MAT 137Y: Calculus with proofs
Assignment 2
Due on Thursday, Oct 17 by 11:59pm via GradeScope

Instructions

This problem set is based on Unit 2: Limits and Continuity. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: ____________________

Student number: ____________________

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I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ____________________

2) ____________________

1. (*Note:* Before you attempt this problem, solve Problem 1 and 3 from Practice problems for Unit 2 on Quercus or Problem 3 and Problem 4 from the practice problems of test 1 on Quercus. Otherwise you may find this question difficult.)

Sketch the graph of a function f that satisfies all 9 conditions below simultaneously. For this question, you do not need to prove or explain your answer, as long as the graph is correct and very clear.

(a) The domain of f is $\mathbb{R}$.

(b) $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} f(f(x))$ does not exist, is not ∞, and is not $-\infty$.

(c) $\lim_{x\to 3} f(x) = -1$ and $\lim_{x\to 3} f(f(x)) = -\infty$.

(d) $\lim_{x\to -3} f(x) = -1$ and $\lim_{x\to -3} f(f(x)) = \infty$.

(e) $f(x)$ is not continuous at -3.

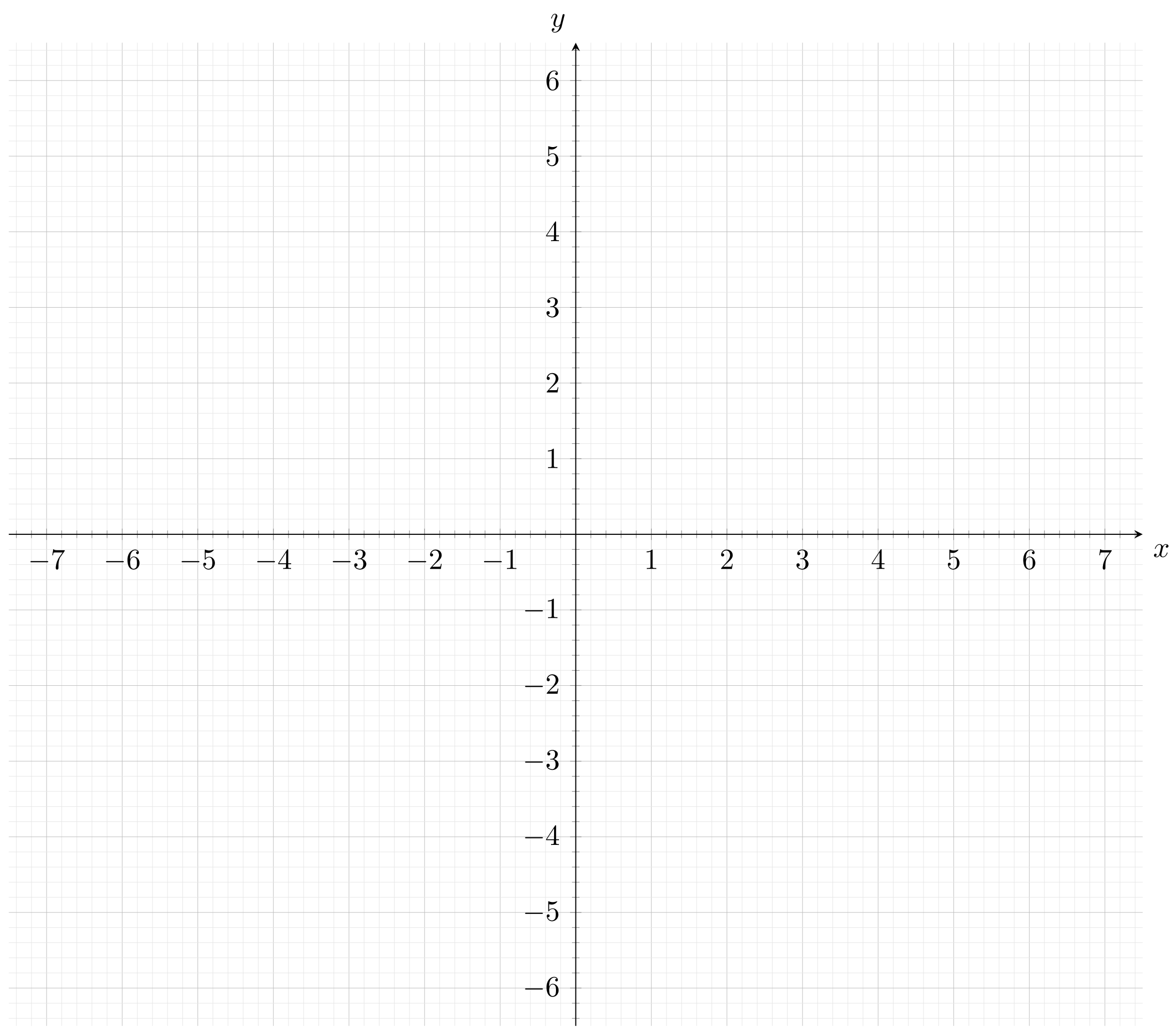
(f) $\lim_{x\to -1} f(f(x)) = 4$

(g) $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} \lfloor f(x) \rfloor$ does not exist

(h) $\lim_{x\to 5} f(x)$ does not exist and $\lim_{x\to 5} [f(x)]^2 = 1$

(i) $\lim_{x\to -5} f(x) = \infty$

To clarify, we want one single function f that satisfies all the conditions in all the parts, all at once. Make your graph tidy and unambiguous.



2. Let $a \in \mathbb{R}$. Let f and g be two functions defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. If it is true, prove it directly from the epsilon-delta definition of a limit. *Hint:* often times, the easiest way to prove something is false is by providing a counter example and explain that counter example satisfies the required conditions.

   (a) IF $\lim_{x \to a} [f(x) \cdot g(x)]$ exists and $\lim_{x \to a} g(x)$ exists, THEN $\lim_{x \to a} f(x)$ exists .

