Computational Fluid Dynamics

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Contents

1 Objective

2 Physics

The Navier-Stokes equations govern the motion of viscous fluids. These equations arise from applying conservation of mass, conservation of momentum, and conservation of energy to the motion of fluids and build upon the notion that fluids are a continuous medium rather than a set of discrete particles. The so-called convective form of the Navier-Stokes equations for an incompressible Newtonian fluid without the presence of body forces is:

Continuity equation:
$$\nabla \cdot \mathbf{u} = 0 \tag{2.1a}$$

Momentum equation:
$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \nabla \right) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \tag{2.1b}$$

where $\mathbf{u}$ is the velocity field, t is the time, ρ is the density, p is the pressure field, and ν is the kinematic viscosity. A more tangible notation of these formidable equations is obtained by writing out the vector components and utilizing Einstein's summation convention:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.2a}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{2.2b}$$

Equations (2.2a) and (2.2b) are dimensional. That is, each parameter such as u_i, p, and ν is expressed in terms of a physical quantity. Non-dimensionalization can reduce the number of free parameters and can help to gain a greater insight into the relative size of the various terms present in the equations. The dimensionless parameters are defined as follows:

$$u_i^* \equiv \frac{u_i}{U} \tag{2.3a}$$

$$x_i^* \equiv \frac{x_i}{L} \tag{2.3b}$$

$$t^* \equiv t\frac{U}{L} \tag{2.3c}$$

$$p^* \equiv \frac{p}{\rho U^2} \tag{2.3d}$$

$$\text{Re} \equiv \frac{\nu}{UL} \tag{2.3e}$$

where U is the velocity scale and L is the length scale of the flow. Multiplication of Equations (2.2a) and (2.2b) by $\frac{L}{U}$ and $\frac{L}{U^2}$, respectively, gives

$$\frac{L}{U} \frac{\partial u_i}{\partial x_i} = 0 \tag{2.4a}$$

$$\frac{L}{U^2} \frac{\partial u_i}{\partial t} + \frac{L}{U^2} u_j \frac{\partial u_i}{\partial x_j} = -\frac{L}{U^2} \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{L}{U^2} \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{2.4b}$$

Rearrangement yields

$$\frac{\partial \frac{u_i}{U}}{\partial \frac{x_i}{L}} = 0 \tag{2.5a}$$

$$\frac{\partial \frac{u_i}{U}}{\partial t \frac{U}{L}} + \frac{u_j}{U} \frac{\partial \frac{u_i}{U}}{\partial \frac{x_j}{L}} = -\frac{1}{\rho} \frac{\partial \frac{p}{U^2}}{\partial \frac{x_i}{L}} + \frac{\nu}{UL} \frac{\partial^2 \frac{u_i}{U}}{\partial \frac{x_j}{L} \partial \frac{x_j}{L}} \tag{2.5b}$$

Substituting Equations (2.3a)–(2.3e), we obtain

$$\frac{\partial u_i^*}{\partial x_i^*} = 0 \tag{2.6a}$$

$$\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{\text{Re}} \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} \tag{2.6b}$$

or
$$\nabla \cdot \mathbf{u}^* = 0 \tag{2.7a}$$

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla) \mathbf{u}^* = -\nabla p^* + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}^* \tag{2.7b}$$

Because it is now understood that the governing equations are dimensionless, the asterisk will be omitted in the remainder of this report.

$$\nabla \cdot \mathbf{u} = 0 \tag{2.8a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} \tag{2.8b}$$

For reasons that will be the subject of Chapter 3, it is convenient to rewrite the Navier-Stokes equations in terms of the divergence, curl, and gradient operator. Consider the following vector identities: if $\mathbf{u}$ is a three-dimensional vector field, then

$$\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \tag{2.9}$$

and
$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} \|\mathbf{u}\|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) \tag{2.10}$$

Using these vector identities, Equation (2.8b) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \|\mathbf{u}\|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla p + \frac{1}{\text{Re}} (\nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})) \tag{2.11}$$

Recall that in a velocity field, the curl of the velocity is equal to the vorticity:

$$\xi = \nabla \times \mathbf{u} \tag{2.12}$$

Substituting Equation (2.12) into Equation (2.11), we have

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \|\mathbf{u}\|^2 \right) - \mathbf{u} \times \xi = -\nabla p + \frac{1}{\text{Re}} (\nabla (\nabla \cdot \mathbf{u}) - \nabla \times \xi)$$

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \xi + \nabla \left(p + \frac{1}{2} \|\mathbf{u}\|^2 \right) = \frac{1}{\text{Re}} (\nabla (\nabla \cdot \mathbf{u}) - \nabla \times \xi) \tag{2.13}$$

In Equation (2.13), denote

$$P \equiv p + \frac{1}{2} \|\mathbf{u}\|^2 \tag{2.14}$$

and recognize that the continuity equation, $\nabla \cdot \mathbf{u} = 0$, appears in the right-hand side. Hence, Equation (2.13) is written as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \xi + \nabla P + \frac{1}{\text{Re}} \nabla \times \xi = 0 \tag{2.15}$$

Thus, the final system of equations is given by:

Continuity equation:
$$\nabla \cdot \mathbf{u} = 0 \tag{2.16a}$$

Velocity-vorticity relation:
$$\xi = \nabla \times \mathbf{u} \tag{2.16b}$$

Momentum equation:
$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \xi + \nabla P + \frac{1}{\text{Re}} \nabla \times \xi = 0 \tag{2.16c}$$

3 Mathematics

We have so far derived a set of partial differential equations that govern the physical problem at hand. In this chapter, we introduce a mathematical toolbox that is fundamental to the conversion of the problem into a format that a computer can understand. Rather than using classical discretization methods such as the finite difference method, we will use elements from discrete exterior calculus. Discrete exterior calculus is a vast mathematical field and a rigorous treatment is far beyond the scope of this assignment. The scope of this chapter is limited to those particular concepts that are required to solve the lid-driven cavity flow problem.

3.1 Cells

Let us first introduce a discrete version of the familiar 2-dimensional Euclidean space. Two-dimensional space supports objects with either zero, one, or two dimensions, such as points, lines, and planes. These k-dimensional objects are named k-cells, represented by $\sigma^{(k)}$, where k denotes the dimension of the cell. Three examples of k-cells are shown in Figure 3.1.
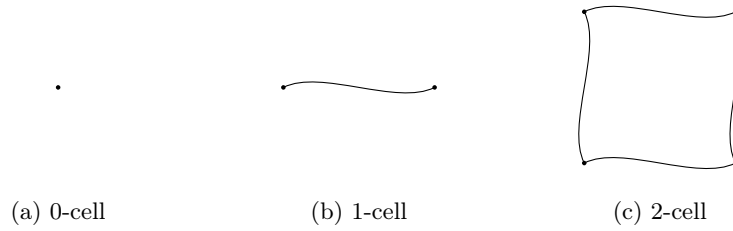

(a) 0-cell　　　　　　　(b) 1-cell　　　　　　　(c) 2-cell

Figure 3.1: Examples of k-cells in two-dimensional space.

