A summary of the course Algebra 3
(with the notations used in class)

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Proposition 1

Let $A \in M_n(\mathbb{K})$ be a *nilpotent matrix*, *i.e.*, there exists $p \in \mathbb{N}^*$ such that $A^p = 0$. Then,

$$\chi_A(t) = (-t)^n.$$

Theorem 1 (Cayley-Hamilton)

Let $A \in M_n(\mathbb{K})$. Then,

$$\chi_A(A) = 0.$$

Proposition 2

Let $A \in M_n(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then,

$$\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad \det(A) = \lambda_1 \cdots \lambda_n.$$

Proposition 3

Let $u \in L(E)$ and $P \in \mathbb{K}[T]$. Then, $\mathrm{Ker}(P(u))$ and $\mathrm{Im}(P(u))$ are invariant subspaces under u. In particular, the eigenspaces E_λ and the generalized eigenspaces N_λ of u are invariant under u.

Proposition 4

Let $u \in L(E)$ and let F be an invariant subspace under u. Then,

$$\chi_{u_{|F}} \quad \text{divides} \quad \chi_u.$$

Corollary 1

Let $u \in L(E)$. Let λ be an eigenvalue of u of multiplicity m. Then,

$$\dim(N_\lambda) \leqslant m.$$

Proposition 5 (Sufficient condition for diagonalisation)

Any matrix $A \in M_n(\mathbb{K})$ with n distinct eigenvalues is diagonalizable on $\mathbb{K}$.

Warning ! A diagonalizable matrix doesn't necessarily have distinct eigenvalues : diagonalization is possible even with repeated eigenvalues.

Lemma 1 (Kernel decomposition Lemma)

Let $u \in L(E)$ and let $P(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$ where $\lambda_1, \ldots, \lambda_p$ are distinct scalars in $\mathbb{K}$. Then, we have the following decomposition

$$\mathrm{Ker}\,(P(u)) = \bigoplus_{i=1}^{p} \mathrm{Ker}\,(u - \lambda_i e)^{m_i}.$$

Put in another way, every element $x \in \mathrm{Ker}\,(P(u))$ writes, in a unique way, as a sum

$$x = x_1 + \cdots + x_p$$

of elements $x_i \in \mathrm{Ker}\,(u - \lambda_i e)^{m_i}$.

Theorem 2 (Characterizing diagonalizable matrices)

Let $A \in M_n(\mathbb{K})$ such that χ_A splits over $\mathbb{K}$, *i.e.*, $\chi_A(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$ where $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of A. Let $u \in L(E)$ such that $A = \mathrm{Mat}\,(u \mid \mathcal{B})$. Then, the following statements are equivalent.

(i) A is diagonalizable on $\mathbb{K}$,

(ii) $\prod_{i=1}^{p} (A - \lambda_i I_n) = 0$, *i.e.*, $\prod_{i=1}^{p} (u - \lambda_i e) = 0$,

(iii) $\forall\, i \in \{1, \ldots, p\}$, $\dim\,(E_{\lambda_i}) = m_i$.

(iv) $\forall\, i \in \{1, \ldots, p\}$, $\mathrm{rk}\,(A - \lambda_i I_n) = n - m_i$.

Corollary 2

Let $A \in M_n(\mathbb{K})$. Then, A is diagonalizable on $\mathbb{K}$ if and only if there exists a polynomial $P \in \mathbb{K}[T]$ which splits over $\mathbb{K}$, has distinct roots and satisfies $P(A) = 0$.

Theorem 3 (Characterizing triangularizable matrices)

Let $A \in M_n(\mathbb{K})$. The matrix A is triangularizable on $\mathbb{K}$ if an only if its characteristic polynomial χ_A splits on $\mathbb{K}$. That is to say, we can factorize χ_A on $\mathbb{K}$ as

$$\chi_A(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i},$$

where the λ_i's are the distinct eigenvalues of A.

