Math for Elementary Teachers

April 29, 2016

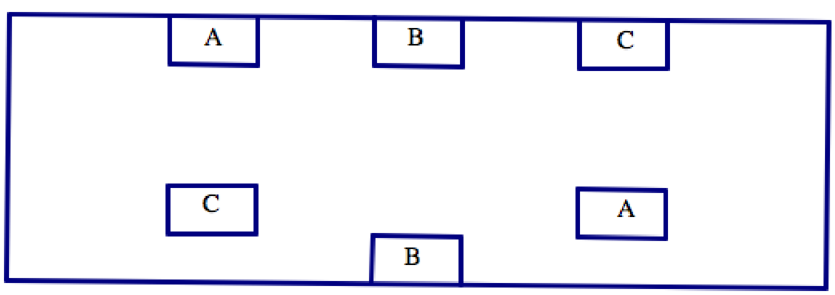
Contents

Chapter 1

Problem Solving

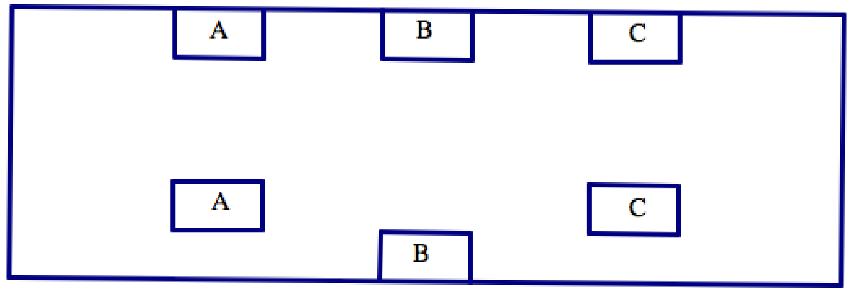
♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

Problem 1 (ABC)**.** Draw curves connecting A to A, B to B, and C to C. Your curves can't cross or even touch each other, and they can't go outside the box.



Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What makes this problem difficult? Can you change the problem slightly so that it would be easier to solve?

Problem Solving Strategy 1 (Wishful Thinking)**.** *Don't you* **wish** *the picture in the problem looked more like this one? Could you solve the problem in that case?*



Can you use a solution to this easier problem to help you solve the original problem? How? Think about moving the boxes around once the lines are already drawn.

♣♣♣ Fellow: [Would be great to have an animation here showing how one solution transforms into the other. Not sure how hard that would be to create...]

The Common Core State Standards for Mathematics (`http://www.corestandards.org/Math/Practice`) identify eight "Mathematical Practices" — the kinds of expertise that all teachers should try to foster in their students, but that go far beyond any particular piece of mathematics content. They describe what mathematics is really about, and why it is so valuable for students to master. The very first Mathematical Practice is:

> "**Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary."

This chapter will help you develop these very important mathematical skills, so that you'll be better prepared to help your future students develop them.

1.1 Problem or Exercise?

The main activity of mathematics is **solving problems**. But what most people experience in most mathematics classrooms is **practice exercises**. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should even look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then *practice* on very similar problems, with the goal of mastering that skill.

Note: What is a **problem** for some people may be an **exercise** for other people who have more background knowledge! For a young student just learning addition, this might be a problem:

Fill in the blank to make a true statement: $\underline{\qquad} + 4 = 7$.

But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice lots of skills — hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court, breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient, keeping control of the ball while making quick turns in soccer, shooting free throws in basketball, catching high fly balls in baseball, and so on — but the whole point of the sport is to *play the game*. You practice the skills so that you're better at playing the game!

The game of math, that's solving problems!

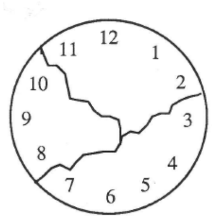
On Your Own

For each question below, decide if it is a **problem** or an **exercise**. (You do not need to solve the problems! Just decide which category it fits for you.)

After you have labeled each one, compare your answers with a partner.

♣♣♣ Fellow: [Add a mix of exercises. Use the problems provided here. Throw in some straightforward computations like adding fractions with unlike denominators, multiplying two-digit numbers, and solving some linear equations. Make a couple of them "word problems" but exercise-y ones like: "What number is 3 more than 20?" You can just flip through the book. Choose about six or seven exercises to go with the problems shows, and intersperse them.]

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?

   Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

   ♣♣♣ Fellow: [The clock picture is scanned & stolen from some long-forgotten source. Any chance we can re-create a version of it? Even take a picture of a real clock and draw some lines on it?]

2. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.

3. Letters stand for digits 0–9. In a given problem: the same letter always represents the same digit, and different letters always represent different digits. There is no relation between problems (so "A" in problem one and "A" in problem 3 might be different).

$$\begin{array}{cccc} & A & B & C \\ + & A & C & B \\ \hline & C & B & A \end{array} \qquad \begin{array}{cccc} & O & N & E \\ + & O & N & E \\ \hline & T & W & O \end{array} \qquad \begin{array}{ccc} & & A \\ & & A \\ + & & A \\ \hline & H & A \end{array}$$

Notes: "O" represents the letter O and not the number zero. Two and three digit numbers never start with 0.

4. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

   ♣♣♣ Fellow: [Can we get a picture of a balance scale? Best if it's one that is in the public sphere or that we create ourselves.]

5. How many squares are on a standard 8×8 chess board?

6. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

1.2 Problem Solving Strategies

Think back to the first problem in this chapter. What did you do to solve it? Even if you didn't figure it out completely by yourself, you probably worked towards a solution and figured out some things that *didn't* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

1.2.1 George Polya

♣♣♣ Fellow: [Get a picture of Polya and write a SHORT bio. You can pull information from the web and from the textbook. But don't overdo it.]

1.2.2 Polya's "How to Solve it"

In 1945, Polya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.

2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually **do** these steps?!?! Steps (1) and (2) are particularly mysterious! How do you "make a plan"? That's where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We'll articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve lots of problems!

We have already seen one problem solving strategy, which we called "Wishful Thinking." Don't be afraid to change the problem! As yourself "what if" questions: What if the picture was different? What if the numbers were simpler? What if I just made up some numbers? You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

Problem Solving Strategy 2 (Try Something!)**.** *If you're really trying to solve a problem, the whole point is that you don't know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what's going on.*

And equally important: If what you tried first doesn't work, try something else! Play around with the problem until you have a feel for what's going on.

1.2.3 Two More Strategies

Problem 2 (Payback)**.** Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had left. Who got the most money from Alex?

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What did you figure out about the problem, even if you haven't solved it completely.

This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before seeing the solution.

Problem Solving Strategy 3 (Draw a Picture)**.** *Some problems are obviously about a geometric situation, and it's clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that isn't geometric, like this one, thinking visually can help! Can you represent something in the situation by a picture?*

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you're totally stuck), you can watch someone else's solution: ♣♣♣ Fellow: [Add an animation of the solution described as shown in Monique's book?]

Problem Solving Strategy 4 (Make Up Numbers)**.** *Part of what makes this problem difficult is that it's about money, but there are no numbers given. That means the numbers must not be important. So just make them up!*

You can work forwards: Assume Alex had some specific amount of money when she showed up at school, say \$100. Then figure out how much he gives to each person. Or you can work backwards: suppose he has some specific amount left at the end, like \$10. Since he gave Chris half of what he had left, that means he had \$20 before running into Chris. Now, work backwards and figure out how much each person got.

Watch the solution only after you tried this strategy for yourself: ♣♣♣ Fellow: [Add an animation of the solution described as shown in Monique's book?]

If you use the "Make Up Numbers" strategy, it's really important to remember what the original problem was asking! You don't want to answer something like "Everyone got \$10." That's not true in the original problem; that's an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and **answer what was asked**!

1.2.4 Four More Strategies

Problem 3 (Squares on a Chess Board)**.** How many squares are on a standard 8×8 chess board? (The answer is *not* 64! It's a lot bigger!)

Remember Polya's first step is to understand the problem. If you're not sure what's being asked, or why the answer is not just 64, be sure to ask someone!

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What did you figure out about the problem, even if you haven't solved it completely.

It's pretty clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you've found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be *absolutely positive* that you found the right answer. You should never ask the teacher, "Is this right?" Instead, you should declare, "Here's my answer, and here's why I know it's correct!"

Problem Solving Strategy 5 (Try a Simpler Problem)**.** *Polya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an* 8×8 *checkerboard is pretty big. Can you solve the problem for smaller boards? Like* 1×1*?* 2×2*?* 3×3*?*

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that's Polya's step (2)).

Problem Solving Strategy 6 (Work Systematically)**.** *If you're working on simpler problems, it's useful to keep track of what you've figured out and what changes as the problem gets more complicated.*

For example, in this problem you might keep track of how many 1×1 squares are on each board, how many 2×2 squares on are each board, how many 3×3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	1×1 squares	2×2 squares	3×3 squares	4×4 squares	...
1×1	1	0	0	0	...
2×2	4	1	0	0	...
3×3					...
$\vdots$					

Problem Solving Strategy 7 (Use Manipulatives to Help You Investigate)**.** *Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!*

For example, in this problem it can be difficult to keep track of which squares you've already counted. You might want to cut out 1×1 squares, 2×2 squares, 3×3 squares, and so on. You can actually move the smaller squares across the checkerboard in a systematic way, making sure that you count everything once and don't count anything twice.

♣♣♣ Fellow: [Make a video showing how to do this on a 5×5 board, using cutouts of a 2×2 and / or a 3×3?]

Problem Solving Strategy 8 (Look for and Explain Patterns)**.** *Sometimes the numbers in a problem are so big, there's no way you will actually count everything up by hand. For example, if the problem in this section were about a* 100×100 *chess board, you wouldn't want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a* 100×100 *chess board just with a calculation.*

Think/Pair/Share. If you haven't done so already, extend the Table above all the way to an 8×8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8×8 chess board. Then:

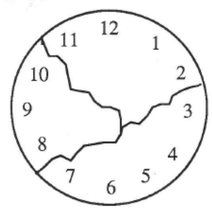
- Describe all of the patterns you see in the table.
- Can you *explain* and *justify* any of the patterns you see? How can you be sure they will continue?

- What calculation would you do to find the total number of squares on a 100×100 chess board?

(We'll come back to this question in Section 1.3. So if you're not sure right now how to explain and justify the patterns you found, that's OK.)

1.2.5 Two More Strategies

Problem 4 (Broken Clock)**.** This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?



Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

♣♣♣ Fellow: [Replace clock picture as before.]

Remember that your first step is to understand the problem. Work out what's going on here. What are the sums of the numbers on each piece? Are they consecutive? (What does that mean?)

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What did you try? What progress have you made?

Problem Solving Strategy 9 (Find the Math, Remove the Context)**.** *Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.*

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
- How do I know when I'm done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it doesn't translate into solving the clock problem.

Problem Solving Strategy 10 (Check Your Assumptions)**.** *When solving problems, it's easy to limit your thinking by adding extra assumptions that aren't in the problem. Be sure you ask yourself: Am I constraining my thinking too much?*

In the clock problem, because the first solution has the clock broken *radially* (all three pieces meet at the center, so it looks like slicing a pie), many people assume that's how the clock must break. But the problem doesn't require the clock to break radially. It might break into pieces like this:

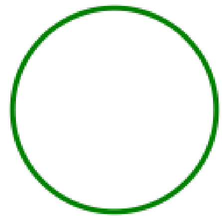
♣♣♣ Fellow: [Add a picture of a clock broken into pieces with the breaks going across, not radial. For example, three pieces: $\{11, 12, 1, 2\}$, $\{10, 9, 3, 4, 5\}$, and $\{6, 7, 8\}$.]

Were you assuming the clock would break in a specific way? Try to solve the problem now, if you haven't already.

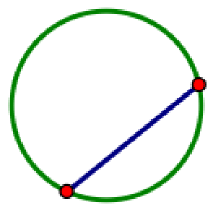
1.3 Beware of Patterns!

The "Look for Patterns" strategy can be particularly appealing, but you have to be careful! Don't forget the **"and Explain"** part of the strategy. Not all patterns are obvious, and not all of them will continue.

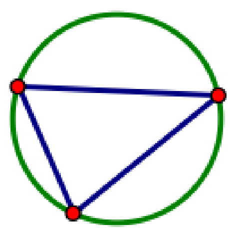
Problem 5 (Dots on a Circle)**.** Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots. How many pieces will you get?

Think/Pair/Share. After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you haven't solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you haven't already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You'll want to draw a *big* circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It's probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and don't count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it *seems* to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 31! The pattern simply doesn't hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we can't explain *why* a pattern would occur, then we aren't willing to just believe it!

For example, if my number pattern starts out

$$2, 4, 8, \ldots$$

I can find *lots* of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- $2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, \ldots.$
  This is a a repeating pattern, cycling through the numbers $2, 4, 8$ and then starting over with 2.

- $2, 4, 8, 32, 256, 8192, \ldots.$
  To get the next number, multiply the previous two numbers together.

- $2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$

- $2, 4, 8, 14, 22, 32, 44, 58, 74 \ldots.$

Think/Pair/Share. Work on your own and then share your ideas with a partner:

1. For the last two patterns above, describe in words how the number sequence is being created.

2. Find at least two other ways to continue the sequence $2, 4, 8, \ldots$ that looks different from all the ones you've seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- $5 \times 5 = 25$ squares of size 1×1.
- $4 \times 4 = 16$ squares of size 2×2.
- $3 \times 3 = 9$ squares of side 3×3.
- $2 \times 2 = 4$ squares of size 4×4.
- $1 \times 1 = 1$ squares of size 5×5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5×5 chess board.

You can probably guess how to continue the pattern to any size board, but how can you be *absolutely sure* the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it's clear why the pattern *must* continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!)**.** *One of the most powerful tools we have is the use of a* variable*. If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.*

Think/Pair/Share. For now, just work on *describing* the pattern with variables.

- Stick with a 5×5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5×5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ *should* fit on a chess board of size 10×10?

- What if you don't know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ *should* fit on a chess board of size $n \times n$?

Now comes the tough part: *explaining* the pattern. Let's focus on an 8×8 board. Since it measures 8 squares on each side, we can see that we get $8 \times 8 = 64$ squares of size 1×1. And since there's just a single board, we get just one square of size 8×8. But what about all the sizes in-between?

♣♣♣ Fellow: [Insert video showing why the count for 2×2 squares should be $7 \times 7 = 49$. Model it on the general solution below.]

Think/Pair/Share. Using the video as a model, work with a partner to carefully explain why the number of 3×3 squares will be $6 \times 6 = 36$, and why the number of 4×4 squares will be $4 \times 4 = 16$.

Here's what a final justification might look like:

Solution (Chess Board Pattern)**.** Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n-k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n-k)$ times, until it hits the top right corner of the chess board. The square is in $(n-k+1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n-k)$ rows below the square, I can shift it down $(n-k)$ times until it hits the bottom row. This makes $(n-k+1)$ total rows that the square moves across, counting the top row.

So there are $(n-k+1)$ rows with $(n-k+1)$ squares in each row. That makes $(n-k+1)^2$ total squares.

Once we're sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10×10 chess board?

- What calculation would you do to solve that problem for a 100×100 chess board?

There *is* a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why* it's the right pattern for this problem, and not just another pattern that seems to work but might not continue.

1.4 Problem Bank

You have several problem solving strategies to work with. Here are the ones we've described in this Section (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help You Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you're stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

♣♣♣ Fellow: [If you have any favorite problems that don't have a lot of mathematical prerequisites — this is the first chapter! — throw them in! The more the merrier!]

Problem 6. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

♣♣♣ Fellow: [Balance scale picture?]

Problem 7. You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8. You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9. Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10. There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and can't see a thing, but you can move the coins around, and you can flip them over. However, you can't tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11. The digital root of a number is the number obtained by adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number. For example, the digital root of 98 is 8, since

$$9 + 8 = 17 \quad \text{and} \quad 1 + 7 = 8.$$

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns? Can you predict the digital root of a number without computing it?

Problem 12. If this lattice were continued, what number would be directly to the right of 98?

	3		6		9		12		...
1	2	4	5	7	8	10	11	13	...

Problem 13. Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

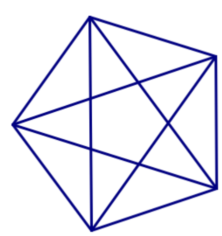
Problem 14. There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high-fives were there?

Problem 15. In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

Problem 17. Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18. How many triangles are in this picture?

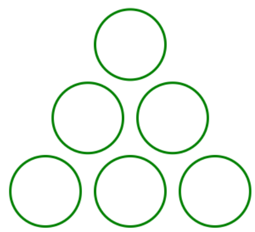


Problem 19. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.
Example: This is a difference triangle, but it doesn't work because it uses 1 twice and doesn't have a 6:

$$\begin{array}{ccccc} 4 & & 5 & & 3 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

Problem 20. Certain pipes are sold in lengths of 6 inch, 8 inch, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21. Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22. Find a way to cut a circular pizza into 11 pieces using just four straight cuts.

1.5 Careful use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it's not. Mathematics is a social endeavor. We don't just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves lots of scratch paper and careful thinking.

2. Convincing *yourself* that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.

3. Convincing *someone else* that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you're not able to do that last step, then you haven't really solved the problem. We'll talk more about how to write up a solution in the next section. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

1.5.1 Mathematical Statements

Definition 1.5.1. A **mathematical statement** is a complete sentence that is either true or false.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It's important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.

Think/Pair/Share. For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not, in what way does it fail?

1. I like the color blue.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. Roses are red.
7. UH Manoa is the best college in the world.
8. $1/2 = 2/4$.
9. Go to bed.
10. There are a total of 204 squares on an 8×8 chess board.

Now, work with a partner. Write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "$1/2 = 2/4$" is a perfectly good mathematical statement. It doesn't look like an English sentence, but read it out loud. The subject is "$1/2$." The verb is "equals." And the object is "$2/4$." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a *paradox*: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements.

1.5.2 Precision

When we use words in an everyday situation, we often rely on context and shared understanding. The American Academy of Dermatology has this sentence on their web page:

"One American dies of melanoma every hour."

Taken literally (as a mathematician would), this statement makes an absurd claim: There is one person in America who keeps dying over and over. In fact, he dies *every single hour*.

A more precise statement would be this: "Every hour, someone in America dies of melanoma."

Think/Pair/Share. Compare the two sentences:

- "One American dies of melanoma every hour."
- "Every hour, someone in America dies of melanoma."

What is the (subtle) difference? Why does that small difference change the meaning so dramatically?

If we're working on mathematical problem, we need to work with clear and correct statements. We cannot make assumptions about context or shared understanding. We have to say exactly what we mean.

On Your Own

Work on the exercises below to reinforce the idea of using precise language.

1. Consider this ambiguous sentence:

   "The man saw the woman with a telescope."

   Find two unambiguous (but natural sounding) sentences equivalent to the sentence above, one in which the man has the telescope, and one in which the woman has the telescope.

2. Here are three ambiguous newspaper headlines. For each one, rewrite it in a way that avoids the unintended second meaning. But keep it short and pithy, like a newspaper headline should be.

    (a) Sisters reunited after 10 years in checkout line of Longs Drugs.

    (b) Large hole appears on H-1. County authorities are looking into it.

    (c) Governor Abercrombie says bus passengers should be belted.

3. This hospital notice says exactly the opposite of what it means to say.

    "No head injury is too trivial to ignore."

    Rewrite the sentence so it would still fit on the sign, but would convey its intended meaning.

1.5.3 And / or

Consider this sentence

"After work, I will go to the beach or I will do my grocery shopping."

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means "either one or the other or both," or whether he means "either one or the other but not both." (Some people use the awkward phrase "and/or" to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition 1.5.2. In mathematics, the word "or" *always means* "either one or the other or both."

Think/Pair/Share. For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.

- Choose a different value of x that makes the statement true (or say why that's not possible).
- Choose a different value of x that makes the statement false (or say why that's not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is negative.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.

1.5.4 Quantifiers

Problem 23 (All About the Benjamins)**.** You are handed an envelope filled with money, and you are told "Every bill in this envelope is a \$100 bill."

- What would convince you *beyond any doubt* that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you *beyond any doubt* that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: "There is a \$100 bill in this envelope."

- What would convince you *beyond any doubt* that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you *beyond any doubt* that the sentence is false? How could you convince someone else that the sentence is false?

Think/Pair/Share. After you have thought about the problem on your own for a while, talk through your ideas with a partner. What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- "Every time…"
- "For all numbers…"
- "For every choice…"
- "It's always true that…"

These are *universal statements*. Such statements claim that something is always true, no matter what.

- To prove a *universal statement* is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a *universal statement* is true, you must either check every single case, or you must find a **logical reason** why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes…"
- "There is some number…"
- "For some choice…"
- "At least once…"

These are *existential statements*. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an *existential statement* is true, you may just find the example where it works.

- To prove an *existential statement* is false, you must either show it fails in every single case, or you must find a **logical reason** why it can't be true. (Sometimes the first option is impossible!)

Think/Pair/Share. For each statement below, do the following:

- Decide if it is a *universal statement* or an *existential statement.* (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)

- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.

2. Every prime number is odd.

3. For all positive numbers x, $x^3 > x$.

4. There is some number x such that $x^3 = x$.

5. The points $(-1, 1)$, $(2, 1)$, and $(3, 0)$ all lie on the same line.

6. Addition (of real numbers) is commutative.

7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That's okay for now! Divide your answers into four categories:

(a) I am confident that the justification I gave is good.

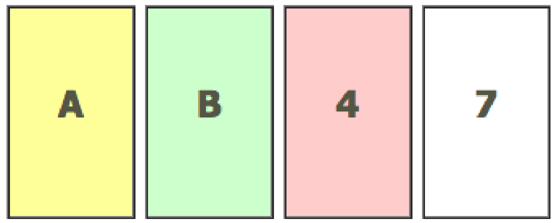
(b) I am not confident in the justification I gave.

(c) I am confident that the justification I gave is *not* good, or I could not give a justification.

(d) I could not decide if the statement was true or false.

1.5.5 Conditional statements

Problem 24 (Card Logic)**.** These cards are on a table.



Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side." Which cards *must* you flip over to be certain that your friend is telling the truth?

♣♣♣ Fellow: [Picture is stolen from a long-forgotten source. Can we re-create it so it's not any kind of copyright violation?]

Think/Pair/Share. After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Problem 25 (IDs at a Party)**.** You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a beer, another has a Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. Unfortunately, you are not sure if they are drinking Seven-up or vodka and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

Think/Pair/Share. After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition 1.5.3. A **conditional statement** can be written in the form

$$\text{If } \underbrace{\textit{some statement}}_{\text{hypothesis}} \text{ then } \underbrace{\textit{some statement}}_{\text{conclusion}}.$$

Think/Pair/Share. These are each conditional statements, though they are not all stated in "if/then" form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent "if/then" statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If $2^n - 1$ is prime, then n is prime.
   n is prime. / $2^n - 1$ is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It's either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the *hypothesis* and the *conclusion* are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 1.5.4. Here is a conditional statement:

$$\text{“If } \underbrace{\text{I win the lottery}}_{\text{hypothesis}}\text{, then } \underbrace{\text{I'll give each of my students \$1,000}}_{\text{conclusion}}\text{.”}$$

There are four things that can happen:

True hypothesis, true conclusion I do win the lottery, and I do give everyone in class \$1,000. I kept my promise, so the conditional statement is TRUE.

True hypothesis, false conclusion I do win the lottery, but I decide **not to** give everyone in class \$1,000. I broke my promise, so the conditional statement is FALSE.

False hypothesis, true conclusion I don't win the lottery, but I'm exceedingly generous, so I go ahead and give everyone in class \$1,000. I didn't break my promise! (Do you see why?) So the conditional statement is TRUE.

False hypothesis, false conclusion I don't win the lottery, so I don't give everyone in class \$1,000. I didn't break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If the statement is not false, it must be true.

Example 1.5.5. Here's another conditional statement:

"If you live in Honolulu, then you live in Hawaii."

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let's think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.

- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it doesn't matter! The statement is true either way. The right way to understand such a statement is as a *universal statement*: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here's how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since

Honolulu is in Hawaii), so the statement is true for that person. I don't need to consider people who don't live in Honolulu. The statement is *automatically* true for those people, because the hypothesis is false!"

Example 1.5.6. How do we show a (universal) conditional statement is false? You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

"If you are a good swimmer, then you are a good surfer."

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think/Pair/Share. For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If the units digit of a number is 4, then the number is even.
6. If a number is even, then the units digit of that number is 4.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.
9. If you are tall, then you have long hair.

Think/Pair/Share (Two truths and a lie). On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but don't say which are true and which is false. See if your partner can figure it out!

1.6 Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you haven't really *solved* the problem until you've explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, and grant proposals. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Polya's step (5).

Writing in mathematics is different from writing poetry or an English paper. The goal of mathematical writing is not florid description, but clarity. If your reader doesn't understand, you haven't done a good job. Here are some tips for good mathematical writing.

Don't Turn in Scratch Work When you are solving *problems* and not *exercises*, you're going to have lots of false starts. You're going to try lots of things that don't work. You're going to make lots of mistakes. You're going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is *not* what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)

(Re)state the Problem Don't assume your reader (even if it's the teacher who assigned the problem!) knows what problem you're solving. If the problem has a very long description, you can summarize it. You don't have rewrite it word-for-word or give all of the details. But make sure the question is clear.

Clearly Give the Answer It's not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you're trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be *clearly* stated somewhere in the writeup, and it should be easy to find.

Be Correct Of course, everyone makes mistakes as they're working on a problem. But we're talking about after you've solved the problem, when you are writing up your solution to share with someone else. The best writing in the world can't save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.

Justify Your Answer You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you *know* your answer is right?"

Be Concise There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.

Use Variables and Equations Often an equation is much easier to read and understand (and is more concise!) than a long paragraph of text describing a calculation. Mathematical writing often has way fewer words (and way more equations) than other kinds of writing.

Define your Variables If you use variables or equations in the solution of your problem, always say what the variable stands for *before* you use it. If you use an equation, say where it comes from and why it applies to this situation. Don't make your reader guess!

Use Pictures If pictures helped you solve the problem, include those pictures in your final solution. Even if you didn't draw a picture to solve the problem, it still might help your reader understand the solution. And that's your goal!

Use Correct Spelling and Grammar Proofread your work. A good test is to read your work aloud. There should be complete, natural-sounding sentences. This includes reading the equations and calculations aloud. They should read naturally and make sense. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.

Format Clearly Don't write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Don't write too small. Don't make your reader struggle to read and understand your work.

Acknowledge Collaborators If you worked with someone else on solving the problem, give them credit!

Here is a problem you've already seen:

Problem. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

Think/Pair/Share. Below you will find several solutions that were turned in by students. Using the criteria above, how would you score these solutions on a scale of 1 to 5? Give reasons for your answers.

Solution (Solution 1)**.** 41312432

This is the largest eight-digit b/c the #s 1, 2, 3, 4 & all separated by the given amount of spaces.

Solution (Solution 2)**.** 41312432

You have to have the 4 in the highest place and work down from there. However unable to follow the rules the 2 and the 1 in the 10k and 100k place must switch.

Solution (Solution 3)**.** 41312432

First, I had to start with the #4 because that is the largest digit I could start with to get the largest #.

Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4s.

Next, I place 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be.

I then placed the 3 because opening spaces showed me that I could fit three digits in between the two 3s.

Lastly, I had to input the final 2s, which worked out because there were two digits separating them.

Solution (Solution 4)**.**

1x1

2xx2

3xxx3

4xxxx4
Answer: 41312432

Solution (Solution 5).

$$\underline{4}\ \underline{3}\ \_\ \underline{2}\ \underline{4}\ \underline{3}\ \underline{2}\ \_$$

$$\underline{4}\ \underline{2}\ \_\ \_\ \underline{2}\ \underline{4}\ \_\ \_$$

$$\underline{4}\ \_\ \underline{1}\ \underline{3}\ \underline{1}\ \underline{4}\ \_\ \underline{3}$$

$$\star\underline{4}\ \underline{1}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{4}\ \underline{3}\ \underline{2}$$

4 needs to be the first # to make it the biggest. Then check going down from next largest to smallest. Ex:

$$\underline{4}\ \underline{3}\ \underline{\hspace{4cm}}\times$$

$$\underline{4}\ \underline{2}\ \underline{\hspace{4cm}}\times$$

$$\underline{4}\ \underline{1}\ \underline{\hspace{4cm}}\checkmark$$

Solution (Solution 6). 41312432

I put 4 at the 10,000,000 place because the largest # should be placed at the highest value.

Numbers 2 & 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly.

So I ended up placing the #1 there. In the 100,000 place I put the #3 because it was the second higehst number.

Solution (Solution 7). 41312432

Since the problem asks you for the largest 8 digit #, I knew 4 had to be the first # since it's the greatest # of the set.

To solve the rest of the problem, I used the guess and test method. I tried many different combinations. First using the #3 as the second digit in the sequence, but came to no answer. Then the # 2, but no combination I found correctly finished the sequence.

I then finished with the #1 in the second digit in the sequence and was able to successfully fill out the entire #.

Solution (Solution 8)**.**

$\underline{4}$ _ _ _ _ $\underline{4}$ _ _

4 has to be the first digit, for the number to be the largest possible. That means the other 4 has to be the 6th digit in the number, because 4s have to be separated by four digits.

$\underline{4}$ _ $\underline{3}$ _ _ $\underline{4}$ $\underline{3}$ _

3 must be the third digit, in order for the number to be largest possible. 3 cannot be the second digit because the other 3 would have to be the 6th digit in the number, but 4 is already there.

$\underline{4}$ $\underline{1}$ $\underline{3}$ $\underline{1}$ _ $\underline{4}$ $\underline{3}$ _

1s must be separated by one digit, so the 1s can only be the 2nd and 4th digit in the number.

$\underline{4}$ $\underline{1}$ $\underline{3}$ $\underline{1}$ $\underline{2}$ $\underline{4}$ $\underline{3}$ $\underline{2}$

This leaves the 2s to be the 5th and 8th digits.

Solution (Solution 9)**.** With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4 then 1 then 3, since I'm going for the highest # possible.

My answer: 41312432

1.7 The Last Step

A lot of people — from Polya to the writers of the Common Core State Standards and lots of people in between — talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example 1.7.1. Recall Problem 3, "Squares on a Chess Board":

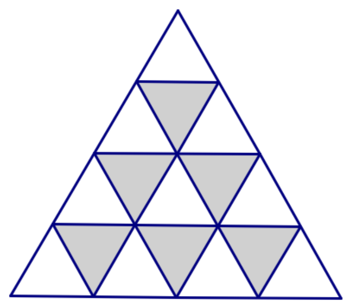
> How many squares are on a standard 8×8 chess board? (The answer is *not* 64! It's a lot bigger!)

We've already talked about some obvious follow-up questions like "What about a 10×10 chess board? Or 100×100? Or $n \times n$?"

But there are lots of interesting (and less obvious ... and harder) questions you might ask:

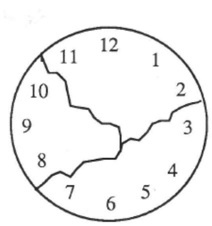
- How many *rectangles* can you find on a standard 8×8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1×2 and 5×3. How could you possibly count them all?)

- How many triangles can you find in this picture?



Example 1.7.2. Recall Problem 4, "Squares on a Chess Board":

> This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?



The original problem only asks if you can find *one* other way. The obvious follow-up question: "Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify that you have found every possibility."

Think/Pair/Share. Choose a problem from the Problem Bank in Section 1.4 (preferably a problem you have worked on, but that's not strictly necessary). What follow-up or similar questions could you ask?

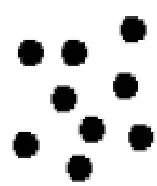
Chapter 2

Place Value

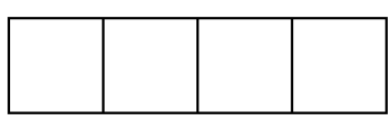
♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

2.1 Dots and Boxes

Here are some dots; in fact there's nine of them.



Here are some boxes:



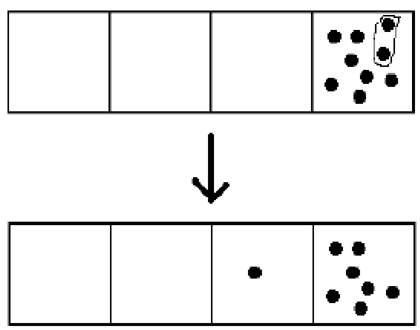
We're going to a play a game in which boxes explode dots and move them around. Here's our first rule:

♣♣♣ Fellow: [Have the rules set off in a box somehow?]

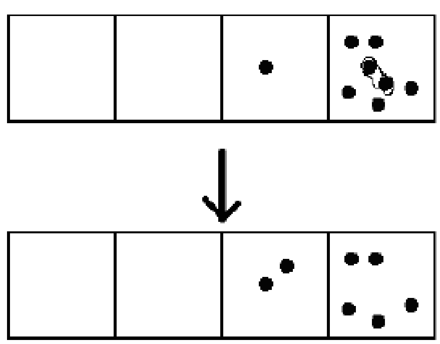
The $1 \leftarrow 2$ **Rule:**
Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

Example 2.1.1 (Nine dots in the $1 \leftarrow 2$ system). We start by placing nine dots in the rightmost box.

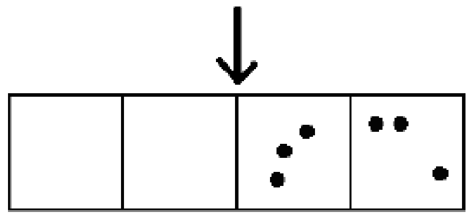
Two dots in that box explode and become one dot in the box to the left.



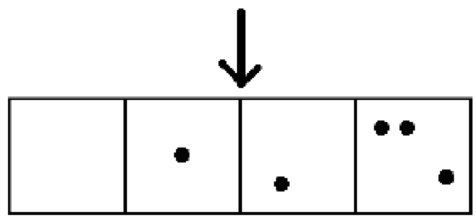
Since there are more than two dots in the rightmost box, it can happen again.



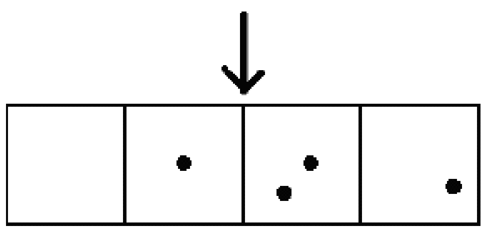
And again!



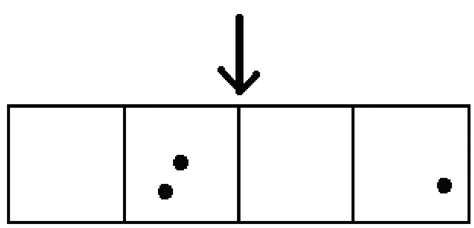
Hey, now we have more than two dots in the second box, so those can explode and move!

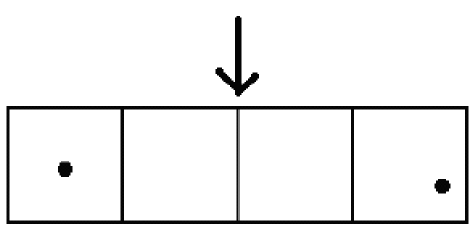


And the rightmost box still has more than two dots.



Keep going, until no box has two dots.



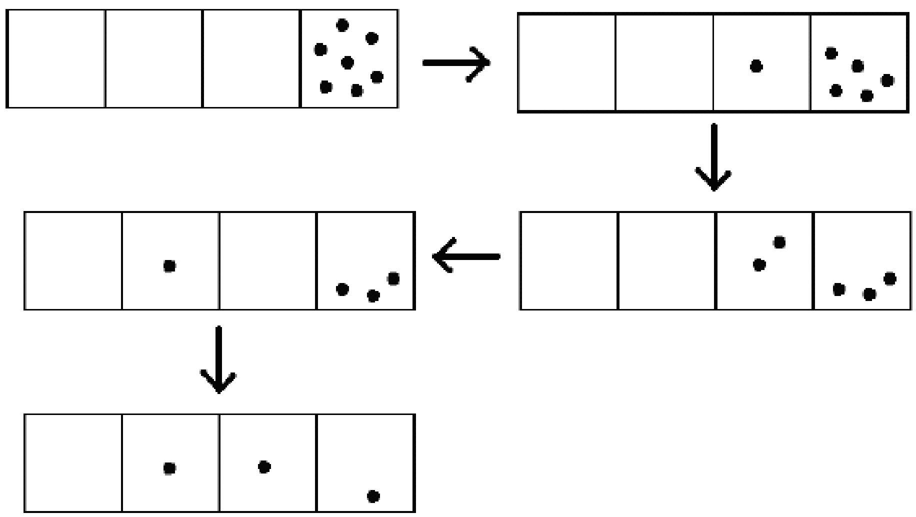


After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

The $1 \leftarrow 2$ code for nine dots is: 1001

On Your Own

Here's a diagram showing what happens for seven dots in a $1 \leftarrow 2$ box. Trace through the diagram, and circle the pairs of dots that "exploded" at each step.



The $1 \leftarrow 2$ code for seven dots is: 111

Problem 26. Note: In solving this problem, you don't need to draw on paper; that can get tedious! Maybe you could use buttons or pennies for dots and do this by hand. What could you use for the boxes?

(a) Draw 10 dots in the right-most box and perform the explosions. What is the $1 \leftarrow 2$ code for ten dots?

(b) Find the $1 \leftarrow 2$ code for thirteen dots.

(c) Find the $1 \leftarrow 2$ code for six dots.

(d) What number of dots has $1 \leftarrow 2$ code 101?

Think/Pair/Share. After you worked on the problem, compare your answer with a partner. Did you both get the same code? Did you have the same process?

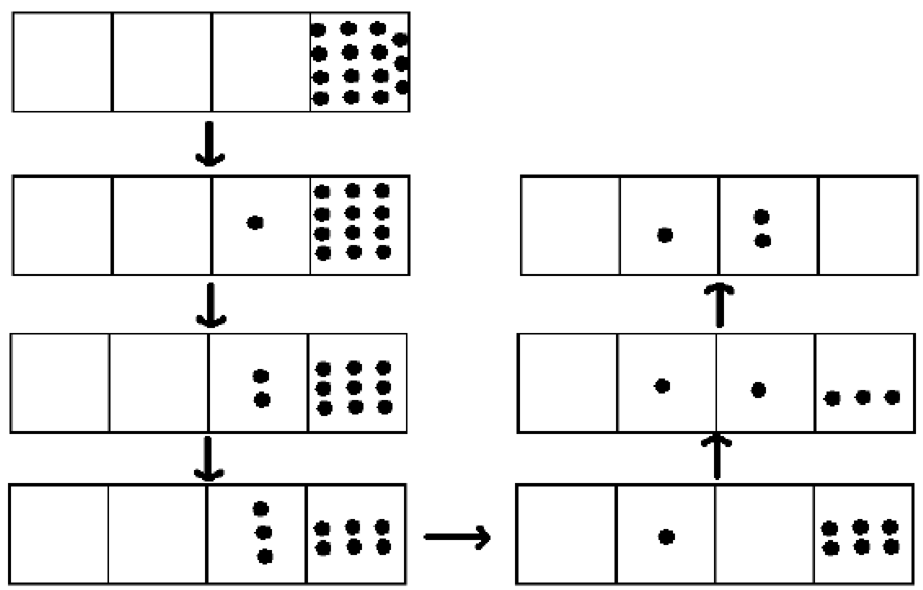
2.2 Other Rules

Let's play the dots and boxes game, but change the rule.

♣♣♣ Fellow: [Have the rules set off in a box somehow?]

> **The $1 \leftarrow 3$ Rule:**
> **Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.**

Example 2.2.1 (Fifteen dots in the $1 \leftarrow 3$ system). Here's what happens with fifteen dots:



The $1 \leftarrow 3$ code for seven dots is: 120

Problem 27.

(a) Show that the $1 \leftarrow 3$ code for twenty dots is 202.

(b) Show that the $1 \leftarrow 3$ code for four dots is 11.

(c) What is the $1 \leftarrow 3$ code for thirteen dots?

(d) What is the $1 \leftarrow 3$ code for twenty-five dots?

(e) What number of dots has $1 \leftarrow 3$ code 1022?

(f) Is it possible for a collection of dots to have $1 \leftarrow 3$ code 2031? Explain your answer.

Problem 28.

(a) Describe how the $1 \leftarrow 4$ rule would work.

(b) What is the $1 \leftarrow 4$ code for the number thirteen?

Problem 29.

(a) What is the $1 \leftarrow 5$ code for the number thirteen?

(b) What is the $1 \leftarrow 5$ code for the number five?

Problem 30.

(a) What is the $1 \leftarrow 9$ code for the number thirteen?

(b) What is the $1 \leftarrow 9$ code for the number thirty?

Problem 31.

(a) What is the $1 \leftarrow 10$ code for the number thirteen?

(b) What is the $1 \leftarrow 10$ code for the number thirty-seven?

(c) What is the $1 \leftarrow 10$ code for the number two hundred thirty-eight?

(d) What is the $1 \leftarrow 10$ code for the number five thousand eight hundred and thirty-three?

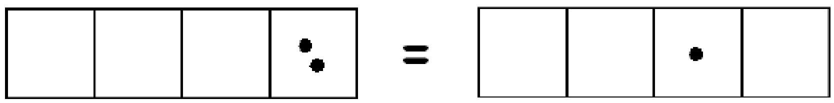
Think/Pair/Share. After you have worked on the problems on your own, compare your ideas with a partner. Can you describe what's going on in Problem 31 and why?

2.3 Binary Numbers

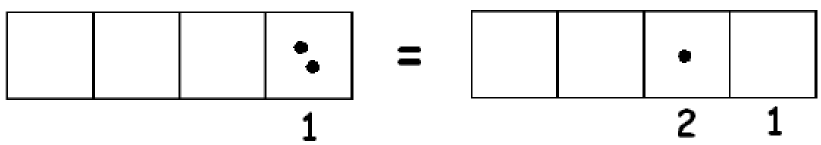
Let's go back to the $1 \leftarrow 2$ rule for a moment:

> **The** $1 \leftarrow 2$ **Rule:**
> **Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.**

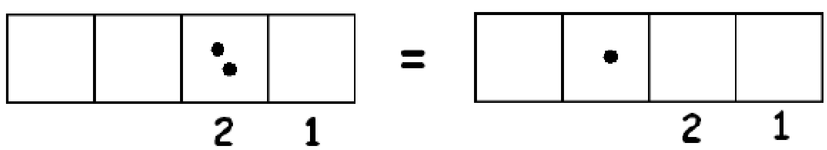
Two dots in the right-most box is worth one dot in the next box to the left.



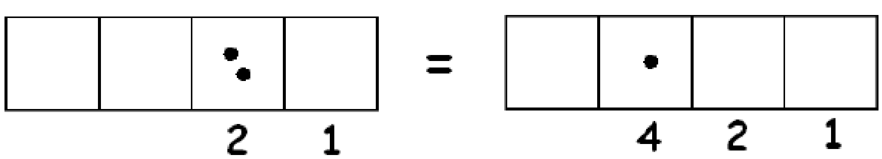
If each of the original dots is worth "one," then the single dot on the left must be worth two.



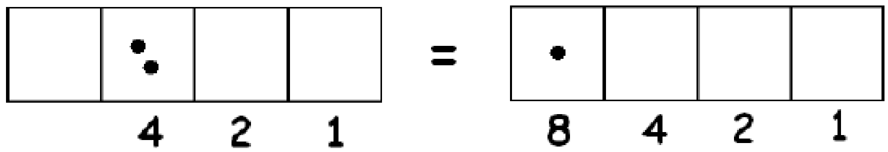
But we also have two dots in the box of value 2 is worth 1 dot in the box just to the left...



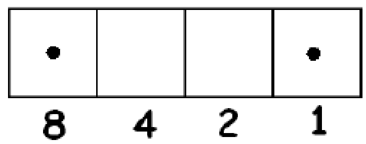
So that next box must be worth two 2s, which is four!



And two of these fours make eight.

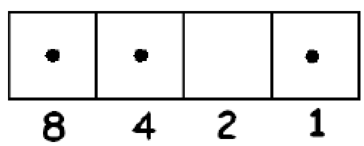


Example 2.3.1. We said earlier that the $1 \leftarrow 2$ code for nine dots was 1001. Lets check:



$$8 + 1 = 9, \text{ so this works!}$$

We also said that thirteen has $1 \leftarrow 2$ code 1101. This is correct.



$$\text{Yep!} \quad 8 + 4 + 1 = 13.$$

Problem 32.

(a) If there were a box to the left of the 8 box, what would the value of that box be?

(b) What would be the value of a box *two* spots to the left of the 8 box? Three spots to the left?

(c) What number has $1 \leftarrow 2$ code 100101?

(d) What is the $1 \leftarrow 2$ code for the number two hundred?

Definition 2.3.2. Numbers written in the $1 \leftarrow 2$ code are called *binary numbers* or *base two* numbers. (The prefix "bi" means "two.") From now on, when we want to indicate that a number is written in base two, we will write a subscript "two" on the number. So 1001_{two} means "the number of dots that has $1 \leftarrow 2$ code 1001," which we already saw was nine.

Important! When we read 1001_{two} we say "one zero zero one base two." We don't say "one thousand and one," because "thousand" is not a binary number.

Think/Pair/Share. Compare you work on problem 32 with a partner.

- Your first goal: come up with a *general method* to find the number of dots represented by any binary number. Clearly describe your method. Test your method out on these numbers, and check your work by actually "unexploding" the dots.

  $$1_{\text{two}} \qquad 101_{\text{two}} \qquad 1011_{\text{two}} \qquad 1111_{\text{two}} \qquad 1101101_{\text{two}}$$

- Explain why binary numbers only contain the digits 0 and 1.

- Here is a new (harder) goal: come up with a *general method* to find the binary number related to any number of dots *without actually going through the "exploding dot" process.* Clearly describe your method. Test your method out on these numbers, and find a way to check your work.

  two dots $=???_{\text{two}}$ seventeen dots $=???_{\text{two}}$ sixty-four dots $=???_{\text{two}}$

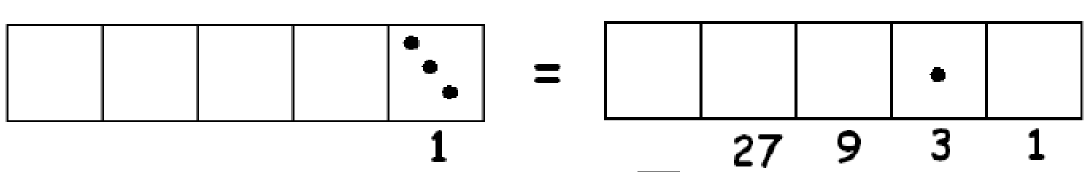
  sixty-three dots $=???_{\text{two}}$ one thousand dots $=???_{\text{two}}$

2.3.1 Binary Numbers and Computers

♣♣♣ Fellow: [Can you write a *short* description of the use of binary numbers in computers? Just a paragraph or two getting across the main ideas.]

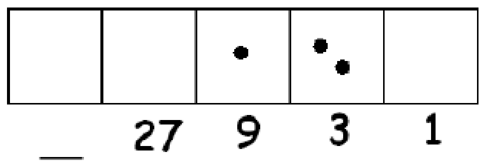
2.4 Other Bases

In the $1 \leftarrow 3$ system, three dots in one box is worth one dot in the box one spot to the left. This gives a new picture:



Each dot in the second box from the left is worth three ones. Each dot in the third box is worth three 3s, which is nine, and so on.

Example 2.4.1. We said that the $1 \leftarrow 3$ code for fifteen is 120. We see that this is correct because



$$9 + 2 \cdot 3 = 9 + 6 = 15.$$

Problem 33. Answer these questions about the $1 \leftarrow 3$ system.

(a) What label should go on the box to the left of the 27 box?

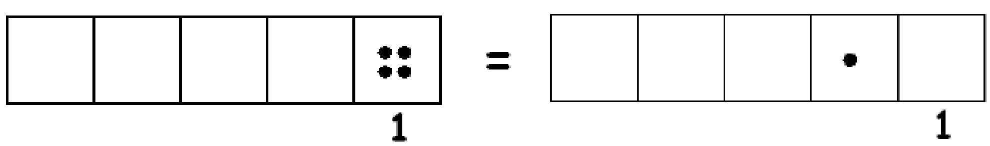
(b) What would be the value of a box *two* spots to the left of the 27 box?

(c) What number has $1 \leftarrow 3$ code 21002?

(d) What is the $1 \leftarrow 3$ code for the number two hundred?

Problem 34. In the $1 \leftarrow 4$ system, four dots in one box are worth one dot in the box one place to the left.

(a) What is the value of each box?

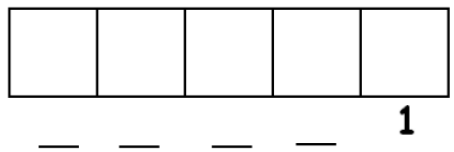


(b) What is the $1 \leftarrow 4$ code for twenty-nine?

(c) What number has $1 \leftarrow 4$ code 132?

Problem 35. In the $1 \leftarrow 10$ system, ten dots in one box are worth one dot in the box one place to the left.

(a) What is the value of each box?



(b) What is the $1 \leftarrow 10$ code for eight thousand four hundred and twenty-two?

(c) What number has $1 \leftarrow 10$ code 95753?

(d) When we write the number 7842 the "7" is represents what quantity? The "4" is four groups of what value? The "8" is eight groups of what value? The "2" is two groups of what value?

(e) Why do human beings like the $1 \leftarrow 10$ system for writing numbers?

Definition 2.4.2. Numbers written in the $1 \leftarrow 3$ system are called *base three numbers*. Numbers written in the $1 \leftarrow 4$ system are called *base four numbers*. Numbers written in the $1 \leftarrow 10$ system are called *base ten numbers*. In general, numbers written in the $1 \leftarrow b$ system are called *base b numbers*.

In a base b number system, each place represents a *power of b*, which means b^k for some positive number k. Remember this means b multiplied by itself k times:

$$b^k = \underbrace{b \cdot b \cdots b}_{k \text{ times}}.$$

- The right-most place is the *units* or *ones* place. (Why is this a *power of b?*)
- The second spot is the "b" place. (In base 10, it's the tens place.)
- The third spot is the "b^2" place. (In base 10, that's the hundreds place, and $100 = 10^2$.)
- The fourth spot is the "b^3" place. (In base 10, that's the thousands place, and $1000 = 10^3$.)
- And so on…the n^{th} spot is the b^{n-1} place.

Notation: Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So 102_{three} is a base three number, 222_{four} is a base four number, and 54321_{ten} is a base ten number. If the base is not written, we assume the number is written in base ten.

Think/Pair/Share.

(a) Find the number of dots represented by:

$$102_{\text{three}}, \quad 222_{\text{four}}, \quad 54321_{\text{ten}}.$$

(b) Represent nine dots in each base:

three, four, five, six, seven, eight, nine, and ten.

(c) Which digits are used in the base two system? The base three system? The base four system? The base five system? The base six system? The base ten system?

(d) What does the *base* tell you about the number system? (Think of as many answers as you can!)

2.4.1 Base b to Base Ten

In Section 2.3, you were asked to come up with *general methods* to translate numbers from base two (binary) to base ten (our standard system). We're now going to describe some general methods for converting from base b to base ten, where b can represent any whole number bigger than one.

If the base is b, that means we're in a $1 \leftarrow b$ system. A dot in the right-most box is worth 1. A dot in the second box is worth b. A dot in the third box is worth $b \times b = b^2$, and so on.

♣♣♣ Fellow: [Can you make a picture of base b boxes with the appropriate labels, say up to b^4?]

So, for example, the number 10123_b represents

$$1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 2 \cdot b + 3 \cdot 1 \text{ dots},$$

because we imagine three dots in the right-most box (each worth one), two dots in the second box (each representing b dots), one dot in the third box (representing b^2 dots), and so on. That means we can just do a short calculation to find the total number of dots, without going through all the trouble of drawing the picture and "unexploding" the dots.

Example 2.4.3. Consider the number 123_{five}. This represents

$$1 \cdot 5^2 + 2 \cdot 5 + 3 = 25 + 10 + 3 = 38 \text{ dots.}$$

On the other hand, the number 123_{seven} represents

$$1 \cdot 7^2 + 2 \cdot 7 + 3 = 49 + 14 + 3 = 66 \text{ dots.}$$

Think/Pair/Share.

- Convert each number to base ten. Compare your answers with a partner to be sure you agree.

$$18_{\text{nine}}, \qquad 547_{\text{eight}}, \qquad 3033_{\text{five}}, \qquad 11011_{\text{three}}.$$

- Which number represents a greater amount of total dots:

$$23,455,443_{\text{six}} \quad \text{or} \quad 23,455,443_{\text{eight}}?$$

  Justify your answer.

2.4.2 Base Ten to Base b

In Section 2.3, you were also asked to come up with *general methods* to translate numbers from base ten to base two. We're now going to describe some general methods for converting from base ten to base b, where b can represent any whole number bigger than one.

We'll work out an example, and then describe the general method.

Example 2.4.4. To convert 321 to a base five number (without actually going through the tedious process of exploding 321 dots in groups of five):

Find the largest power of five that is smaller than 321. We'll just list powers of five:

$$5^1 = 5, \quad 5^2 = 25, \quad 5^3 = 125, \quad 5^4 = 625.$$

So we know that the left-most box we'll use is the 5^3 box. ♣♣♣ Fellow: [add a picture of the appropriately labeled boxes for base 5, up to 5^3?]

How many dots will be in that left-most box? That's the same as asking how many 125s are in 321. Since

$$2 \cdot 125 = 250 \quad \text{and} \quad 3 \cdot 125 = 375,$$

we have two dots in the 5^3 box, representing a total of 250 dots, but rather than drawing dots we'll start writing the digits to represent them.

♣♣♣ Fellow: [picture of the base 5 boxes with a "2" in the 5^3 box?]

How many dots are left unaccounted for? $321 - 250 = 71$ dots are left.

Now just repeat the process: We can put a "2" in the 5^2 box, and that takes care of 50 dots. So so far we have two in the 5^3 box and two in the 5^2 box, so that's a total of

$$2 \cdot 125 + 2 \cdot 25 = 300 \text{ dots.}$$

♣♣♣ Fellow: [picture of the base 5 boxes with 2 in 5^3 box and 2 in the 5^2 box?]

We have 21 dots left to account for. The biggest power of 5 that's less than 21 is just 5. So we can put a "4" in the 5 box, and we have one left over in the one box.

♣♣♣ Fellow: [picture of the base 5 boxes with 2 in 5^3 box and 2 in the 5^2 box 4 in the 5 box and 1 in the 1 box?]

$$2 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 1 = 250 + 50 + 20 + 1 = 321 \text{ dots.}$$

$$\text{So } 321 = 2241_{\text{five}}.$$

The general algorithm to convert from base ten to base b:

1. Start with your base ten number n. Find the largest *power of* b that's less than your number n, say that power is b^k.

2. Figure out how many dots can go in the b^k box without going over the number n. Say that number is a. Put the digit a in the b^k box, and then subtract $n - a \cdot b^k$ to figure out how many dots are left.

3. If your number is now zero, you accounted for all the dots. Put zeros in any boxes that remain, and you have the number. Otherwise, start over at step (1) with the number of dots you have left.

The method seems a little tricky to describe in complete generality. It's probably better to try a few examples on your own to get the hang of it.

Think/Pair/Share. Use the method above to convert 99_{ten} to base three, to base four, and to base five.

The first method we described fills in the boxes from left to right. Here's another method to convert base ten numbers to another base, and this method fills in the digits from right to left. Again, we'll start with an example and then describe the general method:

Example 2.4.5. To convert 712 to a base seven number:

Divide 712 by seven and find the quotient and remainder:

$$712 \div 7 = 101 \text{ R}5.$$

Put the remainder in the ones place:

$$712 = \underline{???}5_{\text{seven}}.$$

Now take the quotient and divide by seven to find the quotient and remainder:

$$101 \div 7 = 14 \text{ R}3.$$

Put the remainder in the sevens place:

$$712 = \underline{???}35_{\text{seven}}.$$

Take the previous quotient and divide by seven again:

$$14 \div 7 = 2 \text{ R}0.$$

Put the remainder in the 7^2 place:

$$712 = \underline{???}035_{\text{seven}}.$$

Since the quotient that's left is less than seven, it goes in the 7^3 place, and we're done.

$$712 = 2035_{\text{seven}}.$$

Of course, we can (and should!) check our calculation by converting the answer back to base ten:

$$2035_{\text{seven}} = 2 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 5 = 686 + 0 + 21 + 5 = 712_{\text{ten}}.$$

So here's a second general method for converting base ten numbers to an arbitrary base b:

1. Divide the base ten number by b to get a quotient and a remainder.
2. Put the remainder in the right-most space in the base b number.
3. If the quotient is less than b, it goes in the space one spot to the left. Otherwise, go back to step (1) and repeat it with the quotient, filling in the remainders from right to left in the base b number.

We can use the dots and boxes system to explain why this method of quotients and remainders works. It's not just a "trick!" We'll stick with the example of converting 712 to base seven, so we have something specific to talk about.

- We imagine 712 dots in the right-most box, since that represents 712 dots total. Since we're converting to base seven, we're in the $1 \leftarrow 7$ system. ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with just the number 721 in the right-most box.]
- Groups of seven dots will explode, and each group of seven becomes one dot in the next box. How many groups of seven dots are there? Well, there are 101 groups of seven, with 5 dots left over out of a group. That's what we figured out with the calculation

  $$712 \div 7 = 101 \text{ R}5.$$

- Imagine we explode all the groups of seven that we can make in the right-most box before we move on. Then we would have 5 dots left in that first box, and 101 dots in the second box.

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 101 in the in the second box.]

- Again, groups of seven dots will explode, and each group becomes one dot in the third box. How many groups of seven dots are there? There are 14 groups with three left over. That's what we computed like this:

  $$101 \div 7 = 14 \text{ R}3.$$

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 3 in the in the second box and 14 in the third box.]

- OK, now there are 5 dots in the right-most box, 3 dots in the second box, and 14 dots in the third box. We do it all again! Groups of seven explode, and each group forms dot in the next box to the left. Fourteen dots gives two equal groups of seven, none left over.

- So we end up with: 5 dots in the right-most box, 3 dots in the second box, zero dots in the third box, and 2 dots in the fourth box. And there's nothing left to explode!

  ♣♣♣ Fellow: [picture of the base seven boxes up to 7^3 with the number 5 in the rightmost box and 3 in the in the second box and 0 in the third box and 2 in the last box.]

- Now we can read off the number left-to-right:

$$712 = 2035_{\text{seven}}.$$

Again, the method probably makes more sense if you try it out a few times.

Think/Pair/Share. Use the method described above to convert 250_{ten} to base three, four, five, and six. For each of the computations, write a careful dots-and-boxes explanation for why it works.

2.5 Number Systems

2.5.1 History

♣♣♣ Fellow: [Can you write a *short* history / description of some different systems like the Egyptian, Mayan, and Roman numerals. What's in the book is totally overkill and too much. No need to have students do any problems. Just a few paragraphs describing additive system versus positional system and giving a couple of examples.]

2.5.2 Fibonacci

♣♣♣ Fellow: [Include a picture and *short* bio of Fibonacci? What he really *should* be famous for is giving us the arabic numerals and showing the ease of computation with a positional system, not the sequence of numbers that came from one little problem in his book... Get across where the base 10 system was created, and that Fibonacci brought it to the Western world from which it spread?]

Problem 36. What is the difference between 5_{nine} and 50_{nine}?

Problem 37. Convert each base-5 number to a base-10 number. Look for a shortcut!

$$4_{\text{five}} \qquad 40_{\text{five}} \qquad 400_{\text{five}} \qquad 4000_{\text{five}}$$

Challenges: 0.4_{five} $\quad 0.04_{\text{five}}$

Think/Pair/Share. Discuss your answers to problems 36 and 37. Discuss:

- When you add zeros to the right of a number in base ten, what does that do to the number? (Think about 2, 20, 200, 2000, etc.).
- When you add zeros to the right of a number in base nine, what does that do to the number?
- When you add zeros to the right of a number in base five, what does that do to the number?
- When you add zeros to the right of a number in base b, what does that do to the number?

2.6 Even Numbers

How do we know if a number is even? What does it mean? Well, some number of dots is *even* if I can divide the dots into pairs, and every dot has a partner. ♣♣♣ Fellow: [Add a picture of pairs of dots grouped together? A fairly large number would be good.]

And some number of dots is *odd* if, when I try to pair up the dots, I always have a single dot left over with no partner. ♣♣♣ Fellow: [Add a picture of an odd number of dots?]

The number of dots is either even or odd. It's a property of the *quantity* and is doesn't change when you write the number in different bases.

Problem 38. Which of these numbers represent an even number of dots? Explain how you decide.

$$22_{\text{ten}} \qquad 319_{\text{ten}} \qquad 133_{\text{five}} \qquad 222_{\text{five}} \qquad 11_{\text{seven}} \qquad 11_{\text{four}}$$

Think/Pair/Share. Compare your answers to problem 38 with a partner. Then try these together:

(a) Count by twos to 20_{ten}.

(b) Count by twos to 30_{four}.

(c) Count by twos to 51_{seven}.

Think/Pair/Share. You know that you can tell if a number in base 10 is even just by looking at the units digit. Which one of the following statements *best* captures the reason for this rule?

1. It works because even and odd numbers alternate, so you only have to look at the ones place.
2. It works if the number ends with an even digit, but it only works for whole numbers and decimals (e.g. 12 and 1.2.).
3. It actually only works if the last digit is 2, 4, 6, or 8.
4. It works because all digits other than the units digit — for example tens, hundreds, and thousands — represent even numbers, and sums of even numbers are even.

Problem 39.

(a) Write the numbers zero through fifteen in base seven:

$$0_{\text{seven}}, 1_{\text{seven}}, 2_{\text{seven}}, \ldots$$

(b) Circle all of the even numbers in your list. How do you know they are even?

(c) Find a rule: how can you tell if a number is even when it's written in base seven?

Problem 40.

(a) Write the numbers zero through fifteen in base four:

$$0_{\text{four}}, 1_{\text{four}}, 2_{\text{four}}, \ldots$$

(b) Circle all of the even numbers in your list. How do you know they are even?

(c) Find a rule: how can you tell if a number is even when it's written in base four?

Think/Pair/Share. Discuss your answers to problems 39 and 40.

- Why are the rules for even numbers different in different bases?
- For either your base four rule or your base seven rule, can you explain *why* it works that way?

2.7 Orders of Magnitude

Problem 41. How old were you when you were one million seconds old? (That's $1,000,000$.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 42. How old were you when you were one *billion* seconds old? (That's $1,000,000,000$.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is $1,000\times$ a million.

So you could have just taken your answer to problem 41 and multiplied it by $1,000$ to get your answer to problem 41. Of course, you would probably still need to do some calculations to make sense of the answer.

Think/Pair/Share. When is your one trillion second birthday? What will you do to celebrate?

Think/Pair/Share. The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* — the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the $10,000,000,000$ place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.
- A headline from April 2013 said, "US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every* quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 43.

1. Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?
2. Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what *order of magnitude*) would you need to stack up to reach the moon? Explain your answers.

2.7.1 Fermi Problems

James Boswell wrote,"Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it."

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself.* In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if..." questions. And eventually, they figure out something new, something no one ever knew before!

Ever for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example 2.7.1. Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have — gasp! — non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about $1,000,000$ people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. Something like this? We'll assume people don't live past 80. (Of course some people do! But we're all about making simplifying assumptions right now. That gives us 8 age categories, with about 125,000 people in each category.

age range	# people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

An even better guess (since we have a large university that draws lots of students) is that there's a "bump" around college age. And some people live past 80, but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# people
0 – 9	125,000
10 – 19	130,000
20 – 29	140,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	120,000
> 70	105,000

So, how many K–5 students are in Hawaii? That covers six years of the 0–9 (maybe 10) range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of $12,500 \times 6 = 75,000$ K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. So a reasonable average might be 25 students per class across all grades K–5 and all schools?

So that makes $75,000 \div 25 = 3,000$ K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

Problem 44. How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process:

- Define your terms.
- Write down what you know.

- Make some reasonable guesses / estimates.
- Do some simple calculations.

It's your turn to try your hand at some Fermi problems.

Problem 45. How much money does UH Manoa earn in parking revenue each year?

Problem 46. How many tourists visit Waikiki in a year?

Problem 47. How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 48. How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 49. How much pizza is consumed by UH Manoa students in a month?

Problem 50. How much would it cost to provide free day care to every 4th grader in the US?

Problem 51. How many books are in Hamilton library?

Problem 52. Make up your own Fermi problem. . . what would you be interested in calculating? Then try to solve it!

2.8 Problem Bank

Problem 53.

(a) If you were counting in base four, what number would you say just before you said 100_{four}?

(b) What number is one more than 133_{four}?

(c) What is the greatest three-digit number that can be written in base four? What numbers come just before and just after that number?

Problem 54. Explain what is wrong with writing 313_{two} or 28_{eight}.

Problem 55.

(a) Write out the base three numbers from 1_{three} to 200_{three}.

(b) Write out the base five numbers from 1_{five} to 100_{five}.

(c) Write the four base six numbers that come after 154_{six}.