A point or scalar is a 0-cell and a line segment or vector is a 1-cell. Just like a 1-cell is a line with a length but with no position, a 2-cell can be thought of as a plane with an area, but no position.

Notice that k-cells are bounded by $(k-1)$-cells and that a cell's shape is arbitrary, as indicated by the wavy lines in Figure 3.1. (*Note:* The boundary of 0-cells is empty by definition.) A set of k-cells is called a cell-complex iff the boundary of each k-cell is also part of the set. For example, the object in Figure 3.1c is a cell-complex because the plane is bounded by lines and the lines are in turn bounded by points, all of which are part of the object.

3.2 Chains and Cochains

A set of k-cells is called a chain, written as

$$\mathbf{c}^{(k)} = \{c_1\sigma_1^{(k)}, \dots, c_N\sigma_N^{(k)}\} \tag{3.1}$$

where N denotes the number of k-cells in the cell complex and c_i denotes a weight $\in \mathbb{R}$. Chains merely differ by their weights, since all chains are composed of the same k-cells. The summation and multiplication of two k-chains is well-defined. In the case of summation, the weights add, and in the case of multiplication by a real number, the weights are multiplied by that real number. Thus, the set of k-chains, denoted $C^{(k)}$, is a linear vector space and the k-cells represent a basis for this vector space.

Because k-cells represent a basis for $C^{(k)}$, we can define linear functionals that assign a value to k-cells. These linear functionals are called k-cochains and the set of all k-cochains is itself a linear vector space. To reiterate the main point: a chain is a set of cells and a cochain is a linear map from a set of cells to a set of numbers.

3.3 Orientation

In addition to their dimension, k-cells have a property that we call orientation. A k-cell either has a positive or a negative orientation that can be freely defined on a per-cell basis, as long as the orientation remains unchanged thereafter.

Figure 3.2 shows three k-cells in two-dimensional space with an intrinsic orientation that we call inner-oriented k-cells. Inner-oriented points for instance, can either represent sources or sinks and are arbitrarily chosen to be sources (i.e. positive) by default, as indicated by the outward pointing arrows in Figure 3.2a. Inner-oriented lines have a positive orientation when pointing to the right, as indicated by the arrows in Figure 3.2b. It is a bit awkward to think of the orientation of a plane, but an inner-oriented plane is defined positive when it is oriented counterclockwise, as indicated in Figure 3.2c.

Inner-oriented k-cells have an outer-oriented counterpart where its orientation is defined around or through the k-cell, rather than within. For example, an inner-oriented surface has a counterclockwise orientation whereas an outer-oriented surface represents a source. Figure 3.3 shows three outer-oriented k-cells in two-dimensional space.
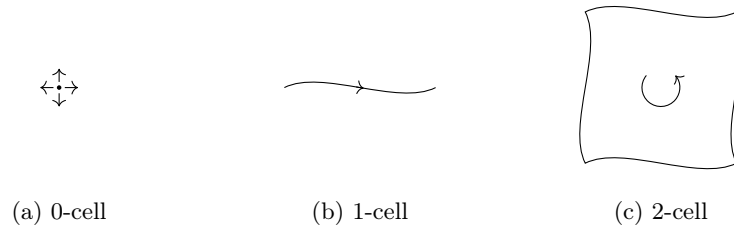

(a) 0-cell   (b) 1-cell   (c) 2-cell

Figure 3.2: Inner-oriented k-cells in two-dimensional space.

(a) 0-cell   (b) 1-cell   (c) 2-cell
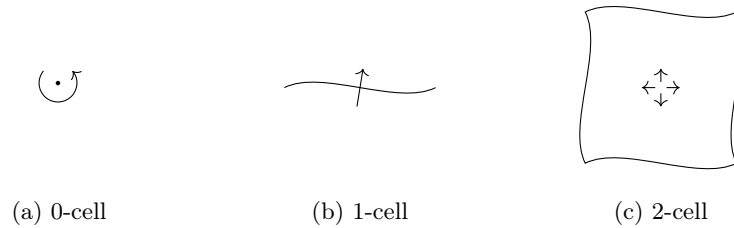
Figure 3.3: Outer-oriented k-cells in two-dimensional space.

The distinction between inner and outer oriented k-cells makes sense from a physical point of view; certain physical quantities are naturally expressed in terms of inner-oriented cochains whereas other physical quantities are naturally expressed in terms of outer-oriented cochains. The circulation along a line naturally resembles an inner-oriented 1-cell whereas the flux through a line naturally resembles an outer-oriented 1-cell.

3.4 The Exterior Derivative

The descrete exterior derivative, δ, is a linear map of a (k)-cochain to a $(k+1)$-cochain. It turns out that the standard treatment of vector calculus hides a number of important things that are applied implicitly. As we shall see in the upcoming

examples, all three of the multivariable derivatives in vector calculus (i.e. gradient, curl, and divergence) are actually one kind of derivative: the exterior derivative.
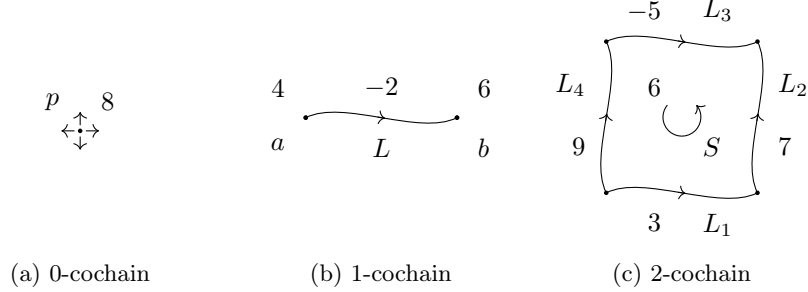
3.4.1 Inner-Oriented Quantities

(a) 0-cochain	(b) 1-cochain	(c) 2-cochain

Figure 3.4: Examples of k-cochains in two-dimensional space.