Theorem 4

Let $u \in L(E)$ such that $\chi_u(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$ where $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of u. Then,

$$E = \bigoplus_{i=1}^{p} N_{\lambda_i} \quad \text{and} \quad \forall i \in \{1, \ldots, p\}, \quad \dim\,(N_{\lambda_i}) = m_i.$$

Corollary 3

Let $A \in M_n(\mathbb{K})$ such that $\chi_A(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$ where $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of A. Then, there exists a matrix $U \in GL_n(\mathbb{K})$ such that

$$U^{-1}AU = \begin{pmatrix} A_1 & & (0) \\ & \ddots & \\ (0) & & A_p \end{pmatrix}$$

where $A_i \in M_{m_i}(\mathbb{K})$ satisfies $(A_i - \lambda_i I_{m_i})^{m_i} = 0$.

Proposition 6

Let $A \in M_n(\mathbb{K})$ such that $A^n = 0$ and $A^{n-1} \neq 0$. Then, there exists an invertible matrix $U \in GL_n(\mathbb{K})$ such that

$$U^{-1}AU = \begin{pmatrix} 0 & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & 0 \end{pmatrix}.$$

Theorem 5 (Jordan reduction)

Let $A \in M_n(\mathbb{K})$ such that $\chi_A(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$ where $\lambda_1, \dots, \lambda_p$ are the distinct eigenvalues of A. Let $\ell = \sum_1^p \dim(E_{\lambda_i})$. Then, the matrix A is similar to a block diagonal matrix of the form

$$J = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J_\ell \end{pmatrix},$$

where every block J_k, $k \in \{1, \cdots, \ell\}$, is a square matrix of the form

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix} \in M_{k_i}(\mathbb{K}), \quad 1 \leq k_i \leq m_i, \tag{1}$$

for some $i \in \{1, \cdots, p\}$.

The matrix J is called the *Jordan (normal) form* or *the Jordan (canonical) form* of A. The matrices J_k in J are called the Jordan blocks of A.

- The (normal) Jordan form J is unique, up to re-ordering the Jordan blocks on the diagonal.

(1) The number of Jordan blocs in J corresponding to the eigenvalue λ_i is equal to $\dim(E_{\lambda_i})$.

(2) The number of Jordan blocs in J corresponding to the eigenvalue λ_i of size at least k is equal to

$$\dim\Big(\ker(A - \lambda_i I_n)^k\Big) - \dim\Big(\ker(A - \lambda_i I_n)^{k-1}\Big).$$

(3) The number of Jordan blocs in J corresponding to the eigenvalue λ_i of size exactly k is equal to

$$2\dim\Big(\ker(A - \lambda_i I_n)^k\Big) - \dim\Big(\ker(A - \lambda_i I_n)^{k+1}\Big) - \dim\Big(\ker(A - \lambda_i I_n)^{k-1}\Big).$$

(4) The eigenvalue λ_i appears exactly m_i times on the main diagonal of J.

(5) The size k_i of the Jordan block in (1) can be $< m_i$. Different Jordan blocks may correspond to the same eigenvalue λ_i. See examples later.

(6) The sum of the sizes of all Jordan blocks corresponding to λ_i is equal to m_i.

(7) The size of the largest Jordan block corresponding to λ_i is the largest $k \in \mathbb{N}^*$ such that

$$\mathrm{Ker}\Big((A - \lambda_i I_n)^k\Big) \supsetneq \mathrm{Ker}\Big((A - \lambda_i I_n)^{k-1}\Big).$$

Case of real symmetric matrices

Proposition 7

Let A be a real symmetric matrix. Then, the following statements hold.

(i) *All the eigenvalues of A are real numbers.*

(ii) *For any eigenvalue of A there exists a corresponding eigenvector in $\mathbb{R}^n$.*

(iii) *If λ and β are two distinct eigenvalues of A, then*

$$E_\lambda \perp E_\beta.$$

This means that, two eigenvectors corresponding to two different eigenvalues of a real symmetric matrix are orthogonal.

Proposition 8 (Characterization of change of basis matrices relating O.N.B)

Let $\mathcal{B}$ be an orthonormal basis of $\mathbb{R}^n$ and $\mathcal{B}'$ be a basis of $\mathbb{R}^n$. Let P be the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. Then, the basis $\mathcal{B}'$ is orthonormal if, and only if, P is orthogonal (*i.e.*, the matrix P satisfies $P^{-1} = {}^tP$. That is to say, the matrix P has orthonormal columns).