Problem 56. Convert each base-4 number to a base-10 number. Explain how you did it.

$$13_{\text{four}} \qquad 322_{\text{four}} \qquad 101_{\text{four}} \qquad 1300_{\text{four}}$$

Challenges: 0.2_{four} $\qquad 0.111..._{\text{four}} = 0.\overline{1}_{\text{four}}$

Problem 57. Convert each base-10 number to a base-4 number. Explain how you did it.

$$13 \qquad 8 \qquad 24 \qquad 49$$

Challenges: 0.125 $\qquad 0.111... = 0.\overline{1}$

Problem 58. In order to use base sixteen, we need sixteen digits — they will represent the numbers zero through fifteen. We can use our usual digits 0 – 9, but we need *new symbols* to represent the *digits* ten, eleven, twelve, thirteen, fourteen, and fifteen. Here's one standard convention:

base 10 number	base 16 digit
10	A
11	B
12	C
13	D
14	E
15	F

(a) Convert these numbers from base sixteen to base ten, and show your work:

$$6\text{D}_{\text{sixteen}} \qquad \text{AE}_{\text{sixteen}} \qquad 9\text{C}_{\text{sixteen}} \qquad 2\text{B}_{\text{sixteen}}$$

(b) Convert these numbers from base ten to base sixteen, and show your work:

97 144 203 890

Problem 59. How many different symbols would you need for a base twenty-five system? Justify your answer.

Problem 60. All of the following numbers are multiples of three.

$$3, \quad 6, \quad 9, \quad 12, \quad 21, \quad 27, \quad 33, \quad 60, \quad 81, \quad 99.$$

(a) Identify the *powers of* 3 in the list. Justify your answer.

(b) Write each of the numbers above in base three.

(c) In base three: how can you recognize a *multiple of* 3? Explain your answer.

(d) In base three: how can you recognize a *power of* 3? Explain your answer.

Problem 61. All of the following numbers are multiples of five.

$$5, \quad 10, \quad 15, \quad 25, \quad 55, \quad 75, \quad 100, \quad 125, \quad 625, \quad 1000.$$

(a) Identify the *powers of* 5 in the list. Justify your answer.

(b) Write each of the numbers above in base five.

(c) In base five: how can you recognize a *multiple of* 5? Explain your answer.

(d) In base five: how can you recognize a *power of* 5? Explain your answer.

Problem 62. Convert each number to the given base.

(a) 395_{ten} into base eight.

(b) 52_{ten} into base two.

(c) 743_{ten} into base five.

Problem 63. What bases makes theses equations true? Justify your answers.

(a) $35 = 120\_$

(b) $41_{\text{six}} = 27\_$

(c) $52_{\text{seven}} = 34\_$

Problem 64. What bases makes theses equations true?

(a) $32 = 44\_$

(b) $57_{\text{eight}} = 10\_$

(c) $31_{\text{four}} = 11\_$

(d) $15_x = 30_y$

Problem 65.

(a) Find a base ten number that is twice the product of its two digits. Is there more than one answer? Justify what you say.

(b) Can you solve this problem in any base other than ten?

Problem 66.

(a) I have a four-digit number written in base ten. When I multiply my number by four, the digits get reversed.

(b) Can you solve this problem in any base other than ten?

Problem 67. Consider this base ten number (I got this by writing the numbers from 1 to 60 in order next to one another):

$$12345678910111213\ldots57585960$$

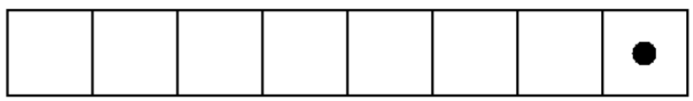
(a) What is the largest number that can be produced by erasing one hundred digits of the number? (When you erase a digit it goes away. For example, if you start with the number 12345 and erase the middle digit, you produce the number 1245.) How do you *know* you got the largest possible number?

(b) What is the smallest number that can be produced by erasing one hundred digits of the number? How do you *know* you got the smallest possible number?

Problem 68. Can you find numbers (not necessarily single digits!) a and b so that $a_b = b_a$? Can you find more than one solution? What must be true of a and b? Justify your answers.

2.9 Exploration

Problem 69. Jay decides to play with a system that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens?



Problem 70. Poindexter decides to play with a system that follows the rule $2 \leftarrow 3$.

(a) Describe what this rule does when there are three dots in the right-most box.

(b) Draw diagrams or use buttons or pennies to find the $2 \leftarrow 3$ codes for the following numbers:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 27, 30, 33, 36, \text{ and } 39$$

Can you find (and *explain*) any patterns?

Problem 71. Repeat problem 70 for your own rule. Choose two numbers a and b and figure out what the code is for your $a \leftarrow b$ system for each of the numbers above.

Chapter 3

Numbers & Operations

♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

When learning and teaching about arithmetic, it helps to have mental and physical *models* for what the operations mean. That way, when you are presented with an unfamiliar problem or a question about why something is true, you can often work it out using the model — this might mean dawning pictures, using physical materials (manipulatives), or just thinking about the model to help you reason out the answer.

Think/Pair/Share. Write down your mental models for each of the four basic operations. What do they actually *mean*? How would you explain them to a second grader? What pictures could you draw for each operation? Think about each one separately, as well as how they relate to each other:

- addition
- subtraction
- multiplication, and
- division.

After writing down your own ideas, share them with a partner. Do you and your partner have the same models for each of the operations or do you think about them differently?

Teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the four basic arithmetic operations. First, let's define some terms:

Definition 3.0.1. *Counting numbers* are literally the numbers we use for counting: $1, 2, 3, 4, 5, \ldots$. These are sometimes called the *natural numbers* by mathematicians, and they are represented by the symbol $\mathbb{N}$.

Whole numbers are the counting numbers together with 0.

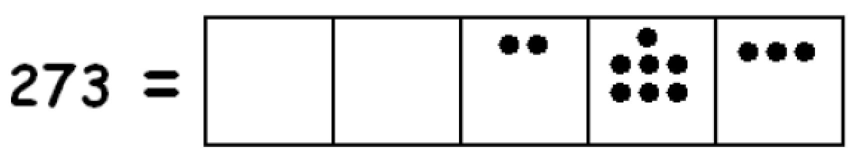
Integers include the positive and negative whole numbers, and mathematicians represent these with the symbol $\mathbb{Z}$. (This comes from German, where the word for "number" is "zählen.")

3.1 Model 1: Dots and Boxes

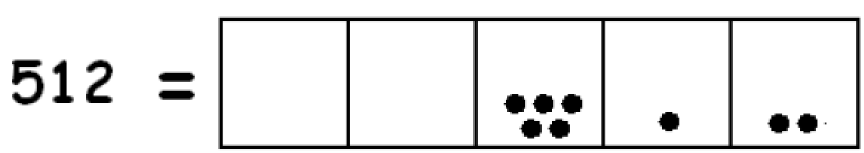
We already have a natural model for thinking about counting numbers: a number is a quantity of dots. Depending on which number system you use, you might write down the number in different ways. But the quantity of dots is a counting number, however you write it down.

3.1.1 Addition as combining

For now, we'll focus on the base-10 system. Here's how we think about the number 273 in that system:

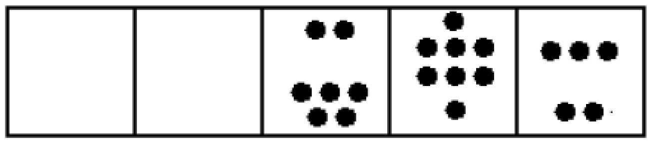


And here is the number 512:



♣♣♣ Fellow: [Most of these addition / subtraction examples might be nicer as animations, if we can do that!]

Example 3.1.1 ($273 + 512$). We can add these in the natural way: just combine the piles of dots. Since they're already in place-value columns, we can combine dots from the two numbers that are in the same place-value box.



We can count up the answer: there are 7 dots in the hundreds box, 8 dots in the tens box, and 5 dots in the ones box.

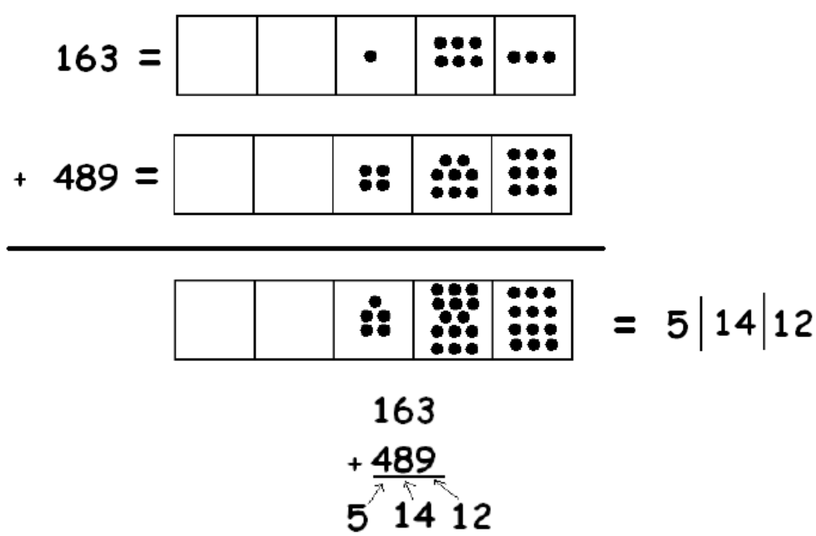
$$\begin{array}{rr} & 273 \\ + & 512 \\ \hline & 785 \end{array}$$

And saying out the long way we have:

- Two hundreds plus five hundreds gives 7 hundreds.
- Seven tens plus one ten gives 8 tens.
- Three ones plus 2 ones gives 5 ones.

This is the answer 785.

Example 3.1.2 ($163 + 489$). Let's do another one. Consider $163 + 489$.

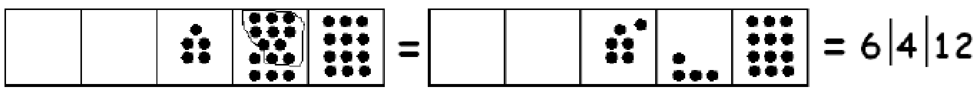


And this is absolutely mathematically correct:

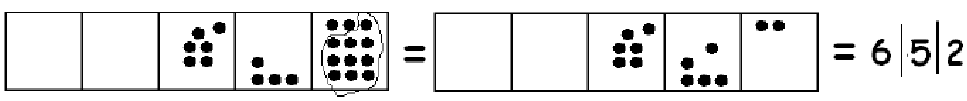
- One hundred plus four hundreds does give 5 hundreds.
- Six tens plus eight tens does give 14 tens.
- Three ones plus nine ones does give 12 ones.

The answer is 5 | 14 | 12, which we might try to pronounce as“five hundred and fourteeny-tenty twelvety!” (Oh my!)

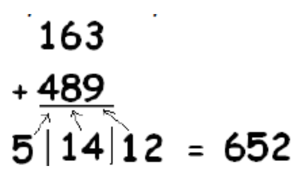
The trouble with this answer is that most of the rest of the world wouldn’t understand what we are talking about! Since this is a base 10 system, we can do some explosions.



The answer now looks like “six hundred forty twelvety”! Still not a familiar number, so let’s do another explosion:



The answer is “six hundred fifty two.” Okay, the world can understand this one!



Think/Pair/Share. Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures, or just imagine them.) Then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ +\quad 323 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ +\quad 271 \\ \hline \end{array} \qquad \begin{array}{r} 377 \\ +\quad 188 \\ \hline \end{array} \qquad \begin{array}{r} 582 \\ +\quad 714 \\ \hline \end{array}$$

$$\begin{array}{r} 310462872 \\ +\quad 389107123 \\ \hline \end{array} \qquad \begin{array}{r} 872637163 \\ +\quad 187782748 \\ \hline \end{array}$$

Problem 72. Use the dots and boxes technique to solve these problems. *Do not convert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

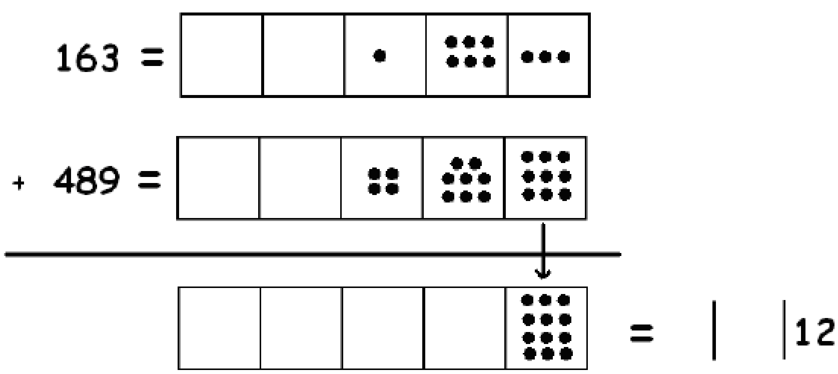
$$\begin{array}{r} 20413_{\text{five}} \\ +\quad 13244_{\text{five}} \\ \hline \end{array} \qquad \begin{array}{r} 4052_{\text{nine}} \\ +\quad 6288_{\text{nine}} \\ \hline \end{array} \qquad \begin{array}{r} 3323_{\text{seven}} \\ +\quad 3555_{\text{seven}} \\ \hline \end{array}$$

3.1.2 The Standard Algorithm for Addition

Let's go back to the example $163 + 489$. Some teachers don't like writing:

$$\begin{array}{r} 163 \\ +\,489 \\ \hline 5\,|\,14\,|\,12 \end{array} = 652$$

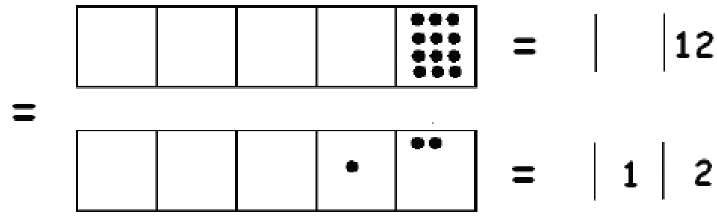
They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct — we got 12 as well.



But they don't want students to write or think "twelvety," so they have their students write something like this:

$$\begin{array}{r} {\scriptstyle 1} \\ 163 \\ +\,489 \\ \hline 2 \end{array}$$

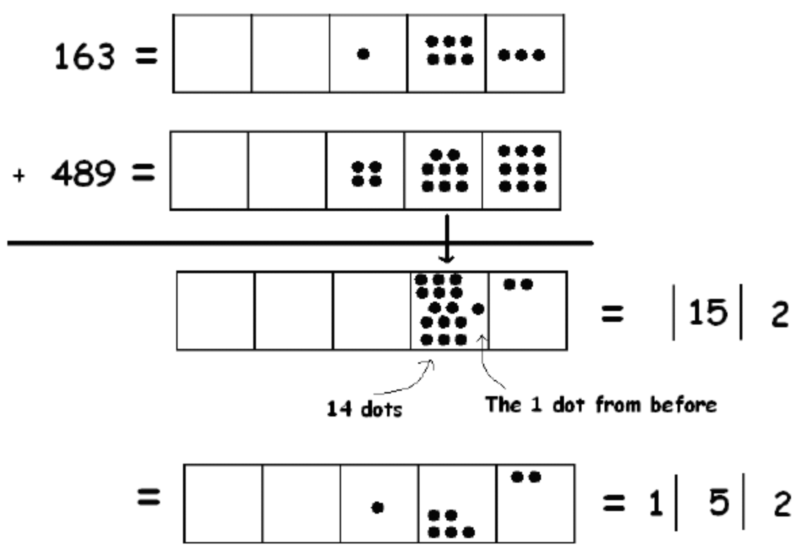
which can seem completely mysterious. What's really going on? They are exploding ten dots, of course!



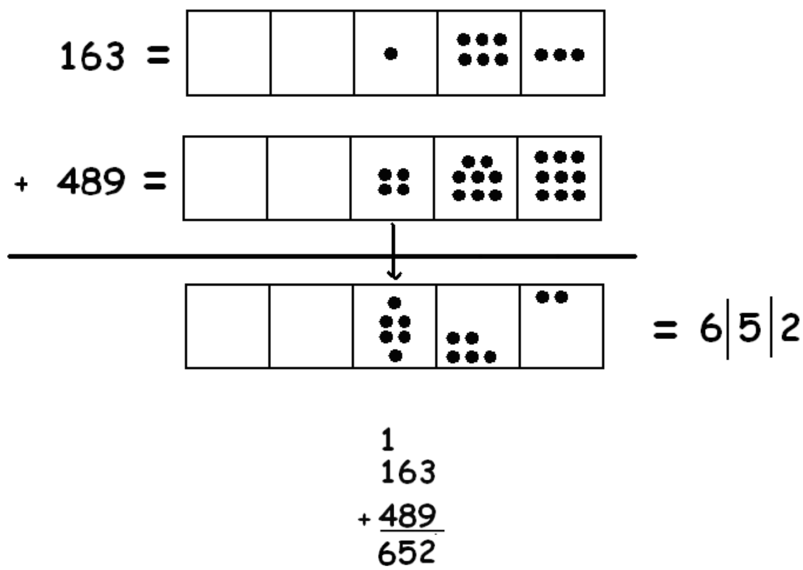
Now we carry on with the problem and add the tens. Students are taught to write:

$$\begin{array}{r} {}^{1} \\ 163 \\ +\underline{489} \\ 52 \end{array}$$

But what this means is better shown in this picture:



And now we finish the problem.



In the standard algorithm, we work from right to left, doing the "explosions" as we go along. This means that we start adding at the ones place and work towards the left-most place value, "carrying" digits that come from the explosions. (This is really not carrying; a better for it is *regrouping.* Ten ones become one ten. Ten tens become one hundred. And so on.)

In the dots and boxes method, we add in any direction or order we like and then we do the explosions at the end.

- **Why do we like the standard algorithm?** Because it is efficient.

- **Why do we like the dots and goes method?** Because it is easy to understand.

Think/Pair/Share. Redo the problems below using the standard algorithm. You will see that it is quicker.

$$\begin{array}{r} 148 \\ +\ \ 323 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ +\ \ 271 \\ \hline \end{array} \qquad \begin{array}{r} 377 \\ +\ \ 188 \\ \hline \end{array} \qquad \begin{array}{r} 582 \\ +\ \ 714 \\ \hline \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ \ 389107123 \\ \hline \end{array} \qquad \begin{array}{r} 872637163 \\ +\ \ 187782748 \\ \hline \end{array}$$

3.1.3 Subtraction as take-away

In addition, we started with two collections of dots (two numbers), and we *combined* them to form one bigger collection. That's pretty much the definition of addition: combining two collections of objects. In subtraction, we start with one collection of dots (one number), and we take some dots away.

Example 3.1.3 ($376 - 125$). Suppose we want to find $376 - 125$ in the dots and boxes model. We start with the representation of 376:

♣♣♣ Fellow: [add picture of dots & boxes for 376?]

Since we want to "take away" 125, that means we take away one dot from the hundreds box, leaving two dots. We take away two dots from the tens box, leaving five dots. And we take away five dots from the ones box, leaving one dot.

♣♣♣ Fellow: [add picture of dots & boxes for 251? Would be cool if we can see shadows or erasure of the 125. If not, maybe an intermediate picture where the appropriate dots are circle with an arrow showing they're being removed? (Sort of like pic at the bottom of p. 95 in the textbook, but using dots & boxes)]

So the answer is:

$$\begin{array}{rr} & 376 \\ - & 125 \\ \hline & 251 \end{array}$$

And saying out the long way we have:

- Three hundreds take away one hundred leaves 2 hundreds.
- Seven tens take away two tens gives 5 tens.
- Six ones take away five ones gives 1 one.

Example 3.1.4 ($921 - 551$). Let's try a somewhat harder example: $921 - 551$. We start with the representation of 921:

♣♣♣ Fellow: [add picture of dots & boxes for 921?]

Since we want to "take away" 551, that means we take away one dot from the hundreds box, leaving four dots.

♣♣♣ Fellow: [add picture of this?]

Now we want to take away five dots from the tens box, but we can't do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can "unexplode" a hundreds dot, and put ten dots in the tens box instead. Then we'll be able to take five of them away, leaving seven.

♣♣♣ Fellow: [add picture of this? show one of the hundreds dots being removed and becoming 10 tens dots, with arrows in the diagram somehow?]

(Notice that we also have one less dot in the hundreds box; there's only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.

♣♣♣ Fellow: [add picture of this?]

So the answer is:

$$\begin{array}{rr} & 921 \\ - & 551 \\ \hline & 370 \end{array}$$

Think/Pair/Share. Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures, or just imagine them.)

$$\begin{array}{rr} & 323 \\ - & 148 \\ \hline \end{array} \qquad \begin{array}{rr} & 567 \\ - & 271 \\ \hline \end{array} \qquad \begin{array}{rr} & 377 \\ - & 188 \\ \hline \end{array} \qquad \begin{array}{rr} & 714 \\ - & 582 \\ \hline \end{array}$$

$$\begin{array}{rr} & 389107123 \\ - & 310462872 \\ \hline \end{array} \qquad \begin{array}{rr} & 872637163 \\ - & 187782748 \\ \hline \end{array}$$

Problem 73. Use the dots and boxes technique to solve these problems. *Do not convert to base 10! Try to work directly in the base given.* It will probably help to actually draw the pictures.

$$\begin{array}{rr} & 20413_{\text{five}} \\ - & 13244_{\text{five}} \\ \hline \end{array} \qquad \begin{array}{rr} & 6288_{\text{nine}} \\ - & 4052_{\text{nine}} \\ \hline \end{array} \qquad \begin{array}{rr} & 3555_{\text{seven}} \\ - & 3323_{\text{seven}} \\ \hline \end{array}$$

3.1.4 The Standard Algorithm for Subtraction

Just like in addition, the standard algorithm for subtraction requires you to work from right to left, and "borrow" (this is really *regrouping*!) whenever necessary. Notice that in the dots and boxes approach, you don't need to go in any particular order when you do the subtraction. You just "unexplode" the dots as necessary when computing.

Here's how the standard algorithm looks with the dots and boxes model for $921 - 551$: Start with 921 dots.

♣♣♣ Fellow: [repeat the picture of 921 in dots & boxes]

Then take away one dot from the ones box.

♣♣♣ Fellow: [show one dot going away]

$$\begin{array}{r} 921 \\ -\quad 551 \\ \hline 0 \end{array}$$

Now we want to take away five dots from the tens box. But there aren't five dots there. So we "unexploded" one of the hundreds dots to get more tens:

♣♣♣ Fellow: [picture of this?]

And here's how we're taught to write that regrouping:

$$\begin{array}{rrrr} & 8 & 12 & \\ & \cancel{9} & \cancel{2} & 1 \\ - & 5 & 5 & 1 \\ \hline & & & 0 \end{array}$$

Now we have enough tens; we can take away five of them.

$$\begin{array}{rrrr} & 8 & 12 & \\ & \cancel{9} & \cancel{2} & 1 \\ - & 5 & 5 & 1 \\ \hline & & 7 & 0 \end{array}$$

Finally, we want to take away five hundreds.

♣♣♣ Fellow: [picture of this?]

$$\begin{array}{rrrr} & 8 & 12 & \\ & \cancel{9} & \cancel{2} & 1 \\ - & 5 & 5 & 1 \\ \hline & 3 & 7 & 0 \end{array}$$

3.1.5 Multiplication as Repeated Addition

Problem 74. Jenny was asked to compute 243192×4. She wrote:

243192 x 4 = 8|16|12|4|36|8

(a) What was Jenny thinking about? Is her answer correct?

(b) Translate Jenny's answer into a number that the rest of the world can understand.

(c) Use Jenny's method to find the answers to these multiplication exercises. Be sure to translate your answers into familiar base 10 numbers.

$156 \times 3 =$ $\qquad$ $2873 \times 2 =$ $\qquad$ $71181 \times 5 =$ $\qquad$ $3726510392 \times 2 =$

Problem 75. Can you adapt Jenny's method to solve these problems? Write your answers in base eight. Try to work directly in base eight rather than converting to base 10 and back again!

$156_{\text{eight}} \times 3_{\text{eight}} =$ $\qquad$ $2673_{\text{eight}} \times 4_{\text{eight}} =$ $\qquad$ $36255772 \times 2_{\text{eight}} =$

Think/Pair/Share. After you have worked on problems 74 and 75, share your ideas with a partner. Can you relate Jenny's method to the standard algorithm for multiplication?

Jenny might have been thinking about multiplication as repeated addition. If we have some number N and we multiply that number by 4, what we mean is:

$$4 \cdot N = N + N + N + N.$$

If we take the number 243192 and add it to itself four times using the "combining method," we get $2+2+2+2 = 8$ ones, $9+9+9+9 = 36$ tens, $1+1+1+1 = 4$ hundreds, and so on.

Notation: Notice that we have used both $\times$ and $\cdot$ to represent multiplication. It's a bit awkward to use $\times$ when you're also using variables. Is it the letter x? Or the multiplication symbol $\times$? It can be hard to tell! In this case, the symbol $\cdot$ is more clear. We can even simplify the notation further, writing $4N$ instead of $4 \cdot N$. But of course we *only* do that when we are multiplying variables by some quantity. (We wouldn't want 34 to mean $3 \cdot 4$, would we?)

Problem 76. Here is a strange addition table. Use it to solve the following problems. *Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!*

$+$	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$A+B$ $\qquad$ $B+C$ $\qquad$ $2A$ $\qquad$ $5C$ $\qquad$ $3A+4B$

Think/Pair/Share. Discuss your answers with a partner. How does an addition table help you solve multiplication problems like $5C$?

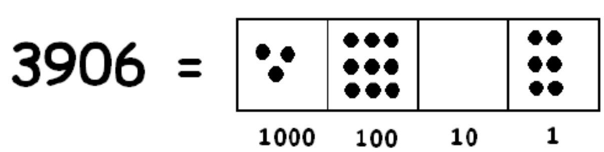
3.1.6 Quotative Model of Division

Suppose you are asked to compute $3906 \div 3$. One way to interpret this question (there are others) is:

> "How many groups of 3 fit into 3906?"

Definition 3.1.5. In the *quotative* model of division, you are given a *dividend* (here it is 3906), and you are asked to split it into equal-sized groups, where the size of the group is given by the *divisor* (here it is 3).

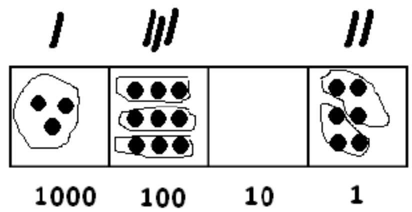
In our dots and boxes model, the dividend 3906 looks like this:



and three dots looks like this: ●●●. So we are really asking:

> "How many groups of ●●● fit into the picture of 3906?"

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



Notice what we have in the picture:

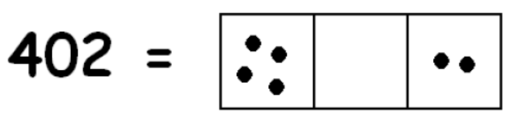
- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

$$3906 \div 3 = 1302.$$

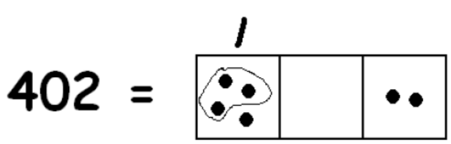
Fun fact: The division sign $\div$ has an unusual name. It is called an *obelus*. Not many people know this.

Let's try a harder one! Consider $402 \div 3$. Here's the picture:



We are still looking for groups of three dots: .

There is certainly one group at the 100s level.

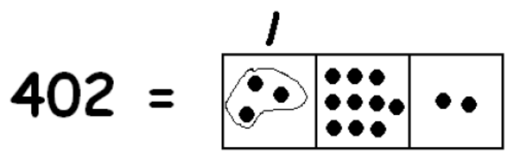


and now it seems we are stuck there are no more groups of three!

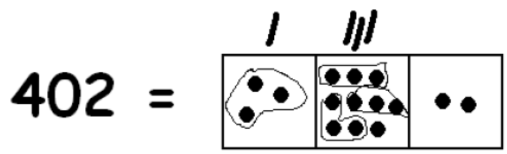
Think/Pair/Share. What can we do now? Are we really stuck? Can you finish the division problem?

Example 3.1.6 ($402 \div 3$). Here are the details worked out for $402 \div 3$. But don't read this until you've thought about it yourself!

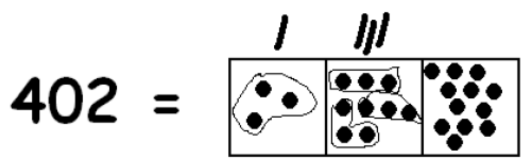
Since each dot is worth ten dots in the box to the right we can write:



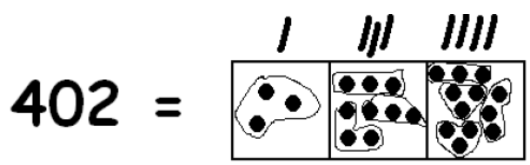
Now we can find more groups of three:



There is still a troublesome extra dot. Let's unexploded it too



This gives us more groups of three:



In the picture we have:

- One group of 3 in the hundreds box.
- Three groups of 3 in the tens box.
- Four groups of 3 in the ones box.

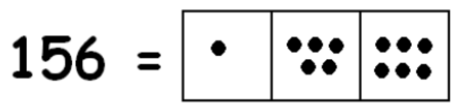
Finally we have the answer!

$$402 \div 3 = 134.$$

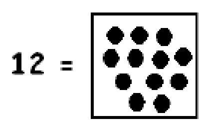
Think/Pair/Share. Solve each of these exercises using the dots and boxes method:

$$62124 \div 3 \qquad\qquad 61230 \div 5$$

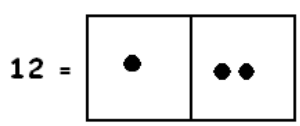
Let's turn up the difficulty a notch. Consider $156 \div 12$. Here we are looking for groups of 12 in this picture:



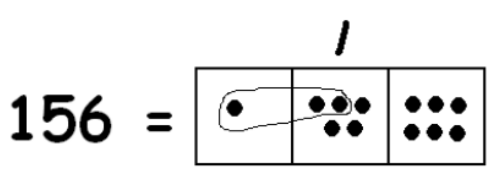
What does 12 look like? It can be twelve dots in a single box:



But most often we would write 12 this way:

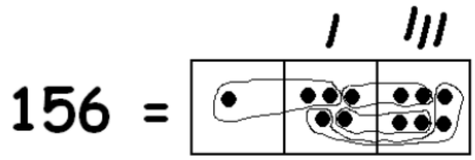


We certainly see some of these in the picture. There is certainly one at the tens level:



(REMEMBER: With an unexplosion this would be twelve dots in the tens box.)

And three at the ones level:



So in the picture we have:

- One group of 12 dots in the tens box.
- Three groups of 12 dots in the ones box.

That means

$$156 \div 12 = 13.$$

Problem 77. Use the dots and boxes model to compute each of the following:

$$13453 \div 11 \qquad 4853 \div 23 \qquad 214506 \div 102$$

Think/Pair/Share.

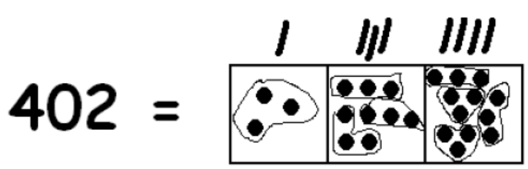
- Compare your solutions to problem 77 with a partner.
- When you agree that you are doing the process correctly, use dots and boxes to compute these:

$$2130 \div 10 \qquad 41300 \div 100$$

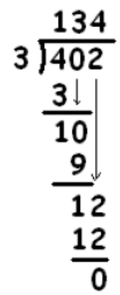
- Discuss: What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100 and *why*?

3.1.7 The Standard Algorithm for Division

We used dots and boxes to show that $402 \div 3 = 134$.

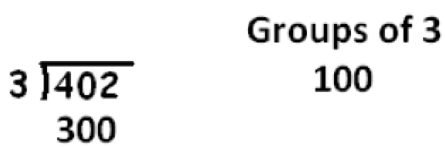


In elementary school, you might have learned to solve this division problem by using a diagram like the following:

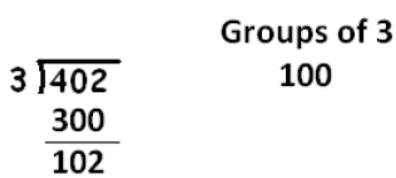


At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

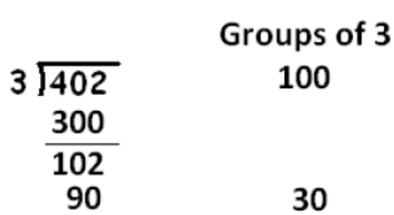
To compute $402 \div 3$, we first make a big estimation as to how many groups of 3 there are in 402. Let's guess that there are 100 groups of three.



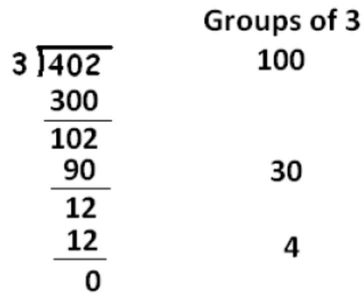
How much is left over after taking away 100 groups of 3?



How many groups of 3 are in 102? Let's try 30:



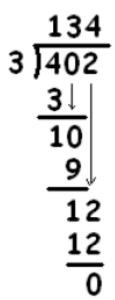
How many are left? There are 12 left and there are four groups of 3 in 12.

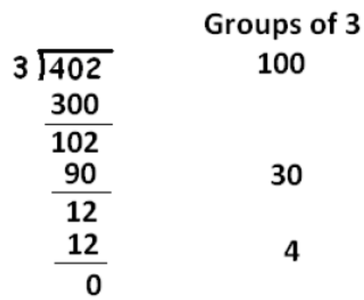


The accounts for entire number 402. And where doe we find the final answer? Just add the total count of groups of three that we tallied:

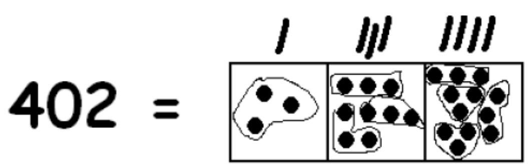
$$402 \div 3 = 100 + 30 + 4 = 134.$$

Think/Pair/Share. Compare these two diagrams. In what way are they the same? In what way are they different?



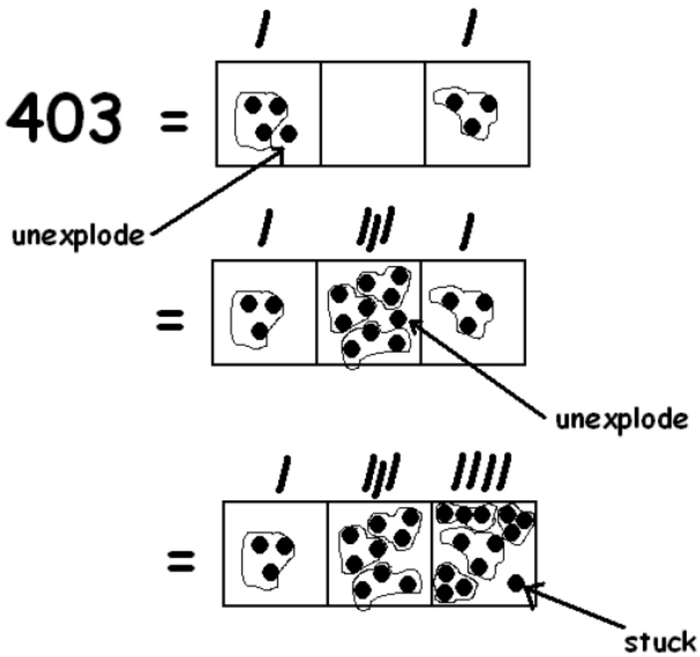


Look at the dots and boxes method. In what way is the same or different from the two tables?



- **Why do we like the standard algorithm?** Because it is quick, not too much to write down, and it works.
- **Why do we like the dots and boxes method?** Because it easy to understand. (And drawing dots and boxes is kind of fun!)

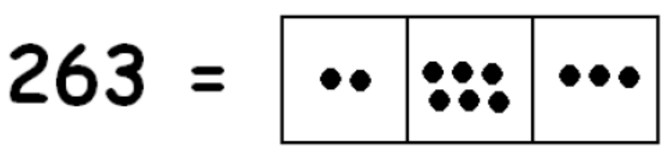
We saw that 402 is evenly divisible by 3: $402 \div 3 = 134$. This means that 403, one more, shouldn't be divisible by three. It should be one dot too big. Do we see the extra dot if we try the dots and boxes method?



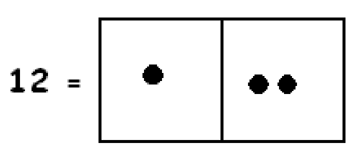
Yes we do! We have one dot left at the end that can't be divided. We say that we have a *remainder* of one and some people like to write:

$$403 \div 3 = 134 \ R1.$$

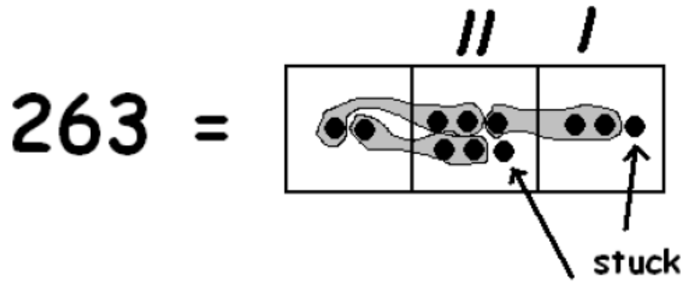
Let's try another one: $263 \div 12$. Here's what we have:



And we are looking for groups like this:



Here goes!



Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position. This means we have a remainder of eleven.

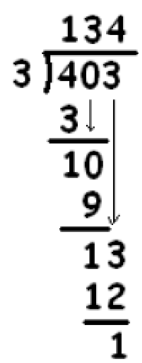
$$263 \div 12 = 21 \; R11.$$

Think/Pair/Share.

- Use the dots and boxes method to compute each quotient:

$$5210 \div 4 \qquad 4857 \div 23 \qquad 31533 \div 101$$

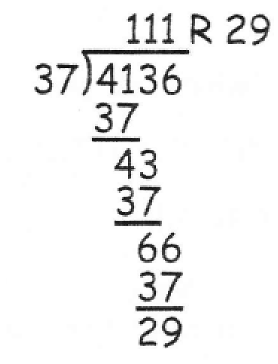
- Now use the standard algorithm (an example is shown below) to compute each of the quotients above.



$$403 \div 3 = 134 \; R1.$$

- Which method do you like better: dots and boxes or the standard algorithm method? Or does it depend on the problem you are doing?

Think/Pair/Share. Which is the *best* explanation for why this long division algorithm works? Think about your choice on your own. Then share your choice with a partner and see if you can agree on the best answer.



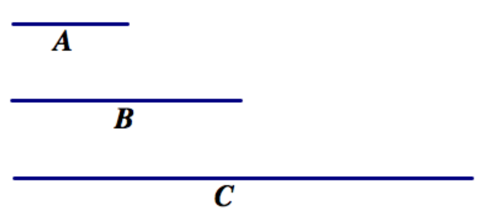
1. It works because you divide 37 into smaller parts of 4136 to make the problem easier to solve.
2. It works because you subtract multiples of powers of ten times 37 from 4136 until you have less than 37 left.
3. It works because if you multiply 111 by 37, and add in 29, you get 4136.
4. It works because you subtract 37s from 4136 until you have less than 37 left.

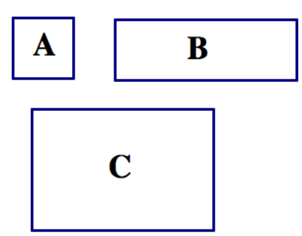
3.2 Model 2: Measurement

Another way we often think about numbers is as abstract quantities that can be measured: length, area, and volume are all examples.

Think/Pair/Share. Answer the following questions about each picture:

- If $A = 1$ unit, then what numbers would you assign to B and C? Why?
- If $B = 1$ unit, then what numbers would you assign to A and C? Why?

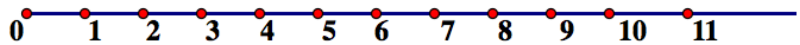




♣♣♣ Fellow: [can you make up a similar picture for volume? maybe that one is too hard? If it doesn't work, don't worry about it.]

In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity — length, area, or volume — that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on the quantity *length*, and we'll work with a number line where the basic unit is already marked off.



3.2.1 Addition and Subtraction on the Number Line

Imagine a person — we'll call him Zed — who can stand on the number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.

♣♣♣ Fellow: [any chance of a picture of a person on the number line? I can do a terrible stick figure, but maybe you can do something better?]

When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). Then what he does depends on the calculation.

If Zed wants to *add* two numbers, he walks forward (to the right of the number line) however many steps are indicated by the first number (the first *addend*). Then he walks forward (to your right on the number line) the number of steps indicated by the second number (the second *addend*). Where he lands is the *sum* of the two numbers.

Example 3.2.1 $(3 + 4)$. If Zed wants to add $3 + 4$, he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

♣♣♣ Fellow: [picture? or even better an animation?]

Zed ends at the number 7, so the sum of 3 and 4 is 7. $3 + 4 = 7$. (But you knew that of course! The point right now is to make sense of the *number line model*.)

When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the *minuend*). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the *subtrahend*). Where he lands is the *difference* of the two numbers.

Example 3.2.2 ($11 - 3$)*.* If Zed wants to subtract $11 - 3$, he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

♣♣♣ Fellow: [picture? or even better an animation?]

Zed ends at the number 8, so the difference of 11 and 3 is 8. $11 - 3 = 8$. (But you knew that!)

Think/Pair/Share.

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$4+5 \qquad 6+9 \qquad 10-7 \qquad 8-1$$

- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as "combining" for addition and "take away" for subtraction"?
- What happens if you do these subtraction problems on a number line? Explain your answers.

$$6-9 \qquad 1-7 \qquad 4-11 \qquad 0-1$$

- Could you do the subtraction problems above with the dots and boxes model?

3.2.2 Multiplication and Division on the Number Line

Since multiplication is really repeated addition, we can adapt our addition model to become a multiplication model as well. Let's think about 3×4. This means to add four to itself three times (that's simply the definition of multiplication!):

$$3 \times 4 = 4 + 4 + 4.$$

So to multiply on the number line, we do the process for addition several times.

To multiply two numbers, Zed starts at 0 as always, and he faces the positive direction. He walks forward the number of steps given by the second number (the second *factor*). He repeats that process the number of times given by the first number (the first *factor*). Where he lands is the *product* of the two numbers.

Example 3.2.3 (3×4). If Zed wants to multiply 3×4, he can think of it this way:

$$\underbrace{3}_{\text{how many times to repeat it}} \times \underbrace{4}_{\text{how many steps to take forward}}$$

Zed starts at 0, facing the positive direction. The he repeats this three times: take four steps forward.

♣♣♣ Fellow: [picture? or animation?]

He ends at the number 12, so the product of 3 and 4 is 12. That is, $3 \times 4 = 12$.

Remember our quotative model of division: One way to interpret $15 \div 5$ (there are others) is:

> "How many groups of 5 fit into 15?"

Thinking on the number line, we can ask it this way:

> "Zed takes 5 steps at a time. If Zed lands at the number 15, how many times did he take 5 steps?"

To calculate a division problem on the number line, Zed starts at 0, facing the positive direction. He walks forward the number of steps given by the second number (the *divisor*). He repeats that process until he lands at the first number (the *dividend*). The number of times he repeated the process gives the *quotient* of the two numbers.

Example 3.2.4 ($15 \div 5$). If Zed wants to divide $15 \div 5$, he can think of it this way:

$$\underbrace{15}_{\text{where he wants to land}} \div \underbrace{5}_{\text{how many steps he takes at a time}}$$

He starts at 0, facing the positive direction.

- Zed takes 5 steps forward. He is now at 5, not 15. So he needs to repeat the process.

- Zed takes 5 steps forward again. He is now at 10, not 15. So he needs to repeat the process.

- Zed takes 5 more steps forward. He is at 15, so he stops.

♣♣♣ Fellow: [animation? or picture for each bullet point? or at least one picture showing the whole process?]

Since he repeated the process three times, we see there are 3 groups of 5 in 15. So the quotient of 15 and 5 is 3. That is, $15 \div 5 = 3$.

Think/Pair/Share.

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$2 \times 5 \qquad 7 \times 1 \qquad 10 \div 2 \qquad 6 \div 1$$

- Can you think of a way to interpret these multiplication problems on a number line? Explain your ideas.

$$4 \times 0 \qquad 0 \times 5 \qquad 3 \times (-2) \qquad 2 \times (-1)$$

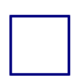
- What happens if you try to solve these division problems on a number line? Can you do it? Explain your ideas.

$$0 \div 2 \qquad 0 \div 10 \qquad 3 \div 0 \qquad 5 \div 0$$

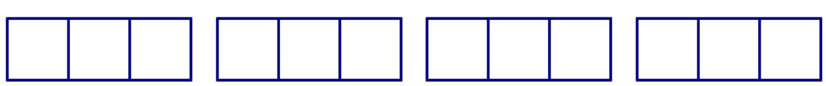
3.2.3 Area Model for Multiplication

So far we have focused on a *linear* measurement model, using the number line. But there's another common way to think about multiplication: using *area*.

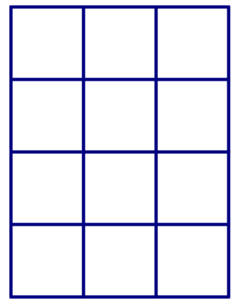
For example, suppose our basic unit is one square:



We can picture 4×3 as 4 groups, with 3 squares in each group, all lined up:

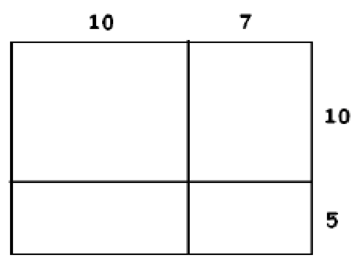


But we can also picture them stacked up instead of lined up. We would have 4 *rows*, with 3 squares in each *row*, like this:



So we can think about 4×3 as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. (That is also the *area* of the rectangle, since each square was one unit!)

Think/Pair/Share. Vera drew this picture as a model for 15×17. Use her picture to help you compute 15×17. Explain your work.



Problem 78. Draw pictures like Vera's for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard algorithm.

23×37 $\qquad 8 \times 43$ $\qquad 371 \times 42$

3.2.4 The Standard Algorithm for Multiplication

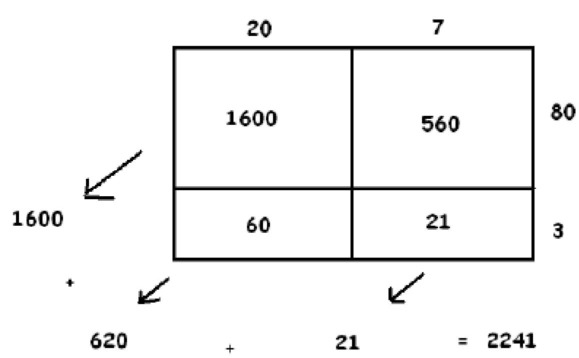
How were you taught to compute 83×27 in school? Were you taught to write something like the following?

$$\begin{array}{rr} & 83 \\ \times & 27 \\ \hline & 21 \\ & 56 \\ & 6 \\ & 16 \\ \hline & 2241 \end{array}$$

Or maybe you were taught to put in the extra zeros rather than leaving them out?

$$\begin{array}{rr} & 83 \\ \times & 27 \\ \hline & 21 \\ & 560 \\ & 60 \\ & 1600 \\ \hline & 2241 \end{array}$$

This is really no different than drawing the rectangle and using Vera's shortcut for calculating!

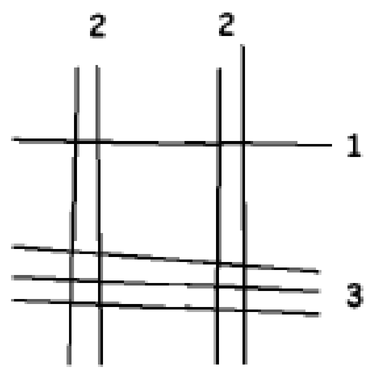


Think/Pair/Share.

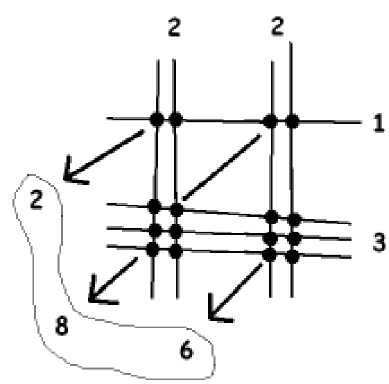
- Use the example above to explain why Vera's rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you're computing the same pieces in each algorithm.

$$23 \times 14 \qquad 106 \times 21 \qquad 213 \times 31$$

Problem 79 (Lines and Intersections)**.** Here's an unusual way to perform long multiplication. To compute 22×13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

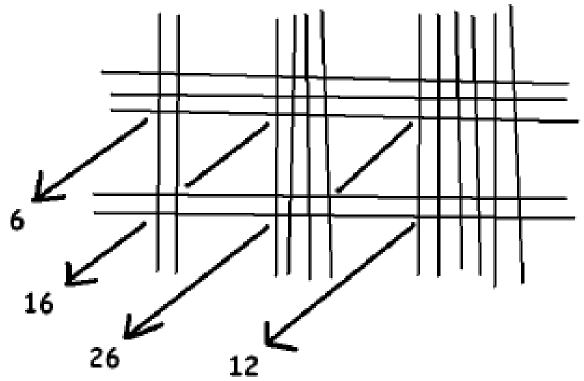


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

There is one possible glitch as illustrated by the computation 246×32:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as $7,872$.

(a) Compute 131×122 via this method. Check your answer using another method.

(b) Compute 15×1332 via this method. Check your answer using another method.

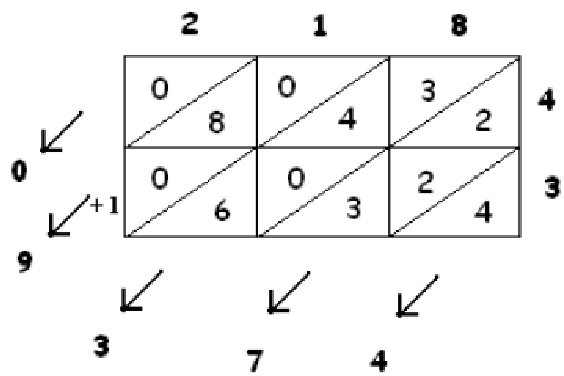
(c) Can you adapt the method to compute 102×3054? (Why is some adaptation necessary?)

(d) Why does the method work in general?

Problem 80 (Lattice Multiplication)**.** In the 1500s in England, students were taught to compute long multiplication using following galley method, now more commonly known as the *lattice method*:

To multiply 43 and 218, for example, draw a 2×3 grid of squares. Write the digits of the first number along the right side of the grid and the digits of the second number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)



Add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary, to see the final answer. In our example, we have $218 \times 43 = 9374$.

(a) Compute 5763×345 via the lattice method.

(b) Explain why the lattice method is really the standard algorithm in disguise.

(c) What is the specific function of the diagonal lines in the grid?

3.3 Operations

So far, you have seen a couple of different *models* for the operations: addition, subtraction, multiplication, and division. But we haven't talked much about the operations themselves — how they relate to each other, what properties they have that make computing easier, and how some special numbers behave. There's lots to think about!

The goal in this section is to use the models to understand why the operations behave according to the rules you learned back in elementary school. We're going to keep asking ourselves "Why does it work this way?"

Think/Pair/Share. Each of these models lends itself to thinking about the operation in a slightly different way. Before we really dig in to thinking about the operations, discuss with a partner:

- Of the models we discussed so far, do you prefer one of them?

- How well do the models we discussed match up with how you usually think about whole numbers and their operations?
- Which models are useful for computing? Why?
- Which models do you think will be useful for *explaining* how the operations work? Why?

3.3.1 Relationships Between the Operations

We defined addition as combining two quantities and subtraction as "taking away." But in fact, these two operations are intimately tied together. These two questions are exactly the same:

$$27 - 13 = \_\_\_\_ \qquad\qquad 27 = 13 + \_\_\_\_.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c - b = a \qquad\qquad c = a + b.$$

In other words, we can think of every subtraction problem as a "missing addend" addition problem. Try it out!

Problem 81. Here is a strange addition table. Use it to solve the following problems. Justify your answers. *Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!*

$+$	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$A+C$ $\qquad$ $B+C$ $\qquad$ $A-C$ $\qquad$ $C-A$ $\qquad$ $A-A$ $\qquad$ $B-C$

Think/Pair/Share. Discuss your answers with a partner. How does an addition table help you solve subtraction problems?

We defined multiplication as repeated addition and division as forming groups of equal size. But in fact, these two operations are also tied together. These two questions are exactly the same:

$$27 \div 3 = \_\_\_\_ \qquad\qquad 27 = \_\_\_\_ \times 3.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c \div b = a \qquad\qquad c = a \cdot b.$$

In other words, we can think of every division problem as a "missing factor" multiplication problem. Try it out!

Problem 82. Rewrite each of these division problems as a "missing factor" multiplication problem. Which ones can you solve and which can you not solve? Explain your answers.

$$9 \div 3 \qquad 100 \div 25 \qquad 0 \div 3 \qquad 9 \div 0 \qquad 0 \div 0$$

Problem 83. Here's a multiplication table.

$\times$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	c	e	b	d
d	a	d	b	e	c
e	a	e	d	c	b

- Use the table to solve the problems below. Justify your answers. *Important: Don't try to assign numbers to the letters. Solve the problems just using what you know about the operations!*

  $c{\times}d$ $c{\times}a$ $a{\times}a$ $c{\div}d$ $d{\div}c$ $d{\div}e$

- Can you use the table to solve these problems? Explain your answers.

  d^2 c^3 $a{\div}c$ $a{\div}d$ $c{\div}a$ $d{\div}a$ $a{\div}a$

Think/Pair/Share. Discuss your answers to problems 82 and 83 with a partner. How does a multiplication table help you solve division (and exponentiation) problems?

Throughout this course, our focus is on explanation and justification. As teachers, you need to know what is true in mathematics, but you also need to know *why* it is true. And you will need lots of ways to explain *why*, since different explanations will make sense to different students.

It's very important not to confuse the fact (or rule) with the reason it is true! Here are some examples:

Example 3.3.1 (Explaining the Connection Between Addition and Subtraction).

Arithmetic Fact: $a + b = c$ and $c - b = a$ are the same mathematical fact.

Why It's True, Explanation 1: First we'll use the definition of the operations.

Suppose we know $c - b = a$ is true. Subtraction means "take away." So

$$c - b = a$$

means we start with quantity c and take away quantity b, and we end up with quantity a. Start with this equation, and imagine adding quantity b to both sides.

On the left, that mans we started with quantity c, took away b things, and then put those b things right back! Since we took away some quantity and then added back the exact same quantity, there's no overall change. We're left with quantity c.

On the right, we would be *combining* (adding) quantity a with quantity b. So we end up with

$$c = a + b.$$

On the other hand, suppose we know the equation $a + b = c$ is true. Imagine *taking away* (subtracting) quantity b from both sides of the equation

$$a + b = c.$$

On the left, we started with a things and *combined* that with b things, but then we immediately take away those b things. So we're left with just our original quantity of a.

On the right, we start with quantity c and *take away* b things. That's the very definition of $c - b$. So we have the equation

$$a = c - b.$$

♣♣♣ Fellow: [Maybe this would be more clear as an animation!]

Why It's True, Explanation 2: Let's use the measurement model to come up with another explanation.

The equation $a+b = c$ means Zed starts at 0, walks forward a steps, and then walks forward b steps, and he ends at c.

If Zed wants to compute $c - b$, he starts at 0, walks forward c steps, and then walks backwards b steps. But we know that to walk forward c steps, he can first walk forward a steps and then walk forward b steps. So Zed can compute $c - b$ this way:

- Start at 0.
- Walk forward a steps.
- Walk forward b steps. (Now at c, since $a + b = c$.)
- Walk backwards b steps.

♣♣♣ Fellow: [picture?]

The last two steps cancel each other out, so Zed lands back at a. That means $c - b = a$.

On the other hand, the equation $c - b = a$ means that Zed starts at 0, walks forward c steps, then walks backwards b steps, and he ends up at a.

If Zed wants to compute $a + b$, he starts at 0, walks forward a steps, and then walks forwards b additional steps. But we know that to walk forward a steps, he can first walk forward c steps and then walk backwards b steps. So Zed can compute $a + b$ this way:

- Start at 0.
- Walk forward c steps.
- Walk backwards b steps. (Now at a, since $c - b = a$.)
- Walk forward b steps.

♣♣♣ Fellow: [picture?]

The last two steps cancel each other out, so Zed lands back at c. That means $a + b = c$.

Think/Pair/Share.

- Read over the two explanations in the example above. Do you think either one is more clear than the other?
- Why is this *not* a good explanation of the mathematical fact?

  > "I can check that this is true! For example, $2 + 3 = 5$ and $5 - 3 = 2$. And $3 + 7 = 10$ and $10 - 7 = 3$. It works for whatever numbers you try."

- Use either the definition of the operations or the measurement model to explain the connection between multiplication and division:

$$c \div b = a \quad \text{is the same fact as} \quad c = a \times b.$$

3.3.2 Properties of Addition and Subtraction

You probably know several properties of addition, but you may never have stopped to wonder: *Why is that true?!* Now's your chance! In this section, you'll use the definition of the operations of addition and subtraction and the models you've learned to explain *why* these properties are always true.

Here are the three properties you'll think about:

- Addition of whole numbers is *commutative*.
- Addition of whole numbers is *associative*.
- The number 0 is an *identity* for addition of whole numbers.

For each of the properties, we don't want to confuse these three ideas:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Notice that *examples* and *explanations* are not the same! These properties are all *universal statements* — statements of the form "for all," "every time," "always," etc. That means that to show they are true, you either have to check every case or find a *reason why* it must be so.

Since there are infinitely many whole numbers, it's impossible to check every case. You'd never finish! Our only hope is to look for *general explanations.*

Example 3.3.2 (Commutativity). We'll work out the explanation for the first of these facts, and you will work on the others.

Property Addition of whole numbers is *commutative.*

What it Means (words) When I add two whole numbers, the order I add them doesn't affect the sum.

What it Means (symbols) For any two whole numbers a and b,

$$a + b = b + a.$$

Examples $3 + 5 = 8$ and $5 + 3 = 8$. $2 + 0 = 2$ and $0 + 2 = 2$.

♣♣♣ Fellow: [dots and boxes pictures of the examples? They can be more interesting examples...]

Now we need a *justification.* Why is addition of whole numbers commutative?

Why It's True, Explanation 1: Let's think about addition as combining two quantities of dots.

- To add $a + b$, we take a dots and b dots, and we combine them in a box. To keep things straight, lets imagine the a dots are colored red and the b dots are colored blue. So in the box we have a red dots, b blue dots and $a + b$ total dots.
- To add $b + a$, let's take b blue dots and a red dots, and put them all together in a box. We have b blue dots, a red dots and $b + a$ total dots.

- But the total number of dots are the same in the two boxes! How do we know that? Well, there are a red dots in each box, so we can match them up. There are b blue dots in each box, so we can match them up. That's it! If we can match up the dots one-for-one, there must be the same number of them!
- That means $a + b = b + a$.

Why It's True, Explanation 2: We can also use the measurement model to explain why $a+b = b+a$ no matter what numbers we choose for a and b. Imagine taking a segment of length a and combining it linearly with a segment of length b. That's how we get a length of $a + b$.

But if we just rotate that segment so it's upside down, we see that we have a segment of length b combined with a segment of length a, which makes a length of $b + a$.

But of course it's the same segment! We just turned it upside down! So the lengths must be the same. That is, $a + b = b + a$.

Problem 84 (Addition is Associative)**.** Your turn! You'll answer the question, "Why is addition of whole numbers associative?"

Property Addition of whole numbers is *associative*.

What it Means (words) When I add three whole numbers in a given order, the way I group them (to add two at a time) doesn't affect the sum.

What it Means (symbols) For any three whole numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

(a) Come up with at least three *examples* to demonstrate associativity of addition.

(b) Use our models of addition to come up with an *explanation*. Why does associativity hold in *every case*?

Problem 85 (Identity for Addition)**.** Why is the number 0 an identity for addition?

Property The number 0 is an *identity* for addition of whole numbers.

What it Means (words) When I add any whole number to 0 (in either order), the sum is the very same whole number I added to 0.

What it Means (symbols) For any whole numbers n,

$$n + 0 = n \quad \text{and} \quad 0 + n = n.$$

(a) Come up with at least three *examples* to demonstrate that 0 is an identity for addition.

(b) Use our models of addition to come up with an *explanation*. Why does this property of 0 hold in *every possible case*?

Since addition and subtraction are so closely linked, it's natural to wonder if subtraction has some of the same properties as addition, like commutativity and associativity.

Example 3.3.3. Justin asked if the operation of subtraction is commutative. That would mean that the difference of two whole numbers doesn't depend on the order in which you subtract them. In symbols: *for every choice* of whole numbers a and b we would have $a - b = b - a$.

Jared says that subtraction is *not* commutative since $4 - 3 = 1$, but $3 - 4 \neq 1$. (In fact, $3 - 4 = -1$.)

Since the statement "subtraction is commutative" is a *universal statement*, one counterexample is enough to show it's not true. So Jared's example lets us say with confidence: subtraction is not commutative.

Think/Pair/Share. Can you find *any* examples of whole numbers a and b where $a - b = b - a$ is true? Explain your answer.

Problem 86. Lyle asked if the operation of subtraction is associative.

(a) State what it would mean for subtraction to be associative. You should use words and symbols.

(b) What would you say to Lyle? Decide if subtraction is associative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 87. Jess asked if the number 0 is an identity for subtraction.

(a) State what it would mean for 0 to be an identity for subtraction. You should use words and symbols.

(b) What would you say to Jess? Decide if 0 is an identity for subtraction or not. Carefully explain how you made your decision and *how you know you're right*.

3.3.3 Properties of Multiplication and Division

Now we're going to turn our attention to familiar properties of multiplication and division, with the focus still on explaining *why* these properties are *always true*.

Here are the four properties you'll think about:

- Mulitplication of whole numbers is *commutative*.
- Multiplication of whole numbers is *associative*.
- Multiplication of whole numbers *distributes over addition*
- The number 1 is an *identity* for multiplication of whole numbers.

For each of the properties, remember to keep straight:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Once again, it's important to distinguish between *examples* and *explanations*. They are not the same! Since there are infinitely many whole numbers, it's impossible to check every case, so examples will never be enough to explain why these properties hold. You have to figure our *reasons* for these properties to hold, based on what you know about the operations.

Example 3.3.4 (Identity). We'll work out the explanation for the last of these facts, and you will work on the others.

Property The number 1 is an *identity* for multiplication of whole numbers.

What it Means (words) When I multiply a number by 1 (in either order), the product is the same as that number I multiplied by 1.

What it Means (symbols) For any whole numbers m,

$$m \times 1 = m \quad \text{and} \quad 1 \times m = m.$$

Examples $1 \times 5 = 5$, $19 \times 1 = 19$, and $1 \times 1 = 1$.

Why does the number 1 act this way with multiplication?

Why It's True, Explanation 1: Let's think first about the definition of multiplication as repeated addition:

- $m \times 1$ means to add the number one to itself m times:

  $$\underbrace{1 + 1 + \cdots + 1}_{m \text{ ones}}.$$

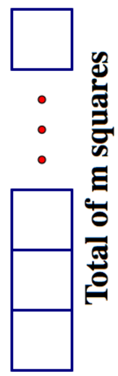
  So we see that $m \times 1 = m$ for any whole number m.

- On the other hand, $1 \times m$ means to add the number m to itself just one time. This might seem a little confusing, since there's no actual addition going on. But if this makes any sense at all, the answer must be m. So $1 \times m = m$ also.

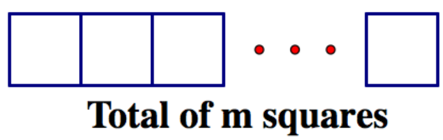
Why It's True, Explanation 2: We can also use the number line model to create a justification. If Zed calculates $1 \times m$, he will start at 0 and face the positive direction. He will then take m steps forward, and he will do it just one time. So he lands at m, which means $1 \times m = m$.

If Zed calculates $m \times 1$, he starts at 0 and faces the positive direction. Then he takes one step forward, and he repeated that m times. So he lands at m. We see that $m \times 1 = m$.

Why It's True, Explanation 3: And what about the area model? In that model, $m \times 1$ represents m rows with one square in each row. That makes a total of m squares. So $m \times 1 = m$.



Similarly, $1 \times m$ represents one row of m squares. That's also a total of m squares . So $1 \times m = m$.



Think/Pair/Share. The example presented several different explanations. Do you think one is more convincing than the others? Or more clear and easier to understand?

Now it's your turn to come up with some explanations.

Problem 88 (Multiplication is Commutative)**.** Why is multiplication of whole numbers commutative?

Property Multiplication whole numbers is *commutative*.

What it Means (words) When I multiply two whole numbers, switching the order in which I multiply them does not affect the product.

What it Means (symbols) For any two whole numbers a and b,

$$a \cdot b = b \cdot a.$$

(a) Come up with at least three *examples* to demonstrate the commutativity of multiplication.

(b) Use our models of multiplication to come up with an *explanation*. Why does commutativity hold in *every case*?

Problem 89 (Multiplication is Associative)**.** Why is multiplication of whole numbers associative?

Property Multiplication of whole numbers is *associative*.

What it Means (words) When I multiply three whole numbers in a given order, the way I group them (to multiply two at a time) doesn't affect the product.

What it Means (symbols) For any three whole numbers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

(a) Come up with at least three *examples* to demonstrate the associativity of multiplication.