Figure 3.4b shows a 0-cochain consisting of two points, a and b, and a 1-cochain consisting of one line segment, L. L points from a to b, so a acts as a source and b acts as a sink. Because a sink is defined negative, application of δ results in a value of -2 assigned to the line segment, since $4 - 6 = -2$. Assuming that points represent samples of a continuous scalar field, ϕ, it is clearly seen that the application of δ on a 0-cochain is analogous to the gradient operator:

$$4 - 6 = \phi(b) - \phi(a)$$
$$= \int_L \nabla \phi \cdot d\mathbf{s} \tag{3.2}$$

Figure 3.4c shows a 1-cochain consisting of four line segments and a 2-cochain consisting of a surface, S. If the orientation of a boundary line segment opposes the orientation of the surface, then its value should be subtracted. Thus, application of δ results in a value of 6 assigned to the surface, since $3 - 7 - (-5) - 9 = 6$. If line segments represent paths through a vector field, $\mathbf{u}$, then the application of δ on a 1-cochain is analogous to the curl operator:

$$4 + 7 - (-8) - 9 = \int_{L_1} \mathbf{u} \cdot d\mathbf{l} + \int_{L_2} \mathbf{u} \cdot d\mathbf{l} + \int_{L_3} \mathbf{u} \cdot d\mathbf{l} + \int_{L_4} \mathbf{u} \cdot d\mathbf{l}$$
$$= \oint_{\partial S} \mathbf{u} \cdot d\mathbf{l} \tag{3.3}$$
$$= \iint_S (\nabla \times \mathbf{u}) \cdot dA$$

where $\partial S \equiv L_1 \cup L_2 \cup L_3 \cup L_4$. This is not the only insight to be gained here. Suppose that the vector field $\mathbf{u}$ represents a velocity field. Recall that the circulation, denoted Γ, is defined as:

$$\Gamma \equiv \oint_C \mathbf{u} \cdot d\mathbf{l} \tag{3.4}$$

Comparing Equations (3.3) and (3.4), we conclude that if a line segment represents a velocity vector, then a surface must represent the circulation around its boundary. Circulation is related to vorticity; the circulation around a closed contour is equal to the integrated vorticity enclosed by that contour. From Stokes' theorem:

$$\Gamma \equiv \oint_C \mathbf{u} \cdot d\mathbf{l} = \iint_A (\nabla \times \mathbf{u}) \, dA = \iint_A \xi \, dA \tag{3.5}$$

Therefore, if the area of S is infinitesimal, then S physically represents vorticity.

Because 3-cochains are undefined in two-dimensional space, the application of δ on a 2-cochain results in the empty set by definition. In a similar vein, 0-cochains

can only be the result of the application of δ on a real number. The results of this section can be summarized in a diagram called the DeRham complex:

$$\mathbb{R} \xrightarrow{\quad \delta \quad} C^{(0)} \xrightarrow[\text{grad}]{\quad \delta \quad} C^{(1)} \xrightarrow[\text{curl}]{\quad \delta \quad} C^{(2)} \xrightarrow{\quad \delta \quad} \emptyset \quad (3.6)$$

3.4.2 Outer-Oriented Quantities

The analogy between the δ operator and the gradient and curl operator applies to both inner and outer oriented k-cochains. Thus, the DeRham complex can be extended to include outer-oriented cochains, marked with a tilde:

$$\mathbb{R} \xrightarrow{\quad \delta \quad} \tilde{C}^{(0)} \xrightarrow[\text{grad}]{\quad \delta \quad} \tilde{C}^{(1)} \xrightarrow[\text{curl}]{\quad \delta \quad} \tilde{C}^{(2)} \xrightarrow{\quad \delta \quad} \emptyset \quad (3.7)$$

3.5 The Hodge-$\star$ Operator

We have established that two-dimensional space supports points, lines, and planes. Let us briefly digress to three-dimensional space for the sake of familiarity; three-dimensional space supports points, lines, planes, and volumes. In three-dimensional space, we can have one linearly independent volume, three linearly independent planes, and three linearly independent lines. It is a bit awkward to think of linearly independent points, but there is just one linearly independent point in three-dimensional space. This 1–3–3–1 sequence suggests a pairing; there exists a vector space isomorphism between k-vectors and $(n-k)$-vectors. The Hodge-$\star$ operator is a linear map between inner-oriented k-cochains and outer-oriented $(n-k)$-cochains or vice versa, where n denotes the dimension of the vector space.

We have introduced the δ operator to map (k)-cochains to $(k+1)$-cochains. However, the Navier-Stokes equations equate both inner and outer oriented cochains, which requires the Hodge-$\star$ operator. The Hodge-$\star$ operator links the DeRham complexes for inner and outer oriented cochains. This linked structure is called the double DeRham complex:

$$(3.8)$$

In a discrete setting, cochains are stored in vectors, δ operators are represented by incidence matrices, $\mathbb{E}$, and Hodge-$\star$ operators are represented by Hodge matrices, $\mathbb{H}$. In such a framework, application of the δ or Hodge-$\star$ operator becomes a matrix vector-multiplication. Hence, the double DeRham complex in Equation (3.8) becomes

$$(3.9)$$
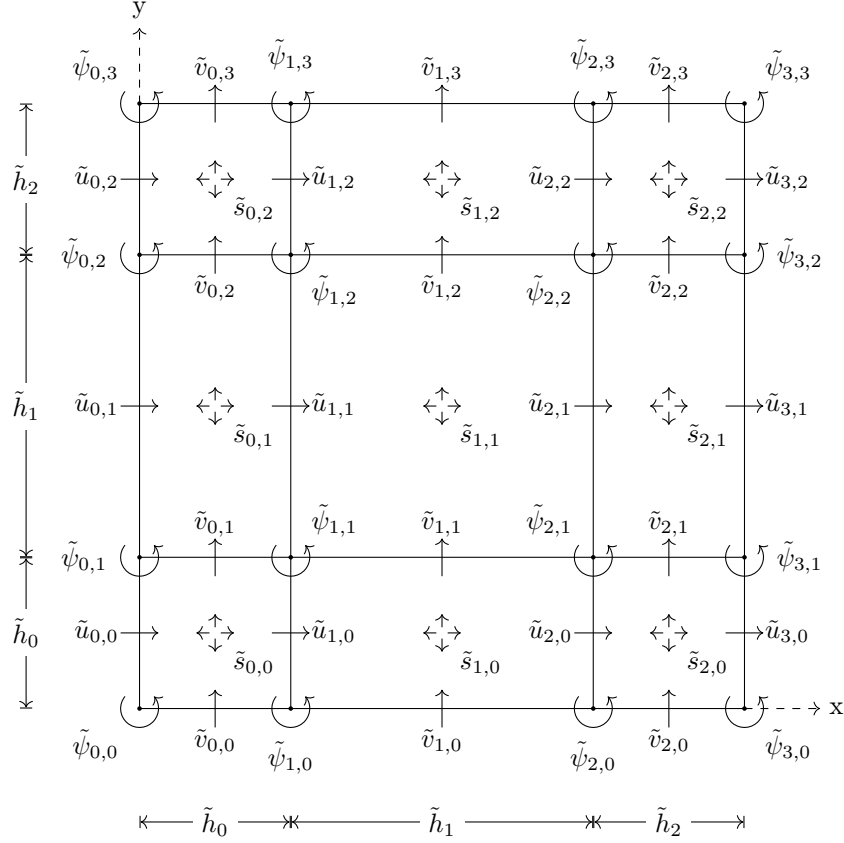
Figure 3.5: The outer-oriented grid.

