven}}.$$

Take the previous quotient and divide by seven again:

$$14 \div 7 = 2 \text{ R}0.$$

Put the remainder in the 7^2 place:

$$712 = \underline{???}035_{\text{seven}}.$$

Since the quotient that's left is less than seven, it goes in the 7^3 place, and we're done.

$$712 = 2035_{\text{seven}}.$$

Of course, we can (and should!) check our calculation by converting the answer back to base ten:

$$2035_{\text{seven}} = 2 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 5 = 686 + 0 + 21 + 5 = 712_{\text{ten}}.$$

So here's a second general method for converting base ten numbers to an arbitrary base b:

1. Divide the base ten number by b to get a quotient and a remainder.
2. Put the remainder in the right-most space in the base b number.
3. If the quotient is less than b, it goes in the space one spot to the left. Otherwise, go back to step (1) and repeat it with the quotient, filling in the remainders from right to left in the base b number.

We can use the dots and boxes system to explain why this method of quotients and remainders works. It's not just a "trick!" We'll stick with the example of converting 712 to base seven, so we have something specific to talk about.

- We imagine 712 dots in the right-most box, since that represents 712 dots total. Since we're converting to base seven, we're in the $1 \leftarrow 7$ system. ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with just the number 721 in the right-most box.]
- Groups of seven dots will explode, and each group of seven becomes one dot in the next box. How many groups of seven dots are there? Well, there are 101 groups of seven, with 5 dots left over out of a group. That's what we figured out with the calculation
$$712 \div 7 = 101 \text{ R5}.$$
- Imagine we explode all the groups of seven that we can make in the right-most box before we move on. Then we would have 5 dots left in that first box, and 101 dots in the second box.

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 101 in the in the second box.]
- Again, groups of seven dots will explode, and each group becomes one dot in the third box. How many groups of seven dots are there? There are 14 groups with three left over. That's what we computed like this:
$$101 \div 7 = 14 \text{ R3}.$$

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 3 in the in the second box and 14 in the third box.]

- OK, now there are 5 dots in the right-most box, 3 dots in the second box, and 14 dots in the third box. We do it all again! Groups of seven explode, and each group forms dot in the next box to the left. Fourteen dots gives two equal groups of seven, none left over.

- So we end up with: 5 dots in the right-most box, 3 dots in the second box, zero dots in the third box, and 2 dots in the fourth box. And there's nothing left to explode!

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 3 in the in the second box and 0 in the third box and 2 in the last box.]

- Now we can read off the number left-to-right:

$$712 = 2035_{\text{seven}}.$$

Again, the method probably makes more sense if you try it out a few times.

Think/Pair/Share. Use the method described above to convert 250_{ten} to base three, four, five, and six. For each of the computations, write a careful dots-and-boxes explanation for why it works.

2.5 Number Systems

2.5.1 History

♣♣♣ Fellow: [Can you write a *short* history / description of some different systems like the Egyptian, Mayan, and Roman numerals. What's in the book is totally overkill and too much. No need to have students do any problems. Just a few paragraphs describing additive system versus positional system and giving a couple of examples.]

2.5.2 Fibonacci

♣♣♣ Fellow: [Include a picture and *short* bio of Fibonacci? What he really *should* be famous for is giving us the arabic numerals and showing the ease of computation with a positional system, not the sequence of numbers that came from one little problem in his book... Get across where the base 10 system was created, and that Fibonacci brought it to the Western world from which it spread?]

Problem 36. What is the difference between 5_{nine} and 50_{nine}?

Problem 37. Convert each base-5 number to a base-10 number. Look for a shortcut!

$$4_{\text{five}} \qquad 40_{\text{five}} \qquad 400_{\text{five}} \qquad 4000_{\text{five}}$$

Challenges: 0.4_{five} 0.04_{five}

Think/Pair/Share. Discuss your answers to problems 36 and 37. Discuss:

- When you add zeros to the right of a number in base ten, what does that do to the number? (Think about 2, 20, 200, 2000, etc.).
- When you add zeros to the right of a number in base nine, what does that do to the number?
- When you add zeros to the right of a number in base five, what does that do to the number?
- When you add zeros to the right of a number in base b, what does that do to the number?

2.6 Even Numbers

How do we know if a number is even? What does it mean? Well, some number of dots is *even* if I can divide the dots into pairs, and every dot has a partner. ♣♣♣ Fellow: [Add a picture of pairs of dots grouped together? A fairly large number would be good.]

And some number of dots is *odd* if, when I try to pair up the dots, I always have a single dot left over with no partner. ♣♣♣ Fellow: [Add a picture of an odd number of dots?]

The number of dots is either even or odd. It's a property of the *quantity* and is doesn't change when you write the number in different bases.