(b) Use our models of addition to come up with an *explanation*. Why does associativity hold in *every case*?

Property Multiplication *distributes over addition*.

What it means The distributive law for multiplication over addition is a little hard to state in words, so we'll jump straight to the symbols. For any three whole numbers x, y, and z:

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

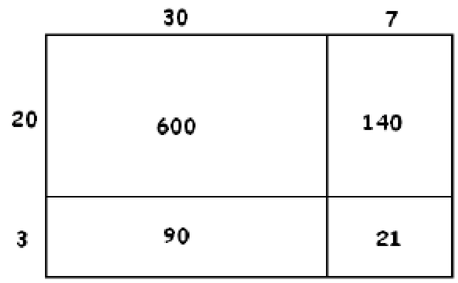
Examples

$$8 \cdot (23) = 8 \cdot (20 + 3) = 8 \cdot 20 + 8 \cdot 3 = 160 + 24 = 184$$
$$5 \cdot (108) = 5 \cdot (100 + 11) = 5 \cdot 100 + 5 \cdot 8 = 500 + 40 = 540$$

Think/Pair/Share. We actually did calculations very much like the examples above, when we looked at the area model for multiplication.

- Draw a picture for each example above.

This picture shows our method for multiplying 23×37.

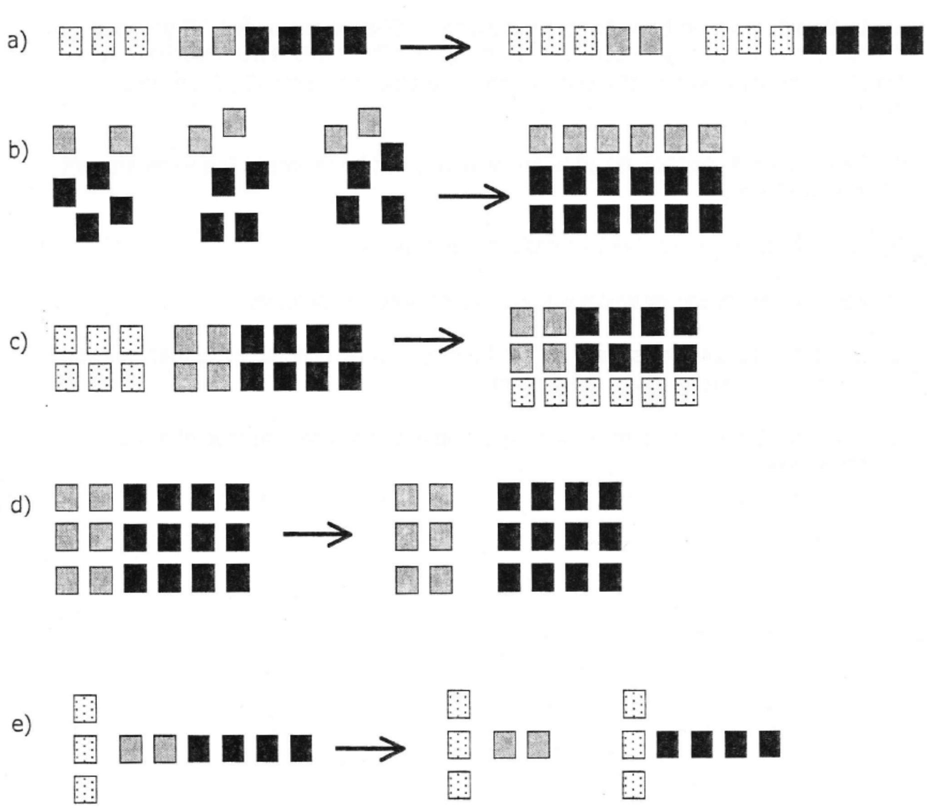


We can also write the calculation this way:

$$23 \cdot 37 = (20+3)(30+7) = 20 \cdot 30 + 20 \cdot 7 + 3 \cdot 30 + 3 \cdot 7 = 600 + 140 + 90 + 21 = 851.$$

- Use the distributive rule (twice!) to explain why this calculation works.

Problem 90. Which of the following pictures *best* represents the distributive law in the equation $3 \cdot (2+4) = 3 \cdot 2 + 3 \cdot 4$? Explain your choice.



♣♣♣ Fellow: [This picture is kind of crappy and stolen from a long-forgotten source. Possible to make a nicer version?]

Problem 91. Use the distributive law to easily compute each of these in your head (no calculators!). Show your work.

$$45 \times 11 \qquad 63 \times 101 \qquad 172 \times 1001$$

Think/Pair/Share. Use one of our models for multiplication and addition to explain why the distributive rule works *every time*.

It's natural to wonder which, if any, of these properties also hold for division (since you know that the operations of multiplication and division are connected).

Example 3.3.5. If division were associative, then for any choice of three whole numbers a, b, and c, we would have

$$a \div (b \div c) = (a \div b) \div c.$$

Remember, the parentheses tell you which two numbers to divide first.

Let's try the example $a = 9$, $b = 3$, and $c = 1$. Then

$$a \div (b \div c) = 9 \div (3 \div 1) = 9 \div 3 = 3 \qquad \text{and}$$
$$(a \div b) \div c = (9 \div 3) \div 1 = 3 \div 1 = 3.$$

So is it true? Is division associative? Well, we can't be sure. This is just one example. But "division is associative" is a *universal statement*. If it's true, it has to work for *every possible example*. Maybe we just stumbled on a good choice of numbers, but it won't always work.

Let's keep looking. Try $a = 16$, $b = 4$, and $c = 2$.

$$a \div (b \div c) = 16 \div (4 \div 2) = 16 \div 2 = 8 \qquad \text{and}$$
$$(a \div b) \div c = (16 \div 4) \div 2 = 4 \div 2 = 2.$$

That's all we need! A single counterexample lets us conclude that division is *not* associative.

Problem 92. What about the other properties? It's your turn to decide!

(a) State what it would mean for division to be commutative. You should use words and symbols.

(b) Decide if division is commutative or not. Carefully explain how you made your decision and *how you know you're right*.

(c) State what it would mean for division to distribute over addition. You definitely want to use symbols!

(d) Decide if division distributes over addition or not. Carefully explain how you made your decision and *how you know you're right*.

(e) State what it would mean for the number 1 to be an identity for division. You should use words and symbols.

(f) Decide if 1 is an identity for division or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 93 (Zero property)**.** You probably know another property of multiplication that hasn't been mentioned yet: If I multiply any number times 0 (in either order), the product is 0. This is sometimes called the *zero property* of multiplication. Notice that the zero property is very different from the property of being an identity!

(a) Write what the zero property means in symbols: For every whole number n ...

(b) Give at least three examples of the zero property for multiplication.

(c) Use one of our models of multiplication to explain why the zero property holds.

Think/Pair/Share (Division by 0). Which of the following *best* explains why division by 0 is undefined? First decide for yourself. Then share your thoughts with a partner and see if you agree.

1. Division by 0 is undefined because you cannot do it.
2. Division by 0 is undefined because you cannot make 0 groups of something.
3. Division by 0 is undefined because there is no single number that, when multiplied by 0, gives the original number.
4. Division by 0 is undefined because every number divided by 0 equals 0.

Think/Pair/Share (More on division by 0).

- For each division problem below, turn it into a multiplication problem. Solve those problems if you can. If you can't, explain what is wrong.

$$5 \div 0 \qquad 0 \div 5 \qquad 7 \div 0 \qquad 0 \div 7 \qquad 0 \div 0$$

- Use your work to explain why we say that *division by* 0 *is undefined*.
- Use one of our models of division to explain why *division by* 0 *is undefined*.

3.3.4 Four Fact Families

In elementary school, students are often encouraged to memorize "four fact families," for example:

$$\begin{aligned} 2+3&=5 \qquad & 5-3&=2 \\ 3+2&=5 \qquad & 5-2&=3 \end{aligned}$$

Here's a different kind of family:

$$\begin{aligned} 2\cdot 3&=6 \qquad & 6\div 3&=2 \\ 3\cdot 2&=6 \qquad & 6\div 2&=3 \end{aligned}$$

Think/Pair/Share.

- In what sense are these groups of equations "families"?
- Write down at least two more addition / subtraction four fact families.
- Use properties of addition and subtraction to explain *why* these four fact families are each really one fact.
- Write down at least two more multiplication / division four fact families.
- Use properties of multiplication and division to explain *why* these four fact families are each really one fact.

Problem 94.

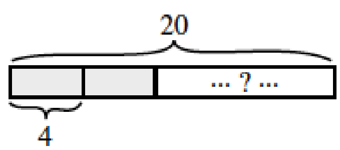
(a) Here's a true fact in base six: $2_{\text{six}} + 3_{\text{six}} = 5_{\text{six}}$. Write the rest of this four fact family.

(b) Here's a true fact in base six: $11_{\text{six}} - 5_{\text{six}} = 2_{\text{six}}$. Write the rest of this four fact family.

3.3.5 Going Deeper with Division

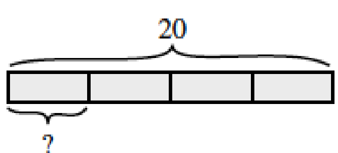
So far we've been thinking about division in what's called the *quotative model*. In the quotative model, we want to make groups of equal size. We know the *size of the group*, and we ask *how many groups*. For example, we think of $20 \div 4$ as:

How many groups of 4 are there in a group of 20?



Thinking about four fact families, however, we realize we can turn the question around a bit. We could think about the *partitive model* of division. In the partitive model, we want to make an equal number of groups. We know *how many groups*, and we ask *the size of the group*. In the partitive model, we think of $20 \div 4$ as:

20 is 4 groups of what size?



When we know the original amount and the number of parts, we use partitive division to find the size of each part. When we know the original amount and the size of each part, we use quotative division to find the number of parts.

Here are some examples in word problems:

Partitive number of groups known find the number in each group	**Quotative** number in each group known find the number of groups
Sylvia makes \$26,000 per year. How much does she make weekly?	Sylvia's makes \$650 weekly. Last year she made \$26,000. How many weeks did she work?
A movie theater made \$6450 in one night of ticket sales. 430 people purchased a ticket. How much does each ticket cost?	A movie theater made \$6450. in one night of ticket sales. Each ticket costs \$12.50. How many people purchased a ticket?

Think/Pair/Share. For each word problem below:

- Draw a picture to show what the problem is asking.
- Use your picture to help you decide if it is a *quotative* or a *partitive* division problem.
- Solve the problem using any method you like.

1. David made 36 cookies for the bake sale. He packaged the cookies in boxes of 9. How many boxes did he use?
2. David made 36 cookies to share with his friends at lunch. There were 12 people at his lunch table (including David). How many cookies did each person get?
3. Liz spent one summer hiking the Appalachin trail. She completed 1,380 miles of the trail and averaged 15 miles per day. How many days was she out hiking that summer?
4. On April 1, 2012, Chase Norton became the first documented person to hike the entire Koʻolau summit in a single trip. (True story!) It took him eight days to hike all 48 miles from start to finish. If he kept a steady pace, how many miles did he hike each day?

Think/Pair/Share. Write your own word problems: Write one partitive division problem and one quotative division problem. Choose your numbers carefully so that the answer works out nicely. Be sure to solve your problems!

Why think about these two models for division? You won't be teaching the words *partitive* and *quotative* to your students. But recognizing the two kinds of division problems (and being able to come up with examples of each) will make you a better teacher. It's important that your students are exposed to both ways of thinking about division, and to problems of both types. Otherwise, they may think about division too narrowly and not really understand what's going on. If you understand the two kinds of problems, you can more easily diagnose and remedy students' difficulties.

Most of the division problems we've looked at so far have come out evenly, with no remainder. But of course, that doesn't always happen!

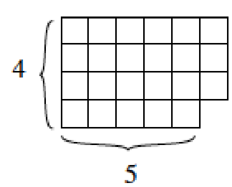
Problem 95. What is $43 \div 4$?

(a) Write a problem that uses the computation $43 \div 4$ and gives 10 as the correct answer.

(b) Write a problem that uses the computation $43 \div 4$ and gives 11 as the correct answer.

(c) Write a problem that uses the computation $43 \div 4$ and gives 10.75 as the correct answer.

We can think about division with remainder in terms of some of our models for operations. For example, we can calculate that $23 \div 4 = 5\ R3$. We can picture it this way:



Think/Pair/Share.

- Explain how the picture above illustrates the equation $23 \div 4 = 5\ R3$.
- Explain the connection between the equations

$$23 \div 4 = 5\ R3 \quad \text{and}$$
$$23 = 5 \cdot 4 + 3.$$

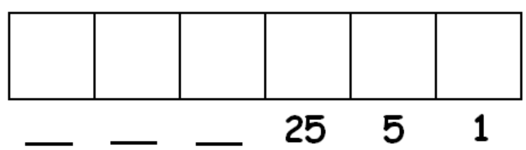
- How could you use the number line model to see that $23 \div 4 = 5\ R3$? What does a "remainder" look like in this model?

- Draw area models for each of these division problems. Find the quotient and remainder.

$$40 \div 12 \qquad 59 \div 10 \qquad 91 \div 16$$

3.4 Division Explorations

Problem 96 (Base 5 Division)**.** Remember that base five numbers are in a $1 \leftarrow 5$ dots-and-boxes system. What are the place values in the $1 \leftarrow 5$ system? Fill in the blanks:



- Draw a dots-and-boxes picture of the number 424_{five}.

- Draw a dots-and-boxes picture of the number 11_{five}.

- Use the dots and boxes method to show that $424_{\text{five}} \div 11_{\text{five}} = 34_{\text{five}}$.

- Rewrite the division sentence $424_{\text{five}} \div 11_{\text{five}} = 34_{\text{five}}$ in base 10, and check that it's correct.

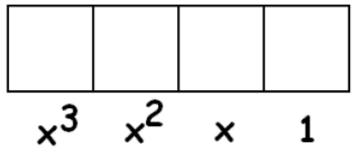
- **Challenge!** Use dots-and-boxes to find $2021_{\text{five}} \div 12_{\text{five}}$. *Don't convert to base 10!*

Problem 97 (Base... x?!?!!)**.** Anu refuses to tell anyone if she is working in a $1 \leftarrow 10$ system, or a $1 \leftarrow 5$ system, or any other system. She makes everyone call it a $1 \leftarrow x$ system but won't tell a soul what number she has in mind for x.

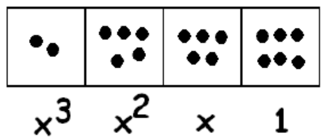
We know that boxes in a $1 \leftarrow 10$ have values that are powers of ten: 1, 10, 100, 1000, 10000, ...

And boxes in a $1 \leftarrow 5$ system are powers of five: 1, 5, 25, 125, 625,...

So Anu's system, whatever it is, must be powers of x:

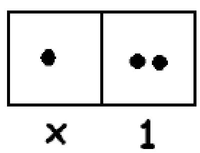


When Anu writes 2556_x she must mean:



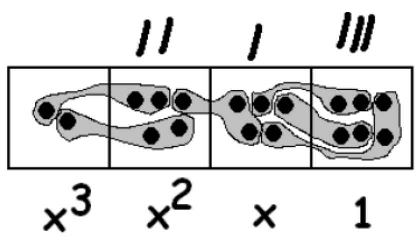
$$2x^3 + 5x^2 + 5x + 6.$$

And when she writes 12_x she means:



$$x + 2.$$

Anu decides to compute $2556_x \div 12_x$. She obtains:



$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3.$$

(a) Check Anu's division by computing

$$(x + 2)(2x^2 + x + 3).$$

Did it work?

(b) Use Anu's method to find $(3x^2 + 7x + 2) \div (x + 2)$.

(c) Use Anu's method to find $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.

(d) Use Anu's method to find $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

Anu later tells use that she really was thinking of a $1 \leftarrow 10$ system so that x does equal ten. Then her number 2556_x really was two thousand, five hundred and fifty six and 12_x really was twelve. Her statement:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3$$

is actually $2556 \div 12 = 213$.

(e) Check that $2556 \div 12 = 213$ is correct in base 10.

(f) What division problems did you actually solve for parts (b), (c), and (d) in the $1 \leftarrow 10$ system? Check that they are correct.

Hard Challenge! Uh Oh! Anu has changed her mind. She now says she was thinking of a $1 \leftarrow 11$ system.

Now 2556_x means $2 \cdot 11^3 + 5 \cdot 11^2 + 5 \cdot 11 + 6 = 3328_{\text{ten}}$. Similarly, 12_x means $1 \cdot 11 + 2 = 13_{\text{ten}}$ and 213_x means $2 \cdot 11 + 1 \cdot 11 + 3 = 256_{\text{ten}}$, and so her computation $2556_x \div 12_x = 213_x$ is actually the statement:

$$3328 \div 13 = 256.$$

(g) Check that $3328 \div 13 = 256$ is also correct.

(h) What division problems did you actually solve for parts (b), (c), and (d) in the $1 \leftarrow 11$ system? Check that they are correct.

Problem 98. This problem continues problem 97 above. Use Anu's method to show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1) = (x^3 + 3x^2 + 3x + 1)$.

(a) What is this saying for $x = 10$? Check that the division is correct.

(b) What is this saying for $x = 2$? Check that the division is correct.

(c) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11? Check that each division is correct.

(d) What is this saying for $x = 0$?

3.5 Problem Bank

Problem 99. Compute the following using dots and boxes:

$$64212 \div 3 \qquad 44793 \div 21 \qquad 6182 \div 11$$

$$99916131 \div 31 \qquad 637824 \div 302 \qquad 2125122 \div 1011$$

Problem 100.

(a) Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the largest possible sum:

$$\begin{array}{rccc} & \square & \square & \square \\ + & \square & \square & \square \\ \hline \end{array}$$

(b) Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the smallest possible (positive) difference:

$$\begin{array}{rccc} & \square & \square & \square \\ - & \square & \square & \square \\ \hline \end{array}$$

Problem 101. Make a base-6 addition table:

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

Use the table to solve the subtraction problems.

$$13_{\text{six}} - 5_{\text{six}} \qquad 12_{\text{six}} - 3_{\text{six}} \qquad 10_{\text{six}} - 4_{\text{six}}$$

Problem 102. Do these calculations in base 4. Try not to translate to base 10 and then calculate there — try to work in base 4!

1. $33_{\text{four}} + 11_{\text{four}}$.
2. $123_{\text{four}} + 22_{\text{four}}$
3. $223_{\text{four}} - 131_{\text{four}}$
4. $112_{\text{four}} - 33_{\text{four}}$

Problem 103. Make a base-5 multiplication table:

$\times$	0	1	2	3	4
0					
1					
2					
3					
4					

Use the table to solve the division problems.

$11_{\text{five}} \div 2_{\text{five}}$ $\qquad$ $22_{\text{five}} \div 3_{\text{five}}$ $\qquad$ $13_{\text{five}} \div 4_{\text{five}}$

Problem 104.

(a) Here is a true fact in base five: $2_{\text{five}} \cdot 3_{\text{five}} = 11_{\text{five}}$. Write the rest of this four fact family.

(b) Here is a true fact in base five: $13_{\text{five}} \div 2_{\text{five}} = 4_{\text{five}}$. Write the rest of this four fact family.

Problem 105 (AlphaMath)**.** Letters stand for digits 0–9. In a given problem: the same letter always represents the same digit, and different letters always represent different digits. There is no relation between problems (so "A" in problem one and "A" in problem 3 might be different).

$$\begin{array}{r} A\ B\ C \\ +\ A\ C\ B \\ \hline C\ B\ A \end{array} \qquad \begin{array}{r} O\ N\ E \\ +\ O\ N\ E \\ \hline T\ W\ O \end{array} \qquad \begin{array}{r} A \\ A \\ +\quad A \\ \hline H\ A \end{array}$$

Notes: "O" represents the letter O and not the number zero. Two and three digit numbers never start with 0.

Problem 106. Here's another AlphaMath problem. (See problem 105 for the rules of AlphaMath.)

$$\begin{array}{r} T\ E\ N \\ +\quad N\ O\ T \\ \hline N\ I\ N\ E \end{array}$$

(a) Solve this AlphaMath problem in base 10.

(b) Now solve it in base 6.

Problem 107. Find all solutions to this AlphaMath problem in base 9. (See problem 105 for the rules of AlphaMath.) Note: even though this is two calculations, it is a *single problem*. All T's in both calculations represent the same digit, all O's represent the same digit, and so on.

$$\begin{array}{r} T\ O \\ -\quad B\ E \\ \hline O\ R \end{array} \qquad \begin{array}{r} N\ O\ T \\ -\quad T\ O \\ \hline B\ E \end{array}$$

Problem 108. Remember that a *perfect square* is a number that can be written as $a \cdot a$ or a^2 (some number times itself).

(a) Which of the following *base seven numbers* are perfect squares? For each part, answer **yes** (it is a perfect square) or **no** (it is not a perfect square) and give a justification of your answer.

$$4_{\text{seven}} \qquad 25_{\text{seven}} \qquad 51_{\text{seven}}$$

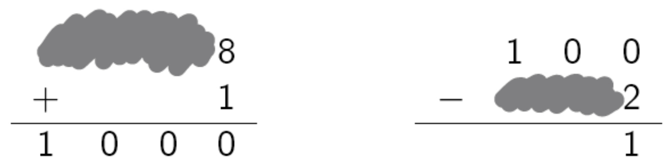
(b) For which choices of base b is the number 100_b a perfect square? Justify your answer.

Problem 109. This is a single AlphaMath problem. (So all G's represent the same digit. All A's represent the same digit. And so on.)

Solve the problem in **base 6**.

$$\text{GALON} = (\text{GOO})^2 \qquad \text{ALONG} = (\text{OOG})^2$$

Problem 110. Some ink was spilled on these problems. The answers were correct. Can you determine the missing digits and the bases?



Problem 111.

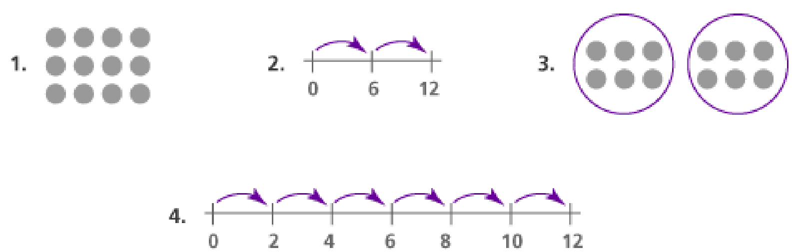
(a) Rewrite each subtraction problem as an addition problem:

$$x - 156 = 279 \qquad 279 - 156 = x \qquad a - x = b$$

(b) Rewrite each division problem as a multiplication problem:

$$24 \div x = 12 \qquad x \div 3 = 27 \qquad a \div b = x$$

Problem 112. Which two of the following models represent the same multiplication problem? Explain your answer.



♣♣♣ Fellow: [stolen picture with a weird spot in the corner. can you make a better one? If not, no problem.]

Problem 113. Show an area model for each of these multiplication problems. Write down the standard computation next to the area model and see how it compares:

$$20 \times 33 \qquad 24 \times 13 \qquad 17 \times 11$$

Problem 114. Suppose the 2 key on your calculator is broken. How could you still use the calculator compute these products? Think about what properties of multiplication might be helpful. (Write out the calculation you would do on the calculator, not just the answer.)

$$1592 \times 3344 \qquad 2008 \times 999 \qquad 655 \times 525$$

Problem 115. Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again? Choose the best answer and justify your choice.

1. Jennifer will always be twice as old as her brother.
2. It will happen every two years.
3. It depends on Jennifer's age.
4. It will happen when Jennifer is twice as old as she is now.
5. It will never happen again.

Problem 116.

(a) Find the quotient and remainder for each problem:

$$7 \div 3 \qquad 3 \div 7 \qquad 7 \div 1 \qquad 1 \div 7 \qquad 15 \div 5 \qquad 8 \div 12$$

(b) How many possible remainders are there when dividing by these numbers? Justify what you say.

$$2 \qquad 12 \qquad 62 \qquad 23$$

Problem 117. Identify each problem as either *partitive* or *quotative* division and say why you made that choice. Then solve the problem.

(a) Adriana bought 12 gallons of paint. If each room requires three gallons of paint, how many rooms can she paint?

(b) Chris baked 15 muffins for his family of five. How many muffins does each person get?

(c) Prof. Davidson gave three straws for each student for an activity. She used 51 straws. How many students are in her class?

Problem 118. Use the digits 1 through 9. Use each digit exactly once. Fill in the goes to make all of the equations true.

$$\begin{array}{ccccc} \square & - & \square & = & \square \\ & & & & \times \\ \square & \div & \square & = & \square \\ & & & & = \\ \square & + & \square & = & \square \end{array}$$

♣♣♣ Fellow: [can we turn this equals vertically?]

Chapter 4

Fractions

♣♣♣ Fellow: [Formatting: can we put things marked "Problem" in a box (maybe with some color?) to set it apart? Same with the Think/Pair/Share (different color?) and Solutions.]

Fractions are one of the hardest topics to teach (and learn!) in elementary school. What is the reason for this? We'll try to provide some insight in this chapter, along with some better ways for understanding, teaching, and learning about fractions. But for now, talk with a partner about what makes this topic so hard.

Think/Pair/Share. You may have struggled learning about fractions in elementary school. Maybe you still find them confusing. Even if you were one of the lucky ones who didn't struggle when learning about fractions, you probably had friends who did struggle.

With a partner, talk about why this is. What is so difficult about understanding fractions? Why is the topic harder than other ones we tackle in elementary schools?

Remember that teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the idea of fractions as well as basic operations on them.

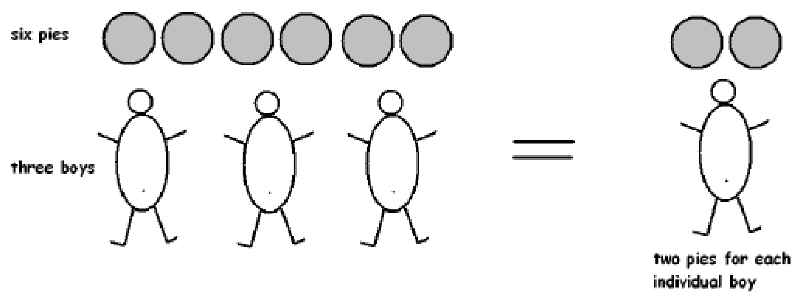
4.1 What is a Fraction?

One of the things that makes fractions such a difficult concept to teach and to learn is that you have to think about them in lots of different ways, depending on the problem at hand. For now, we're going to think of a fraction as the answer to a division problem.

♣♣♣ Fellow: [Throughout: anything you can do to make the pictures better would be awesome.]

Example 4.1.1 (Pies per boy). Suppose 6 pies are to be shared equally among 3 boys. This yields 2 pies per boy. We write:

$$\frac{6}{3} = 2.$$



The fraction $\frac{6}{3}$ is equivalent to the answer to the division problem $6 \div 3 = 2$. It represents the number of pies one whole boy receives.

In the same way ...

sharing 10 pies among 2 boys yields $\frac{10}{2} = 5$ pies per boy,

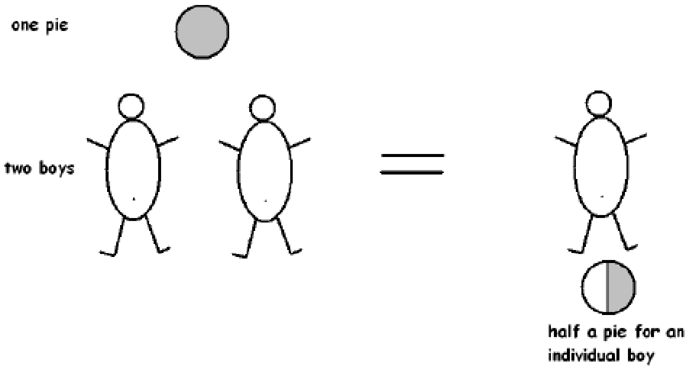
sharing 8 pies among 2 boys yields $\frac{8}{2} = 4$ pies per boy,

sharing 5 pies among 5 boys yields $\frac{5}{5} = 1$ pies per boy, and

the answer to sharing 1 pies among 2 boys is $\frac{1}{2}$, which we call "one-half."

This final example is actually saying something! It also represents how fractions are usually taught to students:

> *If one pie is shared (equally) between two boys, then each boy receives a portion of a pie which we choose to call "half."*



Thus students are taught to associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  is said to represent "one third," that is, $\frac{1}{3}$. (And this is indeed the amount of pie an individual boy would receive if one pie is shared among three.)

The picture  is called "one fifth" and is indeed $\frac{1}{5}$, the amount of pie an individual boy receives when one pie is shared among five.

And the picture  is called "three fifths" to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five boys.

Think/Pair/Share. Carefully explain *why* this is true: If five boys share three pies equally, each boy receives an amount that looks like this: . Your explanation will probably require both words and pictures.

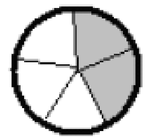
On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a picture associated with the fraction $\frac{1}{6}$.

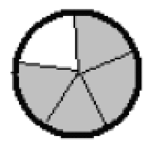
2. Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual boy would receive if three pies are shared among seven boys? Be very clear on this!

3. Let's work backwards! Here is the answer to a division problem:



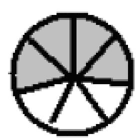
This represents the amount of pie an individual boy receives if some number of pies is shared among some number of boys. How many pies? How many boys? How can you justify your answers?

4. Here is another answer to a division problem:



How many pies? How many boys? How can you justify your answers?

5. Here is another answer to a division problem:



How many pies? How many boys? How can you justify your answers?

6. Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Is three pies shared among five boys three times as much as one pie shared among five boys? Explain your answer.

7. Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.

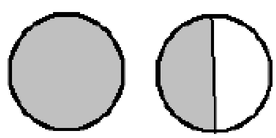
8. Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.

9. What does the division problem $\frac{1}{1}$ represent? How much pie does an individual boy receive?

10. What does the division problem $\frac{5}{1}$ represent? How much pie does an individual boy receive?

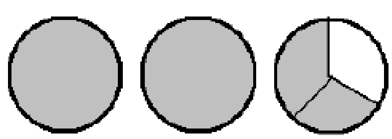
11. What does the division problem $\frac{5}{5}$ represent? How much pie does an individual boy receive?

12. Here is the answer to another division problem. This is the amount of pie an individual boy receives:



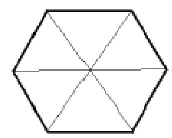
    How many pies were in the division problem? How many boys were in the division problem? Justify your answers.

13. Here is the answer to another division problem. This is the amount of pie an individual boy receives:



    How many pies were in the division problem? How many boys were in the division problem? Justify your answers.

14. Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions
$$\frac{1}{6}, \quad \frac{2}{6}, \quad \frac{3}{6}, \quad \frac{4}{6}, \quad \frac{5}{6}, \text{ and } \quad \frac{6}{6}.$$

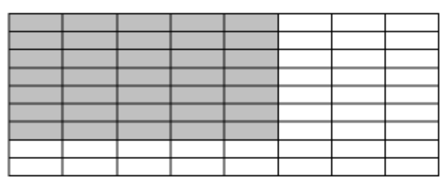


(a) Why is this shape used? What does $\frac{1}{6}$ of a pie look like?

(b) What does $\frac{6}{6}$ of a pie look like?

(c) What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

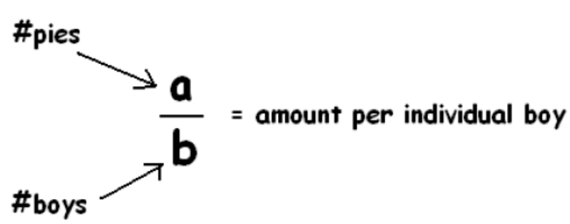
Problem 119. Some rectangular pies are distributed to some number of boys. This picture represents the amount of pie an individual boy receives.



How many pies? How many boys? Carefully justify your answers!

4.1.1 Pies Per Boy Model

In our model, a fraction $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are shared equally by b boys.



Think/Pair/Share.

1. What is $\frac{2}{2}$? What is $\frac{7}{7}$? What is $\frac{100}{100}$? How can you use the "Pies Per Boy Model" to make sense of $\frac{a}{a}$ for any positive whole number a?

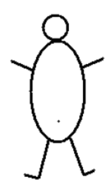
2. What is $\frac{2}{1}$? What is $\frac{7}{1}$? What is $\frac{1876}{1}$? How can you use the "Pies Per Boy Model" to make sense of $\frac{b}{1}$ for any positive whole number b?

3. Write the answer to this division problem: "I have no pies to share among thirteen boys." How can you generalize this division problem to make a general statement about fractions?

Definition 4.1.2. For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the **numerator** of the fraction, and the bottom number b (the number of pies), is called the **denominator** of the fraction.

Most people insist that the numerator and denominator each be whole numbers, but they really don't have to be.

Think/Pair/Share. To understand why the numerator and denominator need not be whole numbers, we must first be a little gruesome. Instead of dividing pies, let's divide boys! Here is one boy:

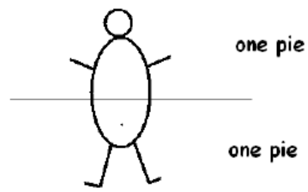


- What would half a boy look like?
- What would one-third of a boy look like?
- What would three-fifths of a boy look like?

So, what would

$$\frac{1}{\left(\frac{1}{2}\right)}$$

represent? This means assigning one pie to each "group" of half a boy. So how much would a whole boy receive? Well, we would have a picture like this:



The whole boy gets two pies, so we have

$$\frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Think/Pair/Share. Draw pictures for these problems if it helps!

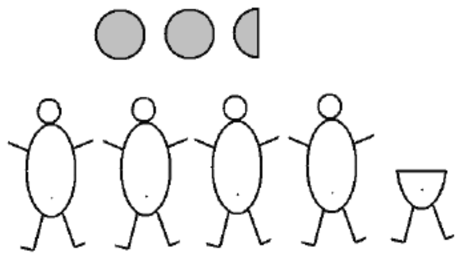
1. What does
$$\frac{1}{\left(\frac{1}{3}\right)}$$
represent? Justify your answer using the "Pies Per Boy Model."

2. What is
$$\frac{1}{\left(\frac{1}{6}\right)}?$$
Justify your answer.

3. Explain why the fraction
$$\frac{5}{\left(\frac{1}{2}\right)}$$
represents the number 10. (How much pie is given to half a boy? To a whole boy?)

4. What is
$$\frac{4}{\left(\frac{1}{3}\right)}?$$
Justify your answer.

5. **Challenge:** Two-and-a-half pies are to be shared equally among four-and-a-half boys. How much pie does an individual (whole) boy receive? Justify your answer.



4.1.2 Jargon

A fraction with a numerator smaller than its denominator is called (in school math jargon) a *proper fraction*. For example, $\frac{45}{58}$ is "proper."

A fraction with numerator larger than its denominator is called (in school math jargon) an *improper fraction*. For example, $\frac{7}{3}$ is "improper." (In the 1800s, these fractions were called *vulgar fractions*. Despite nineteenth-century views they are useful nonetheless!)

For some reason, improper fractions are considered, well, improper by some teachers. So students are often asked to write improper fractions as a combination of a whole number and a proper fraction.

Consider, for example, $\frac{7}{3}$. If seven pies are shared among three boys, then each boy will certainly receive 2 whole pies, leaving one pie over to share among the three boys. Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$.

People write:

$$\frac{7}{3} = 2\frac{1}{3}$$

and call the result $2\frac{1}{3}$ a *mixed number*. (One can also write

$$2 + \frac{1}{3},$$

which is what $2\frac{1}{3}$ really means. But most people choose to suppress the plus sign.)

As another example, consider $\frac{23}{4}$. If 4 boys share 23 pies, we can give them each five whole pies. That uses 20 pies, and there are then 3 pies left over. Those three pies are still be shared equally by the 4 boys. We have:

$$\frac{23}{4} = 5\frac{3}{4}.$$

Mathematically, there is nothing wrong with an improper fraction. (In fact, many mathematicians prefer improper fractions over mixed numbers. They are often easier to use in computations.) Consider, for instance, the mixed number $2\frac{1}{5}$. This is really $2 + \frac{1}{5}$.

For fun, let's write the number 2 as a fraction with denominator five:

$$2 = \frac{10}{5}.$$

So:

$$2\frac{1}{5} \quad = \quad 2 + \frac{1}{5} \quad = \quad \frac{10}{5} + \frac{1}{5} \quad = \quad \frac{11}{5}.$$

We've written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

Think/Pair/Share.

- Write each of the following as a mixed number. Explain how you got your answer.

$$\frac{17}{3}, \qquad \frac{8}{5}, \qquad \frac{100}{13}, \qquad \frac{200}{199}.$$

- Convert each of these mixed numbers into "improper" fractions. Explain how you got your answer.

$$3\frac{1}{4}, \qquad 5\frac{1}{6}, \qquad 1\frac{3}{11}, \qquad 200\frac{1}{200}.$$

Students are often asked to memorize the names "proper fraction," "improper fraction" and "mixed number" so that they can follow directions on tests and problem sets.

But, to a mathematician, these names are not at all important! There is no "correct" way to express an answer (assuming, that the answer is mathematically the right number). We often wish to express our answer in a simpler form, but sometimes the context will tell you what form is "simple" and what form is more complicated.

As you work on problems in this chapter, decide for yourself which type of fraction would be best to work with as you do your task.

4.2 The Key Fraction Rule

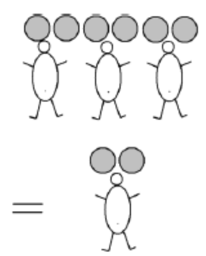
We know that $\frac{a}{b}$ is the answer to a division problem:

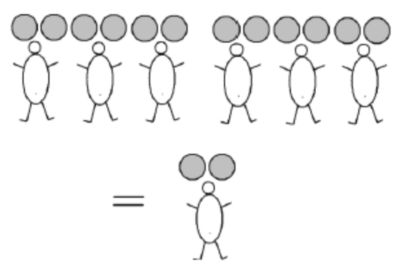
> $\dfrac{a}{b}$ represents the amount of pie an individual boy receives when a pies are distributed among b boys.

What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

$$\frac{2a}{2b} = \frac{a}{b}.$$

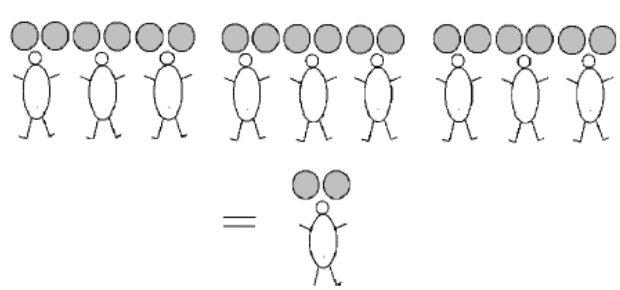
For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each boy.





And tripling the number of pies and tripling the number of boys also does not change the final amount of pie per boy, nor does quadrupling each number, or onetrillion- billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \cdots = \text{ two pies per boy.}$$



This leads us to want to believe:

♣♣♣ Fellow: [put this in a box and centered?]

Key Fraction Rule: $\dfrac{xa}{xb} = \dfrac{a}{b}$ (at least for positive whole numbers x).

Example 4.2.1 (Fractions equivalent to $\frac{3}{5}$). For example, $\frac{3}{5}$ (sharing three pies among five boys) yields the same result as

$$\frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \text{ (sharing six pies among ten boys),}$$

and as

$$\frac{3 \cdot 100}{5 \cdot 100} = \frac{300}{500} \text{ (sharing 300 pies among 500 boys).}$$

Think/Pair/Share. Write down lots of equivalent fractions for $\frac{1}{2}$, for $\frac{10}{3}$, and for 1.

Example 4.2.2 (Going backwards).

$$\frac{20}{32} \text{ (sharing 20 pies among 32 boys)}$$

is the same problem as:

$$\frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8} \text{ (sharing five pies among eight boys).}$$

Most people say we have *cancelled* or taken a common factor of 4 from the numerator and the denominator.

Mathematicians call this process *reducing* the fraction to lowest terms. (We've made the numerator and denominator each smaller, in fact as small as we can make them!)

Teachers tend to say that we are *simplifying* the fraction. (You have to admit that $\frac{5}{8}$ does look simpler than $\frac{20}{32}$.)

Example 4.2.3 (How low can you go?). As another example, $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \cdot 10}{35 \cdot 10} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \cdot 7}{4 \cdot 7} = \frac{4}{5}.$$

Thus, sharing 280 pies among 350 boys gives the same result as sharing just 4 pies among 5 boys!

$$\frac{280}{350} = \frac{4}{5}.$$

Since 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers!).

Think/Pair/Share. Jenny says that $\frac{4}{5}$ does "reduce" further is you are willing to move away from whole numbers. She writes:

$$\frac{4}{5} = \frac{2 \cdot 2}{\left(2\frac{1}{2}\right) \cdot 2} = \frac{2}{\left(2\frac{1}{2}\right)}.$$

Is she right? Does sharing 4 pies among 5 boys yield the same result as sharing 2 pies among $2\frac{1}{2}$ boys? What do you think?

On Your Own

Mix and Match: On the top are some fractions that have not been simplified. On the bottom are the simplified answers, but in random order. Which simplified answer goes with which fraction? (Notice that there are fewer answers than questions!)

1. $\frac{10}{20}$ **2.** $\frac{50}{75}$ **3.** $\frac{24000}{36000}$ **4.** $\frac{24}{14}$ **5.** $\frac{18}{32}$ **6.** $\frac{1}{40}$

a. $\frac{2}{3}$ **b.** $\frac{9}{16}$ **c.** $\frac{12}{7}$ **d.** $\frac{1}{40}$ **e.** $\frac{1}{2}$

Think/Pair/Share. Use the "Pies Per Boy Model" to explain *why* the key fraction rule holds. That is, explain why each individual boy gets the same amount of pie in these two situations:

- if you have a pies and b boys, or
- if you have xa pies and xb boys.

4.3 Adding and Subtracting Fractions

4.3.1 Fractions with the Same Denominator

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might be tempting to say:

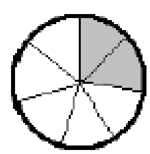
$$\frac{2}{7} \text{ represents 2 pies being shared among 7 boys}$$
$$\frac{3}{7} \text{ represents 3 pies being shared among 7 boys}$$

so $\frac{2}{7}+\frac{3}{7}$ probably represents sharing 5 pies among 14 boys, giving the answer $\frac{5}{14}$. That is, it is very tempting to say that "adding fractions" means "adding pies and adding boys."

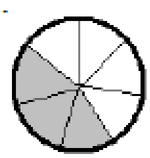
The trouble is that a fraction is not a pie, and a fraction is not a boy. So adding pies and adding boys is **not** actually adding fractions. A fraction is something different. It is related to pies and boys, but something more subtle. A fraction is an amount of **pie per boy**.

One can't add pies, one can't add boys. One must add instead the amounts individual boys receive.

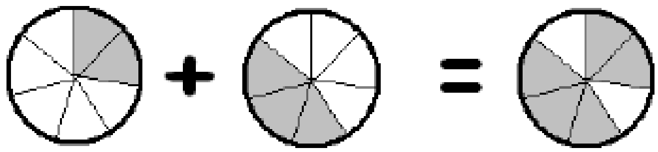
Example 4.3.1 ($\frac{2}{7}+\frac{3}{7}$). Let's take it slowly. Consider the fraction $\frac{2}{7}$. Here is a picture of the amount an individual boy receives when two pies are given to seven boys:



Consider the fraction $\frac{3}{7}$. Here is a picture of the amount an individual boy receives when three pies are given to seven boys:

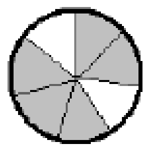


The sum $\frac{2}{7}+\frac{3}{7}$ corresponds to the sum:



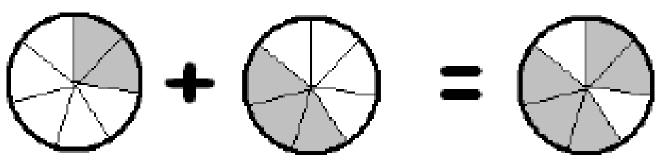
The answer, from the picture, is $\frac{5}{7}$.

Think/Pair/Share. Remember that $\frac{5}{7}$ means "the amount of pie that one boy gets when five pies are shared by seven boys." Carefully explain *why* that is the same as the picture given by the sum above:



Your explanation should use both words and pictures!

Most people read this as "Two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And, in the end, they are right!



$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

This is how the addition of fractions is first taught to students: Adding fractions with the same denominator seems just as easy as adding apples:

$$\begin{aligned} 4 \text{ tenths} + 3 \text{ tenths} + 8 \text{ tenths} &= 15 \text{ tenths} \\ \frac{4}{10} + \frac{3}{10} + \frac{8}{10} &= \frac{15}{10}. \end{aligned}$$

(And, if you like, $\frac{15}{10} = \frac{5 \cdot 3}{5 \cdot 2} = \frac{3}{2}$.)

$$82 \text{ sixty-fifths} + 91 \text{ sixty-fifths} = 173 \text{ sixty-fifths}$$
$$\frac{82}{65} + \frac{91}{65} = \frac{173}{65}.$$

We are really adding **amounts per boy** not amounts, but the answers match the same way.

We can use the "Pies Per Boy Model" to explain *why* adding fractions with like denominators works in this way.

Think about the addition problem $\frac{2}{7} + \frac{3}{7}$:

$$\begin{array}{r} \text{amount of pie each boy gets when 7 boys share 2 pies} \\ + \underline{\text{ amount of pie each boy gets when 7 boys share 3 pies}} \\ \text{???} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \end{array}$$

Since in both cases we have 7 boys sharing the pies, we can imagine that it's the same 7 boys in both cases. First they share 2 pies. Then they share 3 more pies. The total each boy gets by the time all the pie-sharing is done is the same as if the 7 boys had just shared 5 pies to begin with. That is:

$$\begin{array}{r} \text{amount of pie each boy gets when 7 boys share 2 pies} \\ + \underline{\text{ amount of pie each boy gets when 7 boys share 3 pies}} \\ \text{amount of pie each boy gets when 7 boys share 5 pies} \end{array}$$

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

Now let's think about the general case. Our claim is that

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}.$$

Translating into our model, we have d boys. First, they share a pies between them, and $\frac{a}{d}$ represents the amount each boy gets. Then they share b more pies, so the additional amount of pie each boy gets is $\frac{b}{d}$. The total each boy gets is $\frac{a}{d} + \frac{b}{d}$.

But it doesn't really matter that the boys first share a pies and then share b pies. The amount each boy gets is the same as if they had started with all of the pies — all $a + b$ of them — and shared them equally. That amount of pie is represented by $\frac{a+b}{d}$.

Think/Pair/Share.

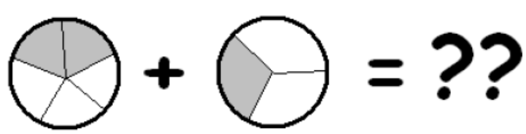
1. How can you *subtract* fractions with the same denominator? For example, what is
$$\frac{400}{903} - \frac{170}{903}?$$

2. Use the "Pies Per Boy" model to *carefully explain why*
$$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d},$$
at least if $b \leq a$ and everything in sight is a positive whole number.

3. Explain why the fact that the denominators are the same is *essential* to this addition and subtraction method. Where is that fact used in the explanations?

4.3.2 Fractions with Different Denominators

This approach to adding fractions suddenly becomes tricky if the denominators involved are not the same common value. For example, what is $\frac{2}{5} + \frac{1}{3}$?



Let's phrase this question in terms of pies and boys:

> *Suppose Poindexter is part of a team of a five boys that receives two pies. Then later he is part of a team of three boys that receives one pie. How much pie does Poindexter receive in total?*

Think/Pair/Share. Talk about these questions with a partner before reading on. It is actually a very difficult problem! What might a student say, if they don't already know about adding fractions? Write down any of your thoughts.

1. Do you see that this is the same problem as computing $\frac{2}{5} + \frac{1}{3}$?

2. What might be the best approach to answering this problem?

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our fraction rule (that is, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on) and to do the same for $\frac{1}{3}$:

$$\begin{array}{ccc} \frac{2}{5} & + & \frac{1}{3} \\ \\ \frac{4}{10} & & \frac{2}{6} \\ \\ \frac{6}{15} & & \frac{3}{9} \\ \\ \frac{8}{20} & & \frac{4}{12} \\ \\ \frac{10}{25} & & \frac{5}{15} \\ \\ \vdots & & \vdots \end{array}$$

We see that the problem $\frac{2}{5} + \frac{1}{3}$ is actually the same as $\frac{6}{15} + \frac{5}{15}$. So we can find the answer using the same-denominator method:

$$\frac{2}{5} + \frac{1}{3} \quad = \quad \frac{6}{15} + \frac{5}{15} \quad = \quad \frac{11}{15}.$$

Example 4.3.2 ($\frac{3}{8} + \frac{3}{10}$). Here's another example of adding fractions with unlike denominators: $\frac{3}{8} + \frac{3}{10}$. In this case, Poindexter is first part of a group of 8 boys who share 3 pies. Later he is part of a group of 10 boys who share 3 different pies. How much total pie did Poindexter get?

$$\begin{array}{ccc} \frac{3}{8} & + & \frac{3}{10} \\ \\ \frac{6}{16} & & \frac{6}{20} \\ \\ \frac{9}{24} & & \frac{9}{30} \\ \\ \frac{12}{32} & & \frac{12}{40} \\ \\ \frac{15}{40} & & \frac{15}{50} \\ \vdots & & \vdots \end{array}$$

$$\frac{3}{8} + \frac{3}{10} = \frac{15}{40} + \frac{12}{40} = \frac{17}{40}.$$

Of course, you don't need to list *all* of the equivalent forms of each fraction in order to find a common denominator. If you can see a common denominator right away (or can think of a faster method that always works), go for it!

Think/Pair/Share. Cassie suggests the following method for the example above:

> *When the denominators are the same, we just add the numerators. So when the numerators are the same, shouldn't we just add the denominators? Like this:* $\frac{3}{8} + \frac{3}{10} = \frac{3}{18}$.

What do you think of Cassie's suggestion? Does it make sense? What would you say if you were Cassie's teacher?

On Your Own

Try these exercises on your own. For each addition exercise, also write down a "Pies Per Boy" interpretation of the problem. You might also want to draw a picture.

1. What is $\frac{1}{2} + \frac{1}{3}$?

2. What is $\frac{2}{5} + \frac{37}{10}$?

3. What is $\frac{1}{2} + \frac{3}{10}$?

4. What is $\frac{2}{3} + \frac{5}{7}$?

5. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?

6. What is $\frac{3}{10} + \frac{4}{25} + \frac{7}{20} + \frac{3}{5} + \frac{49}{50}$?

Now try these subtraction exercises.

7. What is $\frac{7}{10} - \frac{3}{10}$?

8. What is $\frac{7}{10} - \frac{3}{20}$?

9. What is $\frac{1}{3} - \frac{1}{5}$?

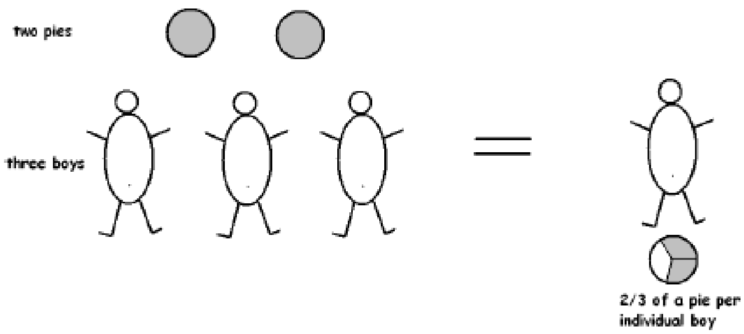
10. What is $\frac{2}{35} - \frac{2}{7} + \frac{2}{5}$?

11. What is $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$?

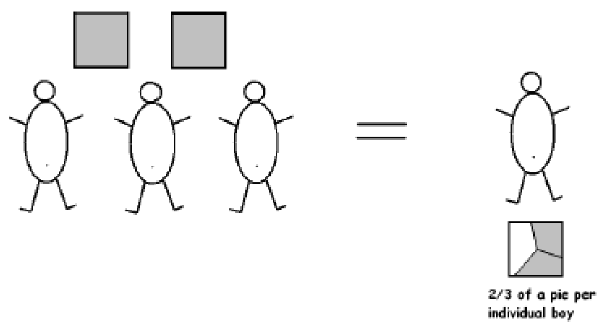
Think/Pair/Share. Which fraction is larger, $\frac{5}{9}$ or $\frac{6}{11}$? Justify your answer. (Oh, and what does this question have to do with the subject of this section: adding and subtracting fractions?)

4.4 What is a Fraction? Revisited

So far, we've been thinking about a fraction as the answer to a division problem. For example, $\frac{2}{3}$ is the result of sharing two pies among three boys.

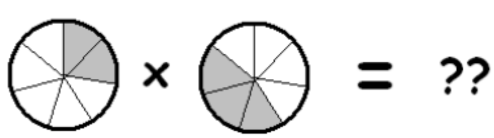


Of course, pies don't have to be round. We can have square pies, or triangular pies or squiggly pies or any shape you please.



This "Pies Per Boy Model" has served us perfectly well in thinking about the meaning of fractions, equivalent fractions, and even adding and subtracting fractions.

However, there's not any way to use this model to make sense of multiplying fractions! What would this mean?

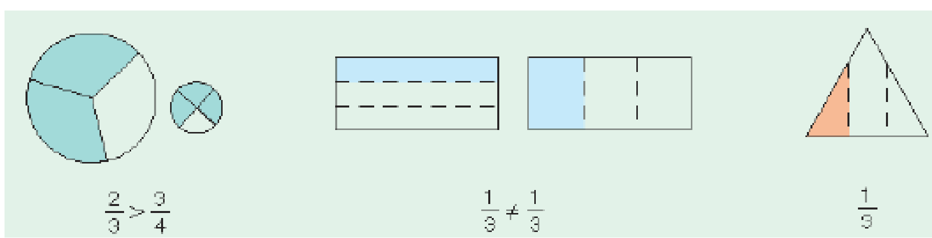


So what are fractions, if we are asked to multiply them? We are forced to switch models and think about fractions in a new way.

This switch of gear is fundamentally perturbing: Does a fraction have anything to do with pie or pies per boy or not? If the answer is that a fraction is more of an abstract concept that applies simultaneously to pies and boys and to something else that we can multiply, then what is that concept exactly?

Think about our poor young students. We keep switching concepts and models, and speak of fractions in each case as though all is naturally linked and obvious. All is not obvious and all is absolutely confusing. This is just one of the reasons that fractions can be such a difficult concept to teach and to learn in elementary school!

Think/Pair/Share (What's wrong here?). For each of the following visual representations of fractions, there is a corresponding *incorrect* symbolic expression. Discuss with your partner: why is the symbolic representation incorrect? What might elementary students find confusing in these visual representations?



♣♣♣ Fellow: [better picture? this is an old scan from a textbook]

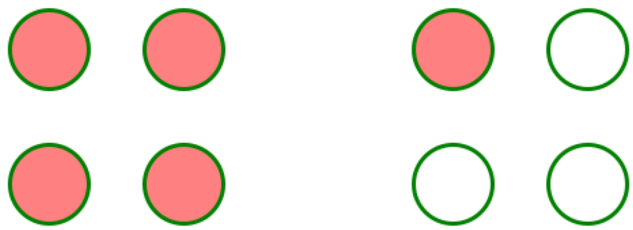
4.4.1 Units and Unitizing

In thinking about fractions, it's important to remember that there are always *units* attached to a fraction, even if the units are hidden. If you see the number $\frac{1}{2}$ in a problem, you should ask yourself "half of what?" The answer to that question is your *unit*, the amount that equals 1.

So far, our units have been consistent: the "whole" (or unit) was a whole pie, and fractions were represented by pies cut into equal-sized pieces. But this is just a model, and we can take anything at all, cut it into equal-sized pieces, and talk about fractions *of that whole*.

One thing that can make fraction problems so difficult is that the fractions in the problem may be given in different *units* (they may be "parts" of different "wholes").

Example 4.4.1 (Everyone is right!). Mr. Li shows this picture to his class and asks what number is shown by the shaded region.



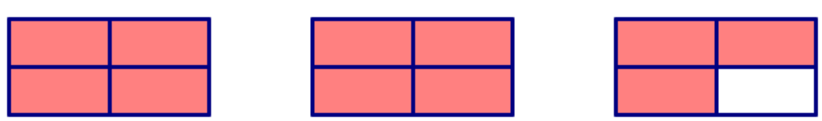
- Kendra says the shaded region represents the number 5.
- Dylan says it represents $2\frac{1}{2}$.
- Kiana says it represents $\frac{5}{8}$.
- Nate says it is $1\frac{1}{4}$.

Mr. Li exclaims, "Everyone is right!"

Think/Pair/Share.

1. How can it be that everyone is right? Justify each answer by explaining what each student thought was the *unit* in Mr. Li's picture.

2. Now look at this picture:

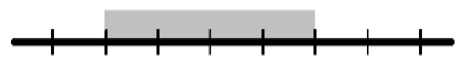


- If the shaded region represents $3\frac{2}{3}$, what is the *unit*?
- Find three other numbers that could be represented by the shaded region.

When we think about multiplying fractions, we will (at least at first) choose to think of them as "portions of line segments," since that fits nicely with our measurement model for numbers. Then we can once again use an area model to make sense of multiplication. (We'll do exactly this in the next section!)

Example 4.4.2 (Segments). This picture  represents $\frac{2}{3}$. The whole segment (the *unit*) is split into three equal pieces by the tick marks, and two of those three equal pieces are shaded.

Think/Pair/Share. For each picture below, say what fraction it represents and how you know you're right.



4.4.2 Ordering Fractions

If we think about fractions as "portions of a segment," then we can talk about their locations on a number line. We can start to treat fractions like numbers. In the back of our minds, we should remember that fractions are always relative to some *unit*. But on a number line, the unit is clear: it is the distance between 0 and 1.

♣♣♣ Fellow: [insert picture of number line with the unit distance from 0 to 1 highlighted?]

This measurement model makes it much easier to tackle questions about the relative size of fractions based on where they appear on the number line. We can mark off different fractions as parts of the unit segment. Just as with whole numbers, fractions that appear farther to the right are larger.

♣♣♣ Fellow: [repeat number line, with $3/5$ and $5/8$ marked on it in the correct positions.]

Think/Pair/Share (Ordering Fractions).

1. What quick method can you use to determine if a fraction is greater than 1?

2. What quick method can you use to determine if a fraction is greater than $\dfrac{1}{2}$?

3. Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1, and justify your choices.

$$\frac{25}{23}, \quad \frac{4}{7}, \quad \frac{17}{35}, \quad \frac{2}{9}, \quad \frac{14}{15}.$$

4. Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

   - $\frac{7}{17}, \quad \frac{4}{17}, \quad \frac{12}{17}.$
   - $\frac{3}{7}, \quad \frac{3}{4}, \quad \frac{3}{8}.$
   - $\frac{5}{6}, \quad \frac{7}{8}, \quad \frac{3}{4}.$
   - $\frac{8}{13}, \quad \frac{12}{17}, \quad \frac{1}{6}.$
   - $\frac{5}{6}, \quad \frac{10}{11}, \quad \frac{2}{3}.$

You probably came up with benchmarks and intuitive methods to think about the relative sizes of fractions. Here are some of these methods. (Did you come up with others?)

♣♣♣ Fellow: [put the following list in a box with some kind of heading? for the pictures, use the number line model, but shade, for example, the missing pieces when those are discussed]

Greater than 1: A fraction is greater than 1 if its numerator is greater than the denominator. How can we see this? Well, the denominator represents how many pieces in one whole (one unit). The numerator represents how many pieces in your portion. So if the numerator is bigger, that means you have more than the number of pieces needed to make one whole. ♣♣♣ Fellow: [picture?]

Greater than $\frac{1}{2}$: A fraction is greater than $\frac{1}{2}$ if the numerator is more than half the denominator. Another way to check (which might be an easier calculation): a fraction is grater than $\frac{1}{2}$ if twice the numerator is bigger than the denominator. Why? Well, if we double the fraction and get something bigger than 1, then the original fraction must be bigger than $\frac{1}{2}$.

Same denominators: If two fractions have the same denominator, just compare the numerators. The fractions will be in the same order as the numerators. For example, $\frac{5}{7} < \frac{6}{7}$. Why? Well, the pieces are the same size since the denominators are the same. If you have more pieces of the same size, you have a bigger number. ♣♣♣ Fellow: [picture?]

Same numerators: If the numerators of two fractions are the same, just compare the denominators. The fractions should be in the reverse order of the denominators. For example, $\frac{3}{4} > \frac{3}{5}$. The justification for this one is a little trickier: The denominator tells you how many pieces make up one whole. If there are *more pieces* in a whole (if the denominator is bigger), then the pieces must be *smaller*. And if you take the same number of pieces (same numerator), then the bigger piece wins. ♣♣♣ Fellow: [picture?]

Numerator = denominator $-$ 1: You can easily compare two fractions whose numerators are both one less than their denominators. The fractions will be in the same order as the denominators. Think of each fraction as a pie with one piece missing. The greater the denominator, the smaller the missing piece, so the greater the amount remaining. For example, $\frac{6}{7} < \frac{10}{11}$, since $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{10}{11} = 1 - \frac{1}{11}$. ♣♣♣ Fellow: [picture?]

Numerator = denominator − constant: You can extend the test above to fractions whose numerators are both the same amount less than their denominators. The fractions will again be in the same order as the denominators, for exactly the same reason. For example, $\frac{3}{7} < \frac{7}{11}$, because both are four "pieces" less than one whole, and the $\frac{1}{11}$ pieces are smaller than the $\frac{1}{7}$ pieces. ♣♣♣ Fellow: [picture?]

Equivalent fractions: Find an equivalent fraction that lets you compare numerators or denominators, and then use one of the above rules.

4.4.3 Arithmetic Sequences

Consider the patterns below

Parttern 1: $5, \quad 8, \quad 11, \quad 14, \quad 17, \quad 20, \quad 23, \quad 26, \ldots$

Parttern 2: $2, \quad 9, \quad 16, \quad 23, \quad 30, \quad 37, \quad 44, \quad 51, \ldots$

Parttern 3: $\frac{1}{5}, \quad \frac{3}{5}, \quad 1, \quad \frac{7}{5}, \quad \frac{9}{5}, \quad \frac{11}{5}, \quad \frac{13}{5}, \quad 3, \ldots$

Think/Pair/Share. Answer these questions about each of the patterns.

1. Can you predict the next 10 numbers?

2. Can you predict the 100th number?

3. What do these sequences have in common? Describe the pattern in words.

The patterns above are called **arithmetic sequences**: a sequence of numbers where the difference between consecutive terms is a constant. Here are some other examples:

Parttern A: $\underbrace{1, \quad 2,}_{+1}\!\!\underbrace{\quad 3,}_{+1}\!\!\underbrace{\quad 4,}_{+1}\!\!\underbrace{\quad 5,}_{+1} \ldots$

Parttern B: $\underbrace{2, \quad 4,}_{+2}\!\!\underbrace{\quad 6,}_{+2}\!\!\underbrace{\quad 8,}_{+2}\!\!\underbrace{\quad 10,}_{+2} \ldots$

Parttern C: $\underbrace{\frac{1}{3}, \quad 1,}_{+\frac{2}{3}}\!\!\underbrace{\quad \frac{5}{3},}_{+\frac{2}{3}}\!\!\underbrace{\quad \frac{7}{3},}_{+\frac{2}{3}}\!\!\underbrace{\quad 3,}_{+\frac{2}{3}} \ldots$

Think/Pair/Share. If you haven't done so already, find the common difference between terms for Patterns 1, 2, and 3. Are they really arithmetic sequences? Then make up your own arithmetic sequence using whole numbers. Exchange sequences with a partner, and check if your partner's sequence is really an arithmetic sequence.

Here are several more number patterns:

Parttern 4: 1, 2, 4, 8, 16, 32, 64, 128, ...

Parttern 5: 1, 3, 6, 10, 15, 21, 28, 36, ...

Parttern 6: $\frac{2}{5}, \frac{7}{10}, 1, \frac{13}{10}, \frac{8}{5}, \frac{19}{10}, \frac{11}{5}, \frac{5}{2}, \ldots$

Parttern 7: $\frac{3}{5}, \frac{6}{5}, \frac{12}{5}, \frac{24}{5}, \frac{48}{5}, \frac{96}{5}, \ldots$

Think/Pair/Share. For each of the sequences above, decide if it is an arithmetic sequence or not. Justify your answers.

Problem 120 (Fractions in-between)**.**

$$\frac{1}{4}, \quad \underline{\qquad}, \quad \underline{\qquad}, \quad \frac{1}{3}$$

1. Find two fractions between $\frac{1}{4}$ and $\frac{1}{3}$.
2. Are the resulting four fractions in an arithmetic sequence? Justify your answer.

Problem 121 (Fractions in-between)**.** Find two fractions between $\frac{1}{6}$ and $\frac{1}{5}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{1}{6}, \quad \underline{\qquad}, \quad \underline{\qquad}, \quad \frac{1}{5}$$

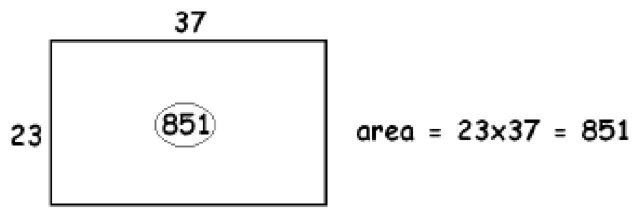
Problem 122 (Fractions in-between)**.** Find three fractions between $\frac{2}{5}$ and $\frac{5}{7}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{2}{5}, \quad \underline{\qquad}, \quad \underline{\qquad}, \quad \underline{\qquad}, \quad \frac{5}{7}$$

Think/Pair/Share (Make your own)*.* Make up two *fraction* sequences of your own, one that **is** an arithmetic sequence and one that **is not** an arithmetic sequence. Exchange your sequences with a partner, but don't tell your partner which is which. When you get your partner's sequences: decide which is an arithmetic sequence and which is not. Check if you and your partner agree.