3.6 Discretization of the Unit Square

It is impossible to derive the incidence and Hodge matrices without any a priori knowledge about the geometry of the grid or cell complex. Let us therefore first discretize the unit square Ω into a grid composed of points, lines, and surfaces. The outer oriented grid shown in Figure 3.5 was constructed for $n = 3$, where n denotes the number of surfaces in the horizontal and vertical direction. The grid has an orthogonal structure, but is not uniform. The points are distributed via cosine spacing at equal angular increments because the smallest flow features are expected to emerge in the vicinity of the walls, which justifies the refinement of the grid. The inner-oriented grid associated with the outer-oriented grid is constructed such that the inner-oriented points are located precisely in the center of the surfaces of the outer-oriented grid.

3.7 Physical Representation

We briefly digress here to discuss the physical meaning of the k-cells in the inner and outer oriented grids. The orientation of the outer-oriented surfaces, $\tilde{s}_{i,j}$, is source-like, as indicated by the outward pointing arrows. That is, an outflow is considered positive and an inflow is considered negative. The line segments, $\tilde{u}_{i,j}$ and $\tilde{v}_{i,j}$, represent mass flow in the form:

$$\tilde{u}_{i,j} = \int_{y_{i,j}}^{y_{i,j+1}} \mathbf{u} \cdot \hat{\mathbf{n}} \, dy \tag{3.10a}$$

and
$$\tilde{v}_{i,j} = \int_{x_{i,j}}^{x_{i+1,j}} \mathbf{u} \cdot \hat{\mathbf{n}} \, dx \tag{3.10b}$$

10
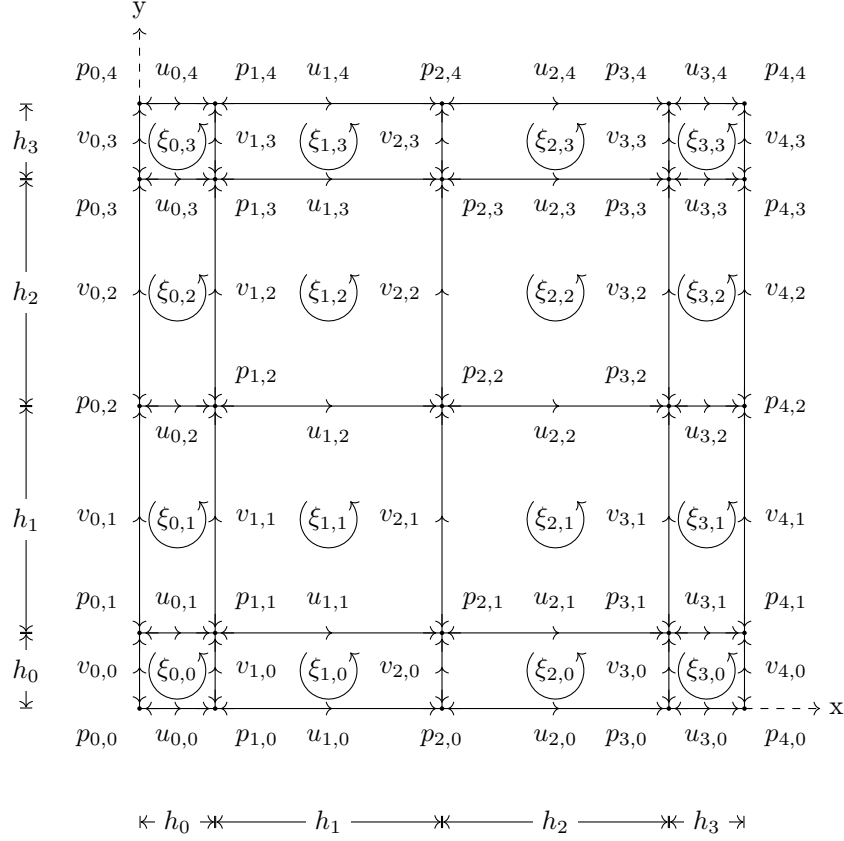
Figure 3.6: The inner-oriented grid.

where $\hat{\mathbf{n}}$ denotes the normal vector along the line segment. (*Note:* it is important to remember that all physical quantities are still dimensionless.) A rightward and upward flow is considered positive, as indicated by the arrows embedded in the lines. If lines represent mass flow, then points must represent samples of the the continuous stream function, ψ. Here is why: the numerical value of ψ is defined such that the difference $\Delta\psi$ between two streamlines is equal to the mass flux between the two streamlines. This fact allows us to determine the stream function up to a constant.

The inner-oriented line segments, $u_{i,j}$ and $v_{i,j}$, represent circulation:

$$u_{i,j} = \int_{x_{i,j}}^{x_{i,j+1}} \mathbf{u} \cdot \hat{\mathbf{t}} \, dx \tag{3.11a}$$

and

$$v_{i,j} = \int_{y_{i,j}}^{y_{i+1,j}} \mathbf{u} \cdot \hat{\mathbf{t}} \, dy \tag{3.11b}$$

where $\hat{\mathbf{t}}$ denotes the tangent vector along the line segment. As discussed in the preceding section, application of the curl operator on a velocity field yields vorticity, assuming that the surfaces are infinitesimal. The inner-oriented surfaces, denoted $\xi_{(i,j)}$, do therefore represent vorticity. The inner-oriented points, denoted $p_{i,j}$, represent pressure because the pressure gradient is related to velocity. Notice that the orientation of the inner-oriented points, $p_{i,j}$, is sink-like, as indicated by the inward pointing arrows.