   ◯ True ◯ False

   (b) IF $\lim_{x \to a} f(x)$ does not exist and $\lim_{x \to a} g(x)$ does not exist, THEN $\lim_{x \to a} [f(x) + g(x)]$ does not exist.

   ◯ True ◯ False

3. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = 4$$

4. Recall that $\mathbb{Q}$ is the set of rational numbers. Define a function f with domain $\mathbb{R}$ as

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is continuous at $x = 0$.
Hint: you may need to use the Squeeze Theorem in your proof.

(b) Prove that f is not continuous at $x = 1$. You may use the following fact:

$$\forall a, b \in \mathbb{R}, \text{ if } a < b, \text{ then } \exists c \notin \mathbb{Q}, \text{ s.t. } a < c < b.$$

Hint: start from writing down the definition of "f is not continuous at $x = 1$".

Extra space for (b):

(c) In fact, f is continuous at $x = 0$ and not continuous everywhere else. Construct a function g by using f such that $g(x)$ is continuous at $x = k$ for any $k \in \mathbb{Z}$ and not continuous at all other points. No justification is needed.

5. Let $\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}$ be the set of planar points. For a planar point P and a subset $S \subset \mathbb{R}^2$, we say P is a *limit point* of S if

$$\forall \varepsilon > 0, \exists Q \in S, \text{ s.t. } 0 < d(P, Q) < \varepsilon.$$

Here $d(P, Q)$ is the distance between P and Q.
Note: The distance between any two points (x_1, y_1) and (x_2, y_2) on $\mathbb{R}^2$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

(a) Let $S = \{(\frac{1}{n} + 1, \frac{1}{n}) : n \in \mathbb{N}\}$. Prove that $(1, 0)$ is a limit point of S.

(b) Let f be a function defined on an open interval D (for example, $D = (1, 2)$). The *graph* of f is defined as the set

$$\Gamma(f) := \{(x, f(x)) : x \in D\}.$$

Prove the following statement: If $\lim_{x \to a} f(x) = L$, then (a, L) is a limit point of $\Gamma(f)$.

(c) (Optional) Prove or disprove the following statement: If (a, L) is a limit point of $\Gamma(f)$, then $\lim_{x \to a} f(x) = L$.