4.5 Multiplying Fractions

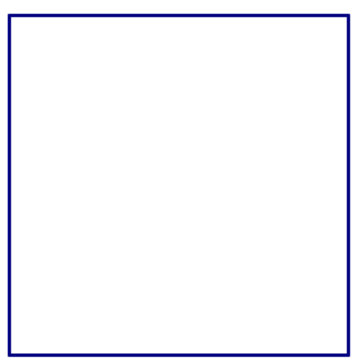
One of our models for multiplying whole numbers was an area model. For example, the product 23×37 is the area (number of 1×1 squares) of a 23-by-37 rectangle:



So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem.

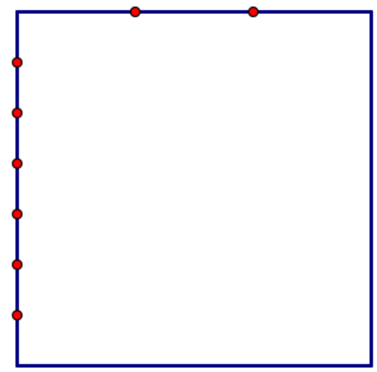
Example 4.5.1 ($\frac{4}{7} \times \frac{2}{3}$). Let's start with a segment of some length that we call 1 unit:

Now, build a square that has one unit on each side:

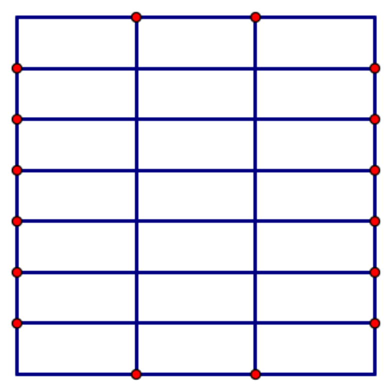


The area of the square, of course, is $1 \times 1 = 1$ square unit.

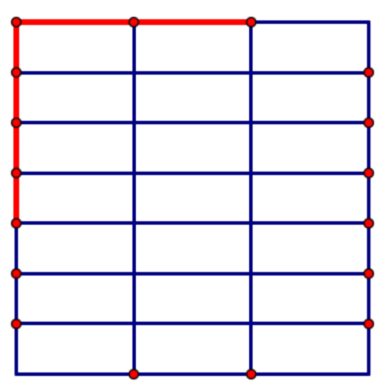
Now, let's divide the segment on top into three equal-sized pieces. (So each piece is $\frac{1}{3}$.) And we'll divide the segment on the side into seven equal-sized pieces. (So each piece is $\frac{1}{7}$.)



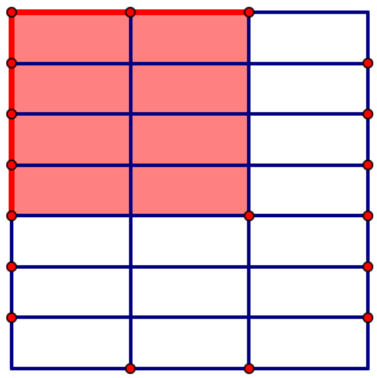
We can use those marks to divide the whole square into small, equal-sized rectangles. (Each rectangle has one side that measures $\frac{1}{3}$ and another side that measures $\frac{1}{7}$.)



We can now mark off four sevenths on one side and two thirds on the other side.



The result of the multiplication $\frac{4}{7} \times \frac{2}{3}$ should be the area of the rectangle with $\frac{4}{7}$ on one side and $\frac{2}{3}$ on the other. What is that area?



Remember, the whole square was one-unit. That one-unit square is divided into 21 equal-sized pieces, and our rectangle (the one with sides $\frac{4}{7}$ and $\frac{2}{3}$) contains eight of those rectangles. Since the shaded area is the answer to our multiplication problem we conclude that

$$\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}.$$

Think/Pair/Share.

1. Use this "unit square method" to compute each of the following products. Draw the picture to see the answer clearly.

$$\frac{3}{4} \times \frac{5}{6}, \qquad \frac{3}{8} \times \frac{5}{10}, \qquad \frac{5}{8} \times \frac{3}{7}.$$

2. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 21 small rectangles. Explain *why* 21 appears as the total number of equal-sized rectangles.

3. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 8 small shaded rectangles. Explain *why* 8 appears as the number of shaded rectangles

Problem 123 (Extend the Model)**.** How can you extend the area model for fractions greater than 1? Try to draw a picture for each of these:

$$\frac{3}{4}\cdot\frac{3}{2}, \qquad \frac{2}{5}\cdot\frac{4}{3}, \qquad \frac{3}{10}\cdot\frac{5}{4}, \qquad \frac{5}{2}\cdot\frac{7}{4}.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Compute the following products, simplifying each of the answers as much as possible. You do not need to draw pictures, but you may certainly choose to do so if it helps!

   $$\frac{5}{11}\times\frac{7}{12}, \qquad \frac{4}{7}\times\frac{4}{8}, \qquad \frac{1}{2}\times\frac{1}{3}, \qquad \frac{2}{1}\times\frac{3}{1}, \qquad \frac{1}{5}\times\frac{5}{1}.$$

2. Compute the following products. (Don't work too hard!)

   $$\frac{3}{4}\times\frac{1}{3}\times\frac{2}{5}, \qquad \frac{5}{5}\times\frac{7}{8}, \qquad \frac{88}{88}\times\frac{541}{788}, \qquad \frac{77876}{311}\times\frac{311}{77876}.$$

3. Try this one. Can you make use of the fraction rule $\frac{xa}{xb}=\frac{a}{b}$ to help you calculate? How?

   $$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\frac{7}{8}\times\frac{8}{9}\times\frac{9}{10}$$

You probably simplified your work in the exercises above by using a multiplication rule like the following.

♣♣♣ Fellow: [put the rule a box, centered?]

Multiplying Fractions:

$$\frac{a}{b} \cdot \frac{c}{d} \quad = \quad \frac{a \cdot c}{b \cdot d}.$$

Of course, you may then choose to simplify the final answer, but the answer is always *equivalent* to this one. Why? The area model can help us explain what's going on.

First, let's clearly write out how the area model says to multiply $\frac{a}{b} \cdot \frac{c}{d}$. We want to build a rectangle where one side has length $\frac{a}{b}$ and the other side has length $\frac{c}{d}$. We start with a square, one unit on each side.

- Divide the top segment into b equal-sized pieces. Shade a of those pieces. (This will be the side of the rectangle with length $\frac{a}{b}$.)

- Divide the left segment into d equal-sized pieces. Shade c of those pieces. (This will be the side of the rectangle with length $\frac{c}{d}$.)

- Divide the whole rectangle according to the tick marks on the sides, making equal-sized rectangles.

- Shade the rectangle bounded by the shaded segments.

♣♣♣ Fellow: [add an animation of this process in a couple of examples?]

If the answer is $\frac{a \cdot c}{b \cdot d}$, that means there are $b \cdot d$ total equal-sized pieces in the square, and $a \cdot c$ of them are shaded. We can see from the model why this is the case:

- The top segment was divided into b equal-sized pieces. So there are b columns in the rectangle.

- The side segment was divided into d equal-sized pieces. So there are d rows in the rectangle.

- A rectangle with b columns and d rows has $b \cdot d$ pieces. (This is exactly our area model for whole-number multiplication!)

Think/Pair/Share. Stick with the general multiplication rule

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

With a partner, write a clear explanation for why $a \cdot c$ of the small rectangles will be shaded.

4.5.1 Multiplying Fractions by Whole Numbers

Often, elementary students are taught to multiply fractions by whole numbers using the fraction rule.

Example 4.5.2 ($2 \cdot \frac{3}{7}$, Multiply Fractions). For example, to multiply $2 \cdot \frac{3}{7}$, we think of "2" as $\frac{2}{1}$, and compute this way:

$$2 \cdot \frac{3}{7} \quad = \quad \frac{2}{1} \cdot \frac{3}{7} \quad = \quad \frac{2 \cdot 3}{1 \cdot 7} \quad = \quad \frac{6}{7}.$$

We can also think in terms of our original "Pies Per Boy Model" to answer questions like this.

Example 4.5.3 ($2 \cdot \frac{3}{7}$, Pies Per Boy). We know that $\frac{3}{7}$ means the amount of pie each boy gets when 7 boys evenly share 3 pies.

If we compute $2 \cdot \frac{3}{7}$, that means we double the amount of pie each boy gets. We can do this by doubling the number of pies. So the answer is the same as $\frac{6}{7}$: the amount of pie each boy gets when 7 boys evenly share 6 pies.

Finally, we can think in terms of units and unitizing.

Example 4.5.4 ($2 \cdot \frac{3}{7}$, Units). The fraction $\frac{3}{7}$ means that I have 7 equal pieces (of *something*), and I take 3 of them.

So $2 \cdot \frac{3}{7}$ means do that twice. If I take 3 pieces and then 3 pieces again, I get a total of 6 pieces. There are still 7 total pieces, so the answer is $\frac{6}{7}$.

Think/Pair/Share.

1. Use all three methods to explain how to find each product:

$$3 \cdot \frac{2}{5}, \qquad\qquad 4 \cdot \frac{3}{8}, \qquad\qquad 6 \cdot \frac{1}{5}.$$

2. Compare these different ways of thinking about fraction multiplication. Are any of them more natural to you? Does one make more sense than the others? Do the particular numbers in the problem affect your answer? Does your partner agree?

Let's think so more about the expression

$$4 \cdot \frac{3}{8}.$$

Using the first method (multiplying fractions), we compute:

$$4 \cdot \frac{3}{8} \quad = \quad \frac{4}{1} \cdot \frac{3}{8} \quad = \quad \frac{12}{8}.$$

And we might notice that this simplifies:

$$\frac{12}{8} \quad = \quad \frac{3 \cdot 4}{2 \cdot 4} \quad = \quad \frac{3}{2}.$$

Here's another example:

$$10 \cdot \frac{2}{15} \quad = \quad \frac{10}{1} \cdot \frac{2}{15} \quad = \quad \frac{10 \cdot 2}{15}.$$

Rather than multiply out the numerator, let's break everything down as far as we can into factors:

$$\frac{10 \cdot 2}{15} \quad = \quad \frac{2 \cdot 5 \cdot 2}{3 \cdot 5} \quad = \quad \frac{2 \cdot 2}{3} \quad = \quad \frac{4}{3}.$$

Here's one more example:

$$8 \cdot \frac{212}{16} \quad = \quad \frac{8 \cdot 212}{16}.$$

We can avoid some work (mathematicians *love* to avoid work and make things easier on themselves!) if we notice that $16 = 8 \cdot 2$:

$$8 \cdot \frac{212}{16} \quad = \quad \frac{8 \cdot 212}{8 \cdot 2} \quad = \quad \frac{212}{2} \quad = \quad 106.$$

On Your Own

Try these exercises on your own or with a partner.

1. Compute each of the following and write your answer in simplified form. Avoid doing extra work if you can!

$$17 \cdot \frac{2}{3}, \qquad 10 \cdot \frac{1}{5}, \qquad \frac{3}{4} \cdot 4, \qquad 11 \cdot \frac{36}{33}, \qquad \frac{13}{12} \cdot 24.$$

2. Compute each of the following and write your answer in simplified form. Look for shortcuts!

$$\frac{3}{7}\cdot\frac{7}{5}, \qquad \frac{133}{112}\cdot 224, \qquad \frac{39}{35}\cdot\frac{14}{13}, \qquad \frac{5}{13}\cdot\frac{4}{7}\cdot\frac{13}{2}\cdot\frac{7}{10}.$$

Think/Pair/Share.

1. Compute the following:

$$6\cdot\frac{5}{6}, \qquad \frac{7}{18}\cdot 18.$$

2. What can you say about these products? Carefully justify your answer using (at least) one of the models for multiplication above.

$$b\cdot\frac{a}{b}, \qquad \frac{c}{d}\cdot d.$$

3. Keo was asked to compute

$$\frac{18}{7}\cdot\frac{70}{36}.$$

Within three seconds, he shouted "The answer is 5!" Is he right? How was he able to compute it so quickly?

Roy says that the fraction rule

$$\frac{xa}{xb}=\frac{a}{b}$$

is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 doesn't change a number:

$$1 \cdot 4 = 4$$
$$1 \cdot 2014 = 2014$$
$$1 \cdot \frac{5}{7} = \frac{5}{7}.$$

So in general

$$1 \cdot \frac{a}{b} = \frac{a}{b}.$$

Now, $\frac{2}{2} = 1$*, so that means that*

$$\frac{2}{2} \cdot \frac{a}{b} = \frac{a}{b}, \text{ which means}$$
$$\frac{2a}{2b} = \frac{a}{b}.$$

By the same reasoning, $\frac{3}{3} = 1$*, so that means that*

$$\frac{3}{3} \cdot \frac{a}{b} = \frac{a}{b}, \text{ which means}$$
$$\frac{3a}{3b} = \frac{a}{b}.$$

Think/Pair/Share. What do you think about Roy's reasoning? Does it make sense? How would Roy explain the general rule for positive whole numbers x:

$$\frac{xa}{xb} = \frac{a}{b}?$$

4.5.2 Fractions of fractions of fractions of fractions of...

Think/Pair/Share. How are these two problems different? Draw a picture of each.

1. Pam had $\frac{2}{3}$ of a cake in her refrigerator, and she ate half of it. How much total cake did she eat?

2. On Monday, Pam ate $\frac{2}{3}$ of a cake. On Tuesday, Pam at $\frac{1}{2}$ of a cake. How much total cake did she eat?

When a problem includes a phrase like "$\frac{2}{3}$ of ...," students are taught to treat "of" as multiplication, and to use that to solve the problem. As the above problems show, in some cases this makes sense, and in some cases it does not. It's important to read carefully and understand what a problem is asking, not memorize rules about "translating" word problems.

If I have 12 circles and I want "$\frac{2}{3}$ of the circles," I can take two out of every three cirlces. ♣♣♣ Fellow: [add a picture: 12 circles, divided into three groups, with two shaded from each group]

I can also take $\frac{2}{3}$ from each individual circle. ♣♣♣ Fellow: [add a picture of 12 circles, each divided into thirds with 2/3 shaded]

In both cases, I can compute the answer as $\frac{2}{3} \times 12$ circles, but the reasoning in each case is a little different.

In the first case, we are really thinking of "$\frac{2}{3}$ of 12" as a sequence of operations:

- Divide my 12 circles groups of three circles each.
- Shade 2 circles in every group.

So I have computed this way:

$$(12 \div 3) \cdot 2 \quad = \quad \frac{12}{3} \cdot 2 \quad = \quad \frac{12 \cdot 2}{3} \quad = \quad 12 \cdot \frac{2}{3}.$$

In the second case, we are really think of $\frac{2}{3}$ of a circle, repeated 12 times, which is also

$$\frac{2}{3} \cdot 12.$$

If we change the numbers, sometimes one of the interpretations is more natural than the other. For example, how can we understand "$\frac{3}{5}$ of 12"? We can interpret this as "take 3 of every 5 circles," but this doesn't really make sense because we can't divide 12 circles into groups of 5 circles each. It's easier to take $\frac{3}{5}$ of each circle.

♣♣♣ Fellow: [add picture?]

Think/Pair/Share.

1. Draw $\frac{5}{8}$ of 4 circles in two different ways. What is $\frac{5}{8}$ of 4?

2. Draw $\frac{3}{4}$ of 2 candy bars in two different ways. What is $\frac{3}{4}$ of 2?

3. Draw a rectangle and shade $\frac{2}{3}$ of $\frac{3}{4}$ of the rectangle. What is $\frac{2}{3} \cdot \frac{3}{4}$?

4.6 Dividing Fractions: Meaning

We had several ways to think about division of whole numbers:

- **Quotatitive model:** Make groups of a given size. For example, for $18 \div 3$, we start with 18 dots (or candy bars or molecules), and we make groups of 3 dots (or 3 whatevers). We ask: how many groups can we make?

- **Partitive model:** Make a given number of groups. For $18 \div 3$, we say start with 18 dots (or people or pencils), and we make three equal-sized groups. We ask: how many objects are in each group?

- **Missing factor model:** Solve a multiplication problem instead. For $18 \div 3$, we rewrite the problem as $3 \cdot \_\_\_\_ = 18$.

We can still think about all of these models when we divide fractions, but actually doing the calculation can be tricky!

Think/Pair/Share. For each problem below, draw a picture of the situation, and label the problem as partitive or quotative. Explain your thinking. Then try to solve each of the problems. Find as many different ways as you can to justify your solutions.

1. It took Mary four bucketfuls of water to fill up her three gallon fish tank. How much water does her bucket hold?

2. You have $\frac{2}{3}$ of a gallon of water in a bucket, and the bucket is $\frac{7}{8}$ full. How many gallons would it take to fill up the whole bucket?

3. $10\frac{1}{2}$ gallons of water fills up $2\frac{1}{3}$ buckets. How many gallons are in one bucket?

4. Mr. Brown has a length of rope that measures $10\frac{1}{2}$ yards long. Each boy in his scout troop needs a piece $2\frac{1}{3}$ yards long. How many pieces

of the required length can he cut?

Most people find problem (2) above quite challenging, and have a hard time both drawing a picture and being certain they have the right answer. (Even if you didn't find it so difficult, certainly you can imagine that some of your future students would be stumped by such a problem!)

If a problem is giving us trouble, what are some things we can do? Solve a simpler problem! Let's rephrase problem (2) in several new ways:

(2a) You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{1}{2}$ of your bucket. How many gallons total would it take to fill up the whole bucket?

(2b) You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{1}{3}$ of your bucket. How many gallons total would it take to fill up the whole bucket?

(2c) You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{1}{4}$ of your bucket. How many gallons total would it take to fill up the whole bucket?

(2d) You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{1}{5}$ of your bucket. How many gallons total would it take to fill up the whole bucket?

(2e) You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{1}{8}$ of your bucket. How many gallons total would it take to fill up the whole bucket?

Think/Pair/Share. Each of the problems above is significantly easier than the original problem (2). Discuss with a partner why these questions are easier. For each one, draw a picture and find the solution. Most importantly, find a general method to answer this question:

> *If $\frac{2}{3}$ of a gallon of water fills my bucket to the $\frac{1}{n}$ mark, how much water does my bucket hold?*

So, back to original problem — what's complicated in that case? The water doesn't fill your bucket to the $\frac{1}{8}$ mark. It fills your bucket to the $\frac{7}{8}$ mark. Here are some helpful questions to think about the next step of the problem:

(2a') You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{3}{4}$ of your bucket. How many gallons would it take to fill up $\frac{1}{4}$ of the bucket? How many total to fill up the whole bucket?

(2b') You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{3}{5}$ of your bucket. How many gallons would it take to fill up $\frac{1}{5}$ of the bucket? How many total to fill up the whole bucket?

(2c') You have $\frac{2}{3}$ of a gallon of water in a bucket, which fills up $\frac{5}{8}$ of your bucket. How many gallons would it take to fill up $\frac{1}{8}$ of the bucket? How many total to fill up the whole bucket?

Think/Pair/Share. Work on the questions above with a partner. Your goal is to be able to answer this question:

> *If $\frac{2}{3}$ gallons of water fills my bucket to the $\frac{a}{b}$ mark, how can I find the total number of gallons that fills my bucket to the $\frac{1}{b}$ mark?*

If you can answer that, you should be able to apply it to answer the original version of problem (2) above.

4.7 Dividing Fractions: Computations

All of the following questions have the same answer! (Why?)

- How many groups of 3 are there in 6?
- How many groups of 3 tens are there in 6 tens?

- How many groups of 3 fives are there in 6 fives?
- How many groups of 3 tenths are there in 6 tenths?
- How many groups of 3 fourths are there in 6 fourths?
- How many groups of 3 @s are there in 6 @s?
- How many groups of 3 anythings are there in 6 anythings (as long as both "anythings" refer to the same unit)?

Think/Pair/Share. With a partner, draw some pictures to illustrate each of the questions above. Do you believe that they all have the same answer? Use a picture or reasoning to solve each of the following fraction division problems:

$$\frac{6}{4} \div \frac{3}{4}, \qquad \frac{6}{10} \div \frac{3}{10}, \qquad \frac{8}{9} \div \frac{4}{9}, \qquad \frac{15}{33} \div \frac{1}{33}, \qquad \frac{10}{9} \div \frac{5}{9}.$$

4.7.1 Common denominator method

This line of reasoning leads to our first fraction division method. If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Think/Pair/Share. What if the fractions *don't* have a common denominator? Is the method useless, or can you find a way to make it work? Can you solve these problems?

$$\frac{3}{5} \div \frac{3}{4}, \qquad \frac{3}{4} \div \frac{8}{7}, \qquad \frac{2}{3} \div \frac{1}{2}, \qquad \frac{5}{8} \div \frac{1}{4}.$$

4.7.2 Missing factor approach

We know that we can always turn a division problem into a "missing factor" multiplication problem. Can that help us compute fraction division? Sometimes!

Think/Pair/Share. For each division problem, rewrite it as a missing factor multiplication question. Then answer that question using what you know about multiplying fractions.

$$\frac{9}{10} \div \frac{3}{5}, \qquad \frac{7}{8} \div \frac{1}{4}, \qquad \frac{6}{7} \div \frac{3}{7}, \qquad \frac{10}{9} \div \frac{2}{3}, \qquad \frac{25}{12} \div \frac{5}{6}.$$

A nasty problem:

> $7\frac{2}{3}$ *pies are shared equally by* $5\frac{3}{4}$ *boys. How much pie does each boy get?*

Technically, we could just write down the answer as

$$\frac{7\frac{2}{3}}{5\frac{3}{4}}$$

and be done! (The answer to this problem is, of course, equivalent to this fraction, so why not?)

Is there a way to make this look friendlier? Recall the key fraction rule:

$$\frac{xa}{xb} = \frac{a}{b}.$$

What might happen if we multiply the numerator and denominator of our answer each by a convenient choice of number? Right now we have the expression:

$$\frac{7\frac{2}{3}}{5\frac{3}{4}} \quad = \quad \frac{\left(7 + \frac{2}{3}\right)}{\left(5 + \frac{3}{4}\right)}.$$

Let's multiply by 3. (Why three?)

$$\frac{\left(7 + \frac{2}{3}\right) \cdot 3}{\left(5 + \frac{3}{4}\right) \cdot 3} \quad = \quad \frac{(21 + 2)}{\left(15 + \frac{9}{4}\right)}.$$

Important Note: We're using some key facts about arithmetic here! First, we used the distributive law for multiplication over addition:

$$(a + b) \cdot c \quad = \quad a \cdot c + b \cdot c. \qquad \text{(Where have we used this fact?)}$$

Second, we used what we know about multiplying fractions by whole numbers. In particular, we used the fact that

$$\frac{a}{b} \cdot b \quad = \quad a. \qquad \text{(Where did we use that fact?)}$$

Let's now multiply numerator and denominator each by 4. (Why four?)

$$\frac{(21+2)\cdot 4}{\left(15+\frac{9}{4}\right)\cdot 4} \quad = \quad \frac{84+8}{60+9} \quad = \quad \frac{92}{69}.$$

We now see that the answer is $\frac{92}{69}$. That means that sharing $7\frac{2}{3}$ pies among $5\frac{3}{4}$ boys is the same as sharing 92 pies among 69 boys. (That is, in both situations, the individual boys get exactly the same amount of pie.)

Example 4.7.1. Let's forget the context now and just focus on the calculations so that we can see what's going on more clearly. Try this one:

$$\frac{3\frac{1}{2}}{1\frac{1}{2}}.$$

Multiplying the numerator and denominator each by 2 should be enough to simplify the expression. (Why?) Let's try it:

$$\frac{3\frac{1}{2}}{1\frac{1}{2}} \quad = \quad \frac{3+\frac{1}{2}}{1+\frac{1}{2}} \quad = \quad \frac{\left(3+\frac{1}{2}\right)\cdot 2}{\left(1+\frac{1}{2}\right)\cdot 2} \quad = \quad \frac{6+1}{2+1} \quad = \quad \frac{7}{3}.$$

On Your Own

Each of the following is a perfectly nice fraction, but it could be written in a simpler form. So do that! Write each of them in a simpler form following the examples above.

$$\frac{4\frac{2}{3}}{5\frac{1}{3}}, \qquad \frac{2\frac{1}{5}}{2\frac{1}{4}}, \qquad \frac{1\frac{4}{7}}{2\frac{3}{10}}, \qquad \frac{\frac{3}{7}}{\frac{4}{5}}.$$

Think/Pair/Share.

1. Jessica calculated the second exercise above this way:

$$\frac{2\frac{1}{5}}{2\frac{1}{4}} \quad = \quad \frac{\cancel{2}\frac{1}{5}}{\cancel{2}\frac{1}{4}} \quad = \quad \frac{\frac{1}{5}}{\frac{1}{4}} \quad = \quad \frac{\frac{1}{5}\cdot 4}{\frac{1}{4}\cdot 4} \quad = \quad \frac{\frac{4}{5}}{1} \quad = \quad \frac{4}{5}.$$

Is her solution correct, or is she misunderstanding something? Carefully explain what's going on with her solution, and what you would do as Jessica's teacher.

2. Isaac calculated the last exercise above this way:

$$\frac{\frac{3}{7}}{\frac{4}{5}} \quad = \quad \frac{\frac{3}{7}\cdot 7}{\frac{4}{5}\cdot 5} \quad = \quad \frac{3}{4}.$$

   Is his solution correct, or is he misunderstanding something? Carefully explain what's going on with his solution, and what you would do as Isaac's teacher.

4.7.3 Simplify an ugly fraction!

Perhaps without realizing it, you have just found another method to divide fractions.

Example 4.7.2 ($\frac{3}{5} \div \frac{4}{7}$). Suppose we are asked about sharing $\frac{3}{5}$ of a pie among $\frac{4}{7}$ of a boy (whatever that would mean!). That is, we are asked to compute:

$$\frac{\frac{3}{5}}{\frac{4}{7}}.$$

Let's multiply numerator and denominator each by 5:

$$\frac{\left(\frac{3}{5}\right)\cdot 5}{\left(\frac{4}{7}\right)\cdot 5} \quad = \quad \frac{3}{\frac{20}{7}}.$$

Let's now multiply top and bottom each by 7:

$$\frac{(3)\cdot 7}{\left(\frac{20}{7}\right)\cdot 7} \quad = \quad \frac{21}{20}.$$

Done! So $\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}$.

Example 4.7.3 ($\frac{5}{9} \div \frac{8}{11}$). Let's do another. Consider $\frac{5}{9} \div \frac{8}{11}$:

$$\frac{\frac{5}{9}}{\frac{8}{11}}$$

Let's multiply top and bottom each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{\left(\frac{5}{9}\right)\cdot 9\cdot 11}{\left(\frac{8}{11}\right)\cdot 9\cdot 11} = \frac{5\cdot 11}{8\cdot 9}.$$

(Do you see what happened here?)

So we have

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5\cdot 11}{8\cdot 9} = \frac{55}{72}.$$

On Your Own

Compute each of the following, using the simplification technique.

$$\frac{1}{2}\div\frac{1}{3}, \qquad \frac{4}{5}\div\frac{3}{7}, \qquad \frac{2}{3}\div\frac{1}{5}, \qquad \frac{45}{59}\div\frac{902}{902}, \qquad \frac{10}{13}\div\frac{2}{13}.$$

4.7.4 Invert and multiply

Consider the problem $\frac{5}{12}\div\frac{7}{11}$. Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12}\cdot 12\cdot 11}{\frac{7}{11}\cdot 12\cdot 11} = \frac{5\cdot 11}{7\cdot 12} = \frac{5}{12}\cdot\frac{11}{7}.$$

She stopped before completing her final step and exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down."

On Your Own

First check each step of Janine's work here and make sure that she is correct in what she did up to this point. Then answer these questions:

1. Do you understand what Janine is saying? Explain it very clearly.

2. Work out

$$\frac{\frac{3}{7}}{\frac{4}{13}}$$

using the simplification method. Is the answer the same as $\frac{3}{7}\cdot\frac{13}{4}$?

3. Work out
$$\frac{\frac{2}{5}}{\frac{3}{10}}$$
using the simplification method. Is the answer the same as $\frac{2}{5} \cdot \frac{10}{3}$?

4. Work out
$$\frac{\frac{a}{b}}{\frac{c}{d}}.$$
using the simplification method. Is the answer the same as $\frac{a}{b} \cdot \frac{d}{c}$?

5. Is Janine right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think? (Don't just think about examples. This is a question if something is *always* true.)

We now have several methods for solving problems that require dividing fractions:

♣♣♣ Fellow: [Put this in a box?]

Dividing fractions:

1. Find a common denominator and divide the numerators.
2. Rewrite the division as a missing factor multiplication problem, and solve that problem.
3. Simplify an ugly fraction.
4. Invert the second fraction (the dividend) and then multiply.

Think/Pair/Share. Discuss your opinions about our four methods for solving fraction division problems with a partner:

- Which method for division of fractions is the easiest to *understand why it works*?
- Which method for division of fractions is the easiest to *use in computations*?
- What are the benefits and drawbacks of each method? (Think both as a teacher and as someone solving math problems here.)

4.8 Fraction Sense

Think/Pair/Share. For each of the following problems, suppose a and b are both fractions that are between 0 and 1, and suppose a is bigger than b. Decide which symbol should go in the $\square$ for each equation: $>$, $<$, or $=$. Justify your answer, and keep in mind that more than one symbol may be possible!

1. Addition:

   $a+b \quad \square \quad a, \qquad a+b \quad \square \quad b, \qquad a+b \quad \square \quad 0 \qquad a+b \quad \square \quad 1.$

2. Subtraction:

   $a-b \quad \square \quad a, \qquad a-b \quad \square \quad b, \qquad a-b \quad \square \quad 0 \qquad a-b \quad \square \quad 1.$

3. Multiplication:

   $a \cdot b \quad \square \quad a, \qquad a \cdot b \quad \square \quad b, \qquad a \cdot b \quad \square \quad 0 \qquad a \cdot b \quad \square \quad 1.$

4. Division:

   $a \div b \quad \square \quad a, \qquad a \div b \quad \square \quad b, \qquad a \div b \quad \square \quad 0 \qquad a \div b \quad \square \quad 1.$

4.8.1 Multiplying and Dividing

Elementary school students are often taught mental shortcuts like "multiplication makes things bigger." But is that necessarily true? You have to be careful as a teacher to make ideas simple for students to understand, but not so simple that you say things that are wrong!

Let's try some examples.

Example 4.8.1 (Multiplying by $\frac{5}{4}$). Let's try it with 100:

$$\frac{5}{4} \cdot 100 \quad = \quad \frac{500}{4} \quad = \quad 125.$$

Yep, that's bigger than 100.

But of course this is only one example. How can we be sure that multiplying *any* (positive) number by $\frac{5}{4}$ gives a result that's bigger than that number? That is, how can we be sure that

$$\frac{5}{4} \cdot X \quad > \quad X \qquad \text{for every choice of } X?$$

This is a *universal* statement, so one example is not enough to be sure it's true. We need an explanation! And here it is.

We can rewrite $\frac{5}{4}$ as $1 + \frac{1}{4}$. So then

$$\frac{5}{4} \cdot X \quad = \quad \left(1 + \frac{1}{4}\right) \cdot X \quad = \quad X + \frac{1}{4} \cdot X \quad = \quad X + \text{ more }.$$

So the answer is bigger than X.

Think/Pair/Share. Go through each step in the series of calculations above, and explain what is going on. Where is the distributive law used? Where do we need the fact that X is a positive number? Then:

1. Write a careful argument that multiplying a (positive) number by $\frac{8}{5}$ gives a result that is larger than the original number.

2. Write a careful argument that multiplying a (positive) number by $\frac{20}{9}$ gives a result that is larger than the original number.

Does this rule hold for other fractions as well? Does multiplication always result in a larger number than the one being multiplied? Let's try another example.

Example 4.8.2 (Multiplying by $\frac{4}{5}$). Again, we'll use 100 as our first test case:

$$\frac{4}{5} \cdot 100 = \frac{400}{5} = 80.$$

So in this case, the result is *smaller* than 100!

This counterexample shows that the following universal statement is **definitely false**: *Multiplying a positive number X by $\frac{4}{5}$ gives a result that is bigger than X.*

We might ask the following:

Is it always true that $\frac{4}{5} \cdot X < X$ for a positive number X?

Notice, this is not the same question! We know that the answer is not *always* bigger than X. But we don't know it it is *always* smaller. It could be *sometimes* bigger and *sometimes* smaller. How can we be sure?

You might have already guessed what to do. We thought about $\frac{5}{4}$ as "one plus a little bit." In a very similar way, we can think about $\frac{4}{5}$ as "a little bit less than one," and use that to explain why, indeed, the result must always be smaller. Here we go:

Notice that $\frac{4}{5} = 1 - \frac{1}{5}$. So we can write

$$\frac{4}{5} \cdot X \quad = \quad \left(1 - \frac{1}{5}\right) \cdot X \quad = \quad X - \frac{1}{5} \cdot X \quad = \quad X - \text{some smallish amount},$$

and the result will be smaller than X.

Think/Pair/Share. Go through each step in the series of calculations above, and explain what is going on. Where is the distributive law used? Where do we need the fact that X is a positive number? Then:

1. Write a careful argument that multiplying a (positive) number by $\frac{7}{8}$ gives a result that is smaller than the original number.

2. Write a careful argument that multiplying a (positive) number by $\frac{5}{9}$ gives a result that is smaller than the original number.

It may seem silly to write such careful arguments for things you already know to be true. Of course multiplying by a number less than one makes your answer smaller!

Well, let's make two comments:

- The fact that this is obvious to you (if it is!) comes from your years of experience with numbers. When students first learn about fractions, it is "obvious" to them that multiplying makes things bigger. In their experience, it has always done so! Our intuition is based on our experiences, and cannot always be trusted! That's why explanation and justification play such a crucial role in mathematics.

- Though many people think the results are obvious when dealing with multiplication, they can get completely turned upside down (so to speak) in dealing with division. And it always helps to work through the relatively simple case first, before tackling the more difficult one.

Claim: If we divide a positive number by some fraction less than one, the result is *bigger* than the original number.

Before trying to justify a claim, we should always check a few examples to see if we even believe that it's true. Testing these ideas out on the number 100 has worked well so far. Let's see what happens when we compute $100 \div \frac{4}{5}$.

$$\frac{100}{\frac{4}{5}} = \frac{(100) \cdot 5}{\left(\frac{4}{5}\right) \cdot 5} = \frac{500}{4} = 125.$$

Indeed, the answer is larger, just as claimed above.

So how can we write a general argument? Well, just replace the 100 by X:

$$\frac{X}{\frac{4}{5}} = \frac{(X) \cdot 5}{\left(\frac{4}{5}\right) \cdot 5} = \frac{5 \cdot X}{4} = \frac{5}{4} \cdot X.$$

And we know from our earlier work that $\frac{5}{4} \cdot X$ is bigger than X whenever X is a positive number.

Think/Pair/Share. Go through each step in the series of calculations above, and explain what is going on. Then:

1. Write a careful argument that dividing a (positive) number by $\frac{7}{9}$ gives a result that is larger than the original number.

2. Write a careful argument that multiplying a (positive) number by $\frac{8}{5}$ gives a result that is smaller than the original number.

4.8.2 Fractions involving zero

Think/Pair/Share. Mr. Kinsella is reviewing equivalent fractions with his class. He asks students for examples of fractions that are equivalent to 1.

One student suggests $\frac{0}{0}$. What is **most** important for him to consider in deciding how to respond? (Choose *one* answer, and be prepared to explain why your choice is the best one.)

(a) Any number divided by itself equals 1. Even though you normally cannot divide by 0, you can divide 0 by 0. So $\frac{0}{0} = 1$.

(b) $\frac{0}{0} = 0$.

(c) $\frac{0}{0}$ is undefined because there is no single number that when multiplied by 0 is 0.

(d) If you multiply the numerator and denominator by the same number, $\frac{0}{0}$ remains the same.

Think/Pair/Share. Some students are talking about the fraction $\frac{0}{11}$.

(a) Cyril says that $\frac{0}{11} = 2$. Carefully explain why he is incorrect.

(b) Ethel says that $\frac{0}{11} = 17$. Carefully explain why she is incorrect.

(c) Wonhi says that $\frac{0}{11} = 887231243$. Carefully explain why he is incorrect.

(d) Duane says that there is no answer to $\frac{0}{11}$. Carefully explain why he is incorrect.

(e) What *is* the correct value for $\frac{0}{11}$?

Sharing zero pies among eleven boys gives zero pie per boy:

$$\frac{0}{11} = 0.$$

The same reasoning would lead us to say:

$$\frac{0}{b} = 0 \quad \text{for any positive number } b.$$

The "Pies Per Boy Model" offers one explanation: If there are no pies for us to share, no one gets any pie. It doesn't matter how many boys there are. No pie is no pie is no pie.

We can also justify this claim by thinking about a missing factor multiplication problem:

$$\frac{0}{b} \quad \text{is asking us to fill in the blank:} \qquad \_\_\_\_ \cdot b = 0.$$

The only way to fill that in and make a true statement is with a 0, so $\frac{0}{b} = 0$.

What if things are flipped the other way round? Does $\frac{a}{0}$ make sense?

Think/Pair/Share. The same students are talking about the fraction $\frac{5}{0}$.

(a) Cyril says that $\frac{5}{0} = 2$. Use a missing factor multiplication problem to explain why he is incorrect.

(b) Ethel says that $\frac{5}{0} = 17$. Use a missing factor multiplication problem to explain why she is incorrect.

(c) Wonhi says that $\frac{5}{0} = 887231243$. Use a missing factor multiplication problem to explain why he is incorrect.

(d) Duane says that there is no answer to $\frac{5}{0}$. Use a missing factor multiplication problem to explain why he is *correct*.

Students often learn in school that "dividing by 0 is undefined." But they learn this as a *rule*, rather than thinking about why it makes sense or how it connects to other ideas in mathematics. In this case, the most natural connection is to a multiplication fact:

$$\text{any number} \cdot 0 = 0.$$

That says we can never find solutions to problems like

$$\_\_ \cdot 0 = 5, \qquad \_\_ \cdot 0 = 17, \qquad \_\_ \cdot 0 = 1.$$

Using the connection between fractions and division, and the connection between division and multiplication, that means there is no number $\frac{5}{0}$. There is no number $\frac{17}{0}$. And there is no number $\frac{1}{0}$. They are all "undefined" because they are not equal to any number at all.

Can we give meaning to $\frac{0}{0}$ at least? After all, a zero does appear on both sides of that equation!

- Cyril says that $\frac{0}{0} = 2$ since $0 \cdot 2 = 0$.
- Ethel says that $\frac{0}{0} = 17$ since $0 \cdot 17 = 0$.
- Wonhi says that $\frac{5}{0} = 887231243$ since $0 \cdot 887231243 = 0$.

Who's right in this case? Can they all be correct?

Cyril says that $\frac{0}{0} = 2$, and he believes he is correct because it passes the check: $2 \cdot 0 = 0$. But $\frac{0}{0} = 17$ also passes the check, and so does $\frac{0}{0} = 887231243$. In fact, I can choose any number for x, and $\frac{0}{0} = x$ will pass the check!

The trouble with the expression $\frac{0}{a}$ (with a not zero) is that there are no meaningful values to assign to it. The trouble with $\frac{0}{0}$ is different: There are too many possible values to give it!

In general, most people would say that dividing by zero is simply too problematic to be done! They say it is best to avoid doing so and never will allow zero as the denominator of a fraction. (But all is fine with 0 as a numerator.)

4.9 Problem Bank

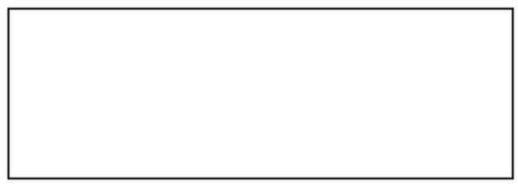
Problem 124 (Who gets more pie?)**.** Joe is with a group of five boys who share four pies. Jeff is with a group of seven boys who share four pies. Chris is in a group of seven boys who share six pies.

(a) Who gets more pie, Joe or Jeff? Justify your answer!

(b) Who best more pie, Joe or Chris? Justify your answer!

(c) Who gets more pie, Jeff or Chris? Justify your answer!

Problem 125 (Leftover Cake)**.** Yesterday was Zoë's birthday, and she had a big rectangular cake. Today, $\frac{2}{5}$ of the cake is left. It is shown here. Draw a picture of the whole cake and explain your work.



Problem 126 (Ordering fractions)**.** Use benchmarks and intuitive methods to arrange the fractions below in ascending order. Explain how you decided. (The point of this problem is to think more and compute less!):

$$\frac{2}{5}, \qquad \frac{1}{3}, \qquad \frac{5}{8}, \qquad \frac{1}{4}, \qquad \frac{2}{3}, \qquad \frac{3}{4}, \qquad \frac{4}{7}.$$

Problem 127. Which of these fractions has the larger value? **Justify** your choice.

$$\frac{10001}{10002} \text{ or } \frac{10000001}{10000002}$$

Problem 128 (Quick!)**.** Solve each division problem. Look for a shortcut, and explain your work.

$$\frac{251 + 251 + 251 + 251}{4}$$

$$\frac{377 + 377 + 377 + 377 + 377}{5}$$

$$\frac{123123 + 123123 + 123123 + 123123 + 123123 + 123123}{3}$$

Problem 129 (Cancellation)**.** Yoko says

$$\frac{16}{64} = \frac{1}{4}$$

because she cancels the sixes:

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}.$$

But note:

$$\frac{16}{64} = \frac{1 \cdot 16}{4 \cdot 16} = \frac{1 \cdot \cancel{16}}{4 \cdot \cancel{16}} = \frac{1}{4}.$$

So is Yoko right? Does her cancelation rule always work? If it doesn't always work, can you find *any other* example where it works? Can you find *every* example where it works?

Problem 130. Jimmy says that a fraction doesn't change in value if you add the same amount to the numerator and the denominator. Is he right? If you were Jimmy's teacher, how would you respond?

Problem 131. Shelly says that if $ab < cd$ then $\frac{a}{b} < \frac{c}{d}$. Is Shelly's claim always true, sometimes true, or never true? If you were Shelly's teacher, what would you say to her?

Problem 132. Jill, her brother, and another partner own a pizza restaurant. If Jill owns $\frac{1}{3}$ of the restaurant and her brother owns $\frac{1}{4}$ of the restaurant, what fraction does the third partner own?

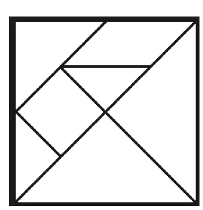
Problem 133. John spent a quarter of his life as a boy growing up, one-sixth of his life in college, and one-half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

Problem 134. Nana was planning to make a red, white, and blue quilt. One-third was to be red and two-fifths was to be white. If the area of the quilt was to be 30 square feet, how many square feet would be blue?

♣♣♣ Fellow: [add a picture of a Hawaiian quilt?]

Problem 135. Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Problem 136 (Tangrams)**.** Tangrams are a seven-piece puzzle, and the seven pieces can be assembled into a big square.



(a) If the large square shown above is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.

(b) The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.

(c) The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.

(d) The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.

(e) The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.

Problem 137. Mikiko said her family made two square pizzas at home. One of the pizzas was 8 inches on each side, and the other was 12 inches on each side. Mikiko ate $\frac{1}{4}$ of the small pizza and $\frac{1}{12}$ of the large pizza. So she said that she ate

$$\frac{1}{4}+\frac{1}{12} \quad = \quad \frac{3}{12}+\frac{1}{12} \quad = \quad \frac{4}{12} \quad = \quad \frac{1}{3}$$

of the pizza. Do you agree with Mikiko's calculation? Did she eat $\frac{1}{3}$ of her family's pizza? Carefully justify your answer.

Problem 138 (Harmonic triangle)**.** Look at the triangle of numbers. There are lots of patterns here! Find as many as you can. In particular, try to answer these questions:

(a) What pattern describes the first number in each row?

(b) How is each fraction related to the two fractions below it?

(c) Can you write down the next two rows of the triangle?

$$
\begin{array}{ccccccccc}
 & & & & \frac{1}{1} & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}
$$

Problem 139 (Let them eat cake!)**.** Marie made a sheet cake at home, but she saved some to bring to work and share with her co-workers the next day. Answer these questions about Marie's cake. (Draw a picture!)

(a) Suppose Marie saved $\frac{1}{2}$ of the cake for her coworkers and the co-workers ate $\frac{3}{4}$ of this. What fraction of the entire cake did they eat?

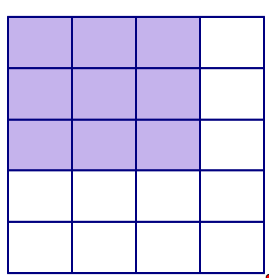
(b) What if Marie saved $\frac{1}{6}$ instead, and they ate $\frac{2}{3}$ of this?

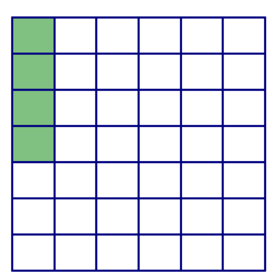
(c) What if she saved $\frac{5}{7}$ of the cake and they ate $\frac{1}{2}$ of this?

Problem 140 (Door prize)**.** An elementary school held a "Family Math Night" event, and 405 students showed up. Two-thirds of the students who showed up won a door prize. How many students won prizes?

Problem 141 (Working Backwards)**.** For each picture shown:

- What multiplication problem is represented?
- What is the product?





Problem 142 (Depreciation)**.** A piece of office equipment was purchased for \$60,000. Each year, it depreciates in value. At the end of each year, the equipment is worth $\frac{9}{10}$ what it was worth at the start of the year. How much is the equipment worth after 1 year? After 2 years? After 5 years?

Problem 143 (How close can you get?)**.** Using only the digits 0, 1, 2,…, 9 at most once each in place of the variables, find the value closest to 1. For each problem, justify your solution. How do you *know* it's closest to 1?

(a) $\dfrac{a}{b}$

(b) $\dfrac{a}{b} \cdot \dfrac{c}{d}$

(c) $\dfrac{a}{b} \cdot \dfrac{c}{d} \cdot \dfrac{e}{f}$

Problem 144 (Community garden)**.** A town plans to build a community garden that will cover $\frac{2}{3}$ of a square mile. They would like to situate it on a pasture of an old farm. One dimension of the garden area will be determined by a fence that is $\frac{3}{4}$ of a mile long. If the garden is a rectangle, how long is the other side?

Problem 145 (Planting wheat)**.** Nate used $90\frac{1}{2}$ pounds of seed to plant $1\frac{1}{4}$ acres of land in spring wheat. How many pounds of seed is he using per acre?

Problem 146. The family-sized box of laundry detergent contains 35 cups of detergent. Your family's machine requires $1\frac{1}{4}$ cup per load. How many loads of laundry can your family do with one box of detergent?

Problem 147. At the start of each semester, $\frac{5}{6}$ of all Math 111 students work out at least three times each week. By the middle of the semester, $\frac{4}{5}$ of those students are still working out regularly. By the time finals rolls around, $\frac{9}{10}$ of those students still hit the gym three times each week. If 36 students are working out regularly during finals, how many were enrolled in Math 111 at the start of the semester?

Problem 148. Jessica bikes to campus every day. When she is one-third of the way between her home and where she parks her bike, she passes a grocery store. When she is halfway to school, she passes a Subway sandwich shop. This morning, Jessica passed the grocery store at 8:30am, and she passed Subway at 8:35am. What time did she get to campus?

Problem 149. If you place a full container of flour on a balance scale and place on the other side a $\frac{1}{3}$ pound weight plus a container of flour (the same size) that is $\frac{3}{4}$ full, then the scale balances. How much does the full container of flour weight?

♣♣♣ Fellow: [add a picture? also if it makes sense for any of the other problems...]

Problem 150. Geoff spent $\frac{1}{4}$ of his allowance on a movie. He spent $\frac{11}{18}$ of what was left on snacks at school. He also spent \$3 on a magazine, and that left him with $\frac{1}{24}$ of his total allowance, which he put into his savings account. How much money did Geoff save that week?

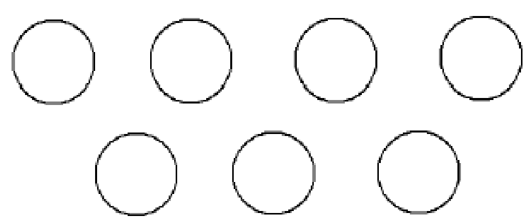
Problem 151. Lily was flying to San Francisco from Honolulu. Halfway there, she fell asleep. When she woke up, the distance remaining was half the distance traveled while she slept. For what fraction of the trip was Lily asleep?

4.10 Egyptian Fractions

Scholars of ancient Egypt (about 3000 B.C.) were very practical in their approaches to mathematics and always sought answers to problems that would be of most convenience to the people involved. This led them to a curious approach to thinking about fractions.

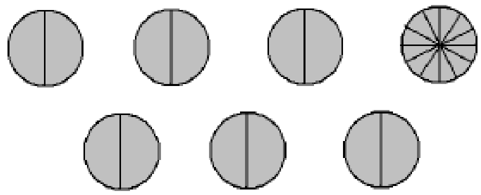
Example 4.10.1 (Egyptian fractions for $\frac{7}{12}$). Consider the problem: Share 7 pies among 12 boys. Of course, given our model for fractions, each boy is to receive the quantity "$\frac{7}{12}$." But this answer has little intuitive feel.

Suppose we took this task as a very practical problem. Here are the seven pies:



Is it possible to give each of the boys a whole pie? No.

How about the next best thing — can each boy get half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 boys. Divide this into twelfths and hand each boy an extra piece.



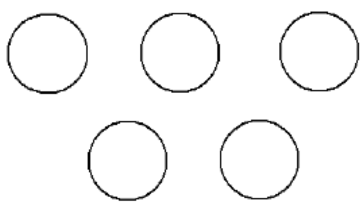
So each boy gets $\frac{1}{2} + \frac{1}{12}$ of a pie, and it is indeed true that

$$\frac{7}{12} = \frac{1}{2} + \frac{1}{12}.$$

(Check that calculation. . . don't just believe it!)

Think/Pair/Share.

1. How do you think the Egyptians might have shared five pies among six boys?



2. How would they have shared seven pies among 12 boys?

The Egyptians (probably) weren't particularly concerned with splitting up pies. But in fact, they did have a very strange (to us) way of expressing fractions. We know this by examining the Rhind Papyrus. This ancient document indicates that fractions were in use as many as four thousand years ago in Egypt, but the Egyptians seem to have worked primarily with *unit fractions*. They insisted on writing all of their fractions as sums of fractions with numerators equal to 1, and they insisted that the denominators of the fractions were all different.

♣♣♣ Fellow: [Can you add a (SHORT) blurb about the Rhind Papyrus and a picture? Like the Polya blurb that Ryan added to the Problem Solving chapter.]

Example 4.10.2 (Egyptian fractions)*.* The Egyptians would not write $\frac{3}{10}$, and they would not even write $\frac{1}{10} + \frac{1}{10} + \frac{1}{10}$. Instead, they wrote

$$\frac{1}{4} + \frac{1}{20}.$$

The Egyptians would not write $\frac{5}{7}$, and they would not even write $\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$. Instead, they wrote

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{70}.$$

(You should check that the sums above give the correct resulting fractions!)

Problem 152 ($\frac{2}{n}$)**.** Write the following as a sum of two *different* unit fractions. Be sure to check your answers.

$$\frac{2}{3}, \qquad \frac{2}{9}, \qquad \frac{2}{15}, \qquad \frac{2}{25}.$$

Can you express this process as a general algorithm?

Problem 153 (Fractions bigger than $\frac{1}{2}$)**.** Write the following as a sum of distinct unit fractions. ("Distinct" means the fractions must have different denominators.) Note that you may need to use more than two unit fractions in some of the sums. Be sure to check your answers.

$$\frac{3}{4}, \qquad \frac{5}{6}, \qquad \frac{3}{5}, \qquad \frac{5}{9}.$$

Can you express this process as a general algorithm?

Problem 154 (Challenges)**.** Write the following fractions as Egyptian fractions.

$$\frac{17}{20}, \qquad \frac{3}{7}.$$

Can you find a general algorithm that will turn *any fraction at all* into an Egyptian fraction?

4.11 Algebra Connections

In an advanced algebra course students are often asked to work with complicated expressions like:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}.$$

We can make it look friendlier by following exactly the same technique of the previous section. In this example, let's multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\cdot x}{\left(\frac{3}{x}\right)\cdot x} \quad = \quad \frac{1+x}{3},$$

and $\frac{1+x}{3}$ is much less scary.

Example 4.11.1. As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab},$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\cdot a\cdot b}{(ab)\cdot a\cdot b} \quad = \quad \frac{b-a}{a^2b^2}.$$

Example 4.11.2. For

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5},$$

it might be good to multiply top and bottom each by $(w+1)^2$. (Why?)

$$\frac{\left(\frac{1}{(w+1)^2}-2\right)\cdot(w+1)^2}{\left(\frac{1}{(w+1)^2}+5\right)\cdot(w+1)^2} \quad = \quad \frac{1-2(w+1)^2}{1+5(w+1)^2}.$$

On Your Own

Can you make each of these expressions look less scary?

$$\frac{2-\frac{1}{x}}{1+\frac{1}{x}}, \qquad \frac{\frac{1}{x+h}+3}{\frac{1}{x+h}}, \qquad \frac{1}{\frac{1}{a}+\frac{1}{b}}, \qquad \frac{\frac{1}{x+a}-\frac{1}{x}}{a}.$$

4.12 What is a Fraction? Part 3

So far, we have no single model that makes sense of fractions in all contexts. Sometimes a fraction is an action ("Cut this in half.") Sometimes it is a quantity ("We each get 2/3 of a pie!") And sometimes we want to treat fractions like *numbers*, like ticks on the number line in-between whole numbers.

We could say that a fraction is just a pair of numbers a and b, where we require that $b \neq 0$. We just happen to write the pair as $\frac{a}{b}$.

But again this is not quite right, since whole infinite collection of pairs of numbers represent the same fraction! For example:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \cdots$$

So a single fraction is actually a whole infinite class of pairs of numbers that we consider "equivalent."

How do mathematicians think about fractions? Well, in exactly this way. They think of pairs of numbers written as $\frac{a}{b}$, where we remember two important facts:

- $b \neq 0$, and
- $\frac{a}{b}$ is really shorthand for a whole infinite class of pairs that look like $\frac{xa}{xb}$.

This is a hefty shift of thinking: The notion of a "number" has changed from being a specific combination of symbols to a whole class of combinations of symbols that are deemed equivalent.

Mathematicians then *define* the addition of fractions to be given by the daunting rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

This obviously motivated by something like the "Pies Per Boy Model." But if we just *define* things this way, we must worry about *proving* that choosing different representations for $\frac{a}{b}$ and $\frac{c}{d}$ lead to the same final answer.

For example, it is not immediately obvious that

$$\frac{2}{3} + \frac{4}{5} \qquad \text{and} \qquad \frac{4}{6} + \frac{40}{50}$$

give answers that are equivalent. (Check that they do!)

They also *define* the product of fractions as:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Again, if we start from here, we have to *prove* that all is consistent with different choices of representations.

Then mathematicians establish that the axioms of an arithmetic system hold with these definitions and carry on from there! (That is, they check that addition and multiplication are both commutative and associative, that the distributive law holds, that all representations of 0 act like an additive identity, and so on. . .)

This is abstract, dry and not at all the best first encounter to offer students on the topic of fractions. And, moreover, this approach completely avoids the question as to what a fraction really means in the "real world." But it is the best one can do if one is to be completely honest.

The definitions are certainly motivated by the type of work we did in this chapter, but in the end one can't explain why these rules are the way they are.

Think/Pair/Share. So. . . what is a fraction, really? How do you think about them? And what is the best way to talk about them with elementary school students?

Chapter 5

Patterns and Algebraic Thinking

Algebra skills are essential for your future students. Why? Here are just a few reasons:

- Mathematics, and especially algebra, is the language of science and modern technology. Thinking algebraically helps you to make sense of the world, to understand and interact with technology more productively, and to succeed in other fields.

- Algebra is a tool for solving problems. This may not be your experience so far, but it is true. If you are able to "algebratize" a problem, that often helps lead you to a solution.

- Algebra helps you to think abstractly. It is a tool for thinking about operations like addition, subtraction, multiplication, and division separate from doing calculations on numbers. Algebra helps you to understand and explain why the operations work the way they do, to describe their properties clearly, and to manipulate expressions to see the bigger picture.

You might wonder why future elementary teachers should master algebra, a topic usually studied (by that name, anyway) in 8th grade and beyond. But the Common Core Standards for School Mathematics[1] has standards in "Operations and Algebraic Thinking" beginning in kindergarten!

[1] `http://www.corestandards.org/Math/Content/OA`

Everyone who shows up to school has already learned a lot about abstraction and generalization — the fundamental ideas in algebra. They are all capable of learning to formalize these ideas. Your job as an elementary school teacher will be to provide your students with even more experiences in abstraction and generalization in a mathematical context, so that these ideas will seem quite natural when they get to a class with the name "Algebra."

Let's start with a problem:

Problem 155. I can use four 4's to make 0:

$$44 - 44 = 0.$$

I can also use four 4's to make the number 10:

$$(4 \times 4) - 4 - \sqrt{4} = 10.$$

Your challenge: Use four 4's to make all of the numbers between 0 and 20. (Try to find different solutions for 0 and 10 than the ones provided.) You can use any mathematical operations, but you can't use any *digits* other than the four 4's.

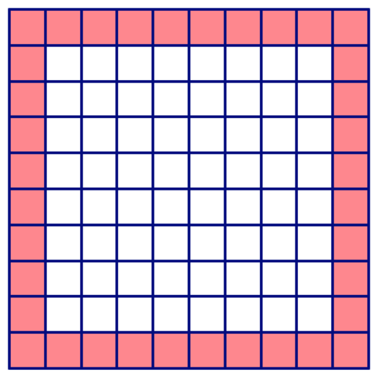
Think/Pair/Share. Share your answers to Problem 155 with a partner. Then talk about these questions together:

- What does "algebra" mean to you?
- What does Problem 155 have to do with "algebra"?
- What do you imagine when you think about using algebra to solve problems in school?
- Have you ever used algebra to solve problems outside of school?
- What is meant by "algebraic thinking," and what kinds of algebraic thinking can be done by elementary school students?

5.1 Borders on a Square

Here's another problem.

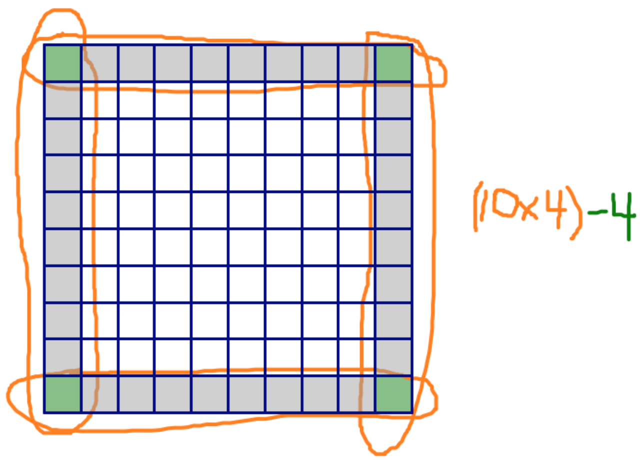
Problem 156. Here is a large square made up of 100 smaller unit squares. The unit squares along the border of the large square are colored red. Without counting one-by-one, can you figure out how many red squares there are in the picture? Clearly describe how you figured out the number of red squares, and how you know your answer is correct.



Justin calculated the number of squares as $(10 \times 4) - 4$. He justified his answer this way:

> *Since the dimensions of the big square are* 10×10*, there are* 10 *squares along each of the four sides. So that gives me* 10×4 *red squares. But then each corner is part of two different sides. I've counted each of the corners twice. So I need to make up for that by subtracting* 4 *at the end.*

Justin showed this picture to justify his work.



Think/Pair/Share. Discuss these questions with your partner:

- What do you think about Justin's solution? Are you convinced? Could he have explained it more clearly?

- Was Justin's solution different from your solution or the same? Discuss Justin's solution and your own solution with a partner.

- Notice the color coding in Justin's picture. What do the colors represent? Why did he use the colors the way he did?

Problem 157. There are lots of different ways to calculate the number of colored squares along the border of a 10×10 square. Below are the calculations several other students did. For each calculation, write a justification and draw a picture to show why it calculates the number of squares correctly. Think about using color in your picture to make your work more clear.

(a) Valerie calculated $10 + 10 + 8 + 8$.

(b) Kayla calculated 4×9.

(c) Linda calculated $(10 \times 10) - (8 \times 8)$.

(d) Mark calculated $(4 \times 8) + 4$.

(e) Allan calculated $10 + 9 + 9 + 8$.

Problem 158. Now suppose that you have a large 6×6 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares. For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 159. Now suppose that you have a large 25×25 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares. For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 160.

(a) Suppose that you have 64 red squares. Can you use all of those squares to make the border of a larger square in a picture like the one above? If yes, what are the dimensions of the larger square? If no, why not?

(b) What if you have 30 red squares? Same questions.

(c) What if you have 256 red squares? Same questions.

Think/Pair/Share. With a partner, see if you can describe some general rules:

- If you have a large $n \times n$ square with the border squares colored red, how many red squares will there be? Justify your answer with words and a picture.

- If you have k red squares, is there a quick test you can do to decide if you can use all of those squares to make the border of a large square? Can you tell how big the square will be?

5.2 Careful use of language in mathematics: =

You have already thought about the careful use of language in mathematics. For example, the word "or" has very specific meanings in math that is slightly different from their everyday use.

The notion of equality is fundamental in mathematics, and especially in algebra and algebraic thinking. The symbol "=" expresses a *relationship*. It is *not* an operation in the way that + and ÷ are operations. It should not be read left-to-right, and it definitely does not mean "...and the answer is ...".

For your work to be clear and easily understood by others, it is essential that you use the symbol = appropriately. And for your future students to understand the meaning of the = symbol and use it correctly, it is essential that you are clear and precise in your use of it.

Let's start by working on some problems.

Problem 161. Akira went to visit his grandmother, and she gave him \$1.50 to buy a treat. He went to the store and bought a book for \$3.20. After that, he had \$2.30 left. How much money did Akira have before he visited his grandmother?

Problem 162. Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.

$(a)\ 5+3=8.$ $\quad (b)\ \frac{2}{3}+\frac{1}{2}=\frac{3}{5}.$ $\quad (c)\ 5+3=y.$ $\quad (d)\ \frac{a}{5}=\frac{5}{a}.$

$(e)\ n+3=m.$ $\quad (f)\ 3x=2x+x.$ $\quad (g)\ 5k=5k+1.$

Problem 163. Consider the equation

$$18-7=\underline{\quad\quad}.$$

(a) Fill in the blank with something that makes the equation *always true*.

(b) Fill in the blank with something that makes the equation *always false.*

(c) Fill in the blank with something that makes the equation *sometimes true and sometimes false.*

Problem 164. If someone asked you to *solve* the equations in Problem 162, what would you do in each case and why?

Think/Pair/Share. Kim solved Problem 161 this way this way:

> *Let's see:*
> $$2.30 + 3.20 = 5.50 - 1.50 = 4,$$
> *so the answer is* 4.

What do you think about Kim's solution? Did she get the correct answer? Is her solution clear? How could it be better?

Although Kim found the correct numerical answer, her calculation really doesn't make any sense. It is true that

$$2.30 + 3.20 = 5.50.$$

It is definitely not true that

$$2.30 + 3.20 = 5.50 - 1.40.$$

She is incorrectly using the symbol "=", and that makes her calculation hard to understand.

Think/Pair/Share.

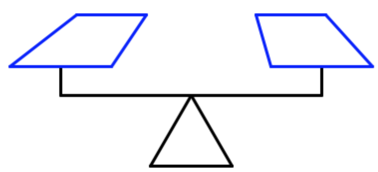
- Can you write a good *definition* of the symbol "="? What does it mean and what does it represent?

- Give some examples: When should the symbol "=" be used, and when should it *not* be used?

- Do these two equations express the same relationships or different relationships? Explain your answer.

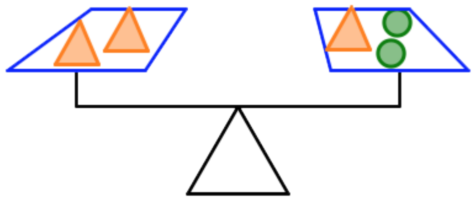
$$x^2 - 1 = (x+1)(x-1) \tag{5.2.1}$$
$$(x+1)(x-1) = x^2 - 1 \tag{5.2.2}$$

This picture shows a (very simplistic) two-pan balance scale. Such a scale allows you to *compare* the weight of two objects. Place one object in each pan. If one side is lower than the other, then that side holds heavier objects. If the two sides are balanced, then the objects on each side weigh the same.