We have constructed the inner and outer oriented grid and we have established what the k-cells physically represent. The double DeRham complex can be updated

incorporating the newly introduced quantities:

$$
\begin{array}{c}
\mathbf{p}^{(0)} \xrightarrow[\text{grad}]{\mathbb{E}^{(1,0)}} \mathbf{u}^{(1)} \xrightarrow[\text{curl}]{\mathbb{E}^{(2,1)}} \xi^{(2)}
\end{array}
$$

$$
\mathbb{H}^{(0,\tilde{2})} \quad \mathbb{H}^{(\tilde{2},0)} \quad \mathbb{H}^{(1,\tilde{1})} \quad \mathbb{H}^{(\tilde{1},1)} \quad \mathbb{H}^{(2,\tilde{0})} \quad \mathbb{H}^{(\tilde{0},2)} \tag{3.12}
$$

$$
\tilde{\mathbf{s}}^{(2)} \xleftarrow[\text{curl}]{\tilde{\mathbb{E}}^{(2,1)}} \tilde{\mathbf{u}}^{(1)} \xleftarrow[\text{grad}]{\tilde{\mathbb{E}}^{(1,0)}} \tilde{\psi}^{(0)}
$$

3.8 Structure of the Navier-Stokes Equations

Recall the system of equations derived in Chapter 2:

Continuity equation: $\qquad\qquad \nabla \cdot \mathbf{u} = 0$ (3.13a)

Velocity-vorticity relation: $\qquad \xi = \nabla \times \mathbf{u}$ (3.13b)

Momentum equation: $\qquad\qquad \dfrac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \xi + \nabla P + \dfrac{1}{\text{Re}} \nabla \times \xi = 0$ (3.13c)

Let us first add superscripts to denote the dimensions of the k-cells:

$$
\nabla \cdot \mathbf{u}^{(1)} = 0 \tag{3.14a}
$$

$$
\xi^{(2)} = \nabla \times \mathbf{u}^{(1)} \tag{3.14b}
$$

$$
\frac{\partial \mathbf{u}^{(1)}}{\partial t} - \mathbf{u}^{(1)} \times \xi^{(2)} + \nabla P^{(0)} + \frac{1}{\text{Re}} \nabla \times \xi^{(2)} = 0 \tag{3.14c}
$$

To make these equations consistent within the framework of descrete exterior calculus, all terms must be k-cells of the same dimension. For example, it is not allowed to equate 2-cochains and 1-cochains.

3.8.1 Continuity Equation

Equation (3.14a) involves the divergence operator. The divergence operator is equivalent to ① mapping velocity to mass flow and a ② subsequent application of the curl operator. These two operations are annotated in the double DeRham complex:

$$
\begin{array}{c}
\mathbf{p}^{(0)} \xrightarrow[\text{grad}]{\mathbb{E}^{(1,0)}} \mathbf{u}^{(1)} \xrightarrow[\text{curl}]{\mathbb{E}^{(2,1)}} \xi^{(2)}
\end{array}
$$

$$
\mathbb{H}^{(0,\tilde{2})} \quad \mathbb{H}^{(\tilde{2},0)} \quad \mathbb{H}^{(1,\tilde{1})} \quad \mathbb{H}^{(\tilde{1},1)} \; ① \quad \mathbb{H}^{(2,\tilde{0})} \quad \mathbb{H}^{(\tilde{0},2)} \tag{3.15}
$$

$$
② \quad
$$

$$
\tilde{\mathbf{s}}^{(2)} \xleftarrow[\text{curl}]{\tilde{\mathbb{E}}^{(2,1)}} \tilde{\mathbf{u}}^{(1)} \xleftarrow[\text{grad}]{\tilde{\mathbb{E}}^{(1,0)}} \tilde{\psi}^{(0)}
$$

Thus, Equation (3.13a) is equivalent to

$$
\tilde{\mathbb{E}}^{(2,1)} \mathbb{H}^{(\tilde{1},1)} \mathbf{u}^{(1)} = 0 \tag{3.16}
$$

3.8.2 Velocity-Vorticity relation

The velocity-vorticity relation states that vorticity, ξ, is equal to the curl of velocity, $\mathbf{u}$. This represents a mapping from an inner-oriented 1-cochain to an inner-oriented 2-cochain. This relation is implicitly present in the double DeRham complex, denoted ① in

$$
\begin{array}{ccccc}
\mathbf{p}^{(0)} & \xrightarrow[\text{grad}]{\mathbb{E}^{(1,0)}} & \mathbf{u}^{(1)} & \xrightarrow[\text{curl}]{\mathbb{E}^{(2,1)}} & \xi^{(2)} \\
\mathbb{H}^{(0,\tilde{2})} \Big\updownarrow \mathbb{H}^{(\tilde{2},0)} & & \mathbb{H}^{(1,\tilde{1})} \Big\updownarrow \mathbb{H}^{(\tilde{1},1)} & & \mathbb{H}^{(2,\tilde{0})} \Big\updownarrow \mathbb{H}^{(\tilde{0},2)} \\
\tilde{\mathbf{s}}^{(2)} & \xleftarrow[\text{curl}]{\tilde{\mathbb{E}}^{(2,1)}} & \tilde{\mathbf{u}}^{(1)} & \xleftarrow[\text{grad}]{\tilde{\mathbb{E}}^{(1,0)}} & \tilde{\psi}^{(0)}
\end{array}
\tag{3.17}
$$

Equation 3.14b can therefore be written as

$$
\xi^{(2)} = \mathbb{E}^{(2,1)}\mathbf{u}^{(1)}
\tag{3.18}
$$

3.8.3 Momentum Equation

The momentum equation involves three terms that must be expressed in terms of incidence and Hodge matrices:

1. $\mathbf{u}^{(1)} \times \xi^{(2)}$

2. $\nabla P^{(0)}$

3. $\nabla \times \xi^{(2)}$

The cross product $\mathbf{u}^{(1)} \times \xi^{(2)}$ represents the nonlinear convective term of the Navier-Stokes equations and is a bit of a special case. We will therefore replace the nonlinear term by a generic vector named "convective", to be derived at a later stage.

The pressure gradient, $\nabla P^{(0)}$, represents a mapping from an inner-oriented 0-cochain to an inner-oriented 1-cochain. This mapping is denoted ① in the double DeRham complex:

$$
\begin{array}{ccccc}
\mathbf{p}^{(0)} & \xrightarrow[\text{grad}]{\mathbb{E}^{(1,0)}} & \mathbf{u}^{(1)} & \xrightarrow[\text{curl}]{\mathbb{E}^{(2,1)}} & \xi^{(2)} \\
\mathbb{H}^{(0,\tilde{2})} \Big\updownarrow \mathbb{H}^{(\tilde{2},0)} & & \mathbb{H}^{(1,\tilde{1})} \Big\updownarrow \mathbb{H}^{(\tilde{1},1)} & & \mathbb{H}^{(2,\tilde{0})} \Big\updownarrow \mathbb{H}^{(\tilde{0},2)} \\
\tilde{\mathbf{s}}^{(2)} & \xleftarrow[\text{curl}]{\tilde{\mathbb{E}}^{(2,1)}} & \tilde{\mathbf{u}}^{(1)} & \xleftarrow[\text{grad}]{\tilde{\mathbb{E}}^{(1,0)}} & \tilde{\psi}^{(0)}
\end{array}
\tag{3.19}
$$