Theorem 6 (Diagonalization of real symmetric matrices)

Every real symmetric matrix A is diagonalizable on $\mathbb{R}$, in a basis $\mathcal{B}$ composed of eigenvectors of A which is orthonormal for the standard scalar product of $\mathbb{R}^n$. The change of basis matrix P, from the standard basis $\mathcal{B}_c$ to $\mathcal{B}$, is orthogonal. We say that the matrix A is orthogonally diagonalizable on $\mathbb{R}$.

Theorem 7 (Case of matrices in $M_n(\mathbb{C})$)

Let $A \in M_n(\mathbb{C})$. If A satisfies

$${}^t(\bar{A}) = A$$

i.e.,

$$A_{ij} = \bar{A}_{ji} \quad \text{(we say that } A \text{ is } \textit{hermitian}\text{)},$$

then A is diagonalizable in an orthogonal basis composed of eigenvectors of A.

Supplement on "the minimal polynomial of a matrix" (paragraph not in the programme)

Theorem 8

Let $A \in M_n(\mathbb{K})$. Then,

(i) *there exists a* **unique** *monic polynomial $\pi_A \in \mathbb{K}[T]$ of least degree such that $\pi_A(A)$ is the zero matrix. i.e., $\pi_A(A) = 0$.*

(ii) *Moreover, if $P \in \mathbb{K}[T]$ satisfies $P(A) = 0$, then π_A divides P.*

Theorem 9

Let $A \in M_n(\mathbb{K})$. Then, A is diagonalizable on $\mathbb{K}$ if and only if its minimal polynomial splits on $\mathbb{K}$ and has distinct roots.

Theorem 10

For any monic polynomial of degree $n \in \mathbb{N}^*$,

$$P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 \in \mathbb{K}[T],$$

there exists a matrix $A \in M_n(\mathbb{K})$ such that $P = \pi_A$.

Theorem 11 (Cauchy)

For any $(t_0, x_0) \in I \times E$, there exists a unique solution $x \in C^1(I, E)$ to the Cauchy problem

$$(C) \qquad \begin{cases} x'(t) = a(t)\,x(t) + b(t), \quad t \in I \\ x(t_0) = x_0. \end{cases}$$

Theorem 12

(i) The set $\mathcal{E}$ of solutions of the homogeneous differential equation (H) is a $\mathbb{K}$-vector space of dimension n.

(ii) Let y be a particular solution to (E). Then, the set of solutions of (E) is given by

$$y + \mathcal{E} := \{y + x \,;\, x \in \mathcal{E}\}.$$

Proposition 9

Let $\varphi_1, \ldots, \varphi_n$ be n solutions of (H). Then, the following statements are equivalent :

(i) $(\varphi_1, \ldots, \varphi_n)$ *is a basis of* $\mathcal{E}$,

(ii) $\exists\, t_0 \in I,\ \Big(\varphi_1(t_0), \ldots, \varphi_n(t_0)\Big)$ *is a basis of* E,

(iii) $\forall\, t \in I,\ \Big(\varphi_1(t), \ldots, \varphi_n(t)\Big)$ *is a basis of* E.

Theorem 13

The wronskian fulfils the following :

$$\forall\, t_0,\ t \in I, \quad w(t) = \exp\left(\int_{t_0}^{t} \mathrm{Tr}(a(s))ds\right) w(t_0),$$

where $\mathrm{Tr(a(s))}$ *denotes the trace of the endomorphism* $a(s)$.

Lemma 2

Let $\varphi \in C^0(I, E)$. Then, there exist n fonctions $\mu_i \in C^0(I, \mathbb{K})$ such that

$$\varphi = \sum_{i=1}^{n} \mu_i\, \varphi_i.$$

For any $A \in M_n(\mathbb{K})$, we define the *exponential of A* by $e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Proposition 10

(a) $\quad \forall\, \lambda \in \mathbb{K}, \quad e^{\lambda I_n} = e^{\lambda} I_n.$

(b) $\quad \forall\, A, B \in M_n(\mathbb{K}), \quad \Big(AB = BA \implies e^{(A+B)} = e^A\, e^B = e^B\, e^A\Big).$

Note that if A and B do not commute (meaning that $AB \neq BA$) the above equality doesn't necessarily hold.