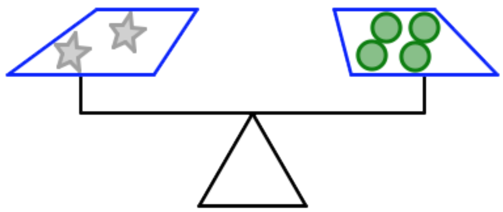


Think/Pair/Share. In the pictures below:

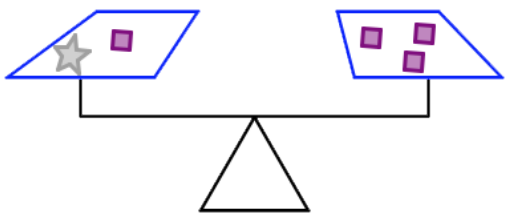
- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.



(a) What do you know about the weights of the triangles and the circles? How do you know it?



(b) What do you know about the weights of the circles and the stars? How do you know it?

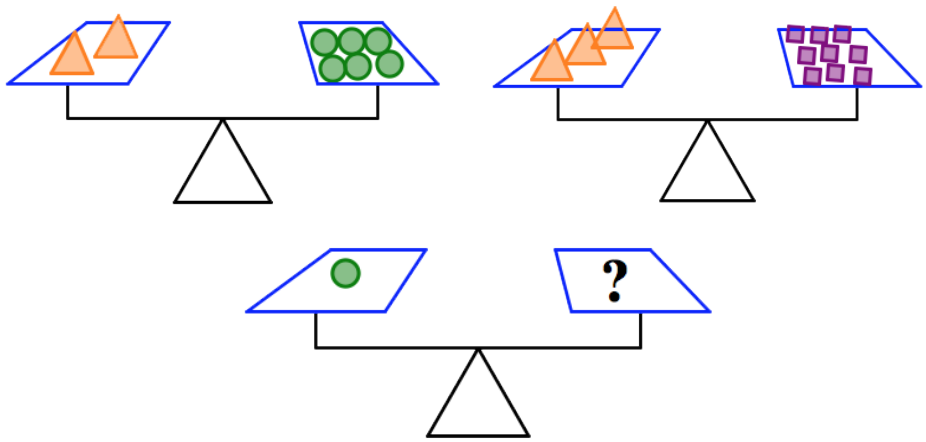


(c) What do you know about the weights of the stars and the squares? How do you know it?

Problem 165. In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.

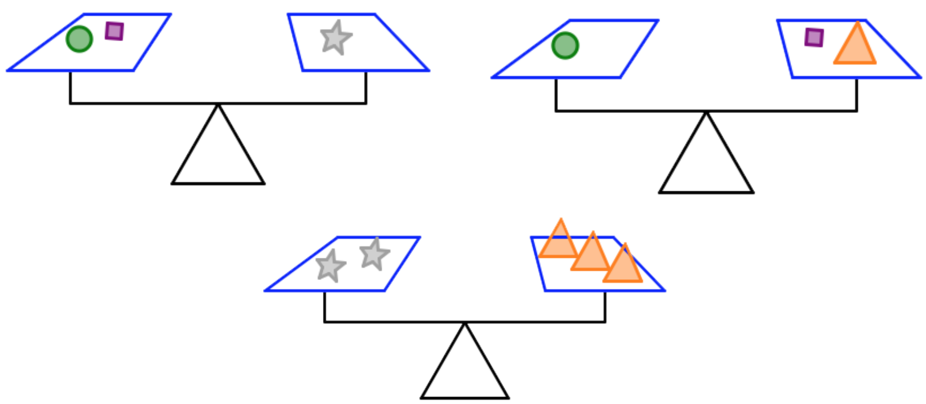
How many purple squares will balance with one circle? Justify your answer.

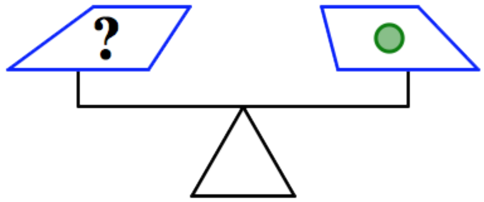


Problem 166. In the pictures below:

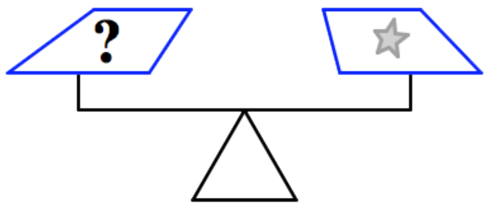
- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

How many purple squares will balance the scale in each case? Justify your answers.

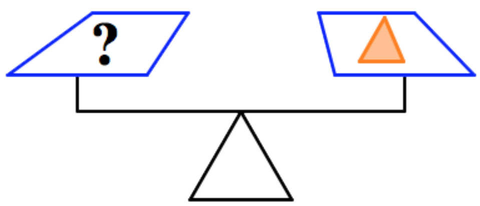




(a)



(b)

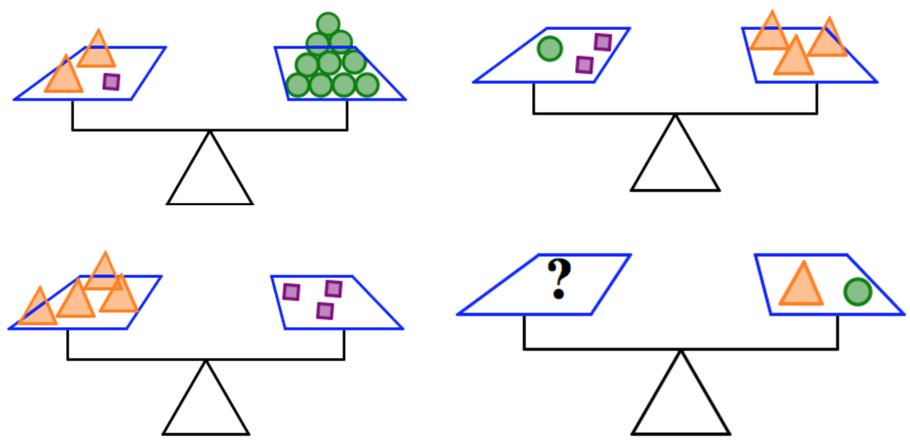


(c)

Problem 167. In the pictures below:

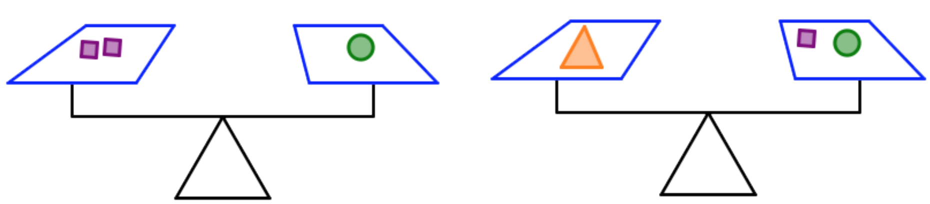
- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.

What will balance the scale? Can you find more than one answer?



Problem 168. In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.



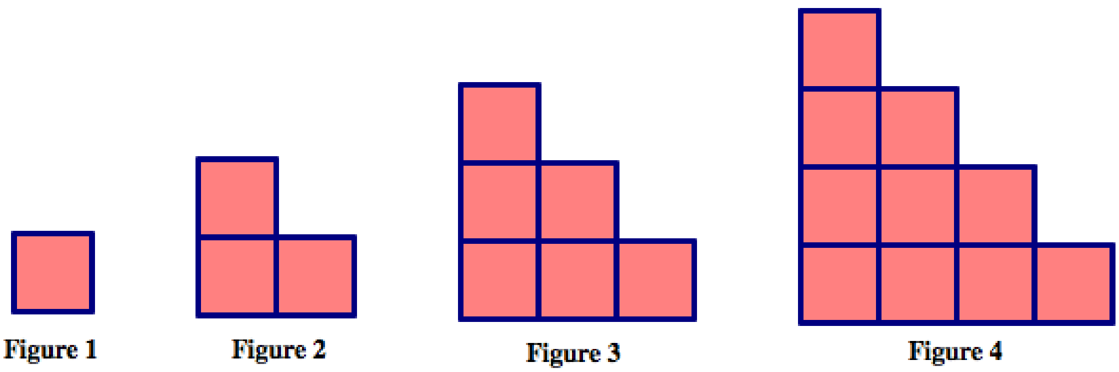
(a) Which shape weighs the most: the square, the triangle, or the circle? Which shape weighs the least? Justify your answers.

(b) Which of the two scales is holding the most total weight? How do you know you're right?

Think/Pair/Share. What do Problems 165–168 above have to do with the "=" symbol?

5.3 Growing Patterns

Here is a pattern made from square tiles.



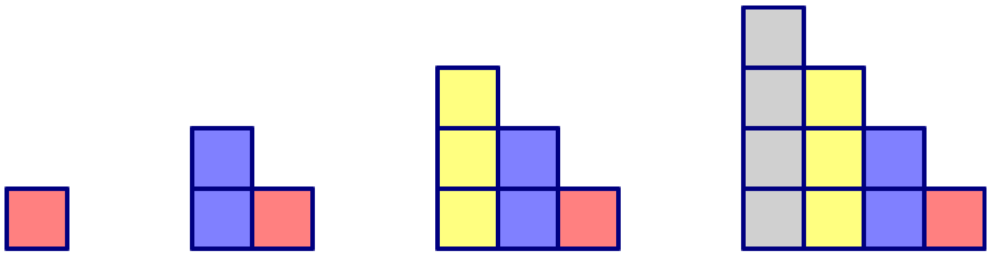
Think/Pair/Share. First on your own and then with a partner, think about these questions:

- Describe how you see this pattern growing. Be as specific as you can. Draw pictures and write an explanation to make your answer clear.

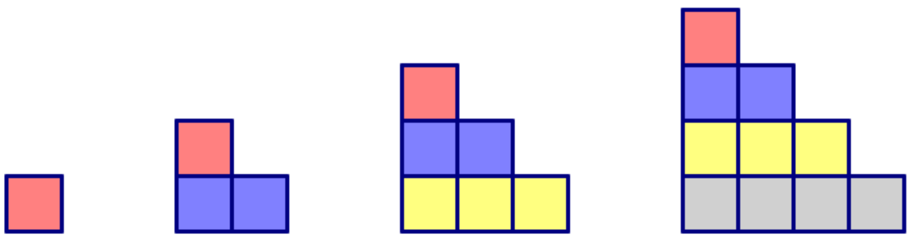
- Say as much as you can about this growing pattern. Can you draw pictures to extend the pattern?

- What mathematical questions can you ask about this pattern? Can you answer any of them?

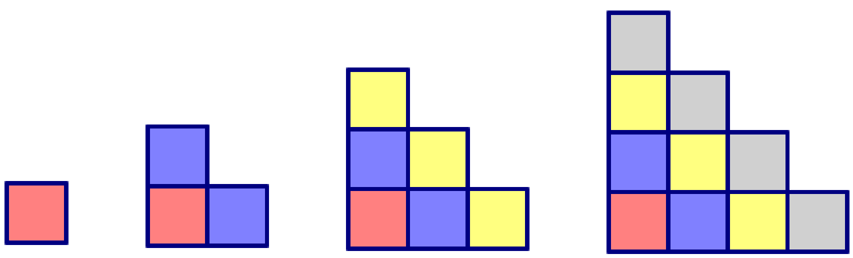
Here are some pictures that students drew to describe how the pattern was growing.



Ali's picture.



Michael's picture.



Kelli's picture.

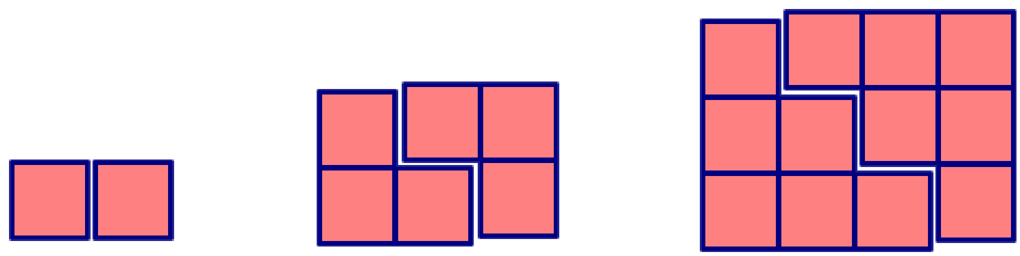
Think/Pair/Share. Describe in words how each student saw the pattern growing. Use the students' pictures above (or your own method of seeing the growing pattern) to answer the following questions:

- How many tiles would you need to build the 5th figure in the pattern?

- How many tiles would you need to build the 10th figure in the pattern?

- How can you compute the number of tiles in any figure in the pattern?

Problem 169. Hy saw the pattern in a different way from everyone else in class. Here's what he drew:



Hy's picture.

(a) Describe in words how Hy saw the pattern grow.

(b) How would Hy calculate the number of tiles needed to build the 10th figure in the pattern?

(c) How would Hy calculate the number of tiles needed to build the 100th figure in the pattern?

(d) How would Hy calculate the number of tiles needed to build any figure in the pattern?

The next few problems present several growing patterns made with tiles. For each problem you work on, do the following:

(a) Describe in words and pictures how you see the pattern growing.

(b) Calculate the number of tiles you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.

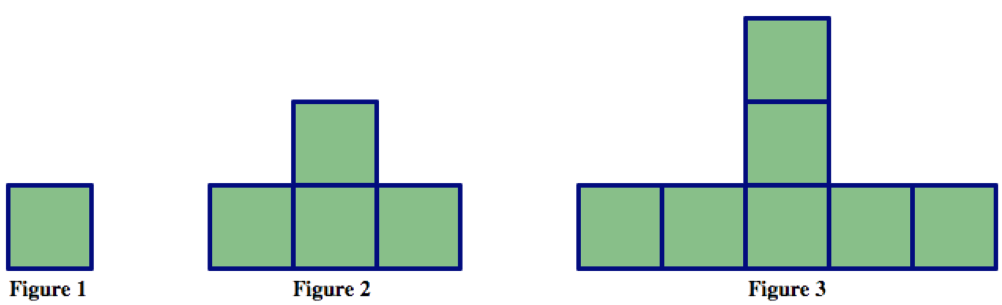
(c) Calculate the number of tiles you would need to build the 100th figure in the pattern.

(d) Describe how you can figure out the number of tiles in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.

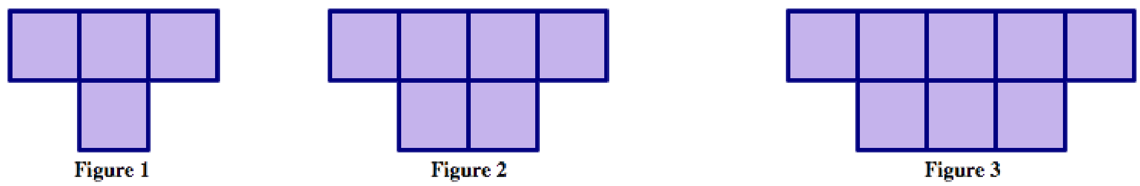
(e) Could you make one of the figures in the pattern using exactly 25 tiles? If yes, which figure? If no, why not? Justify your answer.

(f) Could you make one of the figures in the pattern using exactly 100 tiles? If yes, which figure? If no, why not? Justify your answer.

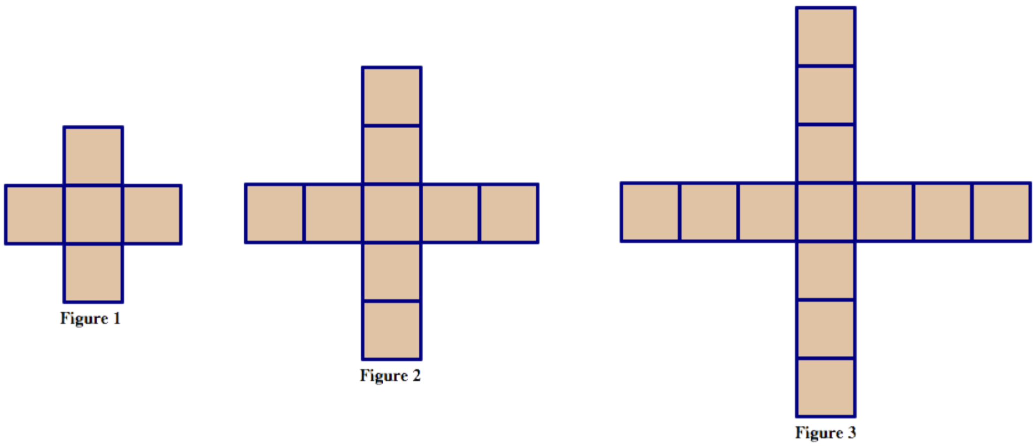
Problem 170.



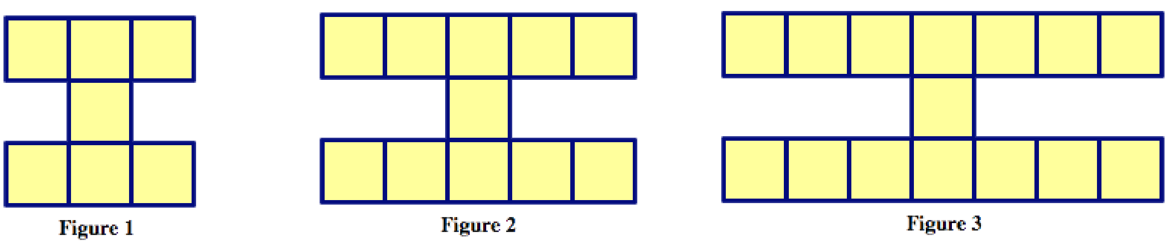
Problem 171.



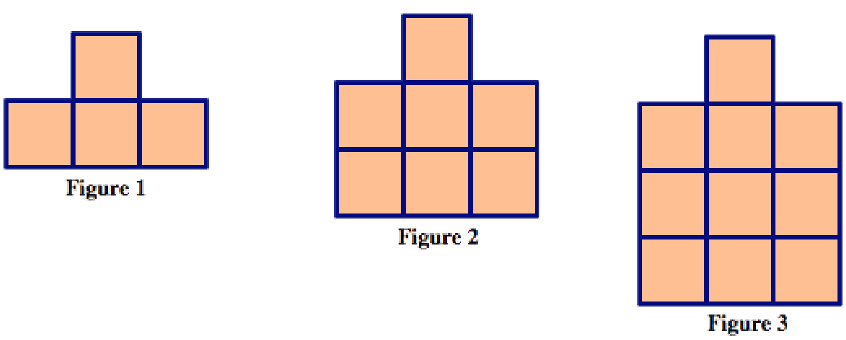
Problem 172.



Problem 173.



Problem 174.



5.4 Matching Game

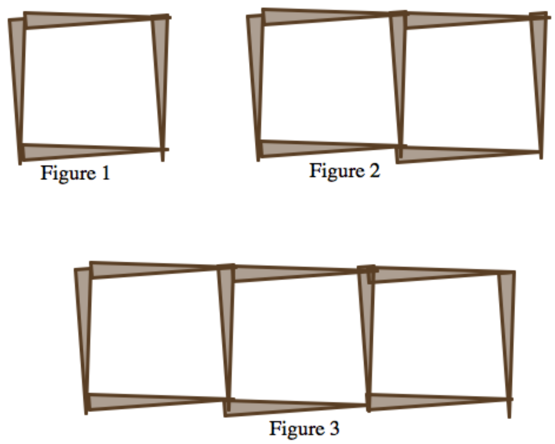
Below, you'll find patterns described in various ways: through visual representations, algebraic expressions, in tables of numbers, and in words. Your job is to match these up in a way that makes sense.

Note: there may be more than one algebraic expression to match a given pattern, or more than one pattern to match a given description. So be ready to justify your answers.

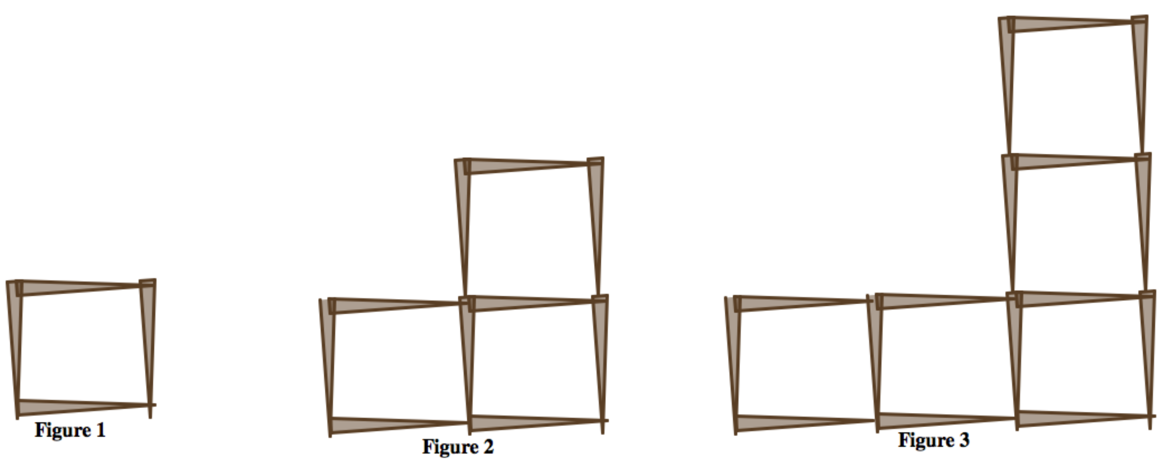
Algebraic expressions

$(a)\ t^2$	$(b)\ 2s+1$	$(c)\ 2k+(k-1)+2k+(k-1)$
$(d)\ 5n+5$	$(e)\ a+a$	$(f)\ 3(\ell-1)+3(\ell-1)+4$
$(g)\ 3b+1$	$(h)\ z+z+1$	$(i)\ m^2-(m-1)^2$
$(j)\ y\cdot y$	$(k)\ 2x-1$	$(\ell)\ 4e-(e-1)$
$(m)\ 6f-2$	$(n)\ 2c$	$(o)\ 5(s+1)$

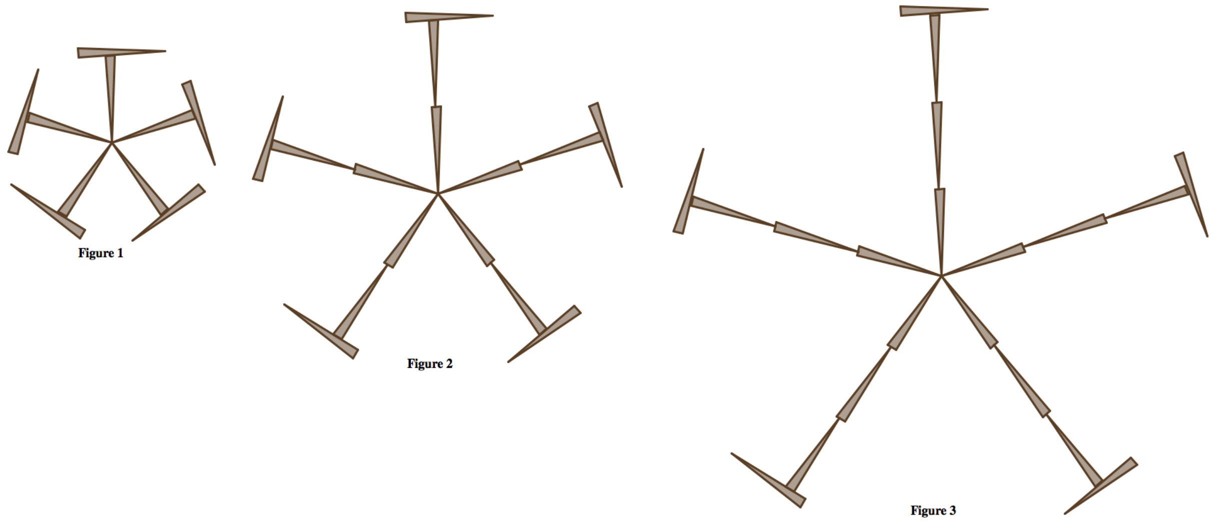
Visual patterns



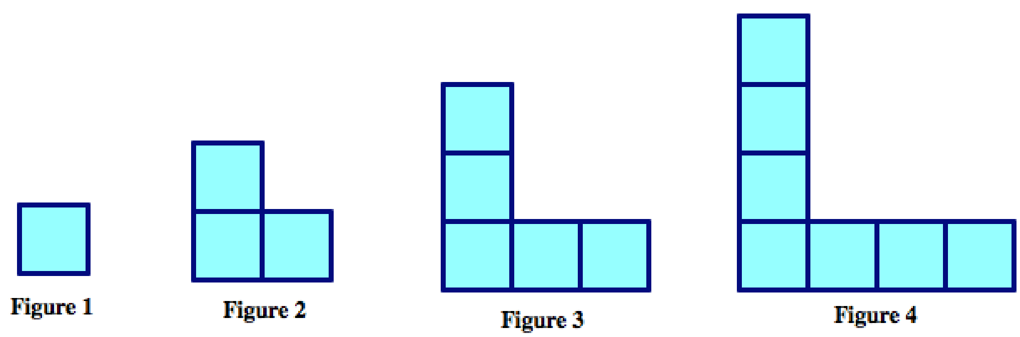
Pattern 1



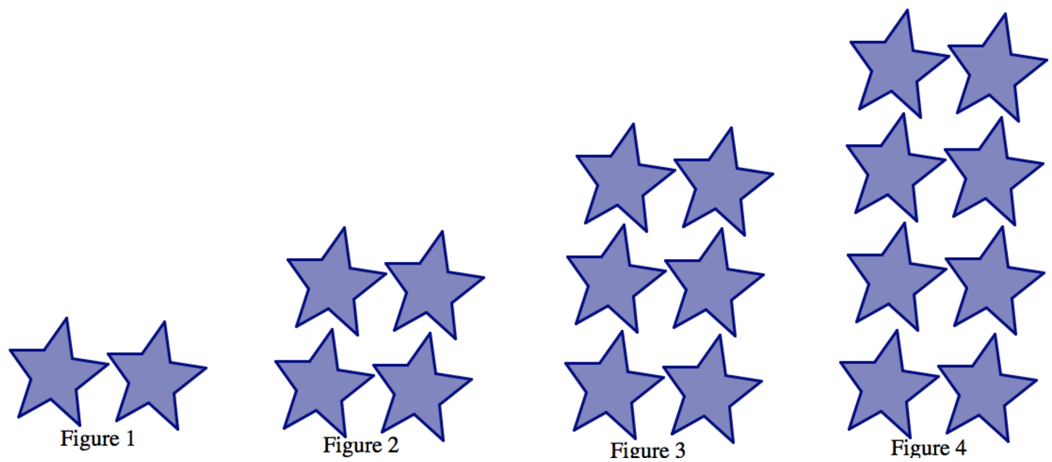
Pattern 2



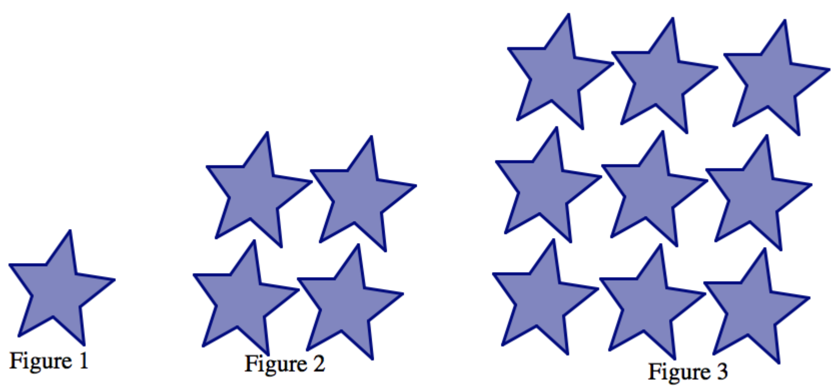
Pattern 3



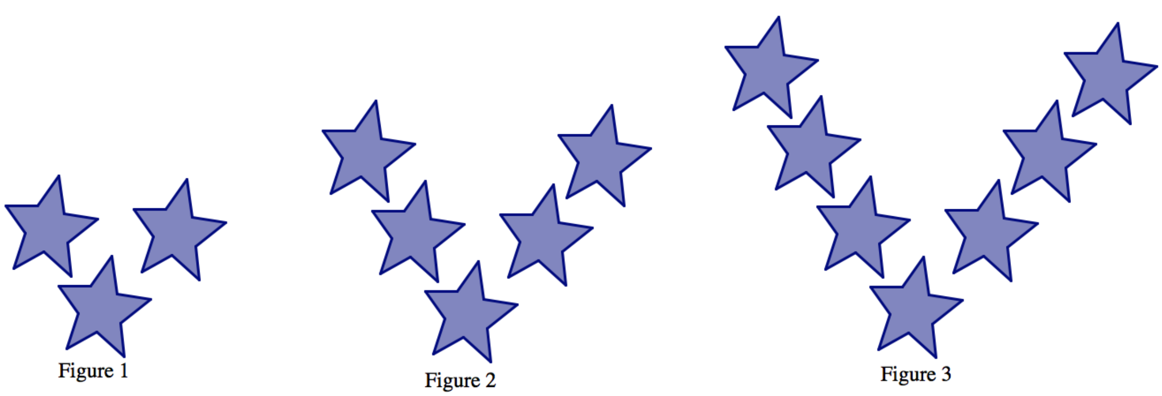
Pattern 4



Pattern 5



Pattern 6



Pattern 7

Tables of Numbers

Table A

Input	1	2	3	4
Output	1	4	9	16

Table B

Input	1	2	3	4
Output	10	15	20	25

Table C

Input	1	2	3	4
Output	1	3	5	7

Table D

Input	1	2	3	4
Output	3	5	7	9

Table E

Input	1	2	3	4
Output	4	7	10	13

Table F

Input	1	2	3	4
Output	4	10	16	22

Table G

Input	1	2	3	4
Output	2	4	6	8

Descriptions in words

(i) Count horizontal and vertical toothpicks separately. Horizaontal: there are two rows of n toothpicks where n is the figure number. There are $n-1$ more of them on the vertical arm. The vertical toothpicks are just the same. There are two colums of n along the vertical arm, and then $n-1$ more of them on the horizontal arm.

(ii) To get a figure from the previous one, you add three toothpicks in a "C" shape on the left side of the figure. The total number of toothpicks is three times the figure number, plus one extra to close off the square on the far right.

(iii) There are five spikes radiating out from the center. Each spike has the same number of toothpicks as the figure number. Each spike is capped off by one additional toothpick.

(iv) Each arm of the "L" shape has the same number of tiles as the figure number. But then we've counted the corner of the "L" twice, so we have to subtract one to get the total number of tiles needed.

(v) The stars are in two equal rows, and each row has the same number of stars as the figure number.

(vi) To make the next figure, you always add five more toothpicks. Each arm has one more than the figure number of toothpicks, and there are five arms.

(vii) The stars are in a square, and the sides of the square have the same number of stars as the figure number.

(viii) Each arm of the "V" shape has the same number of stars as the figure number. Then we need to add one more star for the corner.

(ix) There are the same number of squares as the figure number, and each square uses four toothpicks. But then I've double-counted the toothpicks where the squares touch, so we have to subtract those out. There are one less of those than the figure number.

(x) I can picture a square of tiles filled in. The side length of that square is the same as the figure number, so that's x^2. But then the square isn't really filled in. It's like I took away a square one size smaller from the top right, leaving just the border. What I took away was a square one size smaller, $(x-1)^2$.

(xi) Each time I go from one shape to the next, I add six new toothpicks. Three are added to the left in a "C" shape and three are added to the top in a rotated "C" shape. So the total number will be six times the figure number plus or minus something. I can check to see that the right correction is to subtract 2.

5.5 Structural and Procedural Algebra

When most people think about algebra from school, they think about "solving for x." They imagine lots of equations with varying levels of complexity, but the goal is always the same: find the unknown quantity. This is a *procedural* view of algebra.

Even elementary students can be exposed ideas in procedural algebra. This happens any time they think about unknown quantities and try to solve for them. For example, when first grade students learn to add and subtract numbers "within 10," they should frequently tackle problems like these:

- $3 + \_\_\_\_ = 7$.

- Find several pairs of numbers that add up to 10.

Although procedural algebra is important, it's not the most important skill, and it's certainly not the whole story. You also need to foster thinking about *structural algebra* in your students: using symbols to express meaning in a situation. If there is an x on your page, you should be able to answer, "what does the x mean? What does it represent?"

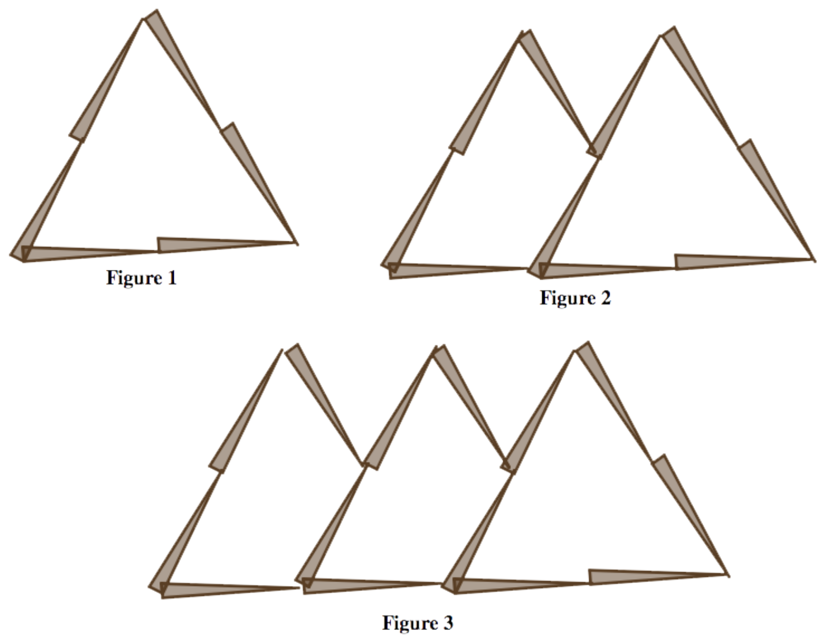
Most of what you've done so far in this chapter is *structural algebra*. You've used letters and symbols not to represent a single unknown quantity, but a *varying* quantity. For example, in Section 6.7 you used letters to represent the "figure number" or "case number" in a growing pattern. The letters could take on different values, and the expressions gave you information: how many tiles or toothpicks or stars you needed to build that particular figure in that particular pattern.

Think/Pair/Share.

- Consider the expression $a+3$. Give a real world situation that could be represented by this expression. Share your answer with your partner. Together, can you come up with even more ideas?

- Suppose the expression $3c + 2$ represents the number of tiles used at any stage of a growing pattern.
  - Evaluate the expression at $c = 1, 2,$ and 3. What do the values tell you about the pattern?
  - Can you describe in words how the pattern is growing?
  - Can you design a pattern with tiles that grows according to this rule?
  - Where do you see the "3" from $3c+2$ in your pattern? Where do you see the "2"? Where do you see the "c"?

Problem 175. Krystal was looking at this pattern, which may be familiar to you from the Problem Bank:



She wrote down the equation

$$y = 4x + 2.$$

In Krystal's equation, what does x represent? What does y represent? How do you know?

Problem 176. Candice was thinking about this problem:

Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again?

She wrote down the equation $2n = m$. In Candice's equation, what does n represent? What does m represent? How do you know?

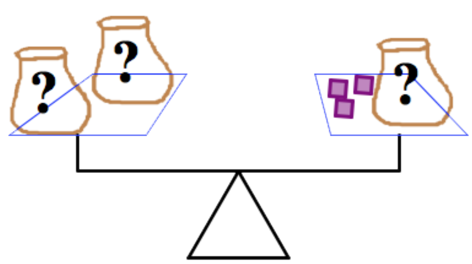
Problem 177. Sarah and David collect old coins. Suppose the variable k stands for the number of coins Sarah has in her collection, and ℓ stands for the number of coins David has in his collection. What would each of these equations say about their coin collections?

(*a*) $k = \ell + 1$ (*b*) $k = \ell$ (*c*) $3k = 2\ell$ (*d*) $k = \ell - 11$

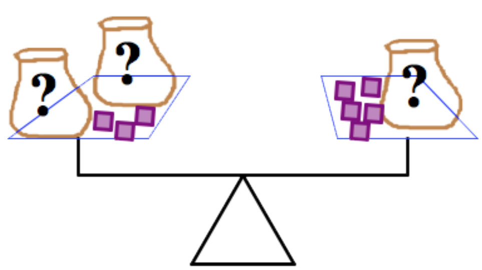
Problem 178. The pictures below show balance scales containing bags and blocks. The bags are marked with a "?" because they contain some unknown number of blocks. In each picture:

- Each bag contains the same number of blocks.
- The scale is balanced.

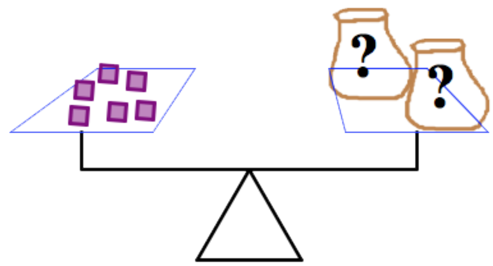
For each picture, determine how many blocks are in each bag. Justify your answers.



(a)



(b)



(c)

Problem 179. When he was working on Problem 178, Kyle wrote down these three equations.

$$(i)\ 2m = 6. \qquad (ii)\ 2x = x + 3. \qquad (iii)\ z + 5 = 2z + 3.$$

Match each equation to a picture, and justify your choices. Then solve the equations, and say (in a sentence) what the solution represents.

Problem 180. Draw a balance puzzle that represents the equation

$$2h + 3 = h + 8.$$

Now solve the balance puzzle. Where is the "h" in your puzzle? What does it represent?

Problem 181. Draw a balance puzzle that represents the equation

$$3b + 7 = 3b + 2.$$

Now solve the equation. Explain what happens.

Problem 182. Which equation below is most like the one in Problem 181 above? Justify your choice.

$$(a)\ 5 + 3 = 8. \qquad (b)\ \frac{2}{3} + \frac{1}{2} = \frac{3}{5}. \qquad (c)\ 5 + 3 = y. \qquad (d)\ \frac{a}{5} = \frac{5}{a}.$$

$$(e)\ n + 3 = m. \qquad (f)\ 3x = 2x + x. \qquad (g)\ 5k = 5k + 1.$$

Problem 183. Draw a balance puzzle that represents the equation

$$4\ell + 7 = 4\ell + 7.$$

Now solve the equation. Explain what happens.

Problem 184. Which equation below is most like the one in Problem 183 above? Justify your choice.

(a) $5 + 3 = 8.$ (b) $\dfrac{2}{3} + \dfrac{1}{2} = \dfrac{3}{5}.$ (c) $5 + 3 = y.$ (d) $\dfrac{a}{5} = \dfrac{5}{a}.$

(e) $n + 3 = m.$ (f) $3x = 2x + x.$ (g) $5k = 5k + 1.$

Problem 185. Create a balance puzzle where the solution is not a whole number of blocks. Can you solve it? Explain your answer.

Problem 186. There are three piles of rocks: pile A, pile B, and pile C. Pile B has two more rocks than pile A. Pile C has four times as many rocks as pile A. The total number of rocks in all three piles is 14.

(a) Use x to represent the number of rocks in pile A, and write equations that describe the rules above. Then find the number of rocks in each pile.

(b) Use x to represent the number of rocks in pile B, and write equations that describe the rules above. Then find the number of rocks in each pile.

(c) Use x to represent the number of rocks in pile C, and write equations that describe the rules above. Then find the number of rocks in each pile.

Think/Pair/Share. Look back at Problems 175–186. Which of them felt like *structural algebraic thinking*? Which felt like *procedural algebraic thinking*? Did any of the problems feel like they involved both kinds of thinking?

5.5.1 Variables and Equations

You have seen that in algebra, letters and symbols can have different meanings depending on the context.

- A symbol could stand for some *unknown quantity*.

- A symbol could stand for some quantity that *varies*. (Hence the term "variable" to describe these symbols.)

In much the same way, *equations* can represent different things.

- They can represent a problem to be solved. This is the traditional procedural algebra type of question.

- They can represent a relationship between two or more quantities. For example, $A = s^2$ represents the relationship between the area of a square and its side length.

- They can represent *identities*: mathematical truths. For example,

  $$x^2 - 1 = (x + 1)(x - 1)$$

  is always true, for every value of x. There is nothing to solve for, and no relationship between varying quantities. (If you do try to "solve for x," you will get the equation $0 = 0$, much like you saw in Problem 183. Not very satisfying!)

Think/Pair/Share. Give an example of each type of equation. Be sure to say what the symbols in the equations represent.

Problem 187. Answer the following questions about the equation

$$x^2 - 1 = (x + 1)(x - 1). \tag{5.5.1}$$

(a) Evaluate both sides of equation 5.5.1 for various values of x:

$$x = 1, 2, 3, 4 \text{ and } 5.$$

What happens?

(b) Use the *distributive property* of multiplication over addition to expand the right side of equation 5.5.1 and simplify it.

(c) Use equation 5.5.1 to compute 99^2 quickly, without using a calculator. Explain how you did it.

5.6 Problem Bank

Problems 188–190 ask you to solve problems about a crazy veterinarian who created three mystifying machines.

Cat Machine Place a cat in the input bin of this machine, press the button, and out jump two dogs and a mouse.

Dog Machine This machine converts a dog into a cat and a mouse.

Mouse Machine This machine can convert a mouse into a cat and three dogs.

Each machine can also operate in reverse. For example, if you have two dogs and a mouse, you can use the first machine to convert them into a cat.

Problem 188. The veterinarian hands you two cats, and asks you to convert them into exactly three dogs (no extra dogs and no other animals). Can you do it? If yes, say what process you would use. If no, say why not.

Problem 189. The veterinarian hands you one dog. He says he only wants cats, but he doesn't care how many. Can you help him? How?

Problem 190. The veterinarian hands you one cat. He says he only wants dogs, but he doesn't care how many. Can you help him? How?

Problems 191–194 present several growing patterns made with toothpicks. For each problem you work on, do the following:

(a) Describe in words and pictures how you see the pattern growing.

(b) Calculate the number of toothpicks you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.

(c) Calculate the number of toothpicks you would need to build the 100th figure in the pattern.

(d) Describe how you can figure out the number of toothpicks in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.

(e) Could you make one of the figures in the pattern using exactly 25 toothpicks? If yes, which figure? If no, why not? Justify your answer.

(f) Could you make one of the figures in the pattern using exactly 100 toothpicks? If yes, which figure? If no, why not? Justify your answer.

Problem 191.

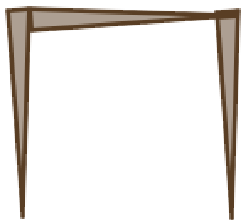


Figure 1

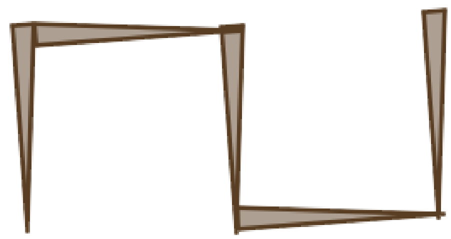


Figure 2

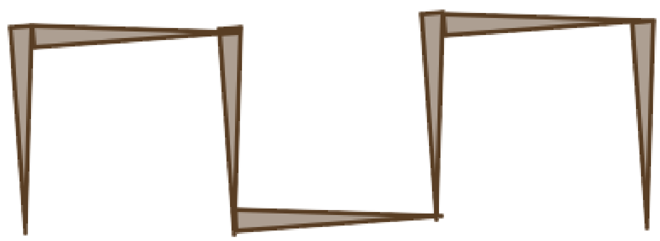
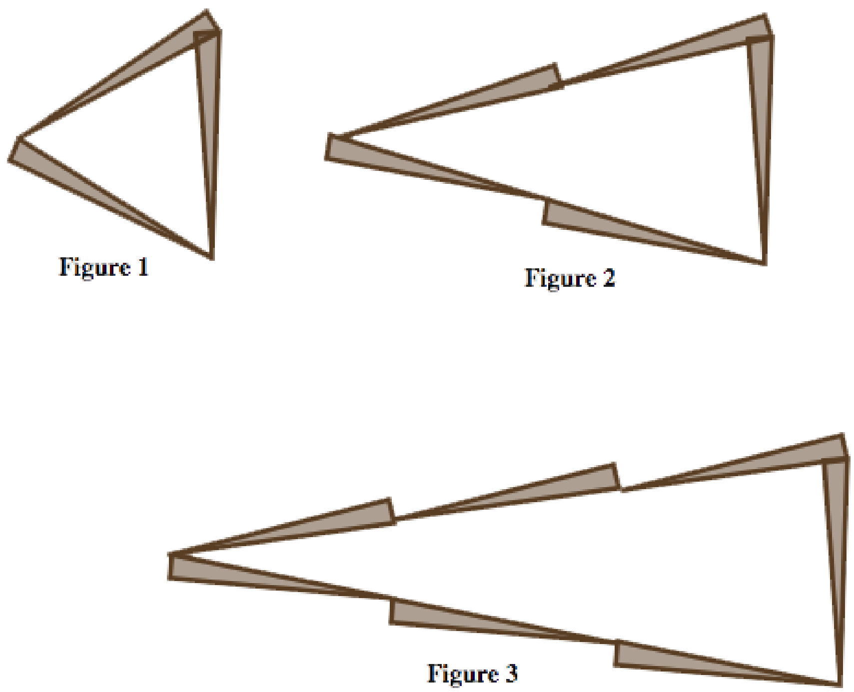


Figure 3

Problem 192.



Problem 193.

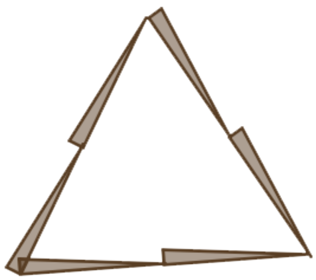


Figure 1

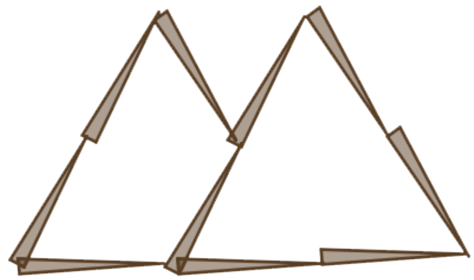


Figure 2

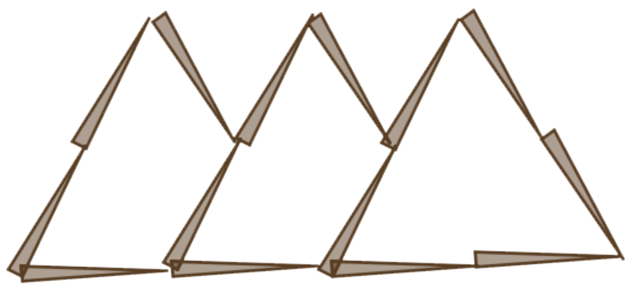


Figure 3

Problem 194.

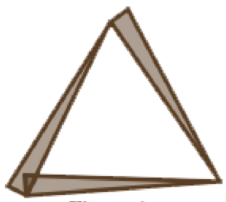


Figure 1

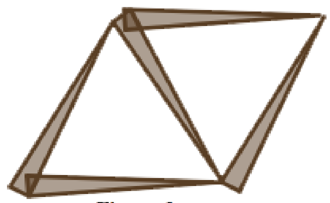


Figure 2

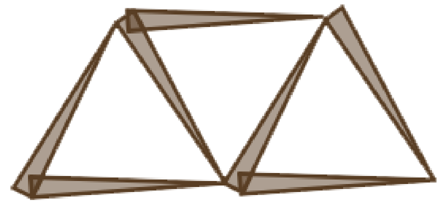


Figure 3

In a *mobile*, the arms must be perfectly balanced for it to hang properly. The artist Alexander Calder was famous for his artistic mobiles.You can view some of his amazing work at `http://calder.org/work/by-category/hanging-mobile`. Click "Explore Works."

Problems 195–196 present you with mobile puzzles. In these puzzles:

- Objects that are the same shape have the same weight. (So all circles weigh the same, all squares weigh the same, etc.)

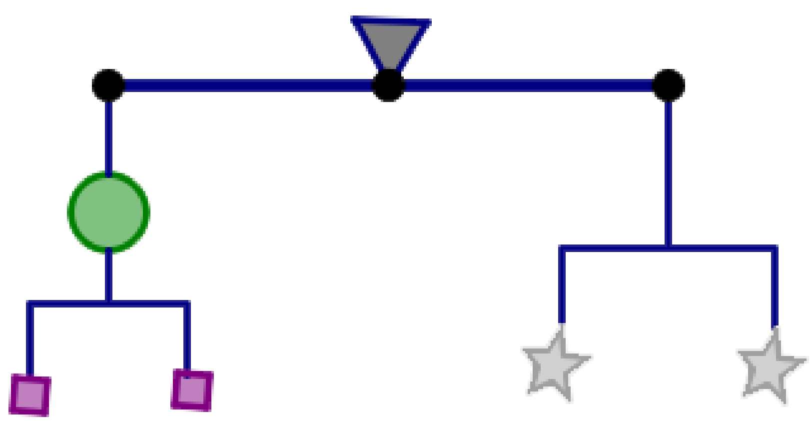
- Assume the strings and rods that hold the objects together don't factor into the total weight.

- Each arm of the mobile must have exactly the same weight.

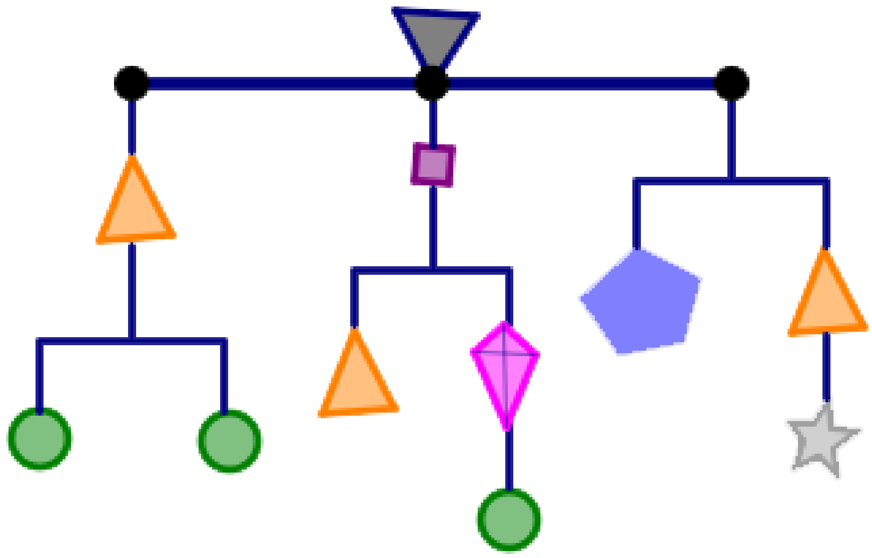
Problem 195. In this puzzle:

- The total weight is 36 grams.
- All shapes weigh less than 10 grams.
- All of the weights are whole numbers.
- One circle weighs more than one square.

Find the weight of each piece. Is there more than one answer? How do you know you are right?



Problem 196. In this puzzle, the total weight is 54 grams. Find the weight of each piece. Is there more than one answer? How do you know you are right?



Chapter 6

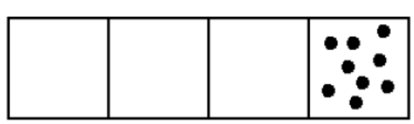
Place Value and Decimals

6.1 Review of Dots & Boxes Model

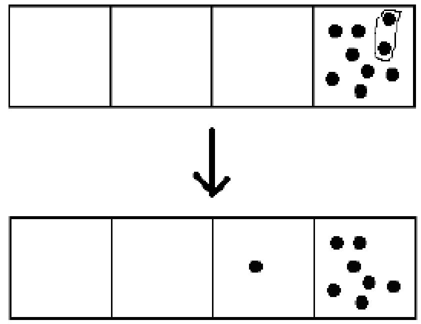
Let's start with a quick review of place value, different bases, and our "Dots & Boxes" model for thinking about these ideas.

> **The $1 \leftarrow 2$ Rule (Base 2):**
> **Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.**

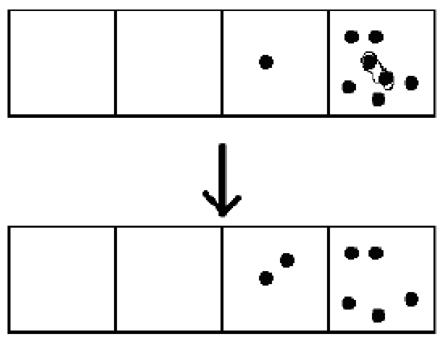
Example 6.1.1 (Nine Dots in the $1 \leftarrow 2$ System). We start by placing nine dots in the rightmost box.



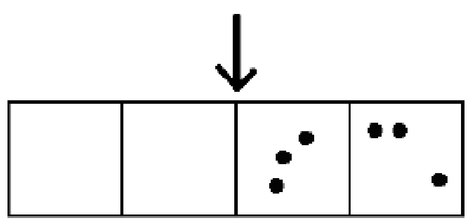
Two dots in that box explode and become one dot in the box to the left.



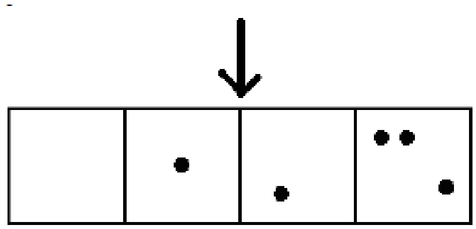
Since there are more than two dots in the rightmost box, it can happen again.



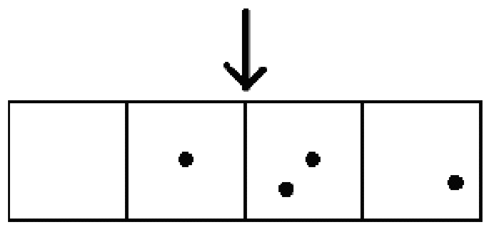
And again!



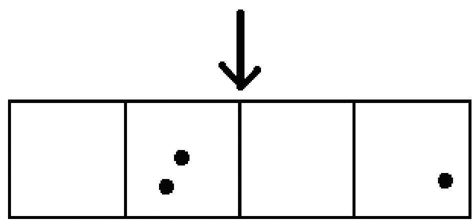
Now we have more than two dots in the second box, so those can explode and move!

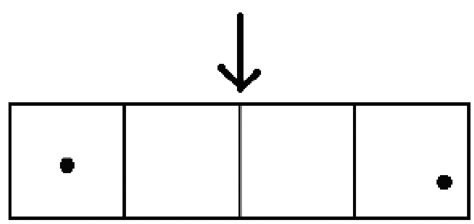


And the rightmost box still has more than two dots.



Keep going, until no box has two dots.



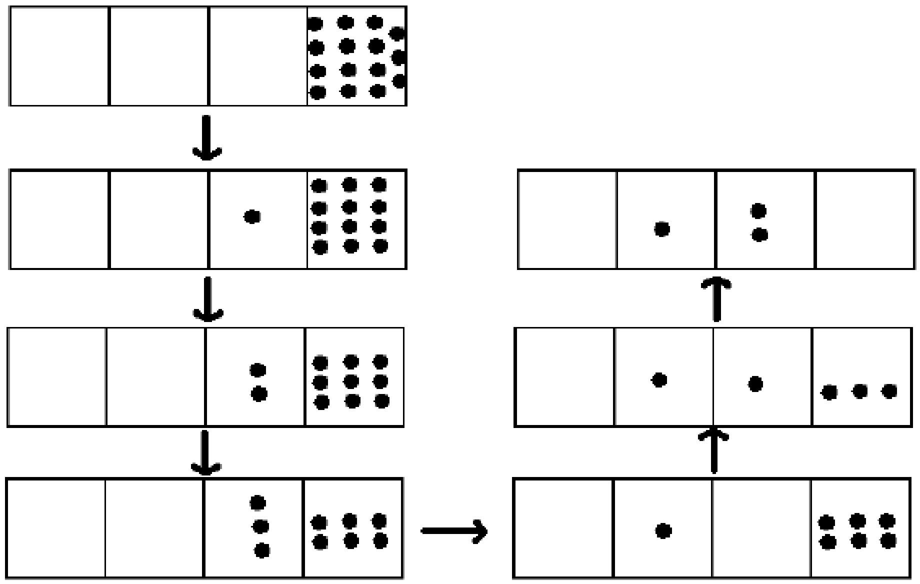


After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot. This process lets us write the number of dots (nine) as a base-two or binary number:

$$9_{\text{ten}} = 1001_{\text{two}}$$

> **The $1 \leftarrow 3$ Rule (Base 3):**
> **Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.**

Example 6.1.2 (Fifteen Dots in the $1 \leftarrow 3$ System). Here's what happens with fifteen dots:



Here's how we write fifteen in base 3:

$$15_{\text{ten}} = 120_{\text{three}}.$$

Think/Pair/Share. Work through the two examples above carefully to be sure you remember and understand how the "Dots & Boxes" model works. Then answer these questions:

- When we write 9 in base 2, why do we write 1001_{two} instead of just 11_{two}?

- When we write 15 in base 3, why do we write 120_{three} instead of just 12_{three}?

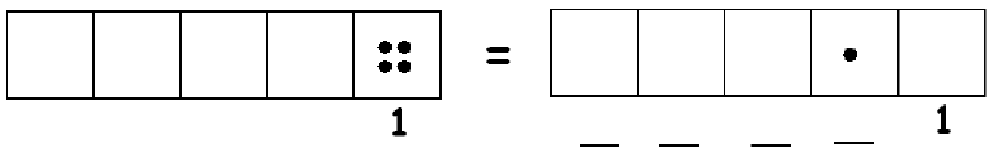
- How many different *digits* do you need in a base 7 system? In a base 12 system? In a base b system? How do you know?

On Your Own

Work on the following exercises on your own or with a partner.

1. In base 4, four dots in one box are worth one dot in the box one place to the left.

   (a) What is the value of each box?

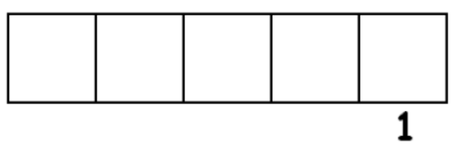
   

   (b) How do you write 29_{ten} in base 4?

   (c) How do you write 132_{four} in base 10?

2. In our familiar base 10 system, ten dots in one box are worth one dot in the box one place to the left.

   (a) What is the value of each box?

   (b) When we write the number 7842, what quantity does the "7" represent? The "4" is four groups of what value? The "8" is eight groups of what value? The "2" is two groups of what value?

3. Write the following numbers of dots in base 2, base 3, base 5, and base 8. Draw the "Dots & Boxes" model if it helps you remember how to do this! (Note: these numbers are all written in base 10. When we don't say otherwise, you should assume base 10.)

(a) 2 (b) 17

(c) 27 (d) 63

4. Convert these numbers to our more familiar base ten system. Draw out dots and boxes and "unexplode" the dots if it helps you remember!

   (a) 1101_{two} (b) 102_{three}

   (c) 24_{five} (d) 24_{nine}

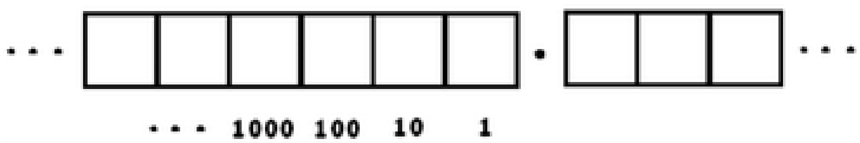
Think/Pair/Share. Quickly compute each of the following. Write your answer in the same base as the problem.

- 131_{ten} times ten
- 263207_{eight} times eight
- 563872_{nine} times nine
- Use the $1 \leftarrow 10$ system to explain why multiplying a whole number in base 10 by 10 results in simply appending a zero to the right end of the number.
- Suppose you have a whole number written in base b. What is the effect of multiplying that number by b? Justify what you say.

6.2 Decimals

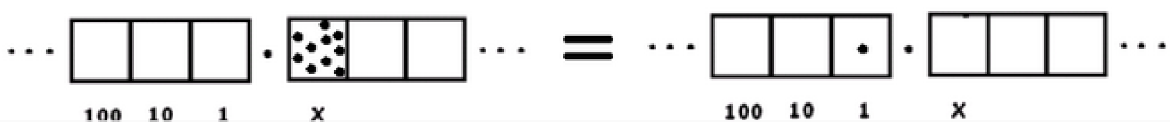
Up to now our "Dots & Boxes" model has consisted of a row of boxes extending infinitely far to the left. Why not have boxes extending to the right as well?

Let's work specifically with a $1 \leftarrow 10$ rule and see what boxes to the right could mean.

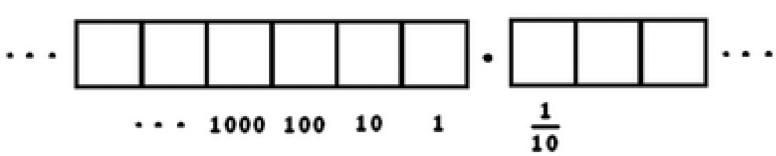


Notation: It has become convention to separate boxes to the left from the ones to the right with a decimal point. (At least, this is what the point is called in the base ten world!)

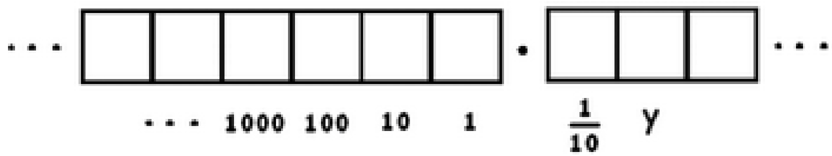
What is the value of the first box to the right of the decimal point? If we denote its value as x , we have that ten x's is equivalent to 1. (Remember, we are using a $1 \leftarrow 10$ rule.)



From $10x = 1$ we get that $x = \frac{1}{10}$.

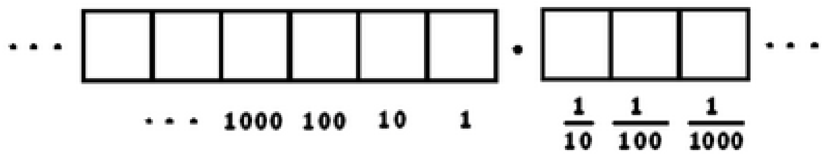


Call the value of the next box to the right of the decimal point y.

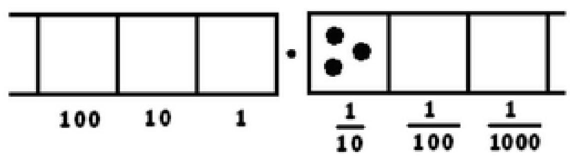


From $10y = \frac{1}{10}$ we get $y = \frac{1}{100}$.

If we keep doing this, we see that the boxes to the right of the decimal point represent the reciprocals of the powers of ten.



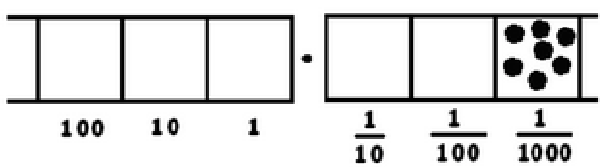
Example 6.2.1 (0.3 in Base Ten)*.* The decimal 0.3 is represented by the picture:



It represents three groups of $\frac{1}{10}$, that is:

$$0.3 = \frac{3}{10}.$$

Example 6.2.2 (0.007 in Base Ten)*.* The decimal 0.007 is represented by the picture:



It represents the fraction $\dfrac{7}{1000}$.

Of course, some decimals represent fractions that can simplify (reduce) further. For example:

$$0.5 \;=\; \frac{5}{10} \;=\; \frac{1}{2}.$$

Similarly, if a fraction can be rewritten to have a denominator that is a power of ten, then it is easy to convert it to a decimal. For example, $\frac{3}{5}$ is equivalent to $\frac{6}{10}$ and so we have:

$$\frac{3}{5} \;=\; \frac{6}{10} \;=\; 0.6.$$

Example 6.2.3 (Decimal Representation of $12\frac{3}{4}$). Can you write $12\frac{3}{4}$ as a decimal? Well,

$$12\frac{3}{4} \;=\; 12 + \frac{3}{4}.$$

We can write the denominator as a power of ten using the key fraction rule:

$$\frac{3}{4} \cdot \frac{25}{25} \;=\; \frac{75}{100}.$$

Thus we can now see that:

$$12\frac{3}{4} \;=\; 12 + \frac{75}{100} \;=\; 12.75.$$

Think/Pair/Share.

- Draw a "Dots & Boxes" picture for each of the following decimals. Then say what fraction each decimal represents:

  0.09   0.003   0.7   0.0000003

- Draw a "Dots & Boxes" picture for each of the following fractions. Then write the fraction as a decimal:

$$\frac{1}{1000} \qquad \frac{7}{100} \qquad \frac{9}{10}$$

- What fractions (in simplest terms) do the following decimals represent?

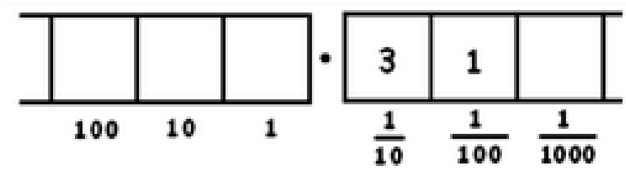
$$0.05 \qquad 0.2 \qquad 0.8 \qquad 0.004$$

- Write the following fractions as decimals.

$$\frac{2}{5} \qquad \frac{1}{25} \qquad \frac{1}{20} \qquad \frac{1}{200} \qquad \frac{1}{1250}$$

- Some people read 0.6 out loud as "point six." Others read it out loud as "six tenths." Which is more helpful for understanding what the number really is? Why do you think so?