Problem 38. Which of these numbers represent an even number of dots? Explain how you decide.

$$22_{\text{ten}} \qquad 319_{\text{ten}} \qquad 133_{\text{five}} \qquad 222_{\text{five}} \qquad 11_{\text{seven}} \qquad 11_{\text{four}}$$

Think/Pair/Share. Compare your answers to problem 38 with a partner. Then try these together:

(a) Count by twos to 20_{ten}.

(b) Count by twos to 30_{four}.

(c) Count by twos to 51_{seven}.

Think/Pair/Share. You know that you can tell if a number in base 10 is even just by looking at the units digit. Which one of the following statements *best* captures the reason for this rule?

1. It works because even and odd numbers alternate, so you only have to look at the ones place.

2. It works if the number ends with an even digit, but it only works for whole numbers and decimals (e.g. 12 and 1.2.).

3. It actually only works if the last digit is 2, 4, 6, or 8.

4. It works because all digits other than the units digit — for example tens, hundreds, and thousands — represent even numbers, and sums of even numbers are even.

Problem 39.

(a) Write the numbers zero through fifteen in base seven:

$$0_{\text{seven}}, 1_{\text{seven}}, 2_{\text{seven}}, \ldots$$

(b) Circle all of the even numbers in your list. How do you know they are even?

(c) Find a rule: how can you tell if a number is even when it's written in base seven?

Problem 40.

(a) Write the numbers zero through fifteen in base four:

$$0_{\text{four}}, 1_{\text{four}}, 2_{\text{four}}, \ldots$$

(b) Circle all of the even numbers in your list. How do you know they are even?

(c) Find a rule: how can you tell if a number is even when it's written in base four?

Think/Pair/Share. Discuss your answers to problems 39 and 40.

- Why are the rules for even numbers different in different bases?
- For either your base four rule or your base seven rule, can you explain *why* it works that way?

2.7 Orders of Magnitude

Problem 41. How old were you when you were one million seconds old? (That's $1,000,000$.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 42. How old were you when you were one *billion* seconds old? (That's $1,000,000,000$.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is $1,000\times$ a million.

So you could have just taken your answer to problem 41 and multiplied it by $1,000$ to get your answer to problem 41. Of course, you would probably still need to do some calculations to make sense of the answer.

Think/Pair/Share. When is your one trillion second birthday? What will you do to celebrate?

Think/Pair/Share. The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* — the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the $10,000,000,000$ place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.

- A headline from April 2013 said, "US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every* quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 43.

1. Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?

2. Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what *order of magnitude*) would you need to stack up to reach the moon? Explain your answers.

2.7.1 Fermi Problems

James Boswell wrote,"Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it."

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself.* In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if..." questions. And eventually, they figure out something new, something no one ever knew before!

Ever for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example 2.7.1. Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have — gasp! — non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about $1,000,000$ people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. Something like this? We'll assume people don't live past 80. (Of course some people do! But we're all about making simplifying assumptions right now. That gives us 8 age categories, with about 125,000 people in each category.

age range	# people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

An even better guess (since we have a large university that draws lots of students) is that there's a "bump" around college age. And some people live past 80, but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# people
$0-9$	125,000
$10-19$	130,000
$20-29$	140,000
$30-39$	125,000
$40-49$	125,000
$50-59$	125,000
$60-69$	120,000
>70	105,000

So, how many K–5 students are in Hawaii? That covers six years of the 0–9 (maybe 10) range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of $12,500 \times 6 = 75,000$ K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. So a reasonable average might be 25 students per class across all grades K–5 and all schools?

So that makes $75,000 \div 25 = 3,000$ K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

Problem 44. How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process:

- Define your terms.

- Write down what you know.

- Make some reasonable guesses / estimates.
- Do some simple calculations.

It's your turn to try your hand at some Fermi problems.

Problem 45. How much money does UH Manoa earn in parking revenue each year?

Problem 46. How many tourists visit Waikiki in a year?

Problem 47. How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 48. How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 49. How much pizza is consumed by UH Manoa students in a month?

Problem 50. How much would it cost to provide free day care to every 4th grader in the US?

Problem 51. How many books are in Hamilton library?

Problem 52. Make up your own Fermi problem…what would you be interested in calculating? Then try to solve it!

2.8 Problem Bank

Problem 53.

(a) If you were counting in base four, what number would you say just before you said 100_{four}?

(b) What number is one more than 133_{four}?

(c) What is the greatest three-digit number that can be written in base four? What numbers come just before and just after that number?

Problem 54. Explain what is wrong with writing 313_{two} or 28_{eight}.

Problem 55.

(a) Write out the base three numbers from 1_{three} to 200_{three}.

(b) Write out the base five numbers from 1_{five} to 100_{five}.

(c) Write the four base six numbers that come after 154_{six}.

