Draft: Z-Property implies confluence for an abstract rewrite system (ARS)

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1 ARS definition

We begin our demonstration by defining an abstract rewrite system (A, $\rightarrow$), where A is a set of objects and $\rightarrow$ is a binary relation (also called "reduction" or "rewriting") over A, i.e., $\rightarrow \subseteq$ A$\times$A.

If (a,b) $\in \rightarrow$ and $a, b \in$ A, we write a $\rightarrow$ b and call it "a reduces (or is rewritten) to b".

2 Reducible object and $\rightarrow$-normal form

An object a $\in$ A is called "reducible" if $\exists$ b $\in$ A, b $\neq$ a and a $\rightarrow$ b.
If there's no such b, a is called "irreducible" and is in $\rightarrow$-normal form.

3 Z-Property

A map $\_^\circ$: A $\rightarrow$ A has the Z-Property for $\rightarrow$ on A if $\forall$ a, b $\in$ A, a $\rightarrow$ b implies b $\twoheadrightarrow$ a$^\circ$ $\twoheadrightarrow$ b$^\circ$. $\twoheadrightarrow$ can be read as "reduces in zero or more steps to".

If $\rightarrow$ admits $\_^\circ$, then $\rightarrow$ has the Z-Property.

4 Confluence

To be written.

5 Z-Property implies Confluence

Theorem: *If $\rightarrow$ has the Z-Property, then $\rightarrow$ is confluent.*

Proof: suppose $\_^\circ$ has the Z-Property for $\rightarrow$.

1. If a is in $\to$-normal form, we define $a^\bullet := a$. Otherwise, $a^\bullet := a^\circ$
2. The proof follows by showing that $\_^\bullet \colon A \to A$ also satisfies the Z-Property for $\to$.

   By hypothesis, if $a \to b$, then $b \twoheadrightarrow a^\circ \twoheadrightarrow b^\circ$ and by (1), we have $a^\bullet = a^\circ$, which gives $b \twoheadrightarrow a^\bullet$. If b is in $\to$-normal form, then by (1) we have $b^\bullet = b$ so that $b^\bullet = b = a^\circ = a^\bullet$, which gives $a^\bullet \twoheadrightarrow b^\bullet$.
3.
4.
5.