Example 6.2.4 (0.31 in Base Ten). Here is a more interesting question: What fraction is represented by the decimal 0.31?



There are two ways to think about this.

Approach 1 From the picture of the $1 \leftarrow 10$ "Dots & Boxes" model we see:

$$0.31 \;=\; \frac{3}{10}+\frac{1}{100}.$$

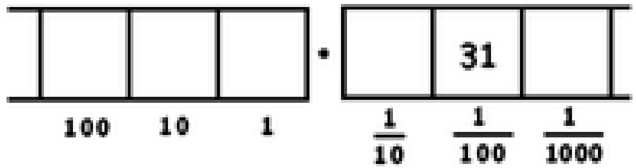
We can add these fractions by finding a common denominator:

$$\frac{3}{10}+\frac{1}{100} \;=\; \frac{30}{100}+\frac{1}{100} \;=\; \frac{31}{100}.$$

So

$$0.31 \;=\; \frac{31}{100}.$$

Approach 2 Let's unexplode the three dots in the $\frac{1}{10}$ position to produce an additional 30 dots in the $\frac{1}{100}$ position.



So we can see right away that

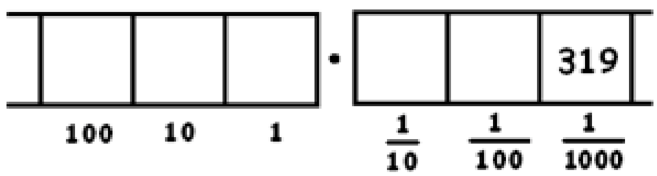
$$0.31 \; = \; \frac{31}{100}.$$

On Your Own

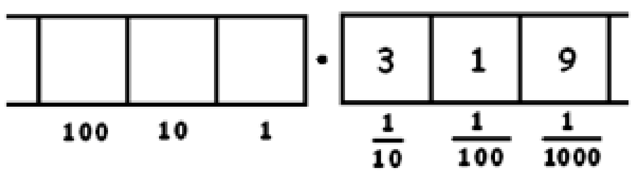
Work on the following exercises on your own or with a partner.

1. Brian is having difficulty seeing that 0.47 represents the fraction $\frac{47}{100}$. Describe the two approaches you could use to explain this to him.

2. A teacher asked his students to each draw a $1 \leftarrow 10$ "Dots & Boxes" picture of the fraction $\frac{319}{1000}$. Jin drew this:



Sonia drew this:



The teacher marked both students as correct.

- Are each of these solutions correct? Explain your thinking.

- Jin said he could get Sonia's solution by performing some explosions. What did he mean by this? Is he right?

3. Choose the best answer and justify your choice. The decimal 0.23 equals:

(a) $\dfrac{23}{10}$

(b) $\dfrac{23}{100}$

(c) $\dfrac{23}{1000}$

(d) $\dfrac{23}{10000}$

4. Choose the best answer and justify your choice. The decimal 0.0409 equals:

   (a) $\frac{409}{100}$ (b) $\frac{409}{1000}$

   (c) $\frac{409}{10000}$ (d) $\frac{409}{100000}$

5. Choose the best answer and justify your choice. The decimal 0.050 equals:

   (a) $\frac{50}{100}$ (b) $\frac{1}{20}$

   (c) $\frac{1}{200}$ (d) None of these

6. Choose the best answer and justify your choice. The decimal 0.000204 equals

   (a) $\frac{51}{250}$ (b) $\frac{51}{2500}$

   (c) $\frac{51}{25000}$ (d) $\frac{51}{250000}$

7. What fraction is represented by each of the following decimals?

   0.567 0.031 0.4077 0.101

8. Write each of the following fractions as decimals. Don't use a calculator!

$$\frac{73}{100} \qquad \frac{519}{1000} \qquad \frac{71}{1000} \qquad \frac{7001}{10000}$$

9. Write each of the following fractions as decimals. Don't use a calculator!

$$\frac{7}{20} \qquad \frac{16}{25} \qquad \frac{301}{500} \qquad \frac{17}{50} \qquad \frac{3}{4}$$

10. Write each of the following as a fraction (or mixed number).

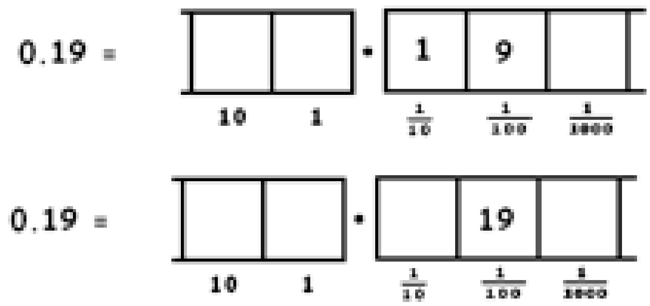
$$2.3 \qquad 17.04 \qquad 1003.1003$$

11. Write each of the following numbers in decimal notation:

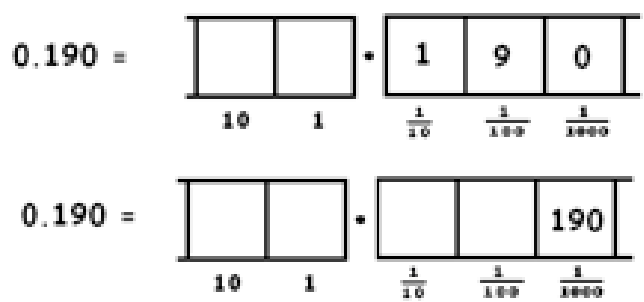
$$5\frac{3}{10} \qquad 7\frac{1}{5} \qquad 13\frac{1}{2} \qquad 106\frac{3}{20}$$

$$\frac{78}{25} \qquad \frac{9}{4} \qquad \frac{131}{40}$$

Do 0.19 and 0.190 represent the same number or different numbers? Here are two dots and boxes pictures for the decimal 0.19.



And here are two dots and boxes picture for the decimal 0.190.



Think/Pair/Share.

- Explain how one "unexplosion" establishes that the first picture of 0.19 equivalent to the second picture of 0.19.

- Explain how several unexplosions establishes that the first picture of 0.190 equivalent to the second picture of 0.190.

- Use explosions and unexplosions to show that all four pictures are equivalent to each other.

- So. . . does 0.190 represent the same number as 0.19?

6.3 x-mals

Just like in base 10, we can add boxes to the right of the decimal point other bases, like base 5.

♣♣♣ Fellow: [Add picture: dots and boxes for base five with 1's, 5's, 25's and 125's labeled. Then a box to the right of the radix point with label x and a box to the right of that labeled y, like on page 6.]

However, the prefix "dec" in "decimal point" means ten. So we really shouldn't call it a decimal point anymore. Maybe a "pentimal point"? In fact, the general term is "radix point."

Think/Pair/Share. Use reasoning like you saw on page 6.2 for the base ten system to think about other number systems:

- Figure out the values of x and y in the picture of the base-5 system above. Be sure you can explain your reasoning.

- Draw a base-4 "Dots & Boxes" model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place.

- Draw a base-6 "Dots & Boxes" model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place.

In general, in a base-b system, the boxes to the left of the ones place represent positive powers of the base b. Boxes to the right of the ones place represent reciprocals of those powers.

♣♣♣ Fellow: [Add picture: dots and boxes for base b with several boxes on each side labeled.]

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a "Dots & Boxes" picture of each number:

   0.03_{five} 0.22_{six} 0.103_{four} 0.002_{three}

2. Find a familiar (base-10) fraction value for each number:

   0.04_{five} 0.3_{six} 0.02_{four} 0.03_{nine}

3. Find a familiar (base-10) fraction value for each number. You might want to re-read Example 6.2.4 first!

   0.13_{five} 0.25_{six} 0.101_{two} 0.24_{seven} 0.55_{eight}

Think/Pair/Share. Tami and Courtney were working on converting 0.44_{five} to a familiar base-10 fraction. Courtney said this:

> *The places in base five to the right of the point are like* $\frac{1}{5}$ *and then* $\frac{1}{25}$. *Since this has two places, the answer should be* $\frac{44}{25}$.

♣♣♣ Fellow: [add picture of 0.44_{five} in a Dots & Boxes model... four dots in each box.]

Tami thought about what Courtney said and replied:

> *I don't know what the right answer is, but I know that* $\frac{44}{25}$ *can't be right. The number* 0.44_{five} *is less than one, since there are no numbers in the ones place and no explosions that we can do. But the fraction* $\frac{44}{25}$ *is more than one. It's almost two. So they can't be the same number.*

- Who makes the most sense, Courtney or Tami? Why do you think so?

- Find the right answer to the problem Courtney and Tami were working on.

Problem 197. Find the "decimal" representation of $\frac{1}{4}$ in each of the following bases. Be sure that you can justify your answer.

base 2   base 4   base 6   base 8   base 10   base 12

Hint: You might want to review Example 6.2.3.

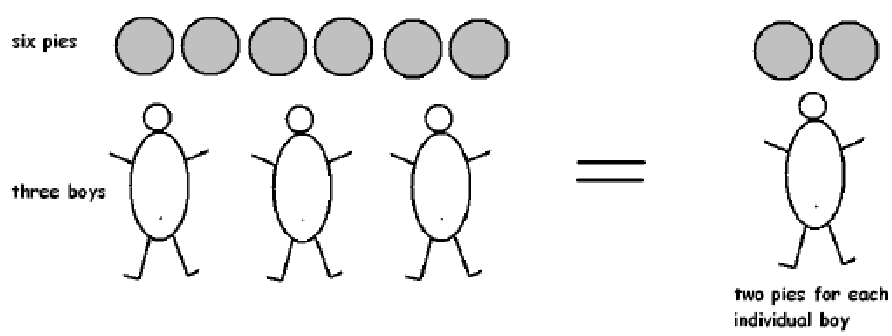
6.4 Division and Decimals

When you studied fractions, you had lots of different ways to think about them. But the first way, and the one we keep coming back to, is to think of a fraction as the answer to a division problem.

♣♣♣ Fellow: [Throughout: pies per kid instead of pies per boy!]

Example 6.4.1 (Pies per boy). Suppose 6 pies are to be shared equally among 3 boys. This yields 2 pies per boy. We write:

$$\frac{6}{3} = 2.$$



The fraction $\frac{6}{3}$ is equivalent to the answer to the division problem $6 \div 3 = 2$. It represents the number of pies one whole boy receives.

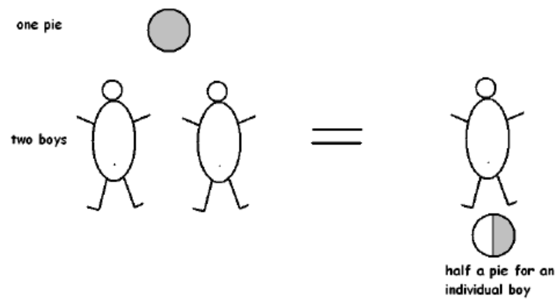
In the same way ...

sharing 10 pies among 2 boys yields $\frac{10}{2} = 5$ pies per boy,

sharing 8 pies among 2 boys yields $\frac{8}{2} = 4$ pies per boy,

sharing 5 pies among 5 boys yields $\frac{5}{5} = 1$ pies per boy, and

the answer to sharing 1 pies among 2 boys is $\frac{1}{2}$, which we call "one-half."



We associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  represents "one third," that is, $\frac{1}{3}$. (This is the amount of pie an individual boy would receive if one pie is shared by three boys.)

The picture  is called "one fifth" and is indeed $\frac{1}{5}$, the amount of pie an individual boy receives when one pie is shared among five.

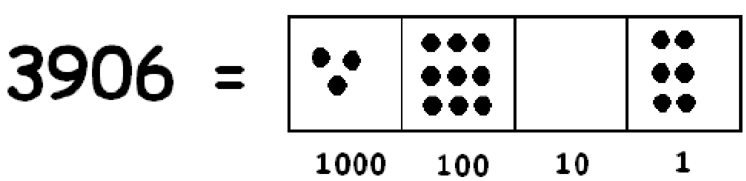
And the picture  is called "three fifths" to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five boys.

We know how to do division in our "Dots & Boxes" model.

Example 6.4.2 ($3906 \div 3$). Suppose you are asked to compute $3906 \div 3$. One way to interpret this question (there are others) is:

> "How many groups of 3 fit into 3906?"

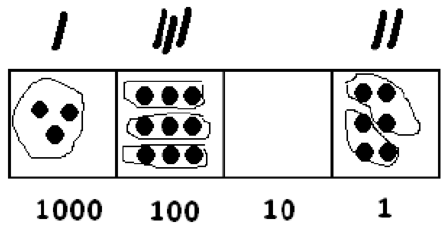
In our "Dots & Boxes" model, the dividend 3906 looks like this:



and three dots looks like this: ●●●. So we are really asking:

> "How many groups of ●●● fit into the picture of 3906?"

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



Notice what we have in the picture:

- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

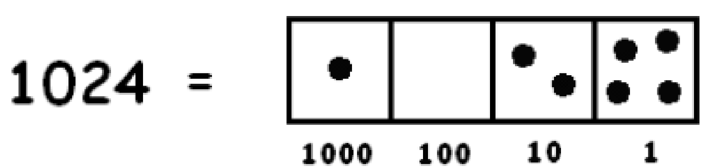
$$3906 \div 3 = 1302.$$

Of course, not every division problem works out evenly! Here's a different example.

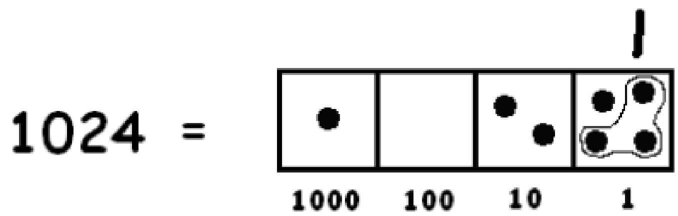
Example 6.4.3 ($1024 \div 3$). Suppose you are asked to compute $1024 \div 3$. One way to interpret this question is:

> "How many groups of 3 fit into 1024?"

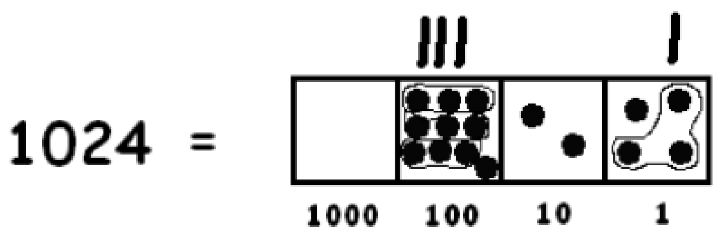
So we're looking for groups of three dots in this picture:



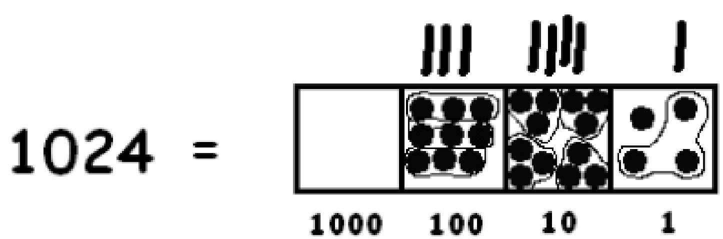
One is easy to spot:



To find more groups of three dots, we must "unexplode" a dot:



And we do it again:

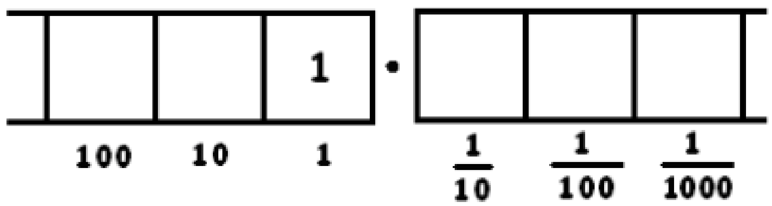


This leaves one stubborn dot remaining in the ones box and no more group of three. So we conclude:

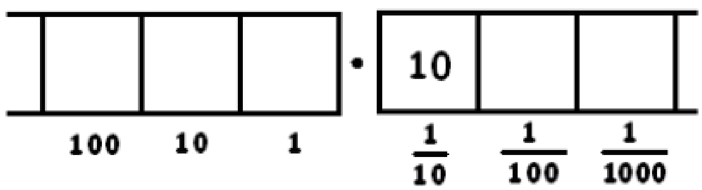
$$1024 \div 3 \; = \; 341 \; R1.$$

We can put these two ideas together — fractions as the answer to a division problem and what we know about division in the "Dots & Boxes" model — to help us think more about the connection between fractions and decimals.

Example 6.4.4 (Decimal Representation of $\frac{1}{8}$). The fraction $\frac{1}{8}$ is the result of dividing 1 by 8. Let's actually compute $1 \div 8$ in a $1 \leftarrow 10$ "Dots & Boxes" model, making use of decimals. We want to find groups of eight in the following picture:

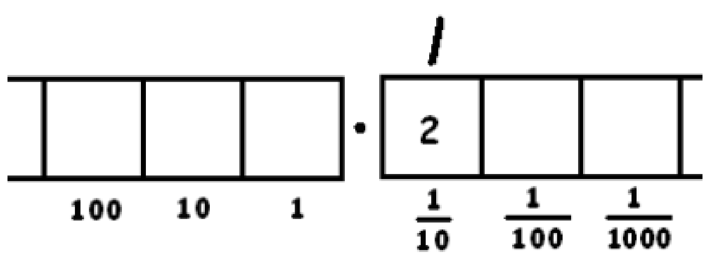


Clearly none are to be found, so let's unexplode:

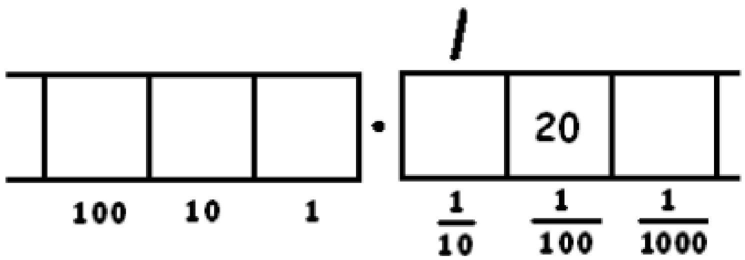


(We're being lazy and not drawing all the dots. As you follow along, you might want to draw the dots rather than the number of dots, if it helps you keep track.)

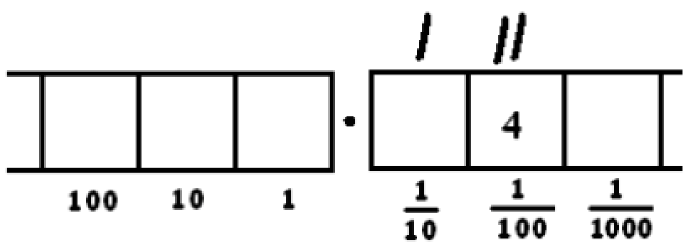
Now there is one group of 8, leaving two behind. We write a tick-mark on top, to keep track of the number of groups of 8, and leave two dots behind in the box.



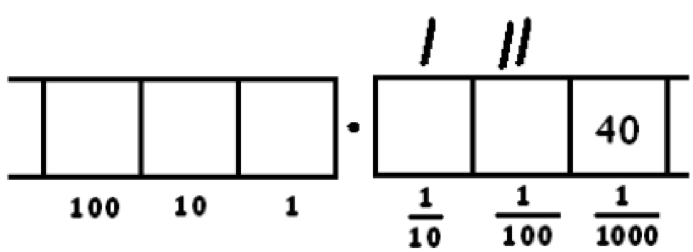
We can unexplode the two dots in the $\frac{1}{10}$ box:



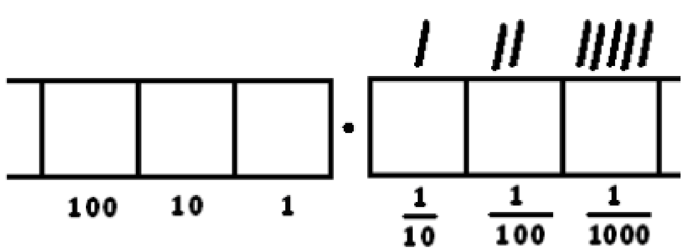
This gives two groups of 8 leaving four behind. Remember: the two tick marks represent two groups of 8. And there are four dots left in the box.



Unexploding those four remaining dots:



Now we have five groups of 8 and no remainder:



Remember: the tick marks kept track of how many groups of eight there were in each box. We have

- One group of 8 dots in the $\frac{1}{10}$ box.
- Two groups of 8 dots in the $\frac{1}{100}$ box.
- Five groups of 8 dots in the $\frac{1}{1000}$ box.

So we conclude that

$$\frac{1}{8} \;=\; 1 \div 8 \;=\; 0.125.$$

Of course, it's a good habit to check our answer:

$$0.125 \;=\; \frac{125}{1000} \;=\; \frac{\cancel{5} \cdot 25}{\cancel{5} \cdot 200} \;=\; \frac{\cancel{5} \cdot 5}{\cancel{5} \cdot 40} \;=\; \frac{\cancel{5} \cdot 1}{\cancel{5} \cdot 8} \;=\; \frac{1}{8}.$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Perform the division in a $1 \leftarrow 10$ "Dots & Boxes" model to show that $\frac{1}{4}$, as a decimal, is 0.25.
2. Perform the division in a $1 \leftarrow 10$ "Dots & Boxes" model to show that $\frac{1}{2}$, as a decimal, is 0.5.
3. Perform the division in a $1 \leftarrow 10$ "Dots & Boxes" model to show that $\frac{3}{5}$, as a decimal, is 0.6.
4. Perform the division in a $1 \leftarrow 10$ "Dots & Boxes" model to show that $\frac{3}{16}$, as a decimal, is 0.1875.

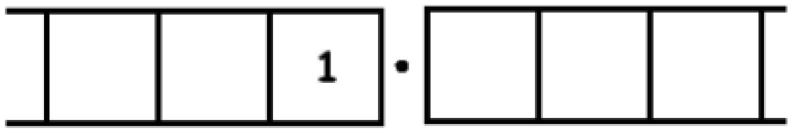
5. In simplest terms, what fraction is represented by each of these decimals?

   0.75 0.625 0.16 0.85 0.0625

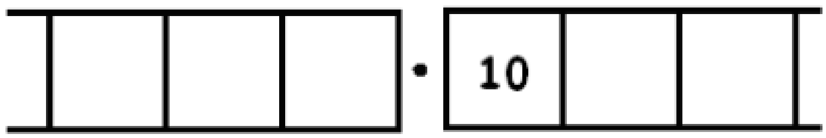
6.4.1 Repeating Decimals

Not all fractions lead to simple decimal representations.

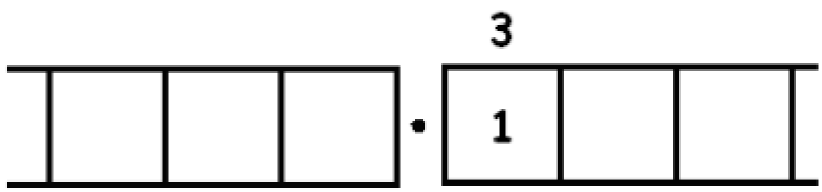
Example 6.4.5 (Decimal Representation of $\frac{1}{3}$). Consider the fraction $\frac{1}{3}$. We seek groups of three in the following picture:



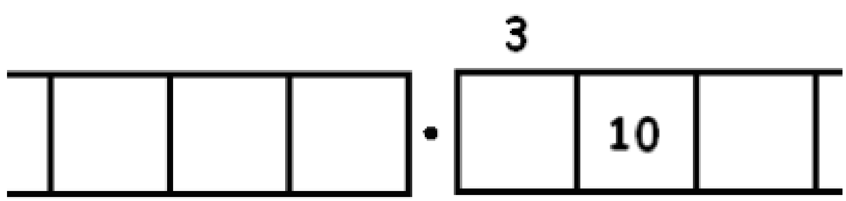
Unexploding requires us to look for groups of 3 in:



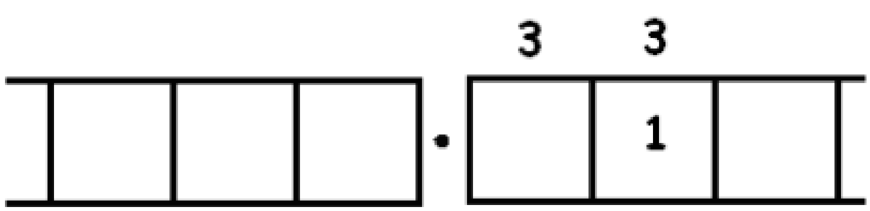
Here there are three groups of 3 leaving one behind:



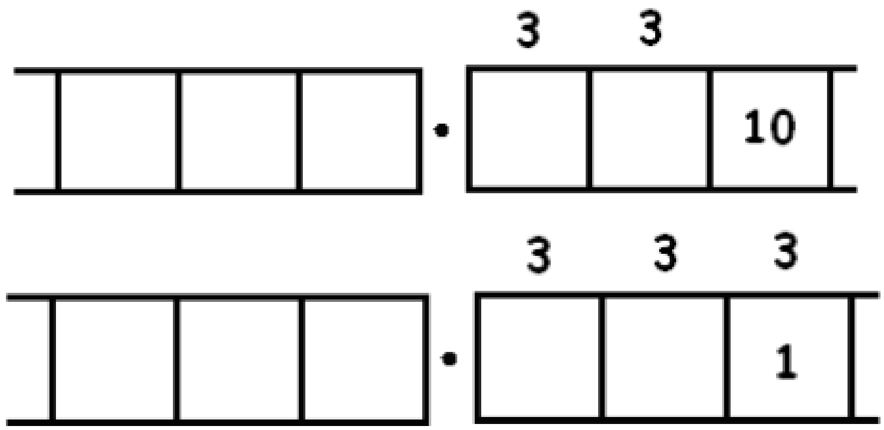
Unexploding gives:



We find another three groups of 3 leaving one behind:



Unexploding gives:



And... we seem to be caught in an infinitely repeating cycle.

We are now in a philosophically interesting position. As human beings, we cannot conduct this, or any, activity an infinite number of times. But it seems very tempting to write:

$$\frac{1}{3} = 0.33333\ldots,$$

with the ellipsis "..." meaning "keep going forever with this pattern." We can *imagine* what this means, but we cannot actually *write down* those infinitely many 3's represented by the ...

Notation: Many people make use of a *vinculum* (horizontal bar) to represent infinitely long repeating decimals. For example, $0.\overline{3}$ means "repeat the 3 forever":

$$0.\overline{3} = 0.33333\ldots,$$

and $0.296\overline{412}$ means "repeat 412 forever":

$$0.296\overline{412} = 0.296412412412412\ldots$$

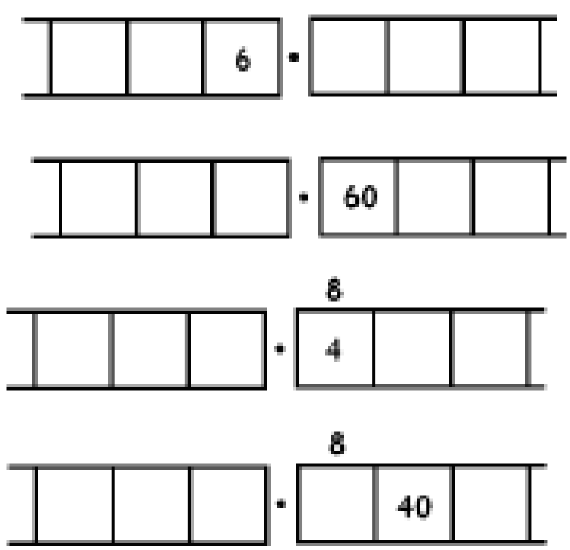
Now we're in a position to give a perhaps more satisfying answer to the question $1024 \div 3$. In Example 6.4.3, we found the answer to be

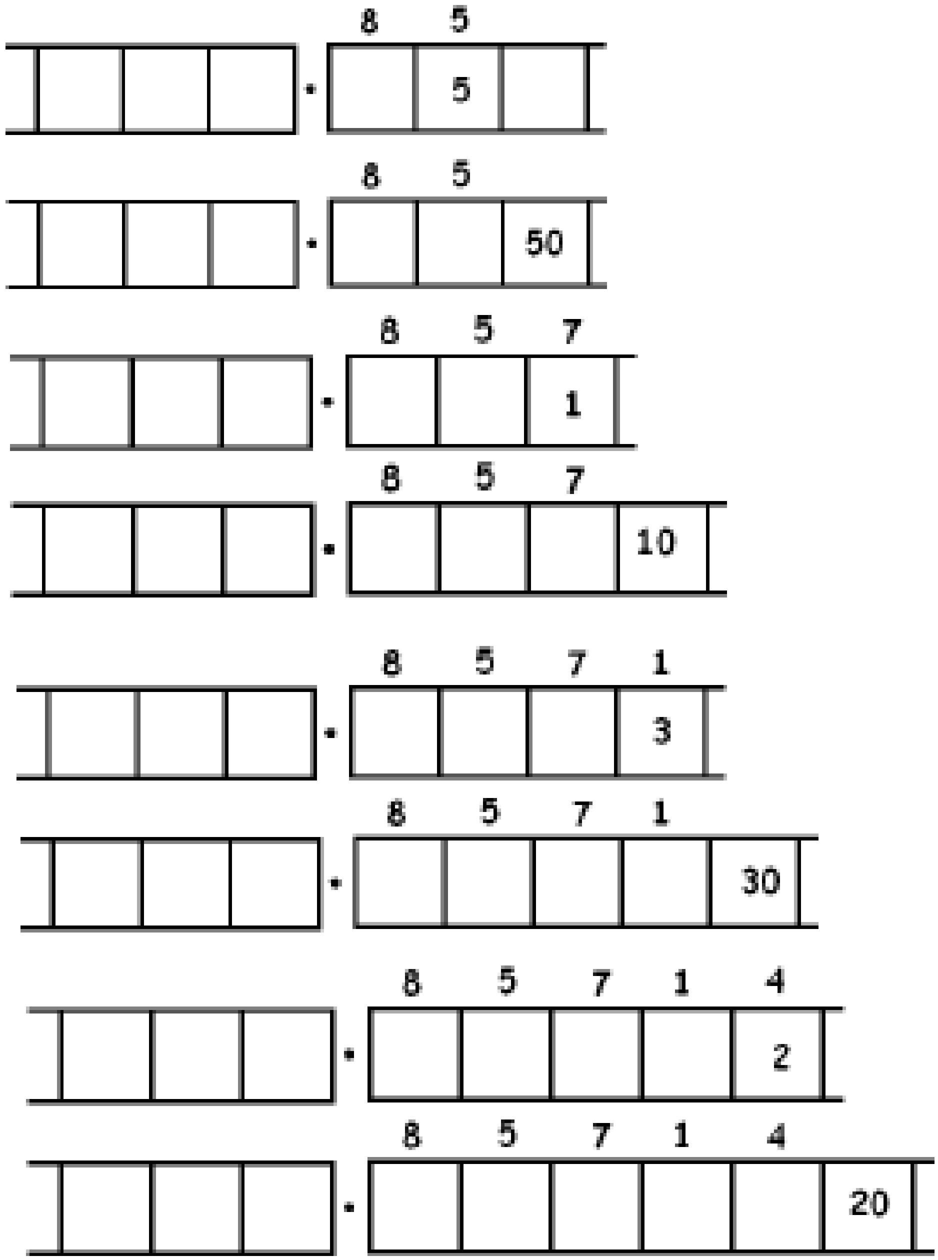
$$1024 \div 3 \;=\; 341\ R1.$$

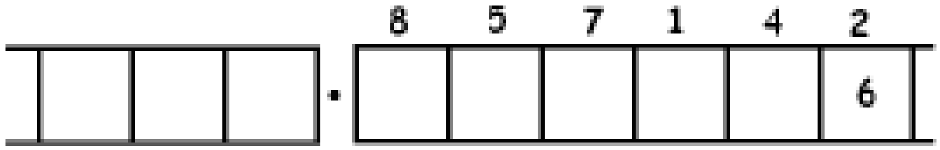
But now we know we can keep dividing that last stubborn dot by 3. Remember, that $R1$ represents a single dot in the ones place, so if we keep dividing by three it really represents $\frac{1}{3}$. So we have

$$1024 \div 3 \;=\; 341\ R1 \;=\; 341\frac{1}{3} \;=\; 341.333333\ldots$$

Example 6.4.6 (Decimal Representation of $\frac{6}{7}$). As another (more complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.




8 5
5
8 5
50
8 5 7
1
8 5 7
10
8 5 7 1
3
8 5 7 1
30
8 5 7 1 4
2
8 5 7 1 4
20



With this 6 in the final right-most box, we have returned to the very beginning of the problem. (Do you see why? Remember, we started with a six in the ones box!)

This means that we will simply repeat the work we have done and obtain the same sequence 857142 of answers, and then again, and then again. We have:

$$\begin{aligned}\frac{6}{7} &= 0.857142857142857142857142\ldots \\ &= 0.\overline{857142}.\end{aligned}$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

2. Compute $\frac{1}{11}$ as an infinitely long repeating decimal.

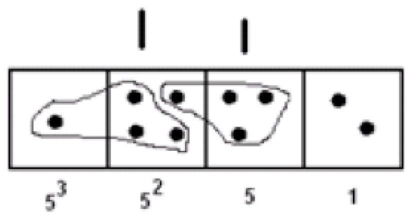
3. Use a $1 \leftarrow 10$ "Dots & Boxes" model to compute $133 \div 6$. Write the answer as a decimal.

4. Use a $1 \leftarrow 10$ "Dots & Boxes" model to compute $255 \div 11$. Write the answer as a decimal.

6.5 More x-mals

It should come as no surprise that we can use this reasoning about division in the "Dots & Boxes" model in other bases as well. The following picture shows that working in base 5,

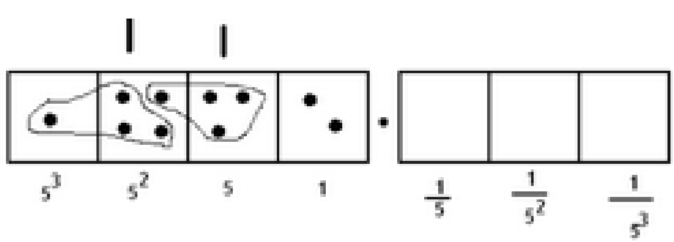
$$1423_{\text{five}} \div 13_{\text{five}} = 110_{\text{five}}\ R2_{\text{five}}. \tag{6.5.1}$$



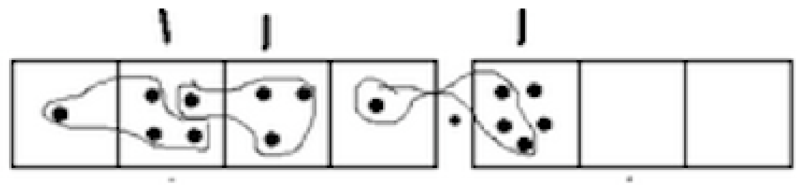
Think/Pair/Share. Carefully explain the connection between the picture and equation (6.5.1) shown above.

- Show in the picture where you see 1423_{five} from the equation.
- Where do you see 13_{five}?
- Where do you see 110_{five} and 2_{five}?

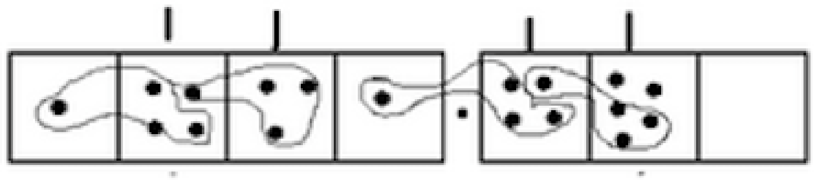
Example 6.5.1 ($1423_{\text{five}} \div 13_{\text{five}}$). Here is the (by now familiar) picture of base 5:



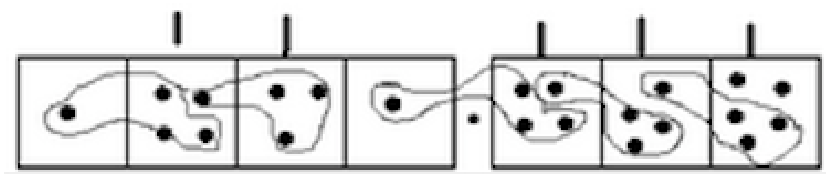
Now we can unexplode one of those two remaining dots. Then we're able to make another group of 13_{five}.



Once again, there are two dots left over, not in any group. So let's unexplode again.



And we still have two dots left over. Why not do it again?



It seems like we're going to be doing the same thing forever:

- Start with two dots in some box.

- Unexplode one one of the dots, so you have one dot in your original box and five in the box to the right.

- Form a group of 13_{five}. That uses the one dot in your original box and three of the five in the box to the right.

- So you have two dots left in a box.

- Unexplode one of the dots, so you have one dot in your original box and five in the box to the right.

- This feels familiar...

We conclude:

$$1432_{\text{five}} \div 13_{\text{five}} = 110.111\ldots_{\text{five}} = 110.\overline{1}_{\text{five}}. \tag{6.5.2}$$

Think/Pair/Share. Equation (6.5.2) is a statement in base five. What is it really saying? "1432_{five}" is the number

$$1 \cdot 125 + 4 \cdot 25 + 3 \cdot 5 + 2 \cdot 1 = 242_{\text{ten}}.$$

- What is 13_{five} in base 10? Be sure to explain your answer.

- What is $110.\overline{1}_{\text{five}}$ in base 10? Explain how you got your answer.

Problem 198.

(a) Draw the pictures to compute $8 \div 3$ in a base 10 system, and show the answer is $2.\overline{6}$.

(b) Draw the pictures to compute $8_{\text{nine}} \div 3_{\text{nine}}$ in a base 9 system, and write the answer as a decimal. (Or is it a "nonimal"?)

Problem 199.

(a) Draw the pictures to compute $1 \div 11$ in a base 10 system, and show the answer is $0.\overline{09}$.

(b) Draw the base 3 pictures to compute $1_{\text{three}} \div 11_{\text{three}}$, and write the answer as a decimal ("trimal"?) number.

(c) Draw the base four pictures to compute $1_{\text{four}} \div 11_{\text{four}}$, and write the answer as a decimal ("quadimal"?) number.

(d) Draw the base six pictures to compute $1_{\text{six}} \div 11_{\text{six}}$, and write the answer as a decimal ("heximal"?) number.

(e) Describe any patterns you notice in the computations above. Do you have a conjecture of a general rule? Can you prove your general rule is true?

Problem 200. Remember that the fraction $\frac{2}{5}$ represents the division problem $2 \div 5$. (This is all written in base 10.)

(a) What is the decimal expansion (in base 10) of the fraction $\frac{2}{5}$?

(b) Rewrite the base-10 fraction $\frac{2}{5}$ as a base 4 division problem. Then find the decimal expansion for that fraction in base 4.

(c) Rewrite the base-10 fraction $\frac{2}{5}$ as a base 5 division problem. Then find the decimal expansion for that fraction in base 5.

(d) Rewrite the base-10 fraction $\frac{2}{5}$ as a base 7 division problem. Then find the decimal expansion for that fraction in base 7.

(e) Barry said that in base 15, the division problem looks like

$$2_{\text{fifteen}} \div 5_{\text{fifteen}},$$

and the decimal representation would be 0.6_{fifteen}. Check Barry's answer. Is he right?

Problem 201. Expand each of the following as a "decimal" number in the base given. (The fraction is given in base 10.)

(a) $\frac{1}{9}$ in base 10

(b) $\frac{1}{2}$ in base 3

(c) $\frac{1}{3}$ in base 4

(d) $\frac{1}{4}$ in base 5

(e) $\frac{1}{5}$ in base 6

(f) $\frac{1}{6}$ in base 7

(g) $\frac{1}{7}$ in base 8

(h) $\frac{1}{8}$ in base 9

Problem 202 (Challenge). What fraction has decimal expansion $0.\bar{3}_{\text{seven}}$? How do you know you are right?

6.6 Terminating or Repeating?

You've seen that when you write a fraction as a decimal, sometimes the decimal *terminates*, like

$$\frac{1}{2} = 0.5 \quad \text{and} \quad \frac{33}{1000} = 0.033.$$

However, some fractions have decimal representations that go on forever in a repeating pattern, like

$$\frac{1}{3} = 0.33333\ldots \quad \text{and} \quad \frac{6}{7} = 0.857142857142857142857142\ldots.$$

It's not totally obvious, but it is true: those are the only two things that can happen when you write a fraction as a decimal.

Of course, you can *imagine* (but never write down) a fraction that goes on forever but doesn't repeat itself, for example:

$$0.101001000100001000001\ldots \quad \text{and} \quad \pi = 3.14159265358979\ldots$$

But these numbers can never be written as a nice fraction $\frac{a}{b}$ where a and b are whole numbers. They are called *irrational numbers.* (The name does not indicate a judgement on their sanity. Rather, fractions like $\frac{a}{b}$ are also called *ratios.* Irrational numbers cannot be expressed as a *ratio* of two whole numbers.)

For now, we'll think about the question: Which fractions have decimal representations that terminate, and which fractions have decimal representations that repeat forever? We'll focus just on *unit fractions.*

Definition 6.6.1. A *unit fraction* is a fraction that has 1 in the numerator. It looks like $\frac{1}{n}$ for some whole number n.

Think/Pair/Share.

- Which of the following fractions have infinitely long decimal representations and which do not?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}$$

- Try some more examples on your own. Do you have a conjecture?

    A fraction $\frac{1}{b}$ has an infinitely long decimal expansion if

    ____.

Problem 203. Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 2. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Fraction	**Denominator**	**Decimal Representation**
$\frac{1}{2}$	2^1	0.5
$\frac{1}{4}$	2^2	0.25
$\frac{1}{8}$	2^3	0.125
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		
$\frac{1}{128}$		
$\frac{1}{256}$		

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\dfrac{1}{2^n}$?

Problem 204. Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Fraction	**Denominator**	**Decimal Representation**
$\frac{1}{5}$	5^1	0.2
$\frac{1}{25}$	5^2	0.04
$\frac{1}{125}$	5^3	
$\frac{1}{625}$		
$\frac{1}{3125}$		
$\frac{1}{15625}$		

Try even more examples until you can make a general statement: What is the decimal representation of the unit fraction $\frac{1}{5^n}$?

Marcus noticed a pattern in the table from Problem 203, but was having trouble explaining exactly what he noticed. Here's what he said to his group:

> *I remembered that when we wrote fractions as decimals before, we tried to make the denominator into a power of ten. So we can do this:*
>
> $$\frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = 0.5$$
>
> $$\frac{1}{4} = \frac{1}{4} \cdot \frac{25}{25} = \frac{25}{100} = 0.25$$
>
> $$\frac{1}{8} = \frac{1}{8} \cdot \frac{125}{125} = \frac{125}{1000} = 0.125$$
>
> *When we only have* 2*'s, we can always turn them into* 10*'s by adding enough* 5*'s.*

Think/Pair/Share.

- Write out several more examples of what Marcus discovered.
- If Marcus had the unit fraction $\frac{1}{2^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?
- Now think about unit fractions with powers of 5 in the denominator. If Marcus had the unit fraction $\frac{1}{5^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?

Problem 205. Marcus has a really good insight, but he didn't explain it very well. He doesn't really mean that we "turn 2's into 10's." And he's not doing any addition, so talking about "adding enough 5's" is pretty confusing.

(a) Complete the statement below by filling in the numerator of the fraction.

> The unit fraction $\frac{1}{2^n}$ has a decimal representation that terminates. The representation will have n decimal digits, and will be equivalent to the fraction
>
> $$\frac{?}{10^n}.$$

(b) Write a better version of Marcus's explanation to justify why this fact is true.

Problem 206. Write a statement about the decimal representations of unit fractions $\frac{1}{5^n}$ and justify that your statement is correct. (Use the statement in Problem 205 as a model.)

Problem 207. Each of the fractions listed below has a terminating decimal representation. Explain how you could know this for sure, without actually calculating the decimal representation.

$$\frac{1}{10} \qquad \frac{1}{20} \qquad \frac{1}{50} \qquad \frac{1}{200} \qquad \frac{1}{500} \qquad \frac{1}{4000}$$

6.6.1 The period of a repeating decimal

If the denominator of a fraction can be factored into just 2's and 5's, you can always form an equivalent fraction where the denominator is a power of ten. For example, if we start with the fraction

$$\frac{1}{2^a 5^b},$$

we can form an equivalent fraction

$$\frac{1}{2^a 5^b} = \frac{1}{2^a 5^b} \cdot \frac{2^b 5^a}{2^b 5^a} = \frac{2^b 5^a}{2^{a+b} 5^{a+b}} = \frac{2^b 5^a}{10^{a+b}}.$$

The denominator is a power of ten, so the decimal expansion is finite with (at most) $a + b$ places.

What about fractions where the denominator has other prime factors besides 2's and 5's? Certainly we *can't* turn the denominator into a power of 10, because powers of 10 have just 2's and 5's as their prime factors. So in this case the decimal expansion will go on forever. But why will it have a *repeating pattern*? And is there anything else interesting we can say in this case?

Definition 6.6.2. The *period* of a repeating decimal is the smallest number of digits that repeat.

For example, we saw that

$$\begin{aligned} \frac{1}{3} &= 0.33333\ldots \\ &= 0.\bar{3}. \end{aligned}$$

The repeating part is just the single digit 3, so the period of this repeating decimal is one.

Similarly, we know that

$$\begin{aligned} \frac{6}{7} &= 0.857142857142857142857142\ldots \\ &= 0.\overline{857142}. \end{aligned}$$

The smallest repeating part is the digits 847142, so the period of this repeating decimal is 6.

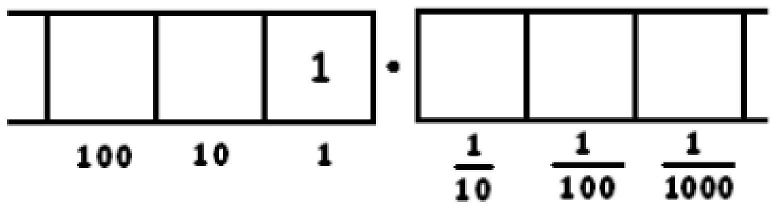
You can think of it this way: the *period* is the length of the string of digits under the vinculum (the horizontal bar that indicates the repeating digits).

Problem 208. Complete the table below which shows the decimal expansion of unit fractions where the denominator has prime factors besides 2 and 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Fraction	**Decimal Representation**	**Period**
$\frac{1}{3}$	$0.\overline{3}$	1
$\frac{1}{6}$	$0.1\overline{6}$	1
$\frac{1}{7}$	$0.\overline{142857}$	6
$\frac{1}{9}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		

Try even more examples until you can make a conjecture: What can you say about the period of the fraction $\frac{1}{n}$ when n has prime factors besides 2 and 5?

Imagine you are doing the "Dots & Boxes" division to compute the decimal representation of a unit fraction like $\frac{1}{6}$. You start with a single dot in the ones box:



To find the decimal expansion, you "unexplode" dots, form groups of six, see how many dots are left, and repeat.

Draw your own pictures to follow along this explanation:

- When you unexplode the first dot, you get 10 dots in the $\frac{1}{10}$ box, which gives one group of six with remainder of 4.

- When you unexplode those four dots, you get 40 dots in the $\frac{1}{100}$ box, which gives six group of six with remainder of 4.

Since the remainder repeated (we got a remainder of 4 again), we can see that the process will now repeat forever:

- unexplode 4 dots to get 40 in the next box to the right,

- make six groups of 6 dots with remainder 4,

- unexplode 4 dots to get 40 in the next box to the right,

- make six groups of 6 dots with remainder 4,

- unexplode 4 dots to get 40 in the next box to the right,

- make six groups of 6 dots with remainder 4,

- and so on forever . . .

On Your Own

Work on the following exercises on your own or with a partner.

1. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{11}$. Explain how you know for sure the process will repeat forever.

2. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{12}$. Explain how you know for sure the process will repeat forever.

3. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{7}$? How can you be sure the process will eventually repeat?

4. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{9}$? How can you be sure the process will eventually repeat?

Problem 209. Suppose that n is a whole number, and it has some prime factors besides 2's and 5's. Write a convincing argument that:

- The decimal representation of $\frac{1}{n}$ will go on forever (it will not terminate).

- The decimal representation of $\frac{1}{n}$ will be an infinite *repeating* decimal.

- The period of the decimal representation of $\frac{1}{n}$ will be less than n.

Problem 210.

(a) Find the "decimal" expansion for $\frac{1}{2}$ in the following bases. Be sure to show your work.

$$2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \text{ and } \quad 10.$$

(b) Make a conjecture: If I write the decimal expansion of $\frac{1}{2}$ in base b, when will that expansion be finite and when will it be an infinite repeating decimal expansion?

(c) Can you prove your conjecture is true?

6.7 Matching Game

Below, you'll find numbers described in various ways: as fractions, as points on a number line, as decimals, and in a picture. Your job is to match these up in a way that makes sense. Note: there may be more than one fraction to match a given decimal, or more than one picture to match a given point on the number line. So be ready to justify your answers.

Fractions

$(a)\ \frac{1}{5}$	$(b)\ \frac{1}{3}$	$(c)\ \frac{2}{3}$
$(d)\ \frac{9}{8}$	$(e)\ \frac{15}{16}$	$(f)\ \frac{25}{100}$
$(g)\ \frac{3}{4}$	$(h)\ \frac{33}{100}$	$(i)\ \frac{3}{25}$
$(j)\ \frac{1}{4}$	$(k)\ \frac{6}{5}$	$(\ell)\ \frac{2}{5}$
$(m)\ \frac{4}{100}$	$(n)\ \frac{2}{10}$	$(o)\ \frac{1}{2}$

Points on a Number Line

Point 1

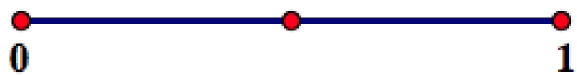
Point 2

Point 3

Point 4

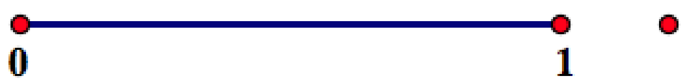
Point 5

Point 6



Point 7

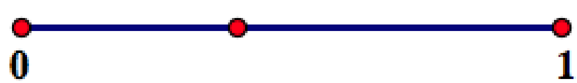
Point 8



Point 9

Point 10

Point 11



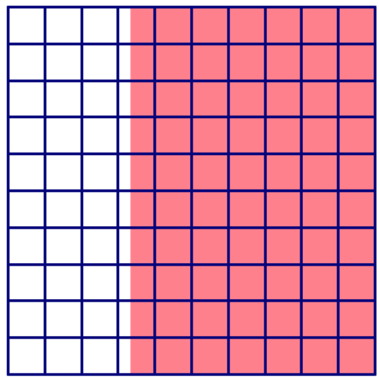
Point 12

Point 13

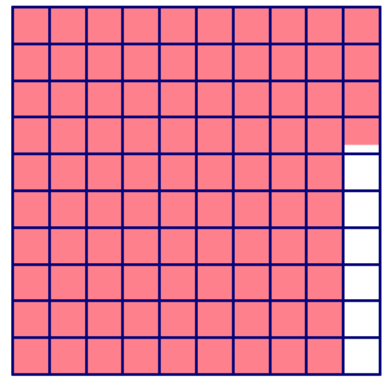
Decimals

(i) 1.20	(ii) $0.\bar{6}$	(iii) 0.33
(iv) 0.25	(v) 0.5	(vi) 0.25
(vii) 0.75	(viii) $0.\bar{3}$	(ix) 0.2
(x) 1.125	(xi) 0.12	(xii) 0.04
(xiii) 0.40	(xiv) 0.20	(xv) 0.9375

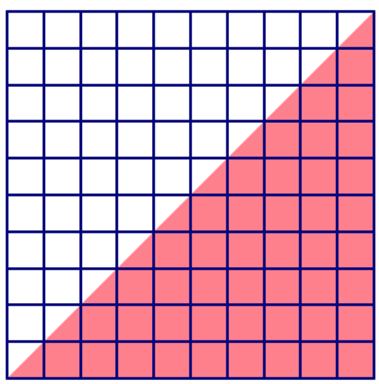
Pictures



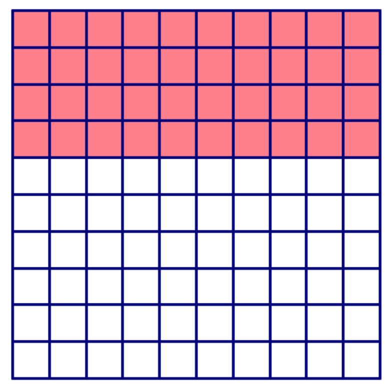
Picture A



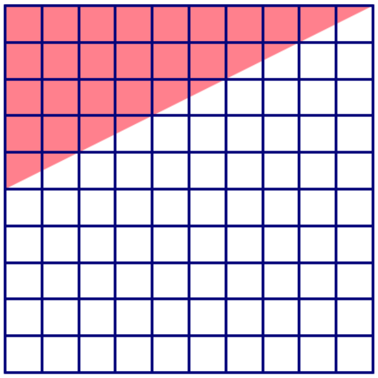
Picture B



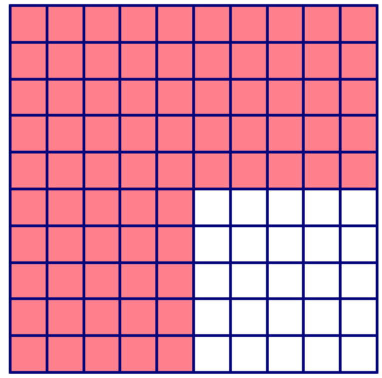
Picture C



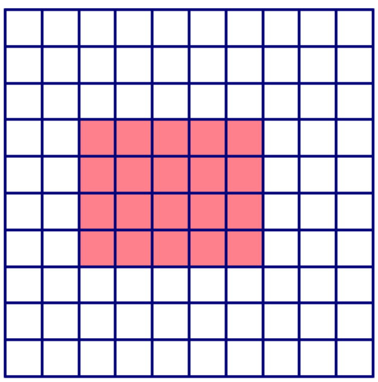
Picture D



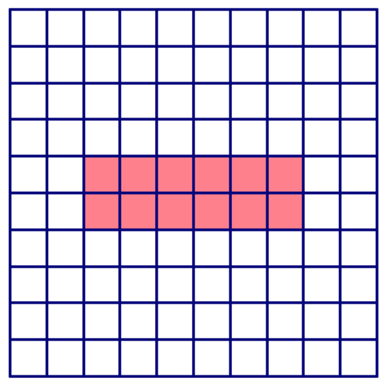
Picture E



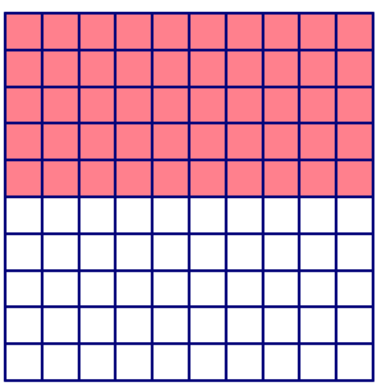
Picture F



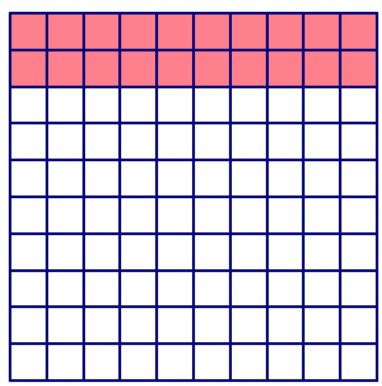
Picture G



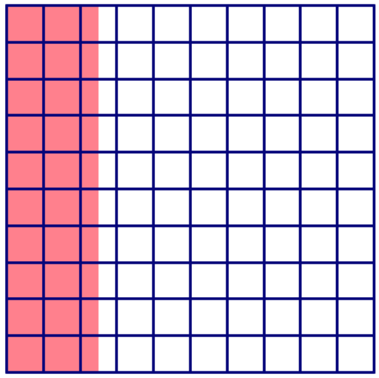
Picture H

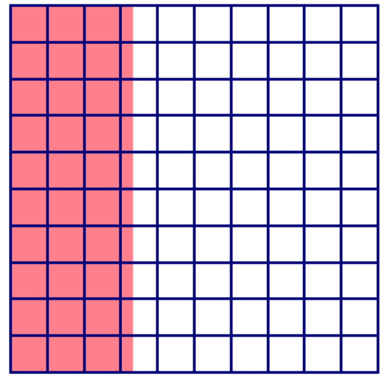


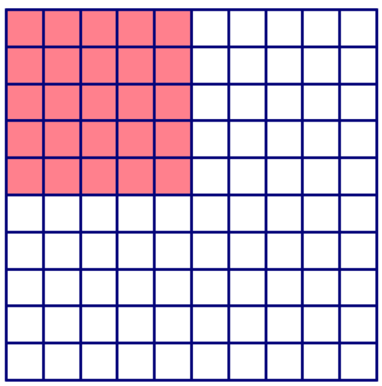
Picture I

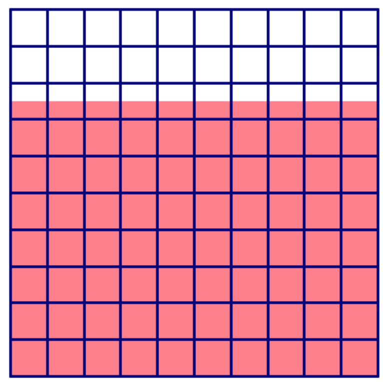


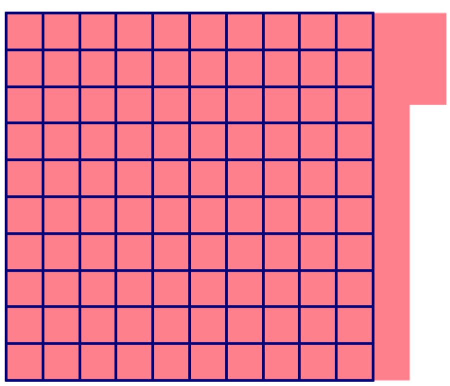
Picture J

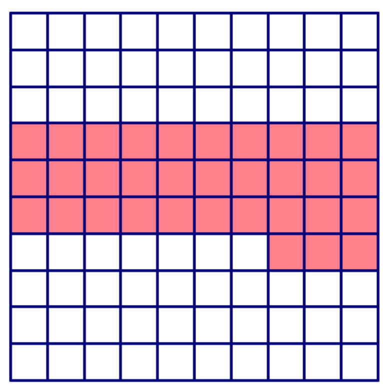

Picture K

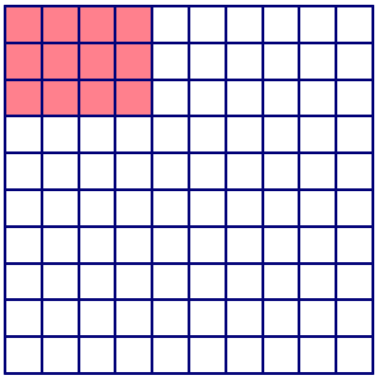

Picture L


Picture M

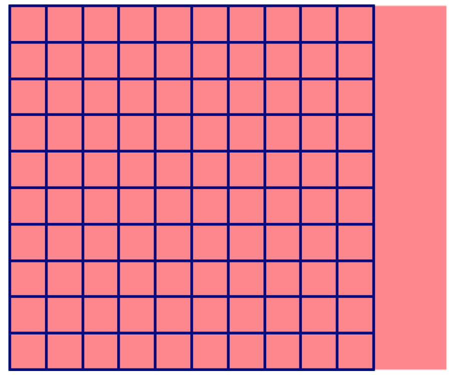

Picture N


Picture O


Picture P



Picture Q



Picture R

6.8 Operations on Decimals

Sometimes computing with decimals is most natural if you think of them as fractions and use what you know there. So let's start by reviewing some computations with fractions.

Think/Pair/Share.

- Compute each of these sums in your head, and explain how you did it:

$$\frac{1}{5}+\frac{2}{5} \qquad \frac{1}{2}+\frac{1}{2} \qquad \frac{4}{7}+\frac{5}{7}.$$

- Solve each missing added computation in your head, and explain how you did it:

$$\frac{1}{5}+\_\_\_\_=\frac{4}{5} \qquad \frac{1}{3}+\_\_\_\_=\frac{11}{3} \qquad \frac{2}{13}+\_\_\_\_=1.$$

You know how to add and subtract fractions with the same denominator; simply add or subtract the numerators and keep the denominator the same. When the fractions have different denominators, you can find equivalent fractions with the same denominator and then proceed.

On Your Own

Work on the following exercise on your own or with a partner.

1. For each computation shown, first rewrite the computation using fractions, then compute. Translate your answer back to a decimal. Show

all of your work.

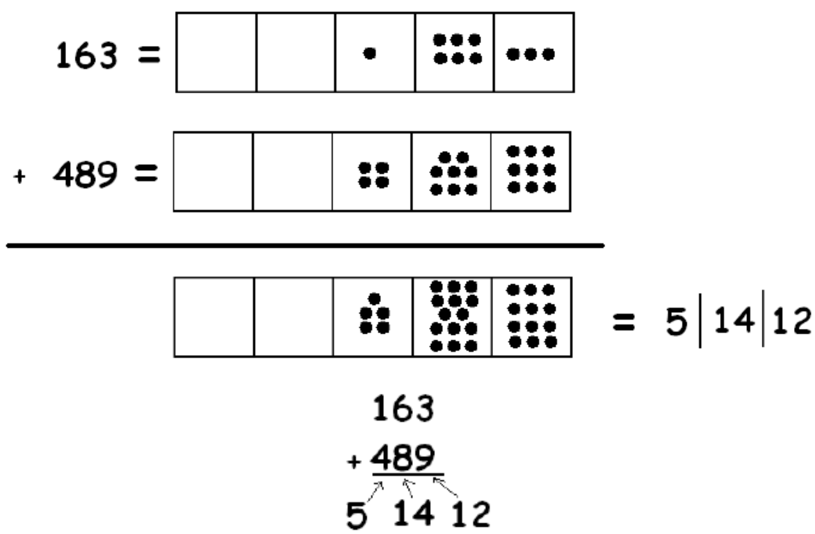
$0.1 + 0.5$	$0.23 + 0.04$	$0.121 + 0.297$
$0.23 + 0.012$	$0.101 + 0.099$	$0.73 + 0.025$
$0.3 + \underline{\qquad} = 0.31$	$0.88 + \underline{\qquad} = 0.93$	$0.49 + \underline{\qquad} = 1$
0.7×0.6	0.002×0.003	5.12×0.3

6.8.1 Adding and Subtracting Decimals

We can always add and subtract decimal numbers by rewriting them as fractions and using the algorithms we know there. Of course, sometimes it is a lot more work to convert to fractions than it is to just add the decimals (as long as you know what you're doing!). So let's think about place value and adding decimals without all of that conversion back and forth.

Remember that when we used the "Dots & Boxes" model to add and subtract, it looked like this.

Example 6.8.1 $(163 + 489)$. Consider $163 + 489$.

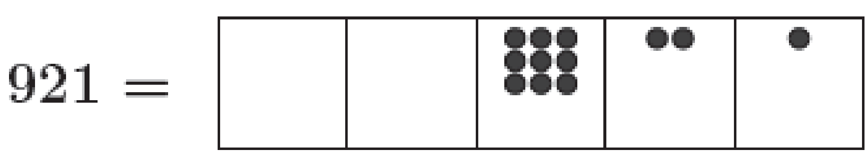


We then perform explosions until there are fewer than ten dots in each box, and we find that

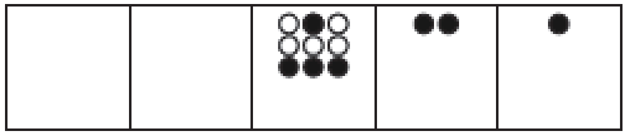
$$163 + 489 = 652.$$

Subtraction was a little more complicated.

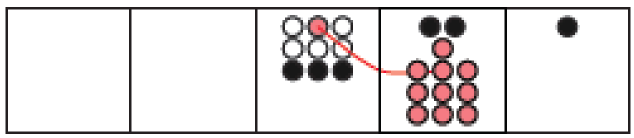
Example 6.8.2 $(921 - 551)$. We start with the representation of 921:



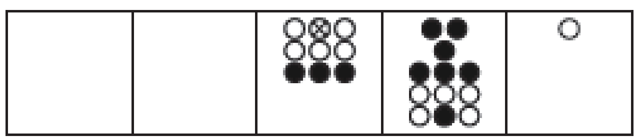
Since we want to “take away” 551, that means we take away five dots from the hundreds box, leaving four dots.

921−551 = 

Now we want to take away five dots from the tens box, but we can’t do it! There are only two dots there. So we can “unexplode” a hundreds dot, and put ten dots in the tens box instead. Then we’ll be able to take five of them away, leaving seven.

921−551 = 

Of course, we now have one less dot in the hundreds box; there’s only three dots left there. Finally, we want to take one dot from the ones box, and that leaves no dots there.

921−551 = 

We conclude that

$$921 - 551 = 370.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. For each calculation, draw a “Dots & Boxes” model and use it to find the result of the calculation.

$$3.56 + 7.95 \qquad 1.452 + 32.27 \qquad 3.0205 + 409.2019$$

$$15.225 - 7.209 \qquad 14.793 - 8.95 \qquad 12.5 - 3.0002$$

2. For each calculation below, add the decimals quickly, and say why your method is faster than converting to fractions and finding a common denominator.

$$0.0066 + 0.9 \qquad 0.25 + 0.0088 \qquad 0.\overline{20} + 0.\overline{01}$$

Think/Pair/Share.