The third term, namely the curl of vorticity or the curl of the curl of velocity, is somewhat less obvious. Both velocity and vorticity are vector fields. Within the realms of ordinary calculus, there would be no difference between the two. However, within the realms of discrete exterior calculus, there is an important distinction; velocity is an integral value associated with lines whereas vorticity is an integral value

associated with surfaces. So the only way to apply the curl operator to velocity a second time is to take a slight detour accross the double DeRham complex. First, apply the curl operator to velocity as usual ①. Second, map vorticity to its outer oriented counterpart, the stream function ψ ②. Third, apply the gradient operator (which is the transpose of the curl operator, $\mathbb{E}^{(2,1)}$, as we shall soon see) to obtain mass flow ③. Last but not least, map the mass flow to velocity since we ultimately want all quantities in the momentum equation to be expressed in terms of velocity ④. This chain of mappings is graphically depicted in the double DeRham complex:

$$(3.20)$$

Hence, Equation 3.14c can be written as

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} + \text{convective}^{(1)} - \mathbb{E}^{(1,0)}P^{(0)} + \frac{1}{\text{Re}}\mathbb{H}^{(1,\tilde{1})}\tilde{\mathbb{E}}^{(1,0)}\mathbb{H}^{(\tilde{0},2)}\mathbb{E}^{(2,1)}\mathbf{u}^{(1)} = 0 \qquad (3.21)$$

3.8.4 Summary

The Navier-Stokes are rewritten in terms of incidence matrices and Hodge matrices, as follows:

$$\tilde{\mathbb{E}}^{(2,1)}\mathbb{H}^{(\tilde{1},1)}\mathbf{u}^{(1)} = 0 \qquad (3.22a)$$

$$\xi^{(2)} = \mathbb{E}^{(2,1)}\mathbf{u}^{(1)} \qquad (3.22b)$$

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} + \text{convective}^{(1)} - \mathbb{E}^{(1,0)}P^{(0)} + \frac{1}{\text{Re}}\mathbb{H}^{(1,\tilde{1})}\tilde{\mathbb{E}}^{(1,0)}\mathbb{H}^{(\tilde{0},2)}\mathbb{E}^{(2,1)}\mathbf{u}^{(1)} = 0 \qquad (3.22c)$$

3.9 The Incidence Matrices and Hodge Matrices

The incidence matrices, $\mathbb{E}$, and the Hodge matrices, $\mathbb{H}$, can be derived using the geometries introduced in the preceding section. The incidence and Hodge matrices derived in this section are only valid for the grids shown in Figures 3.5 and 3.6. That is, the validity is limited to the rather coarse spacing of $n = 3$. However, the format of those matrices belonging to denser grids can be correctly deduced by careful analysis of the matrix structure at $n = 3$, which is of course what we are after.

3.9.1 $\tilde{\mathbb{E}}^{(2,1)}$

Application of the incidence matrix $\tilde{\mathbb{E}}^{(2,1)}$ to line segments that represent mass flow, $\tilde{\mathbf{u}}^{(1)}$, yields the rate of mass production in the surface enclosed by those line segments, $\tilde{\mathbf{s}}^{(2)}$. This implies that

$$\tilde{\mathbb{E}}^{(2,1)}\tilde{\mathbf{u}}^{(1)} = 0 \qquad (3.23)$$

reads as conversation of mass; mass cannot be created nor destoryed and mass that flows into a surface, must flow out, and vice versa. Let us construct a linear equation

for each of the surfaces $\tilde{s}_{(i,j)}$:

$$\tilde{s}_{0,0} = -\tilde{u}_{0,0} + \tilde{u}_{1,0} - \tilde{v}_{0,0} + \tilde{v}_{0,1}$$
$$\tilde{s}_{1,0} = -\tilde{u}_{1,0} + \tilde{u}_{2,0} - \tilde{v}_{1,0} + \tilde{v}_{1,1}$$
$$\vdots$$
$$\tilde{s}_{2,2} = -\tilde{u}_{2,2} + \tilde{u}_{3,2} - \tilde{v}_{2,2} + \tilde{v}_{2,3}$$

(3.24)

Equation (3.24) expressed in matrix notation becomes

$$\tilde{\mathbf{s}}^{(2)} = \tilde{\mathbb{E}}^{(2,1)}\tilde{\mathbf{u}}^{(1)}$$

(3.25)

The mass flow rates $\tilde{u}_{(i,j)}$ and $\tilde{v}_{(i,j)}$ adjacent to the boundary of the unit square are known because the boundary conditions of the problem are known. The matrices in the right-hand side of Equation (3.25) can be split into a matrix of unknowns and into a matrix of knows. Splitting the matrix yields

$$\tilde{\mathbf{s}}^{(2)} = \tilde{\mathbb{E}}^{(2,1)}\tilde{\mathbf{u}}^{(1)} + \tilde{\mathbb{E}}^{(2,1)}_{\text{known}}\tilde{\mathbf{u}}^{(1)}_{\text{known}}$$

(3.26)

where

$$\tilde{\mathbb{E}}^{(2,1)} = \begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1
\end{bmatrix}$$

(3.27)

and

$$\tilde{\mathbb{E}}^{(2,1)}_{\text{known}} = \begin{bmatrix}
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}$$

(3.28)

3.9.2 $\mathbb{E}^{(1,0)}$

The incidence matrix $\mathbb{E}^{(1,0)}$ maps an inner-oriented 0-cochain to an inner-oriented 1-cochain. Let us create a linear equation for each of the 1-cells, $u_{(i,j)}$, in the inner-oriented grid, considering that the 0-cells, $P_{(i,j)}$, are sink-like. The equations are

$$u_{1,1} = -p_{1,1} + p_{2,1}$$
$$u_{2,1} = -p_{2,1} + p_{3,1}$$
$$\vdots$$
$$u_{2,3} = -p_{2,3} + p_{3,3}$$
$$v_{1,1} = -p_{1,1} + p_{1,2}$$
$$v_{2,1} = -p_{2,1} + p_{2,2}$$
$$\vdots$$
$$v_{3,2} = -p_{3,2} + p_{3,3}$$

(3.29)

Equation (3.29) can be written in matrix notation as

$$\mathbf{u} = \mathbb{E}^{(1,0)}\mathbf{p}$$

(3.30)

where

$$\mathbb{E}^{(1,0)} = \begin{bmatrix} -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot \end{bmatrix} \tag{3.31}$$