(c) $\quad \forall\, A \in M_n(\mathbb{K}), \quad \det\left(e^A\right) = e^{\mathrm{Tr}(A)}.$

Proposition 11

Let $A \in M_n(\mathbb{K})$. Then, the function $\left(t \longmapsto e^{tA}\right)$ is differentiable on $\mathbb{R}$ and

$$\left(e^{tA}\right)' = A\, e^{tA} = e^{tA} A.$$

Theorem 14

(a) The solution to the Cauchy problem

$$\begin{cases} X'(t) = A\,X(t), \quad t \in I \\ X(t_0) = X_0, \text{ where } (t_0, X_0) \in I \times \mathbb{K}^n, \text{ is } \quad X(t) = e^{(t-t_0)A}\, X_0. \end{cases}$$

Consider the system of linear differential equations

$$(S) \qquad X'(t) = A\,X(t) + B(t), \quad t \in I, \text{where} B \in C^0(I, \mathbb{K}^n).$$

(b) The solutions to the system (S) are of the form

$$X(t) = e^{(t-t_0)A} X_0 + \int_{t_0}^{t} e^{(t-s)A}\, B(s)ds \quad \text{where } X_0 \in \mathbb{K}^n \text{ and } t_0 \in I.$$

Theorem 15

Let $A \in M_n(\mathbb{K})$.

(a) Suppose that A is diagonalizable on $\mathbb{K}$ and $(V_1, \ldots, V_n)$ is a basis of $\mathbb{K}^n$ consisting of eigenvectors of A corresponding to the eigenvalues $(\lambda_1, \ldots, \lambda_n)$. Then, the set of functions

$$\left(t \longmapsto e^{\lambda_i t}\, V_i\right)_{1 \leqslant i \leqslant n} \tag{2}$$

is a basis of solutions of the homogeneous system $X' = AX$.

In other words, every solution of $X' = AX$ can be written as a linear combination of the functions in (2) :

$$X(t) = \alpha_1 e^{\lambda_1 t} V_1 + \alpha_2 e^{\lambda_2 t} V_2 + \cdots + \alpha_n e^{\lambda_n t} V_n, \quad \alpha_i \in \mathbb{K}, 1 \leq i \leq n.$$

Théorème 15 (continued)

(b) Suppose that A is triangularizable on $\mathbb{K}$, *i.e.* $\chi_A(t) = \prod_{i=1}^{p} (\lambda_i - t)^{m_i}$, where $\lambda_1, \cdots, \lambda_p$ are the distinct eigenvalues of A .

For every $i \in \{1, \ldots, p\}$, let $(V_{ij})_{1 \leqslant j \leqslant m_i}$ be a basis of the corresponding generalized eigenspace $N_{\lambda_i} = \mathrm{Ker}\,(A - \lambda_i I)^{m_i}$. Then, the set of functions

$$\left(t \longmapsto e^{\lambda_i t} \sum_{k=0}^{m_i - 1} \frac{t^k}{k!} (A - \lambda_i I)^k \;\; V_{ij}\right)_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant m_i}}$$

is a basis of solutions of the homogeneous system $X' = AX$.

First order equation

Homogeneous equation

$$x' = a_0 x \quad \text{in} \quad I. \qquad\qquad (H_1)$$

The set of solutions of (H_1) is a $\mathbb{K}$-vector space of dimension 1 a basis of which is given by the function

$$t \longmapsto \exp\left(\int_{t_0}^{t} a_0(s) ds\right), \quad \text{where} \quad t_0 \in I \quad \text{is fixed.}$$

Particular solution for the non-homogeneous equation

$$x' = a_0 x + b \qquad \text{in} \quad I, \qquad\qquad (E_1).$$

By the variation of constants

$$y(t) = \lambda(t) \exp\left(\int_{t_0}^{t} a_0(s) ds\right) \quad \text{where} \quad \lambda'(t) = \exp\left(-\int_{t_0}^{t} a_0(s) ds\right) b(t)$$

is a particular solution of (E_1).