- Chloe added 0.2 and 0.02 and got an answer of 0.04. What was Chloe's likely mistake? As her teacher, how could you help Chloe understand the operation of addition better?

- In elementary school, students are taught to add and subtract decimals by "lining up the decimal points." Use the "Dots & Boxes" model to explain why this shorthand makes sense.

6.8.2 Multiplying and Dividing: Powers of 10

We already reviewed the "Dots & Boxes" model for division of whole numbers. Look back at Example 6.4.2 on page 261 and Example 6.4.3 on page 262. if you need to refresh your memory. Let's quickly review the "Dots & Boxes" model for multiplication of whole numbers before we get back to talking about decimals.

Example 6.8.3 (243192×4). If we want to compute 243192×4, it helps to remember what multiplication *means*. One interpretation is: I want to add 243192 to itself a total of four times. So there will be:

- 2×4 dots in the ones place,
- 9×4 dots in the tens place,
- 1×4 dots in the hundreds place,
- ... and so on.

Here's the start of the computation:

$$243192 \times 4 = 8|16|12|4|36|8$$

To finish the computation, we need to do some explosions to write the result as a familiar base 10 number:

$$243192 \times 4 = 972768.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Do each computation, using reasoning like in the multiplication example above.

$$2.3 \times 10 \qquad 3.56 \times 10 \qquad 1.452 \times 100$$

2. Do each computation, using reasoning like in the division examples on pages 261 and 262.

$$7.1 \div 10 \qquad 98.55 \div 10 \qquad 145.2 \div 100$$

In Section 6.1, you used the "Dots & Boxes" model to explain why why multiplying a whole number by 10 (when it's written is base ten) results in appending a zero to the right end of the number. Your work above should convince this does not work for decimals!

Think/Pair/Share.

- Write a new rule that works for both whole numbers and decimals:

  If I multiply a whole number or a decimal by 10*, a simple way to find the result is* ______________________________.

- Justify the claim you made above!

- One can go much further with this thinking. What is the effect of dividing a number written in decimal notation by ten? By one-hundred? Justify what you say.

6.8.3 Multiplying Decimals

You probably know an algorithm for multiplying decimal numbers by hand. But if you think carefully about the algorithm, it should *make sense* based on what the decimal numbers represent and what it means to multiply. Let's start by using number sense to think about multiplying whole numbers by decimals.

Think/Pair/Share. Consider the equation

$$16 \times \square.$$

Fill in the box with a whole number or decimal so that the product is:

- Greater than 100.
- Greater than 64 but less than 100.
- At least 17, but less than 32.
- Equal to 16.
- Greater than 8 but less than 16.
- Less than 8, but greater than 0.

Be sure to justify your answers. You should use your number sense rather than computing by hand or with a calculator!

Earlier in this Section, you multiplied decimal numbers by converting them to fractions and then using what you know about multiplying fractions. There are other ways to think about multiplying that focus on number sense rather than on the mechanics of computation.

Example 6.8.4 (321×0.4). Suppose a student wanted to compute 321×0.4, but he didn't already know the standard algorithm. What might he do? Here is one idea:

I know that $321 \times 4 = 1284$. *Since I want to multiply by* $0.4 = \frac{4}{10}$ *and not by* 4, *my answer should be* $\frac{1}{10}$ *of this one. So*

$$321 \times 0.4 = 128.4.$$

You should notice that the student is using the *associative property* of multiplication:

$$321 \times 0.4 \;=\; 321 \times \frac{4}{10} \;=\; 321 \times \left(4 \times \frac{1}{10}\right) \;=\; (321 \times 4) \times \frac{1}{10}.$$

Problem 211. For each computation below, the result of the computation is shown correctly, but the decimal point is missing. Use number sense and reasoning to correctly place the decimal point, and briefly justify how you know you're right. (Don't use a calculator, don't work out the multiplication by hand, and don't use the trick of "counting the number of decimal places." Use your number sense!)

(*a*) $855 \times 1.7 = 14535$

(*b*) $549 \times 0.33 = 18117$

(*c*) $2.03 \times 1028 = 208684$

(*d*) $999 \times 0.53 = 52947$

(*e*) $30.02 \times 472 = 1416944$

(*f*) $173 \times 0.09 = 1557$

On Your Own

Work on the following exercises on your own or with a partner.

1. Write each number given as a fraction. (Write them as "improper fractions," not "mixed numbers.")

   15.2   3.43   0.0021   13.02026

2. In exercise (1) above, how does the number of digits to the right of the decimal point compare to the number of zeros in the denominator? Use what you know about place value to explain why your answer is always true (not just for the examples above).

3. Find each product.

   10×10000 $\qquad$ 100×1000 $\qquad$ 100000×1000 $\qquad$ $10^m \times 10^n$

4. In exercise (3) above, how is the number of zeros in the product related to the number of zeros in the two factors? Use what you know about place value to explain why your answer is always true (not just for the examples above).

5.

(a) If you write 0.037 as a fraction, how many zeros would be in the denominator?

(b) What if you write 0.59 as a fraction? How many zeros would be in the denominator?

(c) So how many zeros would be in the denominator of the product of 0.037 and 0.59? (Don't compute the product to answer this question!)

6. Use the fact that $37 \times 59 = 2183$ and your answers to the exercises above to find 0.037×0.59. Explain how you got your answer.

The standard algorithm for multiplying decimal numbers can be described this way:

Step 1 Compute the product as if the two factors were whole numbers. (Ignore the decimal points.)

Step 2 Count the number of digits to the right of the decimal point in each factor, and add those numbers together. Call the result n.

Step 3 The sum n that you found in Step 2 will be the number of digits to the right of the decimal point in the product. So place the decimal point according by counting the appropriate number of places from the right.

Think/Pair/Share.

- Write down two examples of multiplying decimal numbers using the standard algorithm above.

- Use what you know about place value, fractions, and multiplication to *carefully explain why* the standard algorithm described above makes sense.

6.8.4 Dividing Decimals

As you might expect, dividing decimals is perhaps more complicated to explain than any of the other operations. It's hard to adapt our "Dots & Boxes" model for division. Suppose we want to compute $15.37 \div 0.013$. We can certainly draw the picture for 15.37, but how could we make groups of 0.013 dots?

Think/Pair/Share. Let's start by sharing what you already know. Perform this computation (by hand, not with a calculator), showing all of your work. Explain your method to a partner, and see if your partner computed the same way.

$$0.0351 \div 0.074$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Explain why these two fractions are equivalent.
$$\frac{12.33}{44.1} \quad \text{and} \quad \frac{123.3}{441}.$$

2. Explain why these two division computations give the same result.
$$12.33 \div 44.1 \quad \text{and} \quad 123.3 \div 441.$$

3. Explain why these three fractions are equivalent.
$$\frac{325.5}{75.133}, \quad \frac{3255}{751.33}, \quad \text{and} \quad \frac{32550}{7513.3}.$$

4. Explain why these three division computations give the same result.

$$325.5 \div 75.133, \qquad 3255 \div 751.33, \qquad \text{and} \qquad 32550 \div 7513.3.$$

5. Fill in the box to make the equation true. Be sure to justify your answer.

$$\frac{325.5}{75.133} = \frac{\square}{75133}.$$

The standard algorithm for dividing numbers represented by finite decimal expansions is something like this:

Step 1 Move the decimal point of the divisor to the end of the number.

Step 2 Move the decimal point of the dividend the same number of positions (the same distance and direction).

Step 3 Divide the new decimal dividend (from Step 2) by the new whole number divisor (from Step 1). Since we're dividing by a whole number, our standard methods make sense.

This is a pretty mechanical description, and doesn't give a lot of insight into *why* this algorithm works.

Think/Pair/Share. Write down at least two examples of computing with the algorithm described above. (Make up your own numbers to test. Be sure to show every step clearly.) You can do the division by drawing a "Dots & Boxes" picture or by another method (but don't use a calculator). Then answer these more general questions.

- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10 \cdot a) \div (10 \cdot b)$ will give the same result.

- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(100 \cdot a) \div (100 \cdot b)$ will give the same result.

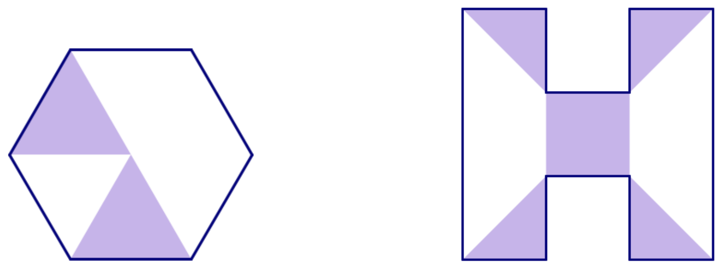
- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10^k \cdot a) \div (10^k \cdot b)$ will give the same result, where k is any whole number.

- Suppose b has a finite decimal expansion. Carefully explain why you can find a power of 10 so that $10^k \cdot b$ is a whole number.

Problem 212. Carefully explain *why* the algorithm described above in three steps works for computing division of decimal numbers. You need to explain what is going on when you "move the decimal point" in Steps 1 and 2, and why the result you compute in Step 3 is the same as the original problem.

6.9 Problem Bank

Problem 213. Express the shaded portion of each figure as both a fraction and as a decimal. Justify your answers.



(a) (b)

Problem 214. Which number is bigger: 0.135 or 0.14? Justify your answer in at least two different ways.

Problem 215. Yu said that 0.21 is bigger than 0.7 since 21 is bigger than 7. Is Yu correct? Carefully explain your reasoning. If you were Yu's teacher, how would you respond to him?

Problem 216. Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible sum. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square}+\frac{\square}{\square}$$

Problem 217. Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible (positive) difference. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square}-\frac{\square}{\square}$$

Problem 218. Use the "Dots & Boxes" model to show that $\frac{1}{9} = 0.\bar{1}$. Then use this fact to answer these questions and justify your answers.

(a) What fraction is given by $0.\bar{2}$?

(b) What fraction is given by $0.\bar{5}$?

(c) What fraction is given by $0.\bar{6}$?

(d) What fraction is given by $0.\bar{8}$?

(e) What fraction is given by $0.\bar{9}$?

Problem 219. In this problem, you will focus on the calculation

$$170 \times \square.$$

Your goal is to get a product that is close to 200.

(a) Will you multiply 170 by a number greater or less than 1? Greater or less than 2? Justify your answers.

(b) Suppose you can use only one decimal place. Fill in the box with a number that gets as close to 200 as possible.

(c) Suppose you can use only two decimal places. Fill in the box with a number that gets as close to 200 as possible.

(d) Suppose you can use only three decimal places. Fill in the box with a number that gets as close to 200 as possible.

Problem 220. Do each computation below *without using a calculator*. Explain your thinking.

(a) $(23 \times 0.1) + (0.001 \times 55)$

(b) $18.45 \div 0.63 \div 0.7$

(c) $22.65 - (0.03 \cdot 10)$

Problem 221. For each question below, choose the correct calculation and explain your choice. Then estimate the answer (don't calculate it exactly) and explain why your estimate is a good one.

(a) A large pizza has eight slices and costs \$15.95. How much does each slice of pizza cost? Should you calculate 15.95×8 or $15.85 \div 8$?

(b) There are 2.54 centimeters in an inch. A standard sheet of notebook paper is $8\frac{1}{2}$ inches wide and 11 inches long. How many centimeters wide is the page? Should you calculate 8.5×2.54 or 11×2.54 or $8.5 \div 2.54$ or $11 \div 2.54$?

(c) In a model train set, 1.38 inches represents one foot in real life. The height of One World Trade Center in New York City is 1776 feet. How tall would a scale model of the building be? Should you calculate 1776×1.38 or $1776 \div 1.38$?

(d) Eight-tenths of a jumprope is 1.75 meters long. How long is the whole rope? Should you compute $0.8 \div 1.75$ or 0.8×1.75 or $1.75 \div 0.8$?

Problem 222. Without actually calculating anything (just use your number sense!), order x, y, and z from smallest to largest. Explain your ordering.

$$x = 0.07 + 0.000001$$

$$y = 0.07 \times 0.000001$$

$$z = 0.07 \div 0.000001$$

Problem 223. Kaimi had no money at all when he cashed his paycheck. As he left the bank, he bought a piece of candy for a nickel from a machine. Later, he realized that the money in his pocket was equal to twice his paycheck. After a quick calculation, he figured out what happened: the teller accidentally switched the dollars and cents. How much was Kaimi supposed to be paid, and what did the teller give him? Justify your answer.

Problem 224. Here's the rules to a card game. Read the rules carefully and then answer the questions below.

- Each player starts with 10 points. The goal is to score as close to 100 points as possible without going over.

- On your turn: draw two cards, which will each have a decimal number on them. Using estimation (no computation), you can choose to multiply or divide your current score by one of the decimal numbers.

- After you decide, compute your new score exactly using a calculator. If your new score is over 100, you lose. If not, the other player takes a turn.

- At the end of your turn, you can decide to end the game. If you do, the other player gets one more turn. Then, the player with the score that is closest to 100 without going over wins the game.

Here are the questions:

(a) On your turn, your score is 50. You draw the cards 0.2 and 1.75. Remember that your choices are:

divide by 0.2	multiply by 0.2
divide by 1.75	multiply by 1.75

What is your best move and why?

(b) On your turn, your score is 88. You draw 1.3 and 0.6. What is your best move and why?

(c) Your partner has a score of 57, and your score is 89. On her turn, your partner draws 0.8 and 1.8. She says she wants to end the game. On your final turn, you draw 0.7 and 1.2. If you both make the best possible move, who will win the game?

Chapter 7

Geometry

The word "geometry" comes from the ancient Greek words "geo" meaning Earth and "metron" meaning measurement. It is probably the oldest field of mathematics, because of its usefulness in calculating measurements of lengths, areas, and volumes of everyday objects.

The study of geometry has evolved a great deal during the last 3,000 years or so. Like all of mathematics, what's really important in geometry is *reasoning, making sense of problems, and justifying your solutions.*

The mathematician Henri Poincaré said that

> *"Geometry is the art of good reasoning from bad drawings."*

This insight should guide your study in this chapter. You should never trust a drawing. You might find that one line segment *appears* to be longer than another, or an angle looks like it *might be* 90 degrees. But "appears to be" and "might be" are simply not good enough. You have to reason through the situation and figure out what you *know for sure* and why you know it.

Think/Pair/Share. Reflect on your learning of geometry in the past. What is geometry really about? Also think about these questions:

- What is a *point*?

- What is a *line*? A *segment*? A *ray*?

- What is a *plane*?

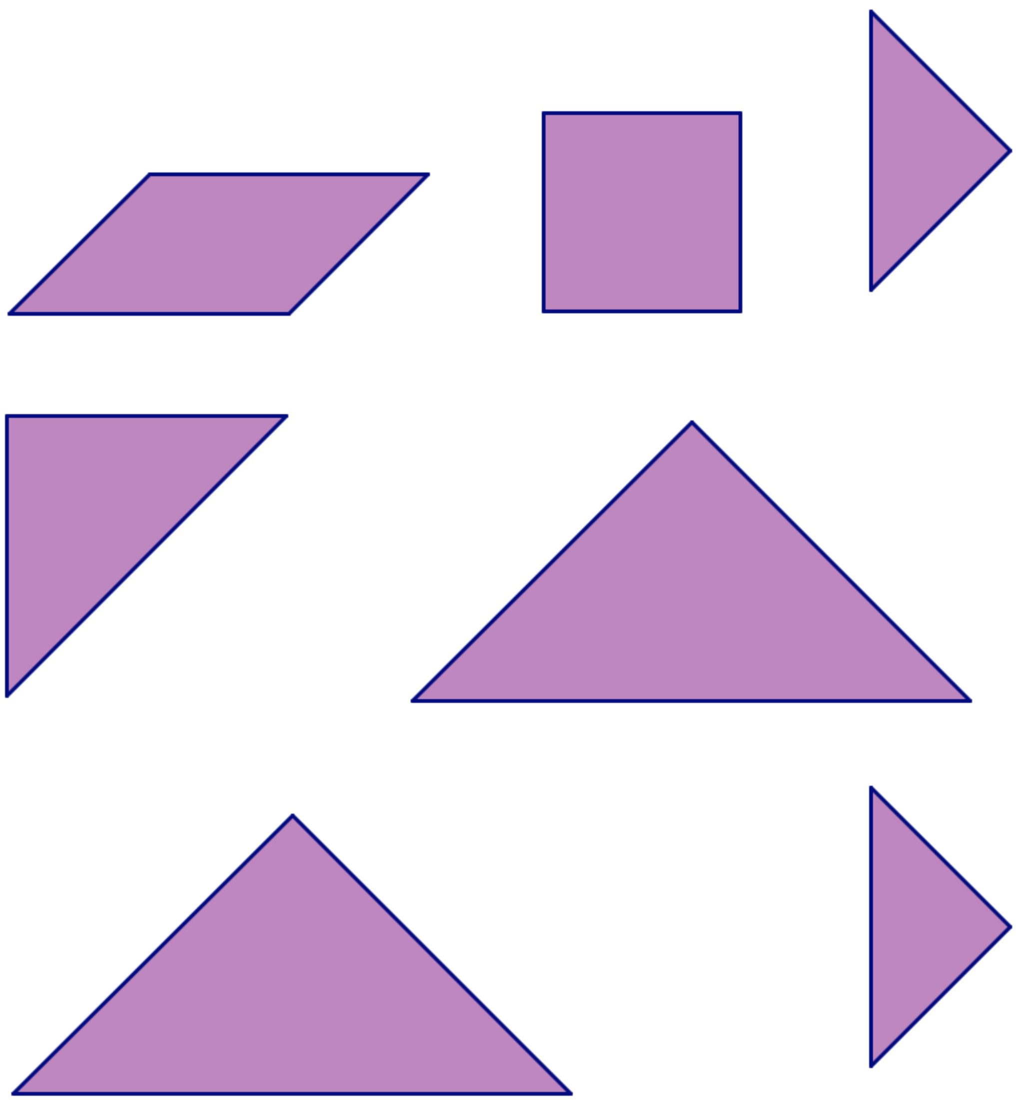
- What is a *circle*?

- What is an *angle*?

- Which of these basic objects can be *measured*? How are they measured? What kinds of tools are useful?

7.1 Tangrams

Tangrams are a seven-piece geometric puzzle that dates back at least to the Song Dynasty in China (about 1100 AD). Below you will find the seven puzzle pieces. Make a careful copy (a photocopy or printout is best), cut out the puzzle pieces, and then use them to solve the problems in this section. You can trace around your solutions to remember what you have done and to have a record of your work.



Whenever you solve a tangram puzzle, your job is to use all seven pieces to form the shape. They should fit together like puzzle pieces, sitting flat on the table; no overlapping of the pieces is allowed.

Problem 225. Use all seven pieces to form a square.

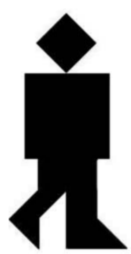
Problem 226. Use all seven pieces to form a rectangle that is not a square.

Problem 227. Use all seven pieces to form a right triangle.

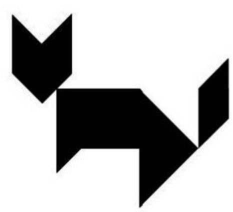
Problem 228. Use your tangram pieces to build the following designs. Remember: You need to use all seven pieces, and they should fit together, not overlap.

(a)

(b)



(c)



(d)

(e)

(f)

Think/Pair/Share. Which tangram problems were easier and which were harder: making "real life" objects like cats and people, or purely mathematical objects like the square? What do you think made one kind of problem easier or harder?

7.2 Triangles and Quadrilaterals

Think/Pair/Share. Follow these directions on your own:

- Draw any triangle on your paper.

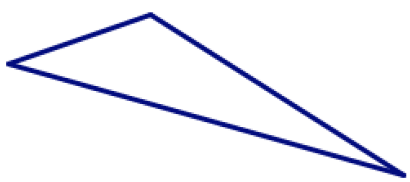
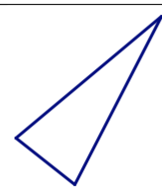
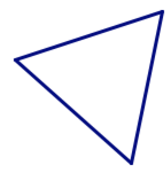
- Draw a second triangle that is different in some way from your first one. Write down a sentence or two to say how it is different.

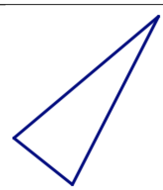
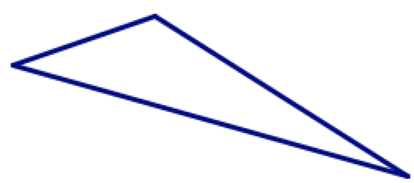
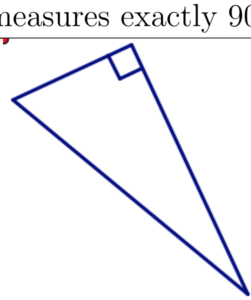
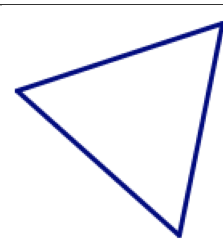
- Draw a third triangle that is different from both of your other two. Describe how it is different.

- Draw two more triangles, different from all the ones that came before.

Compare your triangles and descriptions with a partner. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features that you or your partner changed.

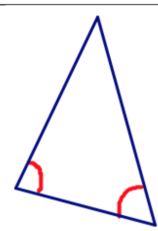
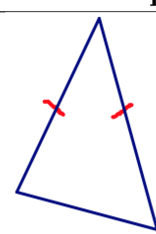
Triangles are classified according to different properties. The point of learning geometry is not to learn a lot of vocabulary, but it's useful to use the correct terms for objects, so that we can communicate clearly. Here's a quick dictionary of some types of triangles.

Classification by Sides		
scalene	**isosceles**	**equilateral**
all sides have different lengths	two sides have the same length	all three sides have the same length
		

Classification by Angles	
acute	**obtuse**
all interior angles measure less than 90°	one interior angle measures more than 90°
	
right	**equiangular**
one interior angle measures exactly 90°	all interior angles have the same measure
	

Remember that "geometry is the art of good reasoning from bad drawings." That means you can't always trust your eyes. If you look at a picture of a triangle and one side *looks like* it's longer than another, that may just mean the drawing was done a bit sloppily.

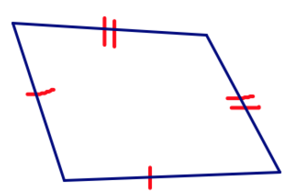
Mathematicians either write down measurements or use *tick marks* to indicate when sides and angles are supposed to be equal. If two sides have the same measurement attached or the same number of tick marks, you must believe they are equal and work out the problem accordingly, even if it doesn't look that way to your eyes. One example of these marks is the little square used to indicate a right angle in the picture above. Here are some other examples.

two angles equal	**two sides equal**
	

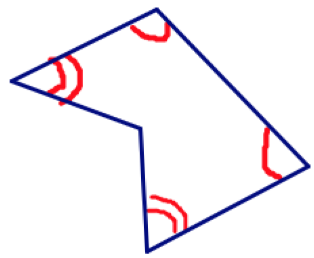
On Your Own

Work on these exercises on your own or with a partner.

1. In the picture below, which sides are understood to have the same length?



2. In the picture below, which angles are understood to have the same measure?



3. Sketch two more scalene triangles, each of which is different from the one shown on page 327 in some way.

4. Sketch two more acute triangles, each of which is different from the one shown on page 327 in some way.

5. Sketch two more obtuse triangles, each of which is different from the one shown on page 327 in some way.

6. Sketch two more right triangles, each of which is different from the one shown on page 327 in some way. Be sure to indicate which angle is 90°.

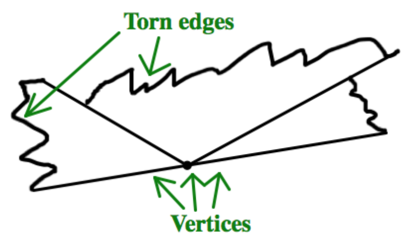
7. Sketch two more isosceles triangles, each of which is different from the one shown on page 327 in some way. Use tick marks to indicate which sides are equal.

7.2.1 Angle sum

Think/Pair/Share. By now, you have drawn several different triangles on your paper. Choose one of your triangles, and follow these directions:

- Using scissors, cut the triangle out.

- Tear (do not cut) off the corners, and place the three vertices together. Your should have something that looks a bit like this picture:



♣♣♣ Fellow: [This might make a nice animation, tearing off the corners and arranging them in this way.]

What do you notice? What does this suggest about the angles in a triangle?

You may remember learning that the sum of the angles in any triangle is 180°. Depending on how many students are in your class, you now have maybe 10–20 *examples* of triangles where the sum of the angles *seems to be* 180°. But remember, our drawings are not exact. How can we be sure that our eyes are not deceiving us? How can we be sure that the sum of the angles in a triangle isn't 181° or 178°, but is really 180° on the nose in every case?

Think/Pair/Share. What would convince you **beyond all doubt** that the sum of the angles in any triangle is 180°? Would testing lots of cases be enough? How many is enough? Could you ever test *every possible triangle*?

♣♣♣ Fellow: [add a picture of Euclid here?] In about 300BC, Euclid was the first mathematician (as far as we know) who tried to write down careful *axioms* and then build from those axioms rigorous proofs of mathematical truths. He had five axioms for geometry, the first four of which seemed pretty obvious to mathematicians. People felt they were reasonable assumptions from which to build up geometric truths:

1. Given two points, you can connect them with a straight line segment.

2. Given a line segment, you can extend it as far as you like in either direction, making a line.

3. Given a line segment, you can draw a circle having that segment as a radius.

4. All right angles are congruent.

The fifth postulate bothered people a bit more. It was originally stated in more flowery language, but it was equivalent to this statement:

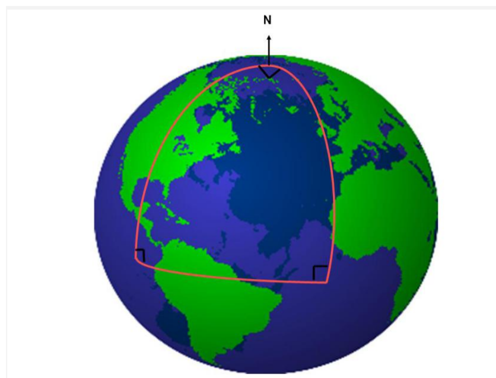
5. The sum of the angles in a triangle is 180°.

Often high school geometry teachers prove the sum of the angles in a triangle is 180°, usually using some facts about parallel lines. But (maybe surprisingly?) it's just as good to take this as an *axiom*, as a given fact about how geometry works, and go from there. Perhaps this is less satisfying than proving it from some other statement, and if you're curious you can certainly find a proof or your instructor can share one with you.

It's easy to see why this fifth axiom caused such a ruckus in mathematics. It seemed much less obvious than the other four, and mathematicians felt like they were somehow cheating if they just *assumed* it rather than *proving* it had to be true. Many mathematicians spent many, many years trying to prove this fifth axiom from the other axioms, but they couldn't do it. And with good reason: There are other kinds of geometries where the first four axioms are true, but the fifth one is not!

For example, if you do geometry on a *sphere* — like a basketball or more importantly on the surface of the Earth — rather than on a flat plane, the first four axioms are true. But triangles are a little strange on the surface of the earth. Every triangle you can you can draw on the surface of the earth has an angle sum strictly greater than 180°. In fact, you can draw a triangle on the Earth that has three right angles, making an angle sum of 270°.

♣♣♣ Fellow: [I would really like this to be a better picture. Maybe just a sphere with a triangle drawn on it, not a globe. It would be nice if the right angles were marked.]

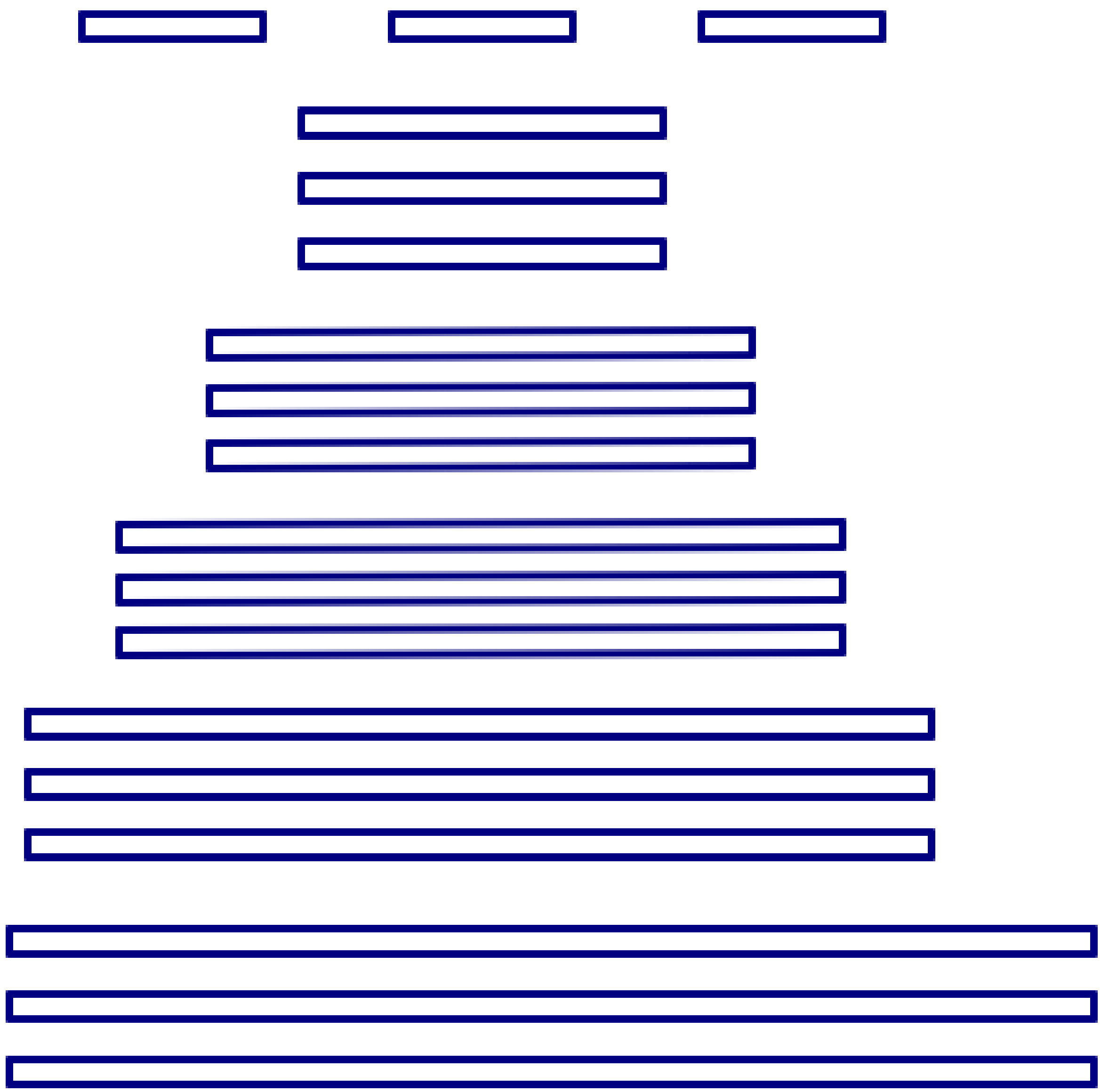


On a sphere like the Earth, the angle sum is not constant among all triangles. Bigger triangles have bigger angle sums, and smaller triangles have smaller angle sums, but even the small triangles have angle sums that are greater than 180°.

The geometry you study in school is called *Euclidean geometry*; it is the geometry of a flat plane, of a flat world. It's a pretty good approximation for the little piece of the Earth that we see at any given time, but it's not the only geometry out there!

7.2.2 Triangle inequality

Make a copy of these strips of paper and cut them out. They have unit lengths from 1 unit to 6 units. You may want to color them, write numbers on them, or do something that makes it easy to keep track of the different units.



Problem 229. Repeat the following process several times (at least 10) and keep track of the results (a table has been started for you on the next page):

- Pick three strips of paper. (The lengths do not have to be all different from each other; that's why you have multiple copies of each length.)

- Try to make a triangle with those three strips, and decide if you think it is possible or not. (Don't overlap the strips, cut them, or fold them. The length of the strips should be the length of the sides of the triangle.)

Your goal is to come up with a **rule** that describes when three lengths will make a triangle and when they will not. Write down the rule in your own words.

length 1	length 2	length 3	triangle?
4	3	2	yes
4	2	1	no

Think/Pair/Share. Compare your rule with other students. Then use your rule to answer the following questions.

- Suppose you were asked to make a triangle with sides 40 units, 40 units, and 100 units units long. Do you think you could do it? Explain your answer. Keep in mind the goal is not to try to build the triangle, but to predict the outcome.

- Suppose you were asked to make a triangle with sides 2.5 units, 2.6 units, and 5 units units long. Do you think you could do it? Explain your answer. Keep in mind the goal is not to try to build the triangle, but to predict the outcome.

Of course, we know that in geometry we should not believe our eyes. You need to look for an *explanation.* Why does your statement make sense?

You probably came up with some version of this statement:

Triangle Inequaltiy
The sum of the lengths of two sides in a triangle is
greater than the length of the third side.

Remember that "geometry is the art of good reasoning from bad drawings." Our materials weren't very precise, so how can we be sure this rule we've come up with is is correct?

Well in this case, the rule is really just the same as the saying "the shortest distance between two points is a straight line." In fact, this is exactly what we mean by the words *straight line* in geometry.

7.2.3 SSS Congruence

We say that two triangles (or any two geometric objects) are *congruent* if they are exactly the same shape and the same size. That means that if you could pick one of them up and move it to put down on the other, they would exactly overlap.

Problem 230. Repeat the following process several times and keep track of the results.

- Pick three strips of paper that will definitely form a triangle.

- Try to make two different (non-congruent) triangles with the same three strips of paper. Record if you were able to do so.

Problem 231. Repeat the following process several times and keep track of the results.

- Pick four strips of paper and form a quadrilateral with them. (If your four strips do not form a quadrilateral, pick another four strips.)

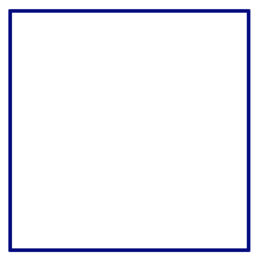
- Try to make two different (non-congruent) quadrilateral with the same four strips of paper. Record if you were able to do so.

Think/Pair/Share. What do you notice from Problems 230 and 231? Can you make a general statement to describe what's going on? Can you explain why your statement makes sense?

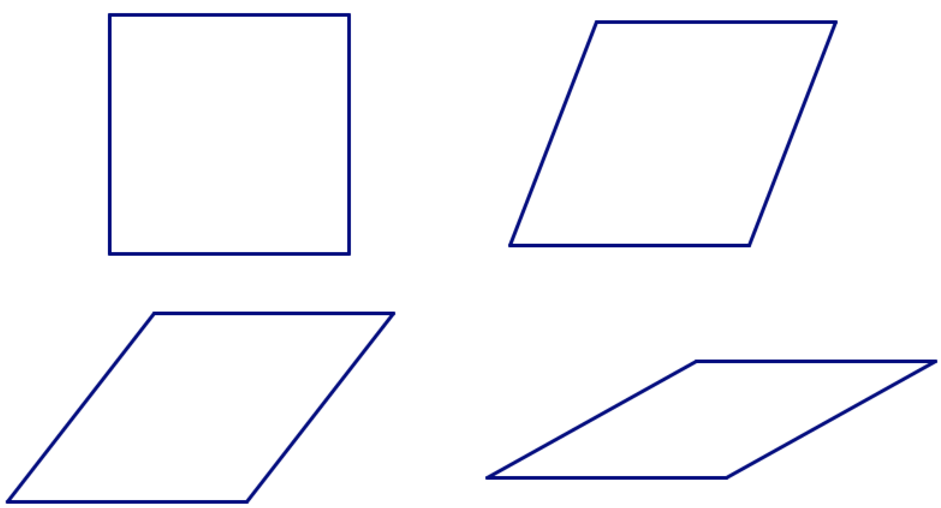
You probably came up with some version of this statement:

SSS (side-side-side) Congruence
If two triangles have the same side lengths,
then the triangles are congruent.

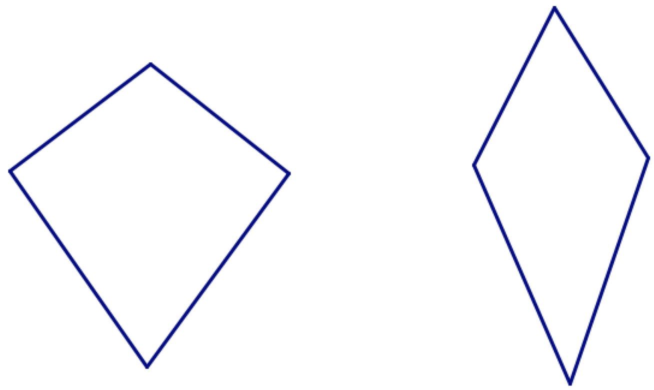
This most certainly is *not* true for quadrilaterals. For example, if you choose four strips that are all the same length, you can make a square:



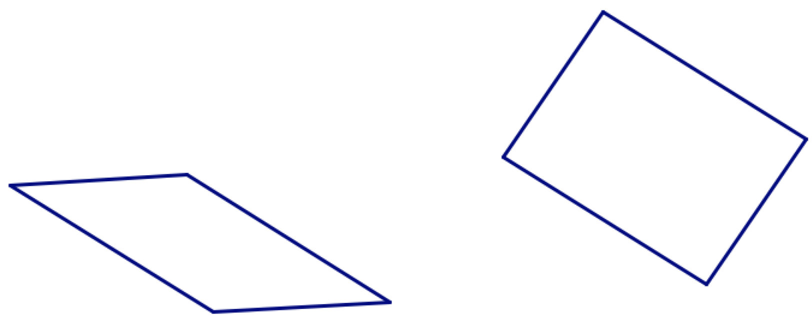
But you can also squish that square into a non-square rhombus. (Try it!)



If you don't choose four lengths that are all the same, in addition to "squishing" the shape, you can rearrange the sides to make different (non-congruent) shapes. (Try it!)



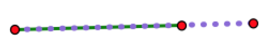
These two shapes have the same four side lengths in the same order.



These two shapes have the same four side lengths as the shapes above, but the sides are in a different order.

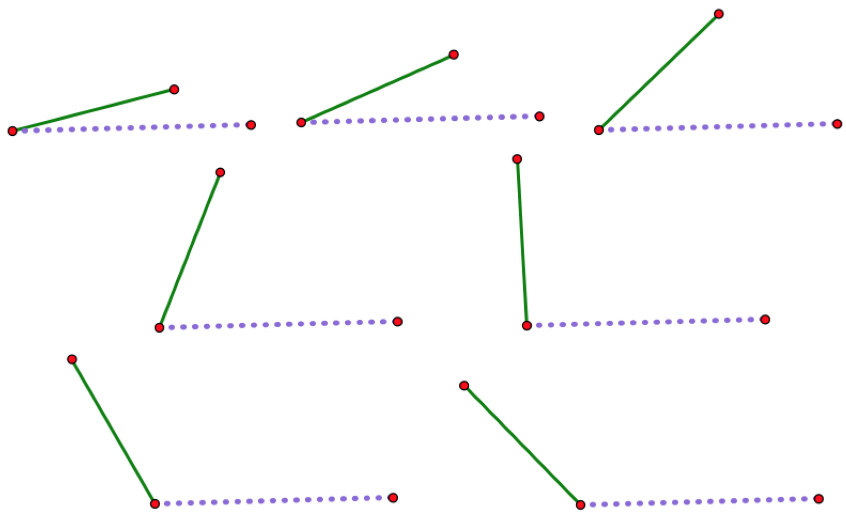
But this can't happen with triangles. Why not? Well, certainly you can't rearrange the three sides. That would be just the same as rotating the triangle or flipping it over, but not making a new shape.

Why can't the triangles "squish" the way a quadrilateral (and other shapes) can? Here's one way to understand it. Imagine you pick two of your three lengths and lay them on top of each other, hinged at one corner.



This shows a longer purple dashed segment and a shorter green segment. The two segments are hinged at the red dot on the left.

Now imagine opening up the hinge a little at a time.



♣♣♣ Fellow: [This is almost certainly better done as an animation!]

As the hinge opens up, the two non-hinged endpoints get farther and farther apart. Whatever your third length is (assuming you are actually able to make a triangle with your three lengths), there is **exactly one** position of the hinge where it will just exactly fit to close off the triangle. No other position will work.

7.3 Polygons

It can seem like the study of geometry in elementary school is nothing more than learning a bunch of definitions and then classifying objects. In this chapter, you'll explore some problem solving and reasoning activities that are based in geometry. But definitions are still important! So let's start with this one.

Definition 7.3.1. A **polygon** is:

1. a *plane* figure
2. that is bounded by a finite number of *straight line segments*
3. in which each segment *meets exactly two others*, one at each of its endpoints.

Think/Pair/Share. Just as the first step in problem solving is to *understand the problem*, the first step in reading a mathematical definition is to *understand the definition*.

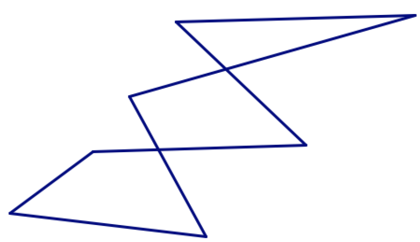
- Use the definition above to draw several examples of figures that are definitely polygons. (You should be able to say why your example fits the definition.)

- Draw several non-examples as well: shapes that are definitely not polygons. (You should be able to say which part of the definition fails for your non-examples.)

A few comments about polygons:

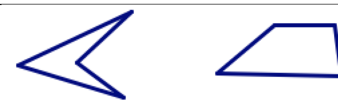
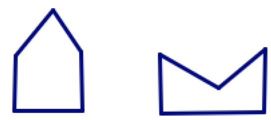
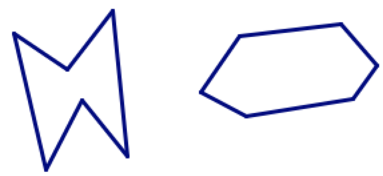
- The line segments that make up a polygon are called its *edges* and the points where they meet are called its *vertices* (singular: vertex).

- Because of properties (2) and (3) in the definition, the boundaries of polygons are not *self-intersecting*.



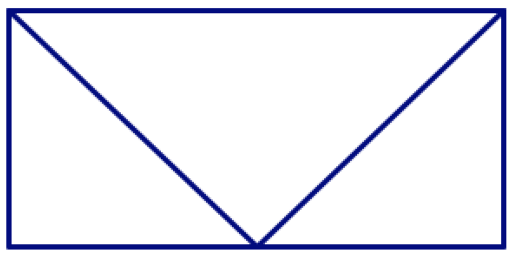
Not a polygon.

- Polygons are named based on the number of sides they have.

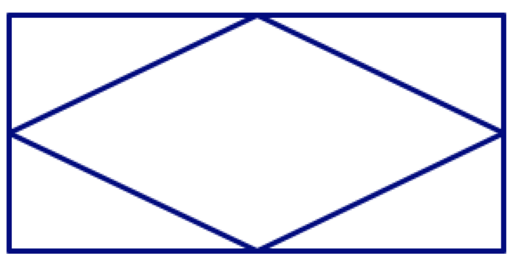
name	**# of sides**	**examples**
triangle	3	
quadrilateral	4	
pentagon	5	
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

- In general, we call a polygon with n sides an n-gon.

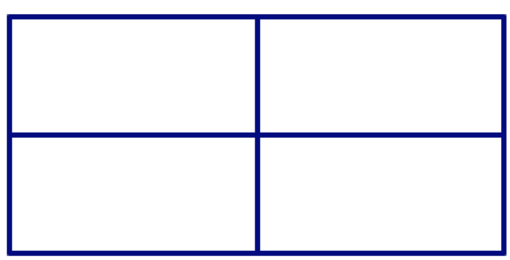
Problem 232. In the pictures below, there are polygons hidden in the design. In each design, find *all* of the triangles, quadrilaterals, pentagons, and hexagons. How can you be sure you've found them all and haven't counted any twice?



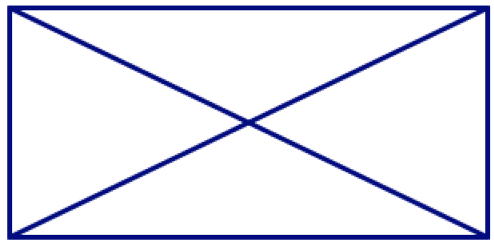
Design 1



Design 2



Design 3



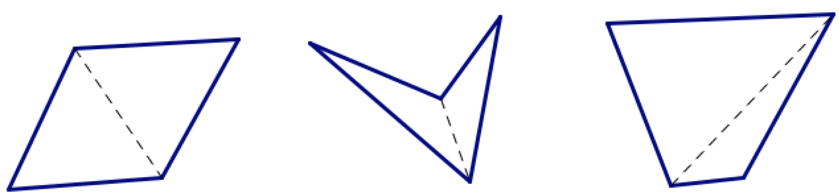
Design 4

7.3.1 Angle sum

You know that the sum of the interior angles in any triangle is 180°. Can you say anything about the angles in other polygons? You probably know that rectangles have four 90° angles. So **if** if all quadrilaterals have the same interior angle sum, it must be $4 \cdot 90° = 360°$.

But notice: We don't necessarily have any reason to believe this constant sum would be true. Remember that SSS congruence is true for triangles, but not for any other polygons. Triangles are special, and we shouldn't *assume* that similar statements will hold for other shapes.

Think/Pair/Share. Any quadrilateral can be split into two triangles, where the vertices of the triangles all coincide with the vertices of the quadrilateral:



Use the pictures above to carefully explain why all quadrilaterals do, indeed, have an angle sum of 360°.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw several different *pentagons* on your paper. Show that each of them can be split into exactly three triangles in such a way that the vertices of the triangles all coincide with the vertices of the pentagon.

2. Use the fact that every pentagon can be split into three triangles in this way to find the sum of the angles in any pentagon.

3. Draw several different *hexagons* on your paper. Show that each of them can be split into exactly four triangles so that the vertices of the triangles all coincide with the vertices of the hexagon.

4. Use the fact that every hexagon can be split into four triangles in this way to find the sum of the angles in any hexagon.

Problem 233. Use your work on the exercises above to complete this general statement:

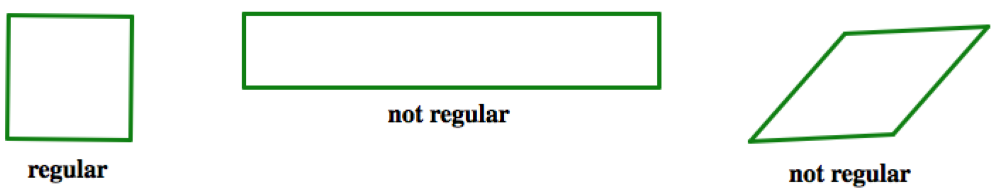
Angle Sum in Polygons:
The sum of the interior angles in an n-gon (a polygon with n sides) is

____

Explain how you know this statement is true.

Definition 7.3.2. A **regular polygon** has all sides the same length and all angles the same measure.

For example, squares are regular quadrilaterals — all four sides are the same length, and all four angles measure 90°. But a non-square rectangle is *not regular*. Even though all of the angles are 90°, the sides are not all the same length. Similarly, a non-square rhombus is *not regular*. Even though the sides of a rhombus are all the same length, the angles can be different.



Problem 234. Since a square is a regular quadrilateral, you know that every angle in a regular quadrilateral measure 90°. What about angles in other regular polygons?

(a) What is the measure of each angle in a regular triangle? Explain how you know you are right.

(b) What is the measure of each angle in a regular pentagon? Explain how you know you are right.

(c) What is the measure of each angle in a regular hexagon? Explain how you know you are right.

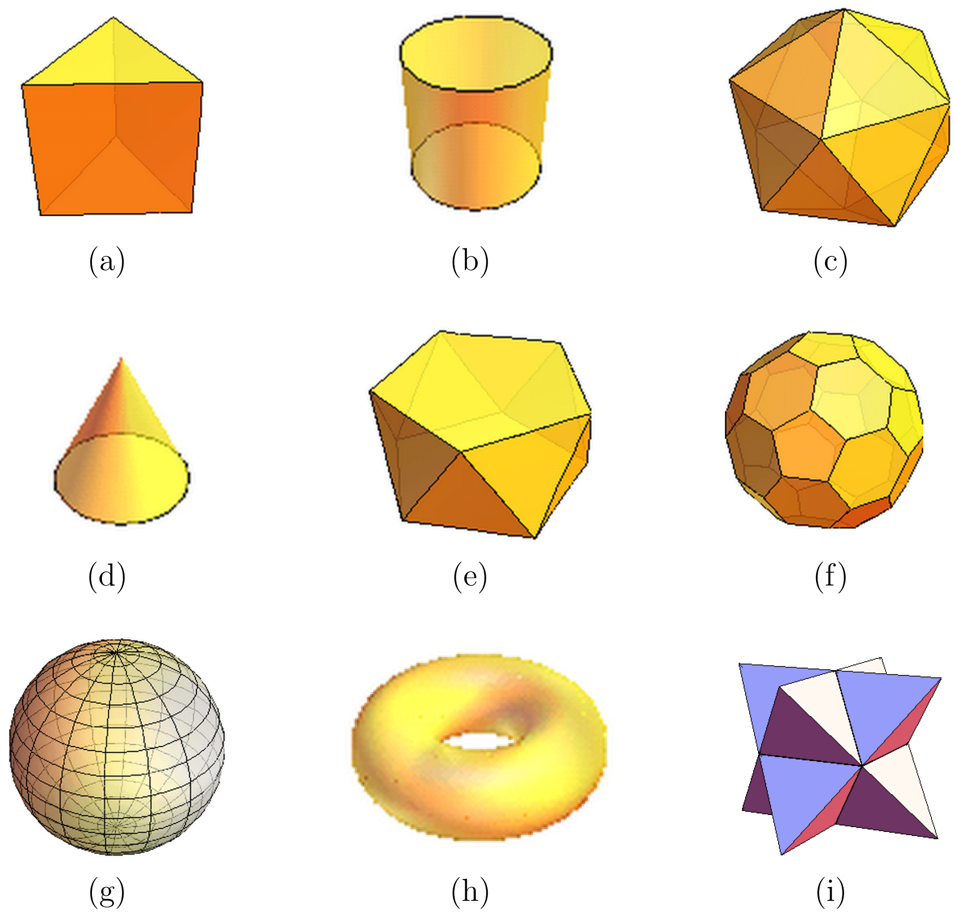
(d) What is the measure of each angle in a regular n-gon? Explain how you know you are right.

7.4 Platonic Solids

Of course, we live in a three-dimensional world (at least!), so only studying flat geometry doesn't make a lot of sense. Why not think about some three-dimensional objects as well?

Definition 7.4.1. A **polyhedron** is as solid (3-dimensional) figure bounded by polygons. A polyhedron has flat polygons for **faces**, straight **edges** where the faces meet in pairs, and **vertices** where three or more edges meet. The plural of polyhedron is **polyhedra**.

Think/Pair/Share. Look at the pictures of solids below, and decide which are polyhedra and which are not. You should be able to say why each figure does or does not fit the defition.


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)

Remember that a *regular polygon* has all sides the same length and all angles the same measure. There is a similar (if slightly more complicated) notion of *regular* for solid figures.

Definition 7.4.2. A **regular polyhedron** has faces that are all *identical (congruent) regular polygons.* All vertices are also identical (the same number of faces meet at each vertex). These are called **Platonic solids** (named for Plato).

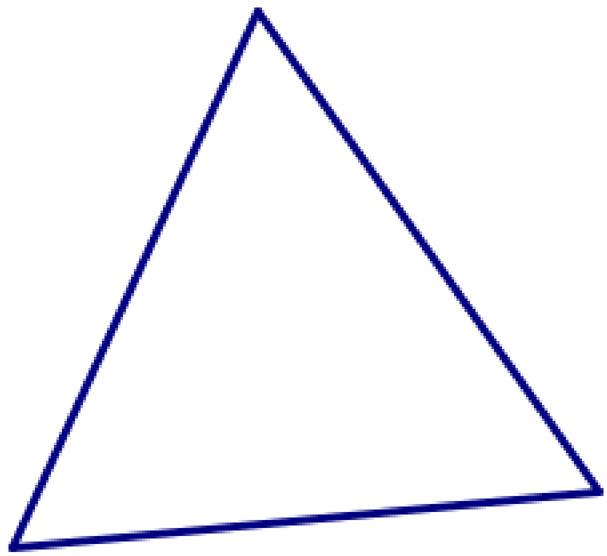
If you fix the number of sides and their length, there is one and only one regular polygon with that number of sides. That is, every regular quadrilateral is a square, but there can be different sized squares. Every regular octagon looks like a stop sign, but it may be scaled up or down. Your job in this section is to figure out what we can say about regular polyhedra.

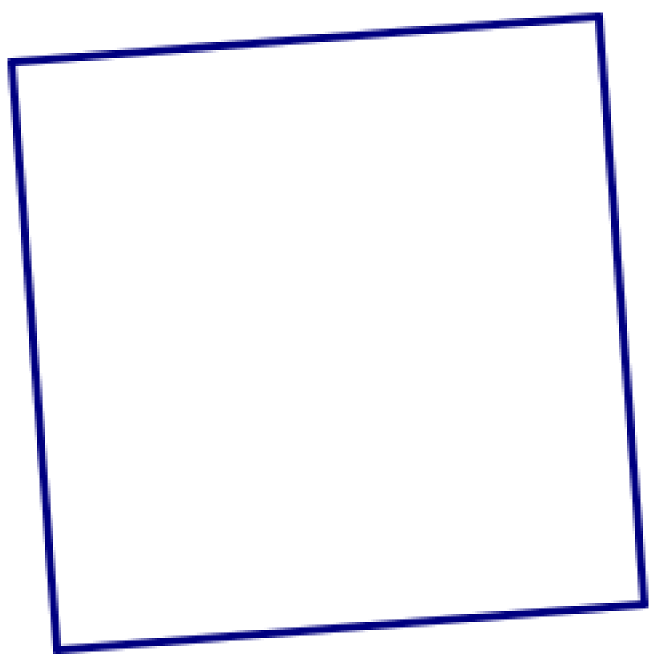
On Your Own

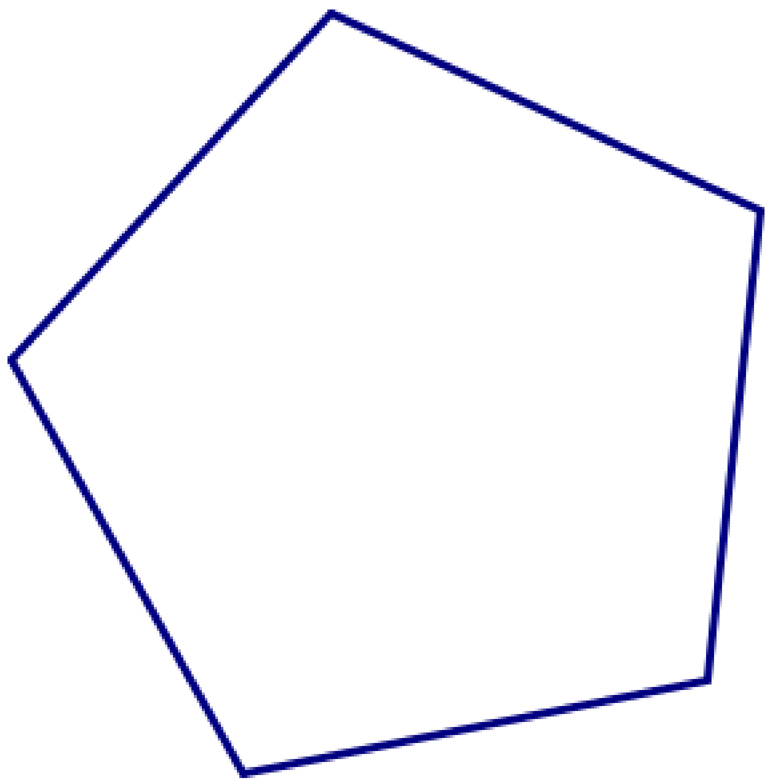
Work on these exercises on your own or with a partner. You will need to make lots of copies of the regular polygons below. Copy and cut out at least:

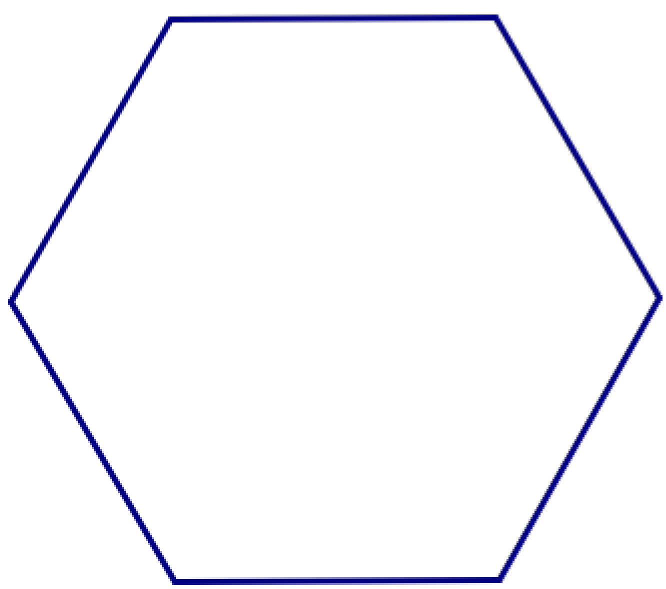
- 40 copies of the equilateral triangle,
- 15 copies of the square,
- 20 copies of the regular pentagon, and
- 10 copies each of the regular hexagon, heptagon, and octagon.

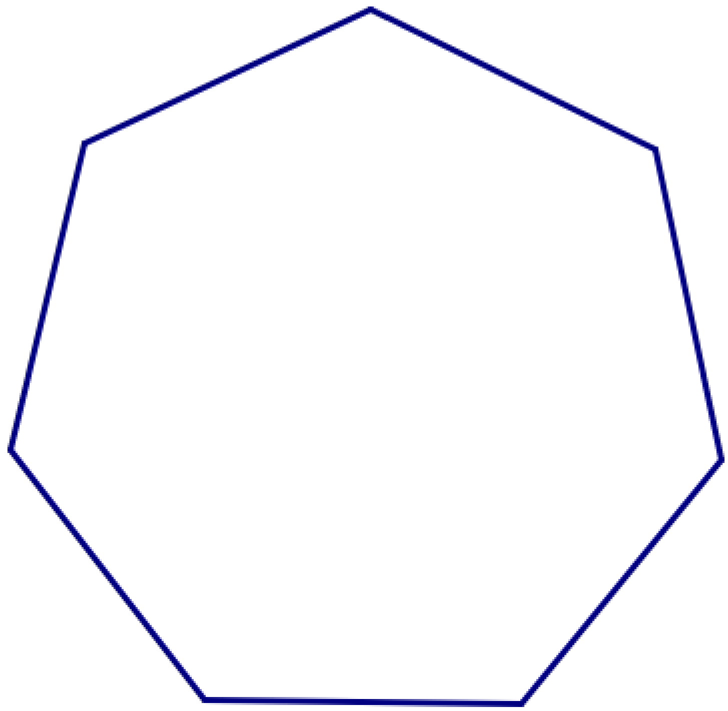
You will also need some tape.

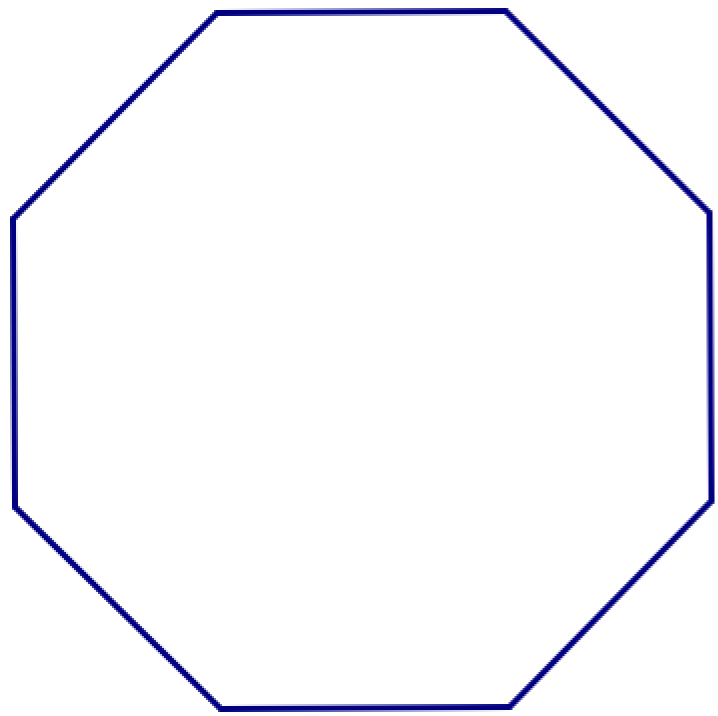












1. In any polyhedron, at least three polygons meet at each vertex. Start with the equilateral triangles: Put three of them together meeting at a vertex and tape them together. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have exactly three triangles meeting.

   ♣♣♣ Fellow: [An animation would really help here. Students had trouble understanding the directions, but once they see it they know what to do. Just show putting three triangles meeting at a vertex and then closing up the gap.]

2. Now repeat this process, but start with *four* equilateral triangles around a single vertex. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have exactly four triangles meeting.

♣♣♣ Fellow: [Another animation?]

3. Repeat this process with five equilateral triangles, then six, then seven, and so on. Keep going until you are convinced you understand what's happening with Platonic solids that have triangular faces.

4. When you are done with triangular faces, move on to square faces. Work systematically: Try to build a Platonic solid with three squares at each vertex, then four, then five, etc. Keep going until you can make a definitive statement about Platonic solids with square faces.

5. Repeat this process with the other regular polygons you cut out: pentagons, hexagons, heptagons, and octagons.

You must have noticed that the situation for Platonic solids is quite different from the situation for regular polygons. There are infinitely many regular polygons (even if you don't account for size). There is a regular polygon with n sides for every value of n bigger than 2. But for solids, we have the following (perhaps surprising) result.

Theorem. *There are exactly five platonic solids.*

The key fact is that for a three-dimensional solid to close up and form a polyhedron, there must be less than 360° around each vertex. Otherwise, it either lies flat or folds over on itself.

Problem 235. Based on your work in the exercises, you should be able to write a convincing justification of the Theorem above. Here's a sketch, and you should fill in the explanations.

(a) If a Platonic solid has equilateral triangles for faces, then fewer than 6 faces must meet at each vertex. Why?

(b) If a Platonic solid has square faces, then three faces can meet at each vertex, but not more than that. Why?

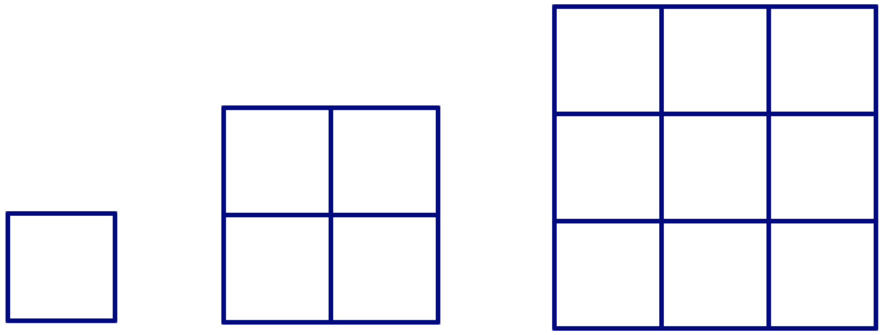
(c) If a Platonic solid has regular pentagons for faces, then three faces can meet at each vertex, but not more than that. Why?

(d) Regular hexagons cannot be used as the faces for a Platonic solid. Why?

(e) Similarly, regular n-gons for n bigger than 6 cannot be used as the faces for a Platonic solid. Why?

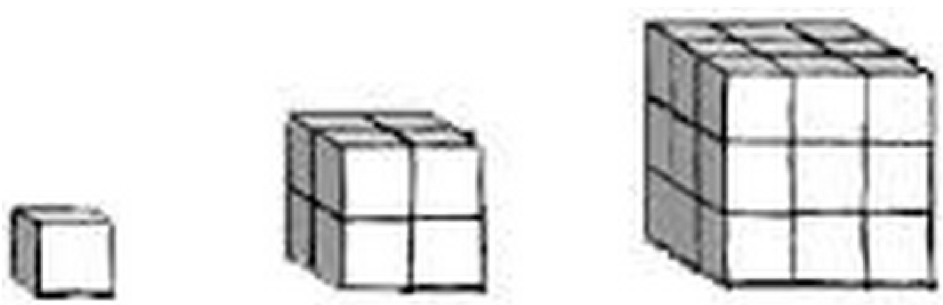
7.5 Painted Cubes

You can build up squares from smaller squares:



1×1 square 2×2 square 3×3 square

In a similar way, you can build up cubes from smaller cubes:



$1 \times 1 \times 1$ cube $2 \times 2 \times 2$ cube $3 \times 3 \times 3$ cube

♣♣♣ Fellow: [Can you make a better picture of cubes?]

Think/Pair/Share. We call a $1 \times 1 \times 1$ cube a *unit cube*.

- How many unit cubes are in a $2 \times 2 \times 2$ cube?

- How many unit cubes are in a $3 \times 3 \times 3$ cube?