It is important to notice that

$$\mathbb{E}^{(1,0)} = -\left(\tilde{\mathbb{E}}^{(2,1)}\right)^T \tag{3.32}$$

3.9.3 $\mathbb{E}^{(2,1)}$

The incidence matrix $\mathbb{E}^{(2,1)}$ maps an inner-oriented 1-cochain to an inner-oriented 2-cochain. That is, it maps circulation along line segments to vorticity in the planes enclosed by those line segments. The derivation of $\mathbb{E}^{(2,1)}$ is straightforward: add the circulation along those line segments sharing a common orientation with the plane and subtract the circulation along those line segments whose orientation opposes the orientation of the plane. Executing this procedure for all planes $\xi_{i,j}$, we have

$$\begin{aligned} \xi_{0,0} &= u_{0,0} - u_{0,1} - v_{0,0} + v_{1,0} \\ \xi_{1,0} &= u_{1,0} - u_{1,1} - v_{1,0} + v_{2,0} \\ &\vdots \\ \xi_{3,3} &= u_{3,3} - u_{3,4} - v_{3,3} + v_{4,3} \end{aligned} \tag{3.33}$$

which is in accordance to the formula

$$\xi_{i,j} = u_{i,j} - u_{i,j+1} - v_{i,j} + v_{i+1,j} \tag{3.34}$$

$$\xi^{(2)} = \mathbb{E}^{(2,1)}\mathbf{u}^{(1)} \tag{3.35}$$

The velocities adjacent to the boundary are again known because the boundary conditions of the problem are known. In a similar fashion as in Section 3.9.1, splitting the incidence matrix $\mathbb{E}^{(2,1)}$ into a matrix of unknowns and into a matrix of knows, we have

$$\xi^{(2)} = \mathbb{E}^{(2,1)}\mathbf{u}^{(1)} + \mathbb{E}^{(2,1)}_{\text{known}}\mathbf{u}^{(1)}_{\text{known}} \tag{3.36}$$

where

$$\mathbb{E}^{(2,1)} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & -1 & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{3.37}$$

and

$$\mathbb{E}^{(2,1)}_{\text{known}} =$$

$$
\begin{bmatrix}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot
\end{bmatrix}
$$

$$(3.38)$$

Evaluating the product $\mathbb{E}^{(2,1)}_{\text{known}}\mathbf{u}^{(1)}_{\text{known}}$, we have

$$\xi^{(2)} = \mathbb{E}^{(2,1)}\mathbf{u}^{(1)} + \mathbf{u}^{(1)}_{\text{prescribed}} \tag{3.39}$$

3.9.4 $\tilde{\mathbb{E}}^{(1,0)}$

$$\tilde{u}_{1,0} = -\tilde{\psi}_{1,0} + \tilde{\psi}_{1,1}$$
$$\tilde{u}_{2,0} = -\tilde{\psi}_{3,0} + \tilde{\psi}_{2,1}$$
$$\vdots$$
$$\tilde{u}_{2,2} = -\tilde{\psi}_{2,2} + \tilde{\psi}_{2,3}$$
$$\tilde{v}_{0,1} = \tilde{\psi}_{0,1} - \tilde{\psi}_{1,1}$$
$$\tilde{v}_{1,1} = \tilde{\psi}_{1,1} - \tilde{\psi}_{2,1}$$
$$\vdots$$
$$\tilde{v}_{2,2} = \tilde{\psi}_{2,2} - \tilde{\psi}_{3,2}$$

$$(3.40)$$

$$
\tilde{\mathbb{E}}^{(1,0)} =
\begin{bmatrix}
\cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot
\end{bmatrix}
$$

$$(3.41)$$

$$\tilde{\mathbf{u}} = \tilde{\mathbb{E}}^{(1,0)}\tilde{\psi} \tag{3.42}$$

$$\tilde{\mathbb{E}}^{(1,0)} = \left(\mathbb{E}^{(2,1)}\right)^T \tag{3.43}$$

3.9.5 $\mathbb{H}^{(\tilde{1},1)}$ and $\mathbb{H}^{(1,\tilde{1})}$

The Hodge matrices $\mathbb{H}^{(\tilde{1},1)}$ and $\mathbb{H}^{(1,\tilde{1})}$ respectively represent a linear map between between the mass flow through a line segment and the circulation along a line segment and vice versa. The cirulation along a line segment is equal to the velocity along that line segment times the lenght of the line segment. Or as a function of mass flow:

$$\text{ciculation along } L_a = \underbrace{\frac{\text{mass flow through } L_b}{\text{length of } L_b}}_{\text{velocity}} \cdot \underbrace{\text{length of } L_a}_{\text{length}} \tag{3.44}$$

Let us look at a specific example. The circulation along the line segment $u_{1,1}$ in Figure 3.6 is given by

$$u_{2,1} = \frac{\tilde{u}_{2,0}}{\tilde{h}_0} h_2 = \frac{h_2}{\tilde{h}_0} \tilde{u}_{2,0} \tag{3.45}$$

In general, the circulation along the line segments $u_{i,j}$ and $v_{i,j}$ can be found in accordance with the formulas

$$u_{i,j} = \frac{h_i}{\tilde{h}_{j-1}} \tilde{u}_{i,j-1} \tag{3.46a}$$

and

$$v_{i,j} = \frac{h_j}{\tilde{h}_{i-1}} \tilde{v}_{i-1,j} \tag{3.46b}$$

Equations (3.46a) and (3.46b) expressed in matrix notation yields

$$\mathbf{u}^{(1)} = \mathbb{H}^{(1,\tilde{1})} \tilde{\mathbf{u}}^{(1)} \tag{3.47}$$

The matrix $\mathbb{H}^{(1,\tilde{1})}$ is a diagonal matrix.

3.9.6 $\mathbb{H}^{(\tilde{0},2)}$ and $\mathbb{H}^{(2,\tilde{0})}$

The Hodge matrix $\mathbb{H}^{(\tilde{0},2)}$ maps the vorticity associated with an inner-oriented 2-cochain to the stream function associated with an outer-oriented 0-cochain. This amounts to simply deviding the inner-oriented 2-cochain by its area, in other words

$$\tilde{\psi}_{i,j} = (h_i h_j)^{-1} \xi_{i,j} \tag{3.48}$$

Which becomes in matrix notation

3.9.7 The Convective Term

The derivation of the convective term, $\mathbf{u}^{(1)} \times \xi^{(2)}$, is not as straightforward. The convective term is an exterior product of a 1-cochain and a 2-cochain. If $\xi^{(2)}$ and $\mathbf{u}^{(1)}$ are given by

$$\xi^{(2)} = \xi \, dx \, dy \tag{3.49}$$

and

$$\mathbf{u}^{(1)} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \tag{3.50}$$

respectively, then the exterior product yields

$$\mathbf{u}^{(1)} \times \xi^{(2)} = u\xi \, dy - v\xi \, dx \tag{3.51}$$

As an example, let us compute the convection through the line segment $u_{1,1}$. Line segment $u_{1,1}$ is shown in Figure 3.7 along with the adjacent lines and planes that are involved in the derivation of the convective term.