Problem 56. Convert each base-4 number to a base-10 number. Explain how you did it.

$$13_{\text{four}} \qquad 322_{\text{four}} \qquad 101_{\text{four}} \qquad 1300_{\text{four}}$$

Challenges: 0.2_{four} $\qquad 0.111..._{\text{four}} = 0.\overline{1}_{\text{four}}$

Problem 57. Convert each base-10 number to a base-4 number. Explain how you did it.

$$13 \qquad 8 \qquad 24 \qquad 49$$

Challenges: 0.125 $\qquad 0.111... = 0.\overline{1}$

Problem 58. In order to use base sixteen, we need sixteen digits — they will represent the numbers zero through fifteen. We can use our usual digits 0 – 9, but we need *new symbols* to represent the *digits* ten, eleven, twelve, thirteen, fourteen, and fifteen. Here's one standard convention:

base 10 number	**base 16 digit**
10	A
11	B
12	C
13	D
14	E
15	F

(a) Convert these numbers from base sixteen to base ten, and show your work:

$$6D_{\text{sixteen}} \qquad AE_{\text{sixteen}} \qquad 9C_{\text{sixteen}} \qquad 2B_{\text{sixteen}}$$

(b) Convert these numbers from base ten to base sixteen, and show your work:

$$97 \qquad 144 \qquad 203 \qquad 890$$

Problem 59. How many different symbols would you need for a base twenty-five system? Justify your answer.

Problem 60. All of the following numbers are multiples of three.

$$3, \quad 6, \quad 9, \quad 12, \quad 21, \quad 27, \quad 33, \quad 60, \quad 81, \quad 99.$$

(a) Identify the *powers of* 3 in the list. Justify your answer.

(b) Write each of the numbers above in base three.

(c) In base three: how can you recognize a *multiple of* 3? Explain your answer.

(d) In base three: how can you recognize a *power of* 3? Explain your answer.

Problem 61. All of the following numbers are multiples of five.

$$5, \quad 10, \quad 15, \quad 25, \quad 55, \quad 75, \quad 100, \quad 125, \quad 625, \quad 1000.$$

(a) Identify the *powers of* 5 in the list. Justify your answer.

(b) Write each of the numbers above in base five.

(c) In base five: how can you recognize a *multiple of* 5? Explain your answer.

(d) In base five: how can you recognize a *power of* 5? Explain your answer.

Problem 62. Convert each number to the given base.

(a) 395_{ten} into base eight.

(b) 52_{ten} into base two.

(c) 743_{ten} into base five.

Problem 63. What bases makes theses equations true? Justify your answers.

(a) $35 = 120\_$

(b) $41_{\text{six}} = 27\_$

(c) $52_{\text{seven}} = 34\_$

Problem 64. What bases makes theses equations true?

(a) $32 = 44\_$

(b) $57_{\text{eight}} = 10\_$

(c) $31_{\text{four}} = 11\_$

(d) $15_x = 30_y$

Problem 65.

(a) Find a base ten number that is twice the product of its two digits. Is there more than one answer? Justify what you say.

(b) Can you solve this problem in any base other than ten?

Problem 66.

(a) I have a four-digit number written in base ten. When I multiply my number by four, the digits get reversed.

(b) Can you solve this problem in any base other than ten?

Problem 67. Consider this base ten number (I got this by writing the numbers from 1 to 60 in order next to one another):

$$12345678910111213\ldots57585960$$

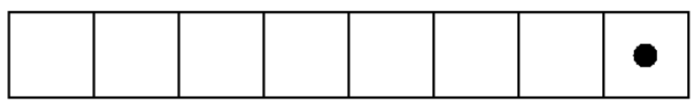
(a) What is the largest number that can be produced by erasing one hundred digits of the number? (When you erase a digit it goes away. For example, if you start with the number 12345 and erase the middle digit, you produce the number 1245.) How do you *know* you got the largest possible number?

(b) What is the smallest number that can be produced by erasing one hundred digits of the number? How do you *know* you got the smallest possible number?

Problem 68. Can you find numbers (not necessarily single digits!) a and b so that $a_b = b_a$? Can you find more than one solution? What must be true of a and b? Justify your answers.

2.9 Exploration

Problem 69. Jay decides to play with a system that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens?



Problem 70. Poindexter decides to play with a system that follows the rule $2 \leftarrow 3$.

(a) Describe what this rule does when there are three dots in the right-most box.

(b) Draw diagrams or use buttons or pennies to find the $2 \leftarrow 3$ codes for the following numbers:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 27, 30, 33, 36, \text{ and } 39$$

Can you find (and *explain*) any patterns?

Problem 71. Repeat problem 70 for your own rule. Choose two numbers a and b and figure out what the code is for your $a \leftarrow b$ system for each of the numbers above.