Binomial Tree

Overview

- Popular Technique for Pricing an Option or other Derivative
- Possible paths followed by underlying asset's price
- Risk-Neutral Valuation
- Cox, Ross and Rubinstein [CRR 1979]

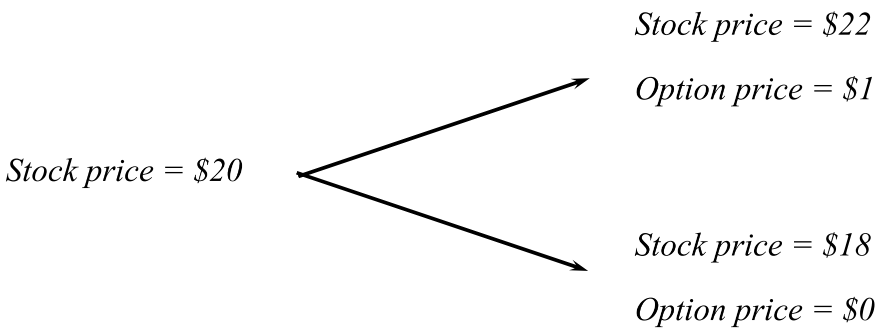
Options

- Call(Put) option gives the holder the right to buy(sell) the underlying asset by a certain date for a certain price
- *European* options can be exercised only on the expiration date
- *American* options can be exercise at any time up to the expiration date

One-Step Binomial Model

Example

European call option to buy stock for \$21 at the end of 3 months



Riskless Portfolio

$$\text{Long: } \quad \Delta \text{ share}$$
$$\text{Short: } \quad 1 \text{ option}$$

The portfolio is riskless if the value of Δ is chosen such that the portfolio is the same for both of the alternative stock prices

$$22\Delta - 1 = 18\Delta$$
$$\Delta = 0.25$$

Option Price

If the stock price moves up to 22 (moves down to 18), the value of the portfolio in 3 months is

$$22 \times 0.25 - 1 = 4.5 \qquad (18 \times 0.25 = 4.5)$$

Suppose risk - free rate is 12% per annum, pv of portfolio is

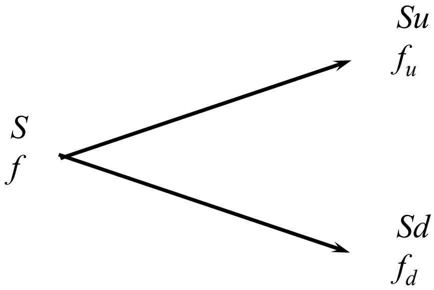
$$4.5e^{-0.12 \times 0.25} = 4.367$$

Thus $\quad 20 \times 0.25 - \text{Option Price} = 4.367$

$$\text{Option Price} = 0.633$$

Generalization

	Current	Up	Down
Stock Price	S	Su	Sd
Derivative Price	f	f_u	f_d



$$Su\Delta - f_u = Sd\Delta - f_d$$

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

Derivative Price

Denoting the risk - free interest rate by r,

pv of portfolio is $\left(Su\Delta - f_u\right)e^{-rT}$

Thus, $S\Delta - f = \left(Su\Delta - f_u\right)e^{-rT}$

$$f = e^{-rT}\left[pf_u + (1-p)f_d\right]$$

where $p = \dfrac{e^{rT} - d}{u - d}$

Determination of *p*, *u* and *d*

p, u and d must give the correct values for

mean and variance of stock price change during Δt

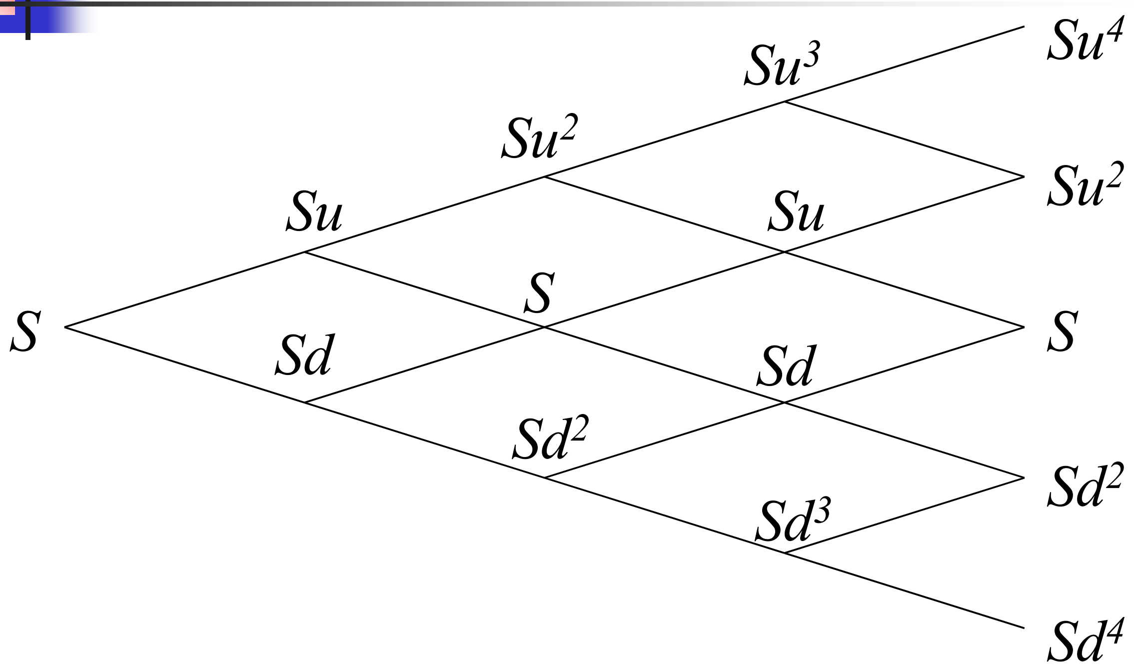
$$Se^{r\Delta t} = pSu + (1-p)Sd$$

$$S^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) = pS^2u^2 + (1-p)S^2d^2 - S^2\left[pu + (1-p)d\right]^2$$

$$u = \frac{1}{d}$$ condition by Cox, Ross and Rubinstein

$$p = \frac{e^{r\Delta t} - d}{u - d} \qquad u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

Multi-steps Tree



Working Backwards

- Derivatives are evaluated by starting at the end of the tree (Time T)
- Value at time (T-ΔT) is the expected value at time T discounted at rate r for a time period ΔT