Because $u_{1,1}$ is a strictly horizontal line segment, we need not to consider the horizontal component of the convection through this line segment. Computing the mean vertical velocity in the plane below $u_{1,1}$ and multiplying it by $\tilde{\psi}_{1,0}$, we have

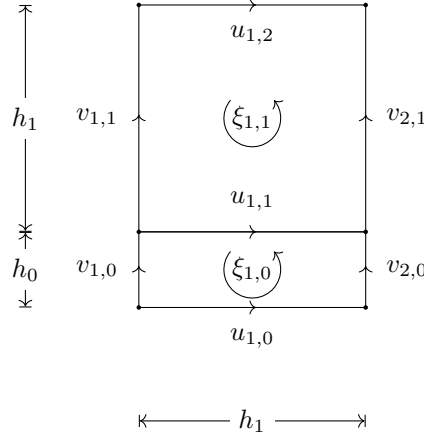$$-\frac{1}{2} \left(\frac{v_{1,0}}{h_0} + \frac{v_{2,0}}{h_0} \right) \tilde{\psi}_{1,0} \tag{3.52}$$

Figure 3.7: Convection through $u_{1,1}$.

For the plane above $u_{1,1}$, we have

$$-\frac{1}{2}\left(\frac{v_{1,1}}{h_1}+\frac{v_{2,1}}{h_1}\right)\tilde{\psi}_{1,1} \tag{3.53}$$

The average of Equations (3.52) and (3.53) multiplied by the length of $u_{1,1}$ yields the convection accross $u_{1,1}$:

$$\frac{1}{2}\left[-\frac{1}{2}\left(\frac{v_{1,0}}{h_0}+\frac{v_{2,0}}{h_0}\right)\tilde{\psi}_{1,0}-\frac{1}{2}\left(\frac{v_{1,1}}{h_1}+\frac{v_{2,1}}{h_1}\right)\tilde{\psi}_{1,1}\right]h_1 \tag{3.54}$$

or
$$-\frac{h_1}{4h_0}\left(v_{1,0}+v_{2,0}\right)\tilde{\psi}_{1,0}-\frac{h_1}{4h_1}\left(v_{1,1}+v_{2,1}\right)\tilde{\psi}_{1,1} \tag{3.55}$$

The multipliplication by the length of $u_{1,1}$ is neseccary because the momentum equation is as a 1-form equation. That is, all terms of the momentum equation must ultimately be expressed as inner-oriented 1-cochains. Repetition of the above procedure for all line segments $u_{i,j}$ and $v_{i,j}$ yields

$$\text{convection} = \begin{bmatrix}
-\frac{\tilde{h}_1}{4\tilde{h}_0}\left(v_{1,0}+v_{2,0}\right)\psi_{1,0}-\frac{\tilde{h}_1}{4\tilde{h}_1}\left(v_{1,1}+v_{2,1}\right)\psi_{1,1}\\[6pt]
-\frac{\tilde{h}_2}{4\tilde{h}_0}\left(v_{2,0}+v_{3,0}\right)\psi_{2,0}-\frac{\tilde{h}_2}{4\tilde{h}_1}\left(v_{2,1}+v_{3,1}\right)\psi_{2,1}\\[6pt]
-\frac{\tilde{h}_1}{4\tilde{h}_1}\left(v_{1,1}+v_{2,1}\right)\psi_{1,1}-\frac{\tilde{h}_1}{4\tilde{h}_2}\left(v_{1,2}+v_{2,2}\right)\psi_{1,2}\\[6pt]
-\frac{\tilde{h}_2}{4\tilde{h}_1}\left(v_{2,1}+v_{3,1}\right)\psi_{2,1}-\frac{\tilde{h}_2}{4\tilde{h}_2}\left(v_{2,2}+v_{3,2}\right)\psi_{2,2}\\[6pt]
-\frac{\tilde{h}_1}{4\tilde{h}_2}\left(v_{1,2}+v_{2,2}\right)\psi_{1,2}-\frac{\tilde{h}_1}{4\tilde{h}_3}\left(v_{1,3}+v_{2,3}\right)\psi_{1,3}\\[6pt]
-\frac{\tilde{h}_2}{4\tilde{h}_2}\left(v_{2,2}+v_{3,2}\right)\psi_{2,2}-\frac{\tilde{h}_2}{4\tilde{h}_3}\left(v_{2,3}+v_{3,3}\right)\psi_{2,3}\\[6pt]
\frac{\tilde{h}_0}{4\tilde{h}_0}\left(u_{0,1}+u_{0,2}\right)\psi_{0,1}+\frac{\tilde{h}_0}{4\tilde{h}_1}\left(u_{1,1}+u_{1,2}\right)\psi_{1,1}\\[6pt]
\frac{\tilde{h}_1}{4\tilde{h}_1}\left(u_{1,1}+u_{1,2}\right)\psi_{1,1}+\frac{\tilde{h}_1}{4\tilde{h}_2}\left(u_{2,1}+u_{2,2}\right)\psi_{2,1}\\[6pt]
\frac{\tilde{h}_2}{4\tilde{h}_2}\left(u_{2,1}+u_{2,2}\right)\psi_{2,1}+\frac{\tilde{h}_2}{4\tilde{h}_3}\left(u_{3,1}+u_{3,2}\right)\psi_{3,1}\\[6pt]
\frac{\tilde{h}_0}{4\tilde{h}_0}\left(u_{0,2}+u_{0,3}\right)\psi_{0,2}+\frac{\tilde{h}_0}{4\tilde{h}_1}\left(u_{1,2}+u_{1,3}\right)\psi_{1,2}\\[6pt]
\frac{\tilde{h}_1}{4\tilde{h}_1}\left(u_{1,2}+u_{1,3}\right)\psi_{1,2}+\frac{\tilde{h}_1}{4\tilde{h}_2}\left(u_{2,2}+u_{2,3}\right)\psi_{2,2}\\[6pt]
\frac{\tilde{h}_2}{4\tilde{h}_2}\left(u_{2,2}+u_{2,3}\right)\psi_{2,2}+\frac{\tilde{h}_2}{4\tilde{h}_3}\left(u_{3,2}+u_{3,3}\right)\psi_{3,2}
\end{bmatrix} \tag{3.56}$$

4 Code

5 Conclusion