Second order equation

Homogeneous equation

$$x'' = a_1 x' + a_0 x \quad \text{in} \quad I. \qquad (H_2)$$

Suppose that a non-zero solution φ_1 of (H_2) is known. We look for a second solution of (H_2) of the form

$$\varphi_2 = \lambda \varphi_1 \qquad \text{where} \qquad \lambda \in C^2(I, \mathbb{K}),$$

such that (φ_1, φ_2) is a basis of solutions of (H_2).

Particular solution for the non-homogeneous equation

$$x'' = a_1 x' + a_0 x + b \quad \text{in} \quad I. \qquad (E_2)$$

By the (method of) variation of constants we obtain a particular solution $y \in C^2(I, \mathbb{K})$ such that

$$y = \lambda_1 \varphi_1 + \lambda_2 \varphi_2 \quad \text{and} \quad \begin{pmatrix} y \\ y' \end{pmatrix} = \lambda_1 \begin{pmatrix} \varphi_1 \\ \varphi_1' \end{pmatrix} + \lambda_2 \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in C^1(I, \mathbb{K})$ satisfy the system

$$\begin{cases} \lambda_1' \varphi_1 + \lambda_2' \varphi_2 = 0 \\ \lambda_1' \varphi_1' + \lambda_2' \varphi_2' = b. \end{cases}$$

Theorem 16 (The principle of superposition of solutions)

If x_1 is a solution of

$$x^{(n)} - \sum_{k=0}^{n-1} a_k x^{(k)} = b_1$$

and x_2 is a solution of

$$x^{(n)} - \sum_{k=0}^{n-1} a_k x^{(k)} = b_2,$$

then, $\lambda_1 x_1 + \lambda_2 x_2$ is a solution of

$$x^{(n)} - \sum_{k=0}^{n-1} a_k x^{(k)} = \lambda_1 b_1 + \lambda_2 b_2$$

for every $(\lambda_1, \lambda_2) \in \mathbb{K}^2$.

Theorem 17

Consider the n-th order homogeneous differential equation

$$x^{(n)} = \sum_{k=0}^{n-1} a_k x^{(k)} \quad \text{in} \quad \mathbb{R}. \qquad (H_n)$$

Suppose that its characteristic polynomial χ splits on $\mathbb{K}$, i.e.,

$$\chi(T) = \prod_{i=1}^{p} (T - \lambda_i)^{m_i},$$

where $\lambda_1, \dots, \lambda_p$ are the distinct roots of χ. Then, the set of functions

$$\left(\varphi_i \ : \ t \longmapsto e^{\lambda_i t} t^j \right)_{\substack{1 \leqslant i \leqslant p \\ 0 \leqslant j \leqslant m_i - 1}}$$

is a basis of solutions of (H_n). In other words, a basis of solutions of (H_n) is

$$\left(e^{\lambda_i t},\ t\, e^{\lambda_i t},\ t^2 e^{\lambda_i t},\ \cdots,\ t^{m_i - 1} e^{\lambda_i t} \right)_{1 \leqslant i \leqslant p}.$$

Proposition 12 (Particular solution when $b(t)$ is of the form $b(t) := e^{\alpha t}\, Q(t)$)

Consider the n-th order linear differential equation

$$x^{(n)} - \sum_{k=0}^{n-1} a_k\, x^{(k)} = b, \quad \textit{in} \quad \mathbb{R}, \qquad (E_n)$$

where $b(t) := e^{\alpha t}\, Q(t)$, $Q \in \mathbb{K}\,[t]$ *and* $\alpha \in \mathbb{K}$.

(a) *Assume that* $\chi(\alpha) \neq 0$. *Then, there exists a particular solution* $y(t)$ *of the equation* (E_n) *of the same form as* $b(t)$. *i.e.,*

$$y(t) = e^{\alpha t} P(t) \quad \textit{with} \quad P \in \mathbb{K}\,[t] \quad \textit{and} \quad \deg(P) = \deg(Q).$$

(b) *Now, assume that* α *is a root of* χ *of multiplicity* $\boldsymbol{m} \in \mathbb{N}^*$.

Then, there exists a particular solution $y(t)$ *of* (E_n) *of the form*

$$y(t) = e^{\alpha t} R(t) \quad \textit{where} \quad R \in \mathbb{K}\,[t] \quad \textit{such that} \quad \deg(R) \leqslant \boldsymbol{m} + \deg(Q).$$