- How many unit cubes are in an $n \times n \times n$ cube?

Explain your answers.

Problem 236. Imagine you build a $3 \times 3 \times 3$ cube from 27 small white unit cubes. Then you take your $3 \times 3 \times 3$ cube and dip it into a bucket of bright blue paint. After the cube dries, you take it apart, separating the small unit cubes.

(a) After you take the cube apart, some of the unit cubes are still all white (no blue paint). How many? How do you know you are right?

(b) After you take the cube apart, some of the unit cubes have blue paint on just one face. How many? How do you know you are right?

(c) After you take the cube apart, some of the unit cubes have blue paint on two faces. How many? How do you know you are right?

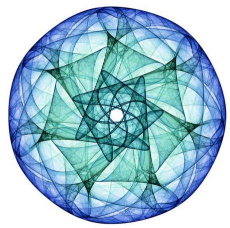
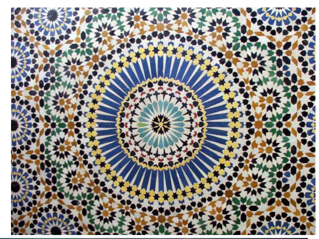
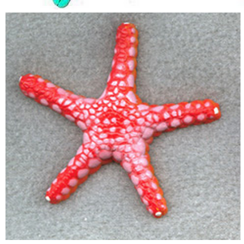
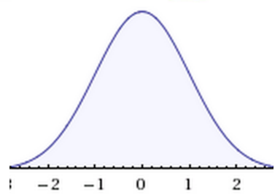
(d) After you take the cube apart, some of the unit cubes have blue paint on three faces. How many? How do you know you are right?

(e) After you take the cube apart, do any of the unit cubes have blue paint on more than three faces? How many? How do you know you are right?

Problem 237. Generalize your work on Problem 236. What if you started with a $4 \times 4 \times 4$ cube? Answer the same questions. What about a $2 \times 2 \times 2$ cube? How about an $n \times n \times n$ cube? Be sure to justify what you say.

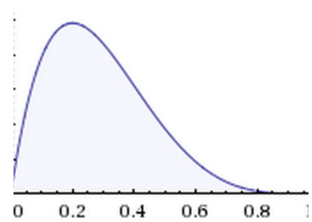
7.6 Symmetry

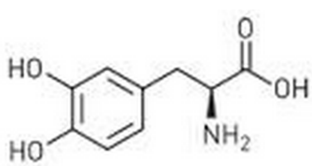
Mathematicians use symmetry in all kinds of situations. There can be symmetry in calculations, for example. But the most recognizable kinds of symmetry are those in geometric designs. Geometric figures can have different kinds of symmetries.

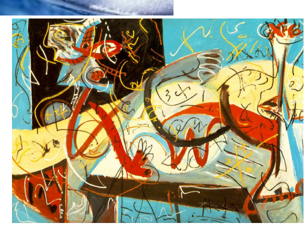
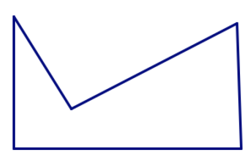






Or they might have no symmetry at all.







Think/Pair/Share.

- What do you already know about the idea of *symmetry*? What does it mean to say a design is *symmetric*?

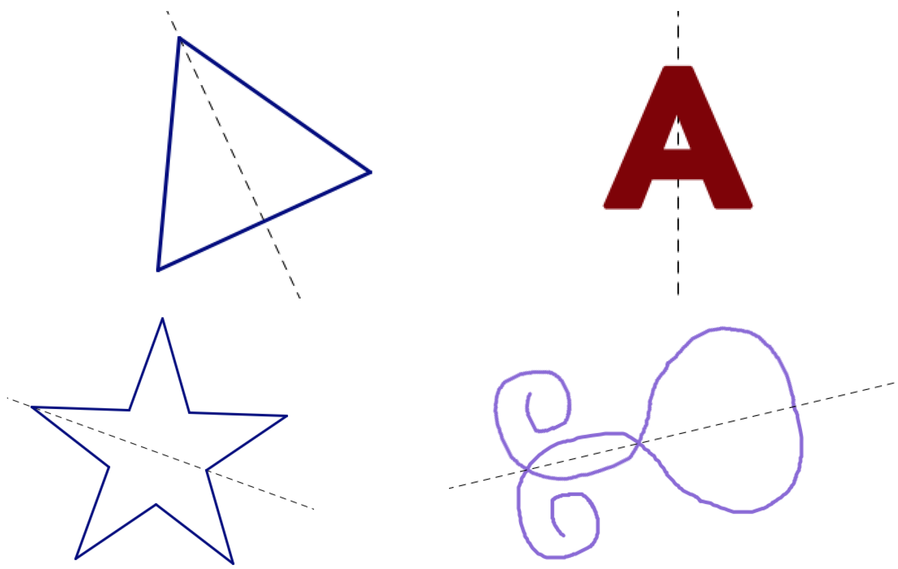
- Do you know about different types of symmetry? What types?

- Can you give examples of real-world objects that are symmetric? What about objects that are not symmetric?

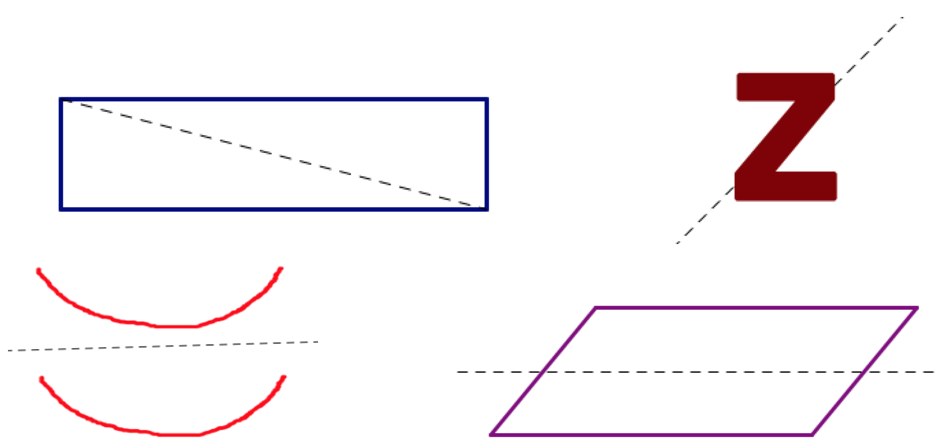
7.6.1 Line Symmetry

If you can flip a figure over a line — this is called *reflecting* the figure — and the it appears unchanged, then the figure has **reflection symmetry** or **line symmetry**. A **line of symmetry** divides an object into two mirror-image halves. The dashed lines below are lines of symmetry:

♣♣♣ Fellow: [Any improvements tot hess pictures would be fantastic.]



Compare with the dashed lines below. Though they do cut the figures in half, they don't create mirror-image halves. These are not lines of symmetry.

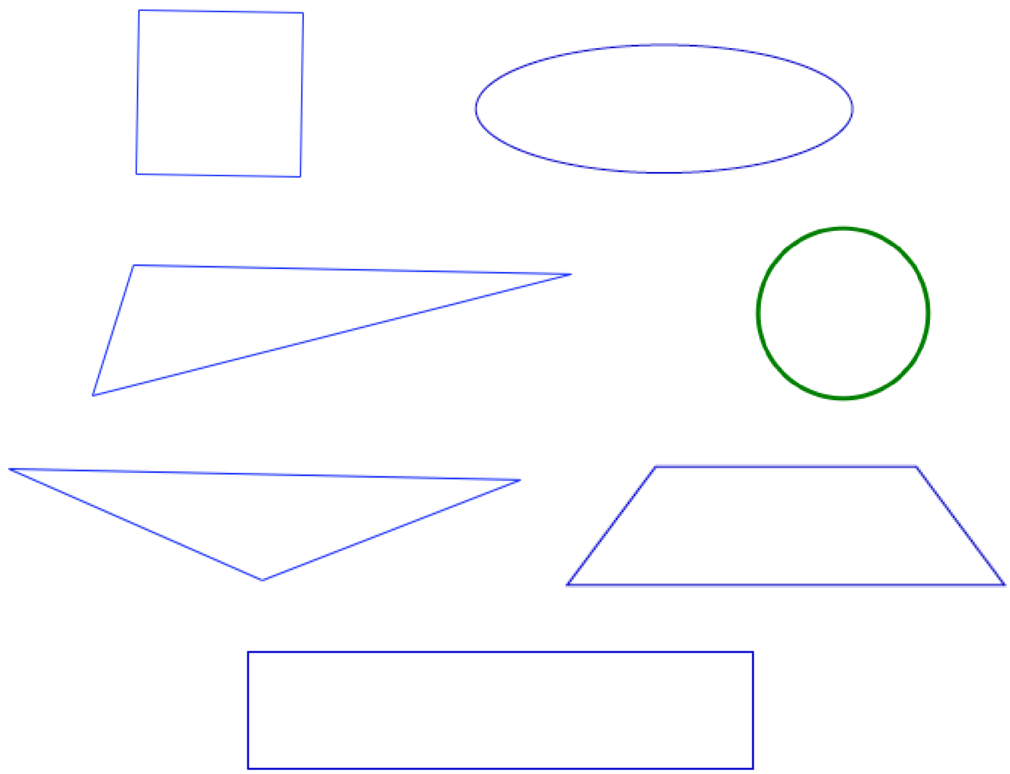


Think/Pair/Share. Look at the first set of pictures on page 360. Do any of them have lines of symmetry? How can you tell?

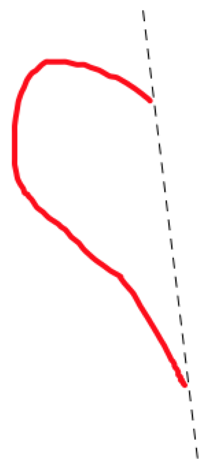
Problem 238. For each of the figures below:

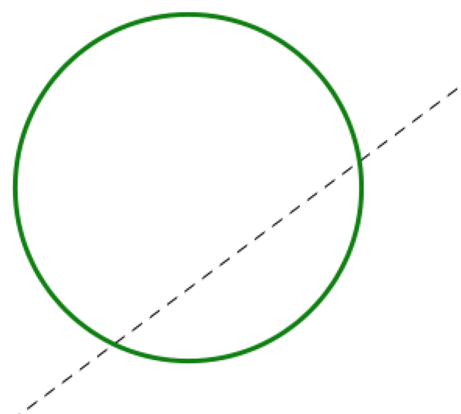
(a) Decide if it has any lines of symmetry. If not, how do you know?

(b) If it does have one or more lines of symmetry, find / describe *all* of them. Explain how you did it.



Problem 239. Each picture below shows **half** of a design with line symmetry. The line of symmetry (dashed) is shown. Can you complete the design? Explain how you did it.

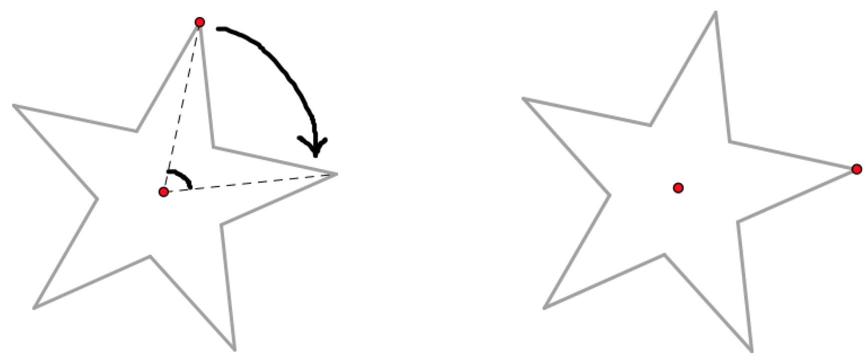




7.6.2 Rotational Symmetry

If you can turn a figure around a center point less than a full circle — this is called a *rotation* — and the figure appears unchanged, then the figure has **rotational symmetry**. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

This star has rotation symmetry of 72°, and the center of rotation is the center of the star. One point is marked to help you visualize the rotation.



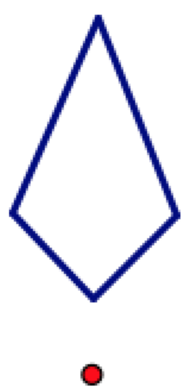
Think/Pair/Share.

- How can you be certain that the angle of rotation for the star is exactly 72°?

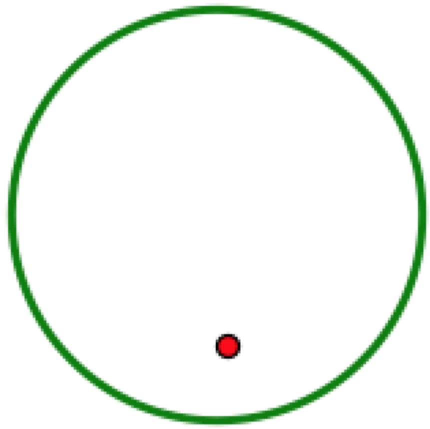
- Look at the first set of pictures on page 360. Do any of them have rotational symmetry? How can you tell?

Problem 240. Each of the figures below has rotational symmetry. Find the center of rotation and the angle of rotation. Explain your thinking.

Problem 241. Each picture below shows part of a design with a marked center of rotation and an angle of rotation given. Can you complete the design so that it has the correct rotational symmetry? Explain how you did it.



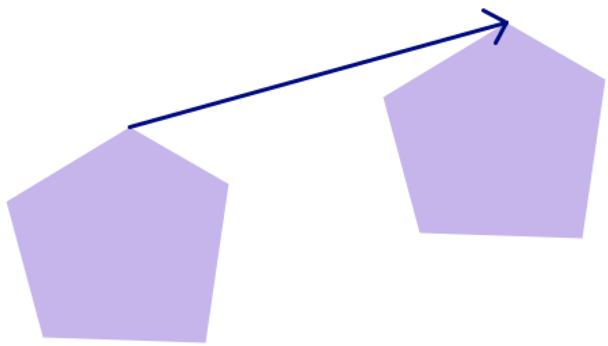
Use an angle of 90°.



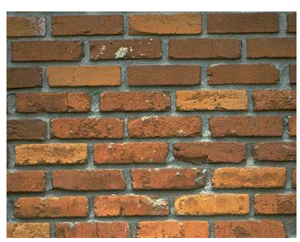
Use an angle of 60°.

Translation symmetry

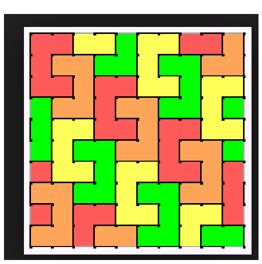
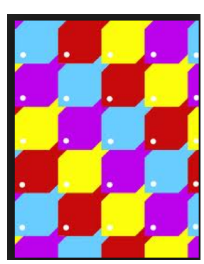
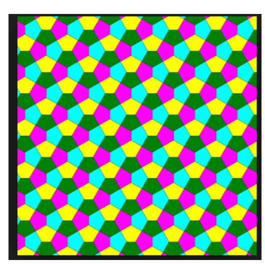
A **translation** (also called a slide) involves moving a figure in a specific direction for a specific distance. A **vector** (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).

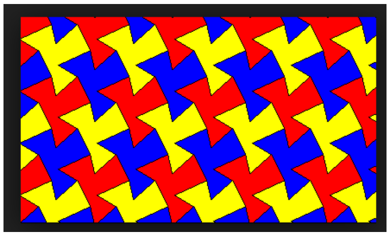


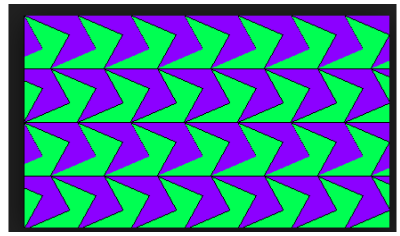
A design has **translational symmetry** if you can perform a translation on it and the figure appears unchanged. A brick wall has translational symmetry in lots of directions!



The brick wall is one example of a *tessellation*, which you'll learn more about in Section 7.7.

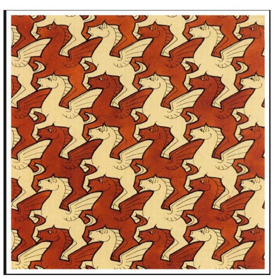


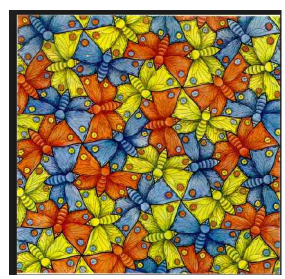


You can see translation symmetry in lots of places. It's in architecture and design.



It's in art, most famously that by M.C. Escher.





And it appears in traditional Hawaiian and other Polynesian tattoo designs.



Think/Pair/Share.

- On each of the pictures with translational symmetry shown on pages 369–371, sketch a vector to indicate the direction and distance of the translational symmetry.

- Create your own design with translational symmetry. Explain how you did it.

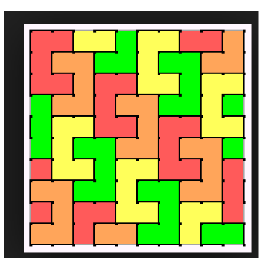
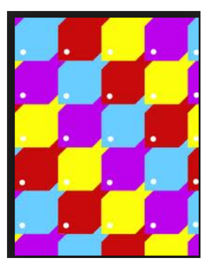
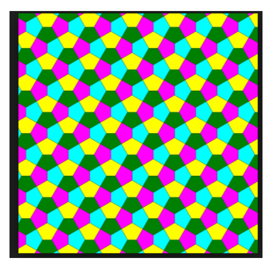
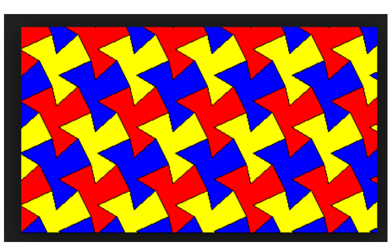
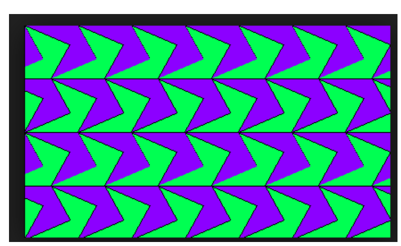
7.7 Geometry in Art and Science

How much have you thought about the geometry of the world around you? When you look at a picture that you find beautiful, is the beauty because of symmetry? Or from lack of symmetry that grabs your interest and surprises you?

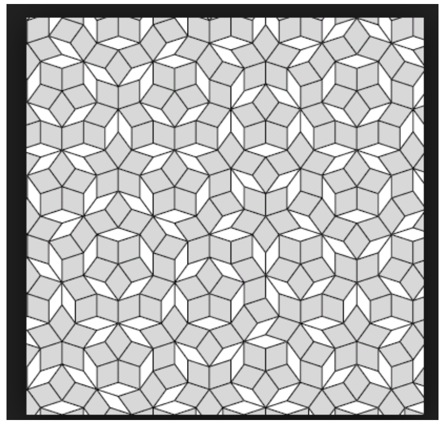
You can imagine that many scientists, engineers, and architects must think about geometric structure as a regular part of their job. So, of course, do visual artists.

7.7.1 Tessellations

A *tessellation* is a design using one ore more geometric shapes with no overlaps and no gaps. The idea is that the design could be continued infinitely far to cover the whole plane (though of course we can only draw a small portion of it).

Many tessellations have translational symmetry, but it's not strictly necessary. The *Penrose tiling* shown below does not have any translational symmetry.

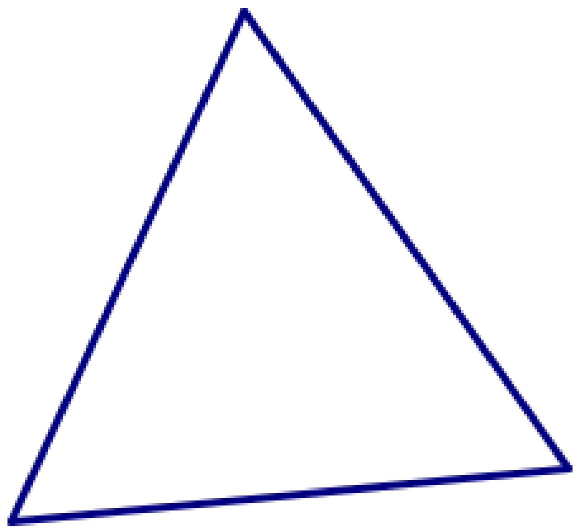


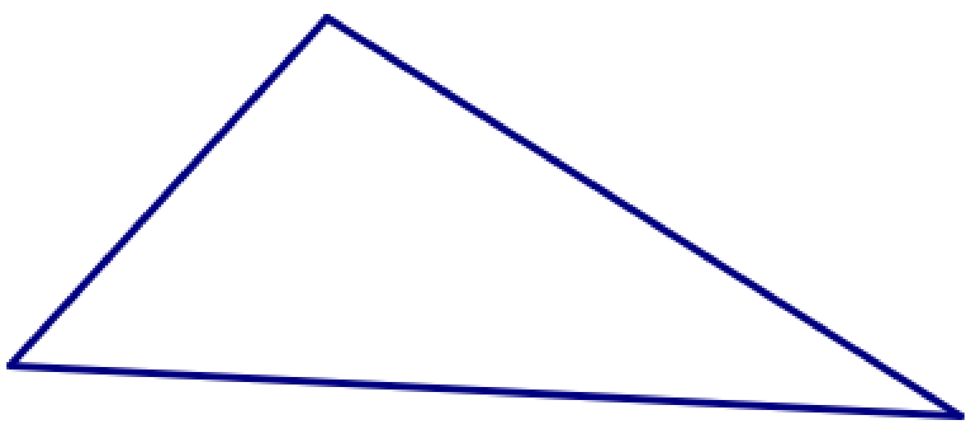
It's actually much harder to come up with these "aperiodic" tessellations than to come up with ones that have translational symmetry. So we'll focus on how to make symmetric tessellations.

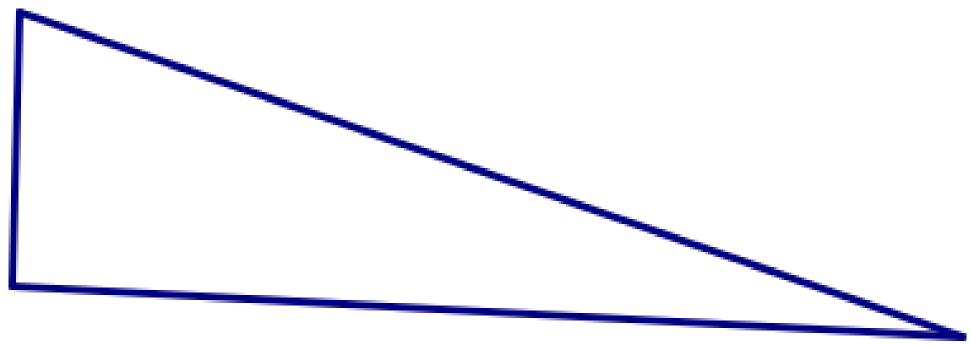
The first set of tessellations above were all made with a single geometric shape (called a *tile*) designed so that they can fit together without gaps or overlaps. Tessellations are often called *tilings*, and that's what you should think about: If I had tiles made in this shape, could I use them to tile my kitchen floor? Or would it be impossible?

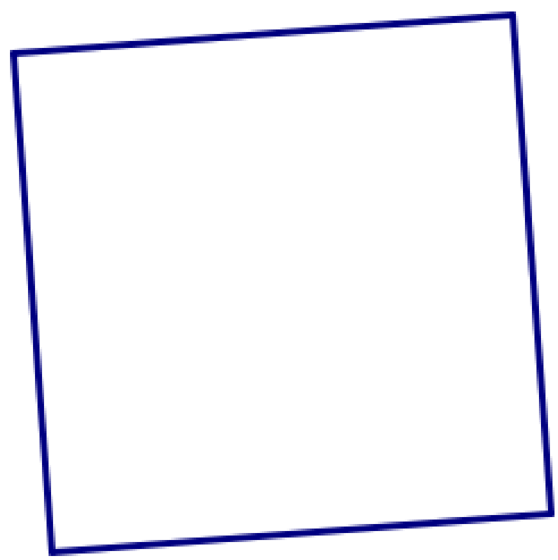
On Your Own

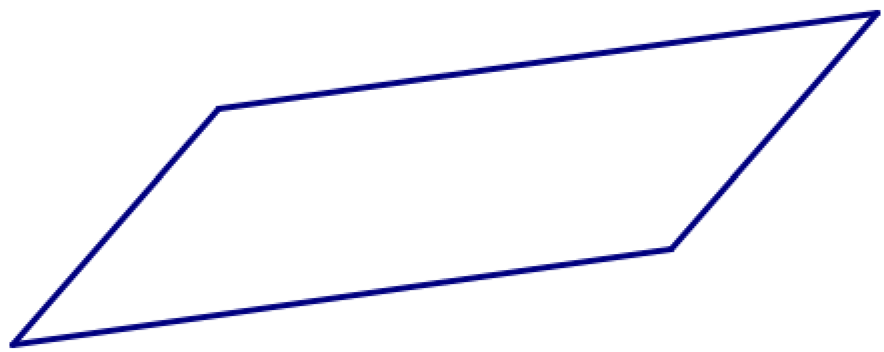
Work on these exercises on your own or with a partner. You will need lots of copies (maybe 10–15 each) of each shape below. In each problem, focus on just a *single tile* for making your tessellation.

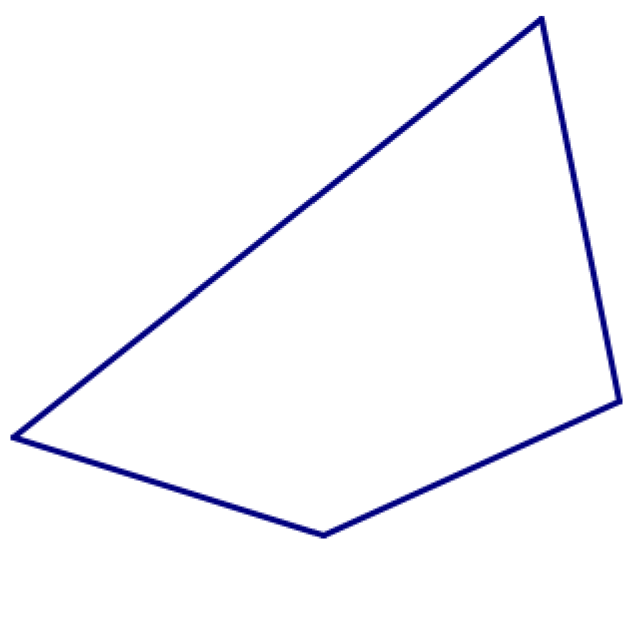


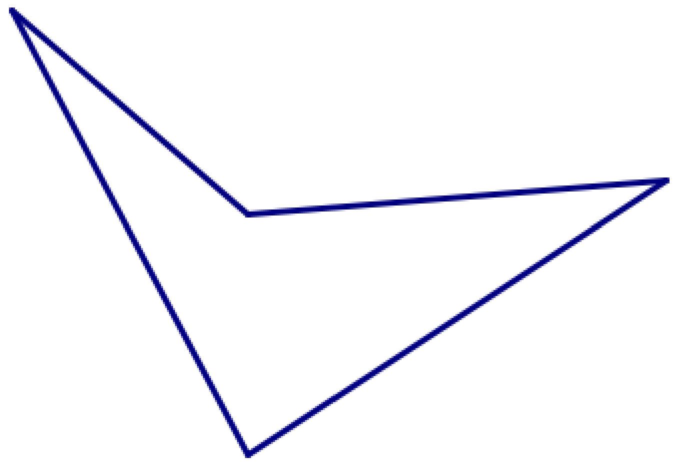


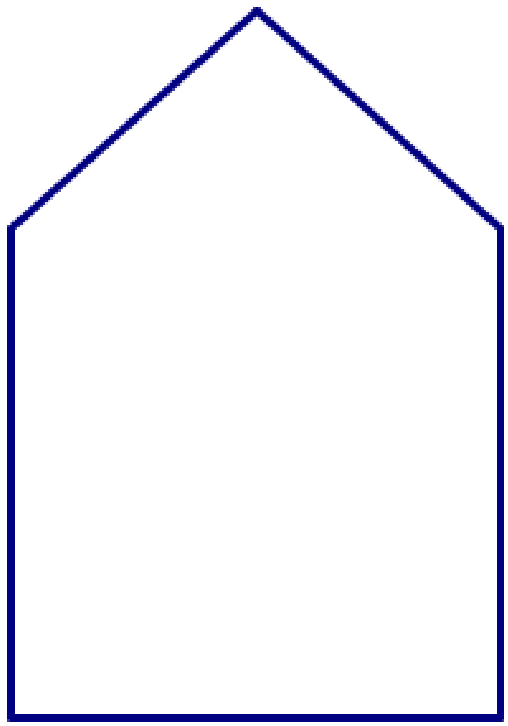


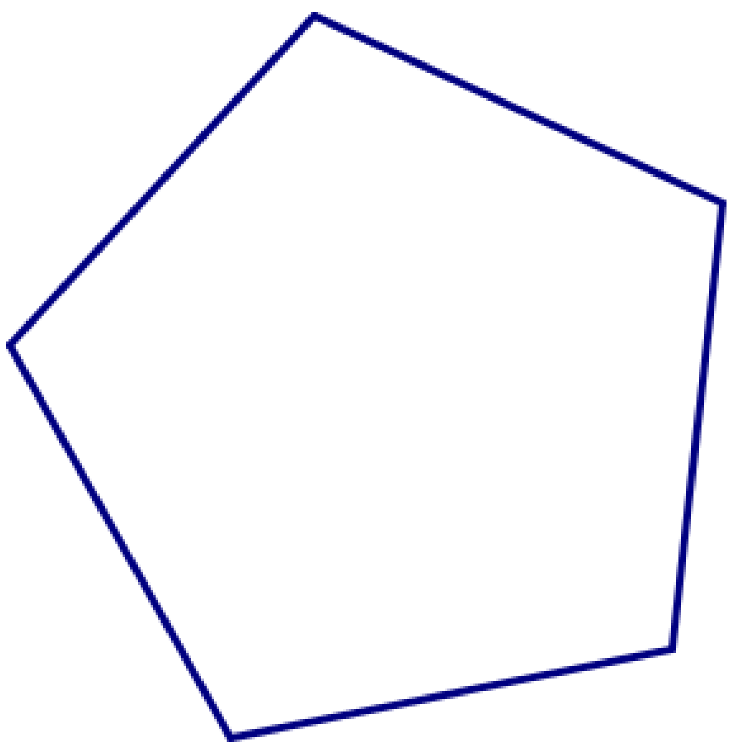


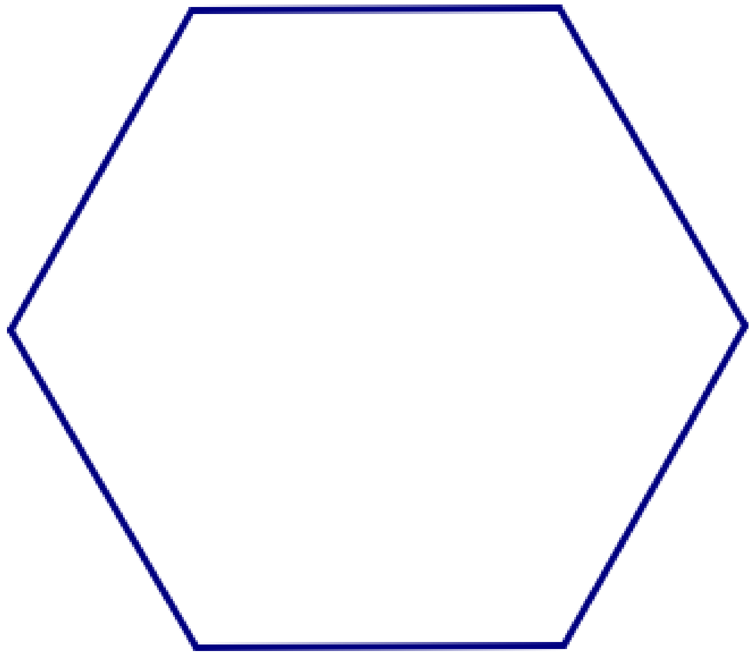


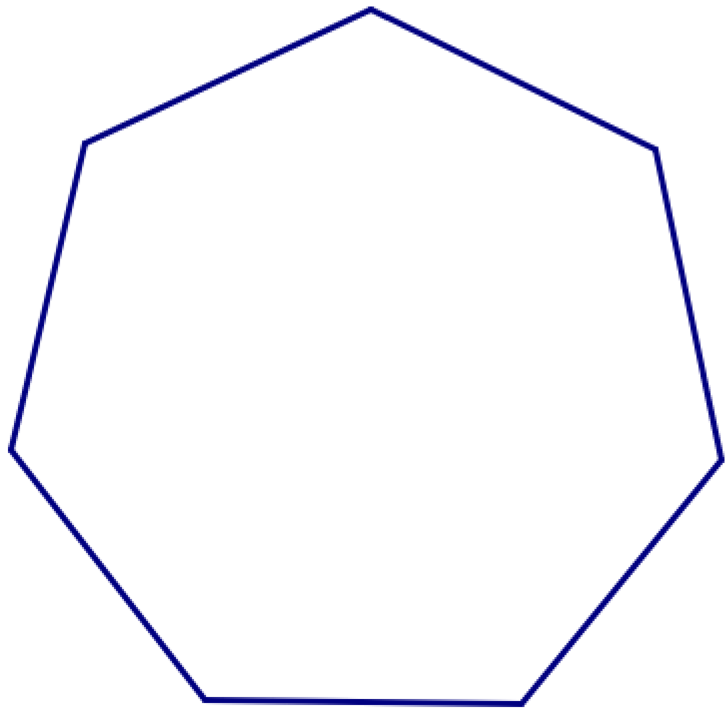


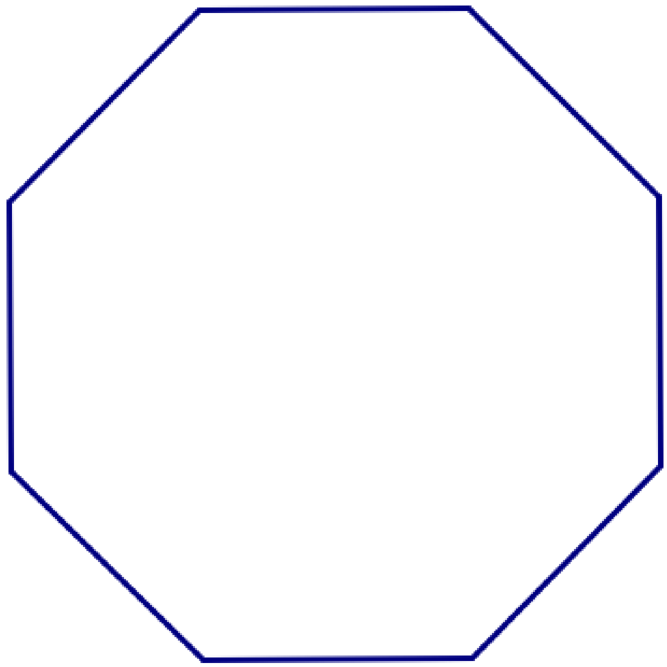












1. Start with the square tile. Can you fit the squares together in a pattern that could be continued forever, with no gaps and no overlaps? Can you do it in more than one way?

2. Now try one of the triangular tiles. Can you use many copies of a single triangle to tessellate the plane?

3. Repeat this process with each of the other tiles. Keep track of your findings.

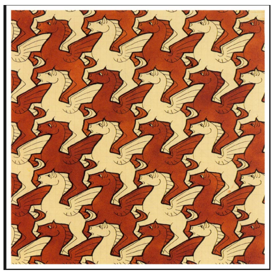
Think/Pair/Share. Share what you learned:

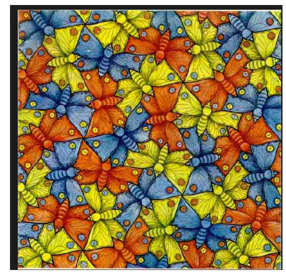
- Which shapes did tile the plane, and which did not?

- Do you have any conjectures based on this experience, about which shapes will tile the plane and which will not?

7.7.2 Escher Drawings

The artist M.C. Escher created many works of art inspired by mathematics, including some very beautiful tessellations.

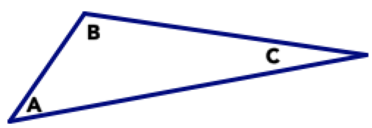




You can make your own Escher-like drawings using some facts that you learned while studying tessellations.

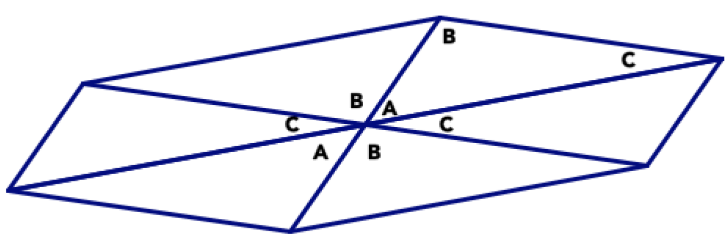
Theorem. *Any triangle will tessellate. So will any quadrilateral.*

The explanation for this comes down to what you know about the sums of angles. The sum of the angles in a triangle is 180°.



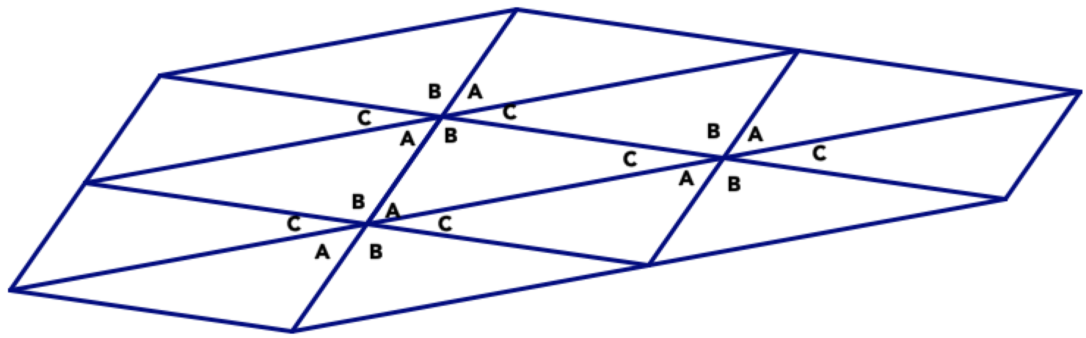
$$A + B + C = 180^\circ$$

So if you make six copies of a single triangle and put them together at a point so that each angle appears twice, there will be a total of 360° around the point, meaning the triangles fit together perfectly with no gaps and no overlaps.



$$A + B + C + A + B + C = 360^\circ$$

You can then repeat this at every vertex, using more and more copies of the same triangles.



Think/Pair/Share.

- Use the fact that the sum of the angles in any quadrilateral is 360° to explain why every quadrilateral will tessellate.

- Use angles to explain why regular hexagons will tessellate.

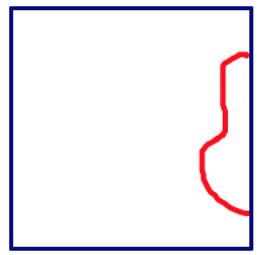
- Explain why regular pentagons will not tessellate.

On Your Own

Work on these exercises on your own or with a partner. Here's how you can create your own Escher-like drawings.

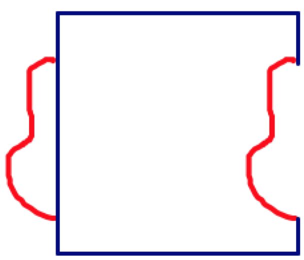
1. The first time you do this, it's easiest to start with a simple shape that you know will tessellate, like an equilateral triangle, a square, or a regular hexagon.

2. Draw a “squiggle” on one side of your shape.

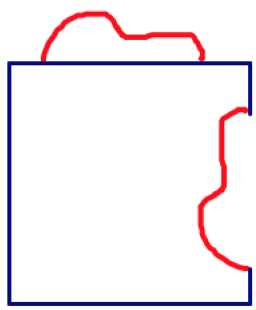


A square with a “squiggle” drawn on one side.

Cut out the squiggle, and move it to another side of your shape. You can either translate it straight across or rotate it.



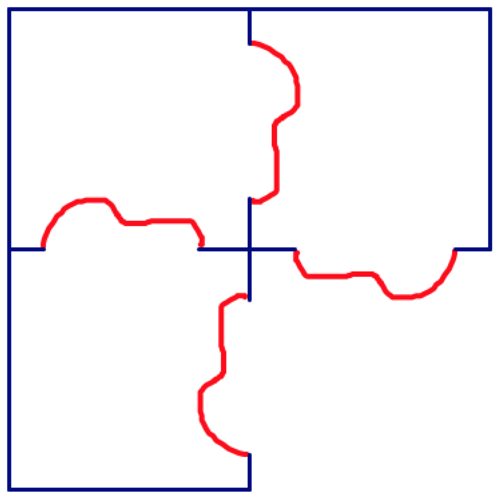
A translation.



A rotation.

The important thing is the cut-out lines up along the new edge in the same place that it appeared on its original edge.

3. Tape the squiggle into its new location. This is your basic tile. On a large piece of paper, trace around your tile. Then move it the same way you moved the squiggle (translate or rotate) so that the squiggle fits in exactly where you cut it out.



A few rotations of the basic tile.

The shape will still tessellate, so go ahead and fill up your paper.

4. Now get creative. Color in your basic shape to look like something — an animal? a flower? a colorful blob? Add color and design throughout the tessellation to transform it into your own Escher-like drawing.

5. If you want to try a more complicated version, cut two different squiggles out of two different sides, and move them both.

7.7.3 Building towers

For this activity, you will need some construction materials:

- You'll need lots of toothpicks.

- You'll also need something to connect the toothpicks together. The best material for this is mini marshmallows; you can stick the ends of the toothpicks into the marshmallows to connect them. You can also use pieces of clay, bits of gummy candies, or other similar (sticky) material.

Problem 242. Try this as a warm-up activity. Grab exactly six toothpicks. Your job is to make four triangles using all six toothpicks. You cannot break any of the toothpicks or add any other materials besides the marshmallow connectors.

Problem 243. Now comes the main challenge. You have ten minutes to build the *tallest free-standing structure* that you can make. Free-standing means that it will stand up on its own. You can't have it lean against a wall or hold it up. When the ten minutes are up, back away from your tower and measure its height.

Think/Pair/Share. Look at your own tower and at other students' towers. Talk about these questions:

- What design choices led to taller free-stranding structures? Why do you think that is?

- If you had another ten minutes to try this activity again, what would you do differently and why?

7.8 Problem Bank

Problem 244 (Tangrams)**.** In Problem 225, you assembled all seven tangram pieces into a large square.

(a) If the large square you made with all seven pieces is one whole, assign a (fractional) value to each of the seven tangram pieces. Justify your answers.

(b) The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

(c) The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

(d) The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

(e) The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

Problem 245. If possible sketch an example of the following triangles. If it is not possible, explain why not.

(a) A right triangle that is scalene.

(b) A right triangle that is isosceles.

(c) A right triangle that is equilateral.

Problem 246. If possible sketch an example of the following triangles. If it is not possible, explain why not.

(a) An acute triangle that is scalene.

(b) An acute triangle that is isosceles.

(c) An acute triangle that is equilateral.

Problem 247. If possible sketch an example of the following triangles. If it is not possible, explain why not.

(a) An obtuse triangle that is scalene.

(b) An obtuse triangle that is isosceles.

(c) An obtuse triangle that is equilateral.

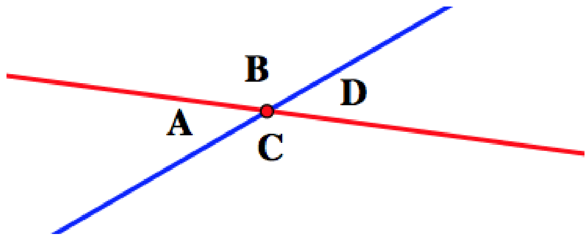
Problem 248. If possible sketch an example of the following triangles. If it is not possible, explain why not.

(a) An equiangular triangle that is scalene.

(b) An equiangular triangle that is isosceles.

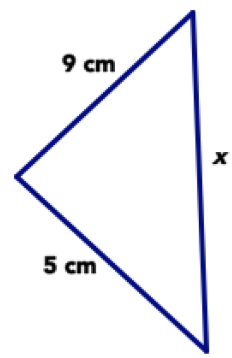
(c) An equiangular triangle that is equilateral.

Problem 249. Look at the picture below, which shows two lines intersecting. Angles A and D are called "vertical angles," and so are angles B and C.



Use this drawing to explain why vertical angles must have the same measure. (Hint: what is the sum of the measures of angle A angle B? How do you know?)

Problem 250. Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like*.

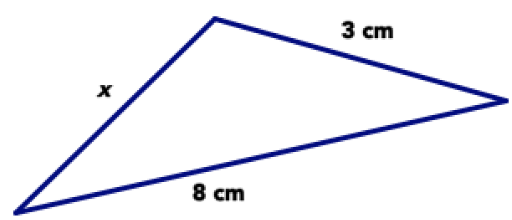


(a) Could it be true that $x = 4$ cm? Explain your answer.

(b) Could it be true that $x = 20$ cm? Explain your answer.

(c) Give three possible values of x, based on the information in the picture.

Problem 251. Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like*.

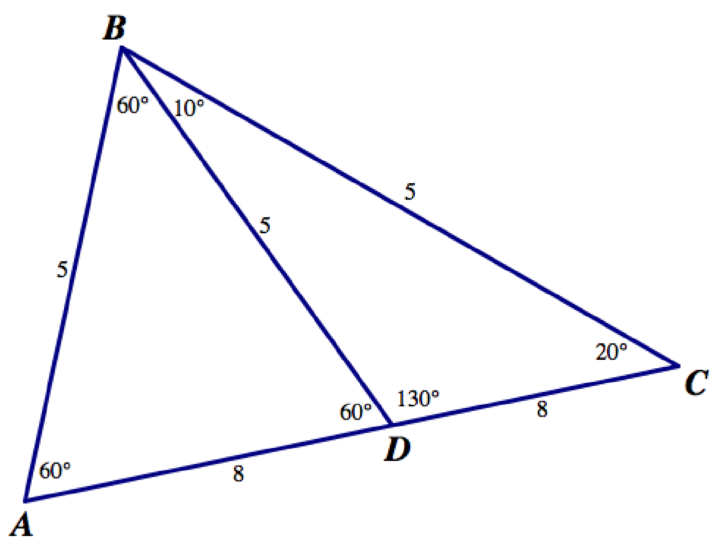


(a) If $x = 3$ cm, the triangle is isosceles. Is this possible? Explain your answer.

(b) If $x = 8$ cm, the triangle is isosceles. Is this possible? Explain your answer.

(c) Give three *impossible* values of x, based on the information in the picture.

Problem 252. Prof. Faber drew this picture on the board, saying it showed three triangles: $\triangle ABC$, $\triangle ABD$, and $\triangle CBD$. Side lengths and angle measurements are shown for each of the triangles.



There are **lots of mistakes** in this picture. Use what you know about side lengths and angles in triangles to find all the mistakes you can. For each mistake, say what is wrong with the picture, and why it's a mistake. Explain your thinking as clearly as you can.

Problem 253. Because of SSS congruence, triangles are exceptionally sturdy. This means they are used frequently in architecture and design to provide supports for buildings, bridges, and other man-made objects. Take your camera with you, and find several places in your neighborhood or near your campus that use triangular supports. Snap a picture, and describe what the structure is and where you see the triangles.

Problem 254. It is possible to create designs that have multiple symmetries. See if you can find images (or create your own!) that have both:

(a) reflection symmetry and rotational symmetry,

(b) reflection symmetry and translational symmetry, or

(c) rotational symmetry and translational symmetry.

Chapter 8

Voyaging on Hōkūleʻa

> "When we voyage, and I mean voyage anywhere, not just in canoes, but in our minds, new doors of knowledge will open, and that's what this voyage is all about…it's about taking on a challenge to learn. If we inspire even one of our children to do the same, then we will have succeeded."
> –Nainoa Thompson, September 20, 1999, the day of departure for the challenge of navigating from Mangareva to Rapa Nui, the remotest, most difficult island to navigate to in Polynesia.

In the 1950s and 1960s, historians couldn't agree on how the Polynesian islands — including the Hawaiian islands — were settled. Some historians insisted that Pacific Islanders sailed around the Pacific Ocean, relocating as necessary, and settling the islands with purpose and planning. Others insisted that such a navigational and voyaging feat was impossible thousands of years ago, before European sailors would leave the sight of land and sail into the open ocean. These historians believed that the Polynesian canoes were caught up in storms, tossed and turned, and eventually washed up on the shores of faraway isles.

Think/Pair/Share. Talk about these questions with a partner:

- How could such a debate ever be settled one way or the other, given that we can't go back in time to find out what happened?

- What kinds of evidence would support the idea of "intentional voyages"? What kinds of evidence would support the idea of "accidental drift"?

- What do you already know about how this debate was eventually settled?

8.1 Hōkūleʻa

The Polynesian Voyaging Society (PVS) was founded in 1973 for scientific inquiry into the history and heritage of Hawaiʻi: How did the Polynesians discover and settle these islands? How did they navigate without instruments, guiding themselves across ocean distances of 2500 miles?

In 1973–1975, PVS built a replica of an ancient double-hulled voyaging canoe to conduct an experimental voyage from Hawaiʻi to Tahiti. The canoe was designed by founder Herb Kawainui Kāne and named Hōkūleʻa, Star of Gladness.

On March 8th, 1975, Hōkūleʻa was launched. Mau Piailug, a master navigator from the island of Satawal in Micronesia, navigated her to Tahiti using traditional navigation techniques (no modern instruments at all).

Think/Pair/Share. Brainstorm with a partner:

- What are some *mathematical* questions you can ask about voyaging on Hōkūleʻa?

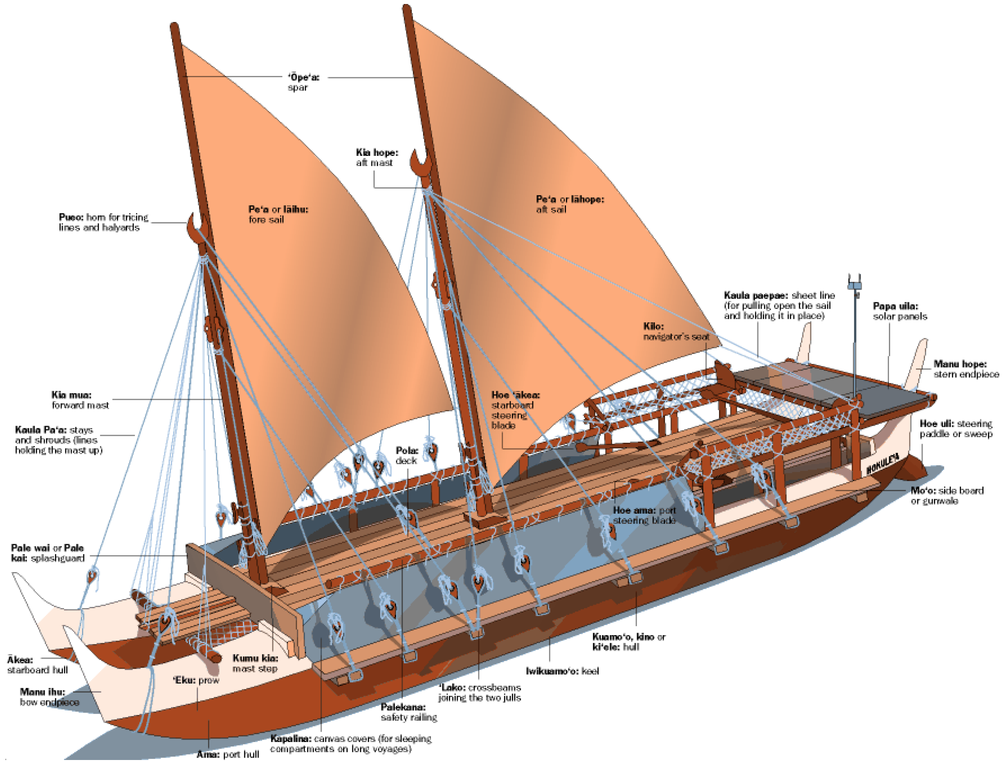
- What kinds of problems (especially mathematics problems) did the crew have to solve before setting off on the voyage to Tahiti?

- What are you curious about, with respect to voyaging on Hōkūleʻa?

When you teach elementary school, you will mostly likely be teaching all subjects to your students. One thing you should think about as a teacher: How can you connect the different subjects together? How can you see mathematics in other fields of study, and how can you draw out that mathematical content?

In this chapter, you'll explore just a tiny bit of the mathematics involved in voyaging on a traditional canoe. You will apply your knowledge of geometry to create scale drawings and make a star compass. And you'll use you knowledge of operations and algebraic thinking to plan the supplies for the voyage. The focus here is on applying your mathematical knowledge to a new situation.

One of the first things to know about Hōkūle‘a is what she looks like. This picture and drawing of Hōkūle‘a come from the PVS website: `http://hokulea.org`.



Problem 255. Here's some information about the dimensions of Hōkūle‘a. Your job is to draw a good scale model of the canoe, like a floorplan. Imagine you are above the canoe looking down at it. Draw a scale model of what you would see. Do not include the sails or any details; you are aiming to convey the overall shape in a scale drawing. You will use this scale drawing several times in the rest of this unit, so be sure to do a good job and keep it somewhere that you can find it later.

- Hōkūle‘a is 62 feet 4 inches long. (This is "LOA" or "length overall" in navigation terms. It means the maximum length measured parallel to the waterline.)

- Hōkūle‘a is 17 feet 6 inches wide. (This is "at beam" meaning at the widest point.)

- You can see from the picture that Hōkūleʻa has two hulls, connected by a rectangular deck. The deck is about 40 feet long and 10 feet wide.

Note: You don't have all the information you need! So you either need to find out the missing information or make some reasonable estimates based on what you do know.

Problem 256. Crew for a voyage is usually 12–16 people. During mealtimes, the whole crew is on the deck together. About how much space does each person get when they're all together on the deck?

8.2 Worldwide Voyage

To prepare for the next activity:

- Read this description of the daily life on Hōkūleʻa: `http://pvs.kcc.hawaii.edu/ike/canoe_living/daily_life.html`.

- Watch the video about the Worldwide Voyage: `http://vimeo.com/51118047`.

From the webpage above, you learned:

> "The quartermaster is responsible for provisioning the canoe — loading food, water and all needed supplies, and for maintaining Hōkūleʻa's inventory. While this is not an on board job, it is critical to the safe and efficient sailing of the canoe."

Problem 257. Imagine that you are part of the crew for the Worldwide Voyage, and you are going to help the quartermaster and the captain with provisioning the canoe for one leg of the voyage. You need to write a preliminary report for the quartermaster, documenting:

- Which leg of the trip are you focused on? (See the map below.)

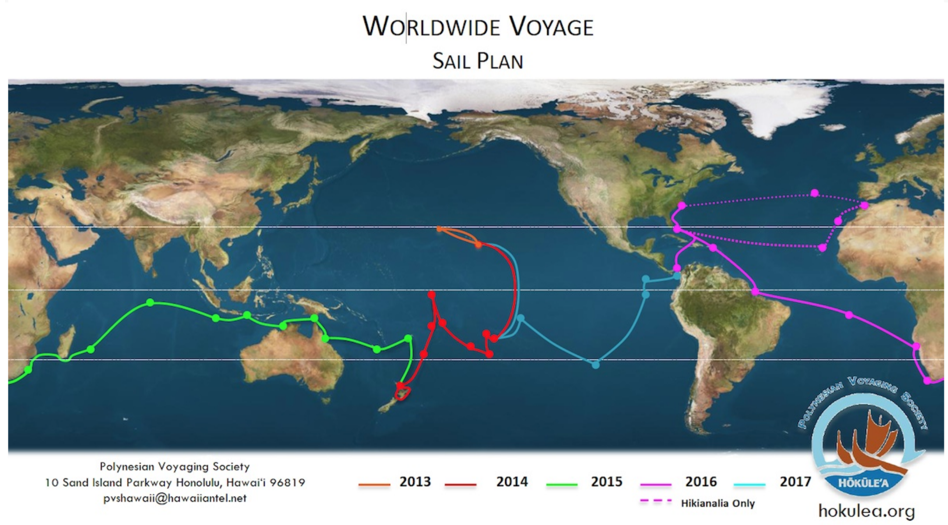
- How long will that leg of the trip take? Explain how you figured that out.

- How much food and water will you need for the voyage? Explain how you figured that out.

The rest of this section contains pointers to information that may or may not be helpful to you as you make your plans. Your job is to do the relevant research and then write your report. You should include enough detail about how you came to your conclusions that the quartermaster can understand your reasoning. Note: during Hōkūleʻa's voyage, you can track the progress here: `http://www.hokulea.com/track-the-voyage/`.

Pick a leg of the route

Here's a picture of the route planned for the Worldwide Voyage, which you can find at the Worldwide Voyage website `http://hokulea.org/world-wide-voyage/`. Note that on the map, the different colors correspond to different years of the voyage. A "leg" means a dot-to-dot route on the map.



After you pick a leg of the voyage, you'll need to figure out the total distance of that leg. This tool might help (or you can find another way): `http://www.acscdg.com/`. Here is some relevant information to help you figure out how long it will take Hōkūleʻa to complete that leg:

- Fully loaded with the maximum weight, Hōkūleʻa can travel at speeds of 4–6 knots, and even 10–12 knots in strong winds. (One knot means one nautical mile per hour.)

- For comparison, the first trip from Hawaiʻi to Tahiti in 1976 took a total of 34 days. (You probably want to use the tool above to compute the number of nautical miles.)

Plan the provisions

Here is some information about provisions.

- Hokulea can carry about 11,000 pounds, including the weight of the crew, provisions, supplies, and personal gear.

- The supplies (sails, cooking equipment, safety equipment, communications equipment, etc.) account for about 3,500 pounds.

- The crew eats three meals per day and each crew member gets 0.8 gallons of water per day.

- For a trip that is expected to take 30 days, the quartermaster plans for 40 days' worth of supplies, in case of bad weather and other delays.

8.3 Navigation

The following is from `http://pvs.kcc.hawaii.edu/ike/hookele/modern_wayfinding.html`:

> “A voyage undertaken using modern wayfinding has three components:
>
> 1. Design a course strategy, which includes a reference course for reaching the vicinity of one’s destination, hopefully upwind, so that the canoe can sail downwind to the destination rather than having to tack into the wind to get there. (Tacking involves sailing back and forth as closely as possible into the wind to make progress against the wind; its very arduous and time-consuming, something to be avoided if at all possible, particularly at the end of a long, difficult voyage.)
> 2. During the voyage, holding as closely as possible to the reference course while keeping track of (1) distance and direction traveled; (2) one’s position north and south and east and west of the reference course and (3) the distance and direction to the destination.
> 3. Finding land after entering the vicinity of the destination, called a target screen or ‘the box’.”

So how is the navigation done — especially component (2) — through thousands of miles of open ocean? You can’t see land. How can you hold closely to the reference course? How can you keep track of distance and direction traveled? How can you even know if you’re going in the right direction?

By day, the navigators use their deep knowledge of the oceans. Which way do the winds blow? Which way do the prevailing currents move? Clouds in the sky, flotsam in the water, and animal behaviors can give you great insight into where land might be, and where you are in relation to it.

By night, they use the stars. In this section, you’ll learn just a tiny fraction of what these master navigators know about the stars.

Think/Pair/Share. Here is a time-lapse picture of the stars in the night sky in Hawai'i.

Photography by Ashley Deeks.

- Describe what you see happening in this picture.

- What can you conclude about how the stars move through the night sky?

- How might that help a navigator find his way?

8.3.1 Star Compass

A fundamental tool for navigators on Hōkūle‘a and other voyaging canoes is a star compass. Here's a picture of Mau Piailug building a star compass to teach navigation.

Picture from
http://pvs.kcc.hawaii.edu/holokai/2007/mau_1_intro.html.

The object in the center of the circle represents the canoe. The rocks along the outside represent directional points. The idea is to imagine the stars rising up from the horizon in the east, traveling through the night sky, and setting past the horizon in the west. They move like they're on a sphere surrounding the Earth (it's called the *celestial sphere*).

Problem 258. Nainoa Thompson developed a star compass with 32 equidistant points around a circle. (Note this is more points than in Mau's star compass above.) You will try first to make a rough sketch of Nainoa's star compass based on this information:

- Place 32 points around the circle so they are equally spaced.

- The arcs between these equidistant points are called "houses." You will label each house with its Hawaiian name. Start with the four cardinal points:

North ‘Akau.

South Hema.

East Hikina.

West Komohana.

- The four quadrants also get names. (These cover all of the houses in the quadrant, so label them in the appropriate place inside the compass.)

Ko‘olau northeast.

Malani southeast.

Kona southwest.

Ho‘olua northwest.

- Moving from ‘Akau to Hikina (clockwise), there are seven houses. They are labeled in order as you move away from ‘Akau:

Haka “empty,” describing the skies in this house.

Nā Leo “the voices” of the stars speaking to the navigator.

Nālani “the heavens.”

Manu “bird,” the Polynesian metaphor for a canoe.

Noio the Hawaiian tern (a bird)

‘Āina “land.”

Lā “sun,” which stays in this house most of the year.

- The compass has a vertical line of symmetry, so there are the same seven houses in the same order as you move from ‘Akau to Komohana (counterclockwise).

- The compass also has a horizontal line of symmetry. Use that fact to label the houses from Hema to Hikina (counterclockwise) and from Hema to Komohana (clockwise).

How is the star compass used in navigation? There are lots of ways. Here's a (very!) quick overview:

♣♣♣ Fellow: [I don't know if there's any way to do an animation of this description, but if so it would be cool!]

- The canoe is pictured in the middle of the star compass, with all of the houses around.

- Winds and ocean swells move directly *across* the star compass from north to south or vice versa.
  - If the swells are coming from ʻĀina Koʻolau, they will be heading in the direction ʻĀina Kona. (Look at your star compass and trace out this path.)
  - If the wind is coming from Nālani Malani, it will be heading towards Nālani Hoʻolua. (Look at your star compass and trace out this path.)

- Stars stay in their houses, but also in their hemisphere. Just like the sun, they rise in the east and set in the west; they do not move across the center of the circle.
  - Hōkūleʻa rises in ʻĀina Koʻolau and sets in ʻĀina Hoʻolua. (Look at your star compass and trace out this path.)
  - ʻAʻā (Sirius) rises in Lā Malanai and sets in Lā Kona.

A navigator memorizes the houses of over 200 stars. At sunrise and sunset (when the sun or the stars are rising), the navigator can use the star compass to memorize which way the wind is moving and which way the currents are moving. He can then use that information throughout the day or night to ensure the canoe stays on course.

Think/Pair/Share. Look again at the time-lapse picture of the stars.

Photography by Ashley Deeks.

- Describe how this shows that stars "stay in their houses" and in their hemisphere as they move through the night sky.

- The star Ke ali'i o kona i ka lewa (Canopus), rises in Nālani Malanai. Where does it set?

When teaching navigation while sitting on land, it's perfectly fine to have a rough sketch or model of the star compass. But if you really have to do the navigation, you need to make a very, very precise star compass.

Imagine Nainoa Thompson, who navigated Hōkūleʻa on the final leg of her journey from Hawaiʻi to Rapa Nui, an island even smaller and lower than Niihau. You have to be within 30 miles of Rapa Nui to see it. But a mistake of even one degree would have led to Hōkūleʻa being 60 miles off course. And if you end up drifting in the open ocean and supplies run out? Well…

Problem 259. Now that you have a rough sketch of the star compass and know what it should look like, your job is to draw one that's as perfect as possible. That means you want to draw:

- A perfect circle (well, as perfect as possible). What tools can you use to do that? What tools would ancient Polynesian navigators have had to use?

- Thirty-two points around the circle that are *exactly* evenly spaced apart. (What tools would help you? What tools would ancient Polynesian navigators have had to use?)

- When you have finished, label your perfectly drawn star compass with the houses.

Of course, a star compass on a piece of paper isn't so useful when you're out on a canoe. How do you position it properly? And how do you keep it from getting lost, damaged, or soaking wet? You paint it on the rails of the canoe, permanently!

Look back at the drawing of Hōkūleʻa on page 399. Find the "kilo" (navigator's seat) in the rear (aft) of the canoe. There is actually one navigator's seat on either side of the deck.

Problem 260. Go back to the scale drawing of Hōkūleʻa that you made in Problem 255. Add the navigator's seats to your drawing. You will then add the star compass to the rails as follows:

- Start with the seat on the left (port side) of the canoe. That will be the center of your star compass. Imagine looking to the right. You want to see the star compass markings on the rails when you look to the right. Of course, the Hōkūleʻa is not a circular canoe, and the navigator doesn't sit at the center. So how can you make the markings in the right places?

- Now repeat that process, using the seat on the right (starboard) side of the canoe.

Nainoa Thompson has said:

> "Initially, I depended on geometry and analytic mathematics to help me in my quest to navigate the ancient way. However as my ocean time and my time with Mau have grown, I have internalized this knowledge. I rely less on mathematics and come closer and closer to navigating the way the ancients did."

Really he is still doing a lot of mathematics; it's just mathematics that he has internalized and is now second nature to him. The ancient navigators may not have spoken of their navigation techniques in the same modern language we've been using — compass points and perfect circles and degrees. But their mathematical understanding was truly astonishing.