Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER Applied Mathematics and Computation 148 (2004) 443–452

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

The decomposition method applied to systems of Fredholm integral equations of the second kind

E. Babolian [a], J. Biazar [b,*], A.R. Vahidi [c]

[a] *Department of Mathematics, Faculty of Science, Teacher Training University, Tehran, Iran*
[b] *Department of Mathematics, Guilan University, P.O. Box 1914, Rasht, Iran*
[c] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

Abstract

In this paper, the Adomian decomposition method is applied to solve systems of linear and nonlinear Fredholm integral equations of the second kind. Convergence of the method is proved and some examples are presented to illustrate the method.

Keywords: Adomian decomposition method; Systems of Fredholm integral equations

1. Introduction

Recently a great deal of interest has been focused on the applications of the Adomian decomposition method to solve a wide variety of stochastic and deterministic problems [1,2]. The solution is the sum of an infinite series which converges rapidly to the accurate solutions.

The Adomian decomposition method for solving linear and nonlinear integral equations is known as a subject of extensive analytical and numerical studies [3,4]. Recently, the Adomian decomposition method has been applied for solving systems of linear and nonlinear Volterra integral equations of the second kind [5,6]. In this paper, we extend the method to solve systems of linear

* Corresponding author.
E-mail addresses: babolian@saba.tmu.ac.ir (E. Babolian), biazar@guilan.ac.ir (J. Biazar), alrevahidi@yahoo.com (A.R. Vahidi).

0096-3003/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0096-3003(02)00859-7

and nonlinear Fredholm integral equations of the second kind. A nonlinear system of Fredholm integral equations can be written as the following:

$$F(t) = G(t) + \int_a^b V(s,t,F(s))\,\mathrm{d}s, \quad t \in [a,b], \tag{1}$$

where,

$$\begin{aligned} &F(t) = (f_1(t),\ldots,f_n(t))^t,\\ &G(t) = (g_1(t),\ldots,g_n(t))^t,\\ &V(s,t,F(s)) = (v_1(s,t,F(s)),\ldots,v_n(s,t,F(s)))^t. \end{aligned}$$

We suppose that the system (1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of the system (1) could be found in [7].

2. The decomposition method applied to (1)

Consider the ith equation of (1):

$$f_i(t) = g_i(t) + \int_a^b v_i(s,t,f_1(s),\ldots,f_n(s))\,\mathrm{d}s. \tag{2}$$

With reference to (2), the *canonical form* of the Adomian equations can be written as:

$$f_i(t) = g_i(t) + N_i(t), \tag{3}$$

where:

$$N_i(t) = N_i(f_1,\ldots,f_n)(t) = \int_a^b v_i(s,t,f_1(s),\ldots,f_n(s))\,\mathrm{d}s. \tag{4}$$

To use the Adomian decomposition method, let $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$, and $N_i(t) = \sum_{m=0}^{\infty} A_{im}$ where A_{im}, $m = 0,1,\ldots$, are polynomials depending on $f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}$ and they are called Adomian polynomials. Hence, (3) can be rewritten as:

$$\sum_{m=0}^{\infty} f_{im}(t) = g_i(t) + \sum_{m=0}^{\infty} A_{im}(f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}). \tag{5}$$

From (5) we define:

$$\begin{cases} f_{i0}(t) = g_i(t) \\ f_{i,m+1}(t) = A_{im}(f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}), \\ \quad i = 1,\ldots,n, \;\; m = 0,1,2,\ldots \end{cases} \tag{6}$$

In practice, all terms of the series $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$ cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t), \quad \text{with } \lim_{k\to\infty} \varphi_{ik}(t) = f_i(t). \tag{7}$$

To determine the Adomian polynomials, we write:

$$f_{i\lambda}(t) = \sum_{m=0}^{\infty} f_{im}(t)\lambda^m, \tag{8}$$

$$N_{i\lambda}(f_1,\ldots,f_n) = \sum_{m=0}^{\infty} A_{im}\lambda^m, \tag{9}$$

where, λ is a parameter introduced for convenience. From (9) we obtain:

$$A_{im} = \frac{1}{m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} N_{i\lambda}(f_1,\ldots,f_n)\right]_{\lambda=0}. \tag{10}$$

Two cases will be considered as the following:

(i) $v_i(s,t,f_1(s),\ldots,f_n(s))$ is a linear function,
(ii) $v_i(s,t,f_1(s),\ldots,f_n(s))$ is a nonlinear function.

In case (i) Eq. (4) would be in the following form:

$$N_i(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_j(t)\,\mathrm{d}s \tag{11}$$

and from (8), (10) and (11) we get:

$$\begin{aligned} A_{im}(f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}) &= \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\left[\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\sum_{l=0}^{\infty} f_{jl}\lambda^l\right]_{\lambda=0} \mathrm{d}s \\ &= \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm}\,\mathrm{d}s. \end{aligned} \tag{12}$$

So (6), for the linear systems of Fredholm integral equations, will be as follows:

$$\begin{cases} f_{i0}(t) = g_i(t) \\ f_{i,m+1}(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm}(t)\,\mathrm{d}s, \quad i = 1,\ldots,n, \;\; m = 0,1,2,\ldots \end{cases} \tag{13}$$

In case (ii), from (4), (8) and (10) we have:

$$A_{im}(f_{10},\dots,f_{1m},\dots,f_{n0},\dots,f_{nm}) = \frac{1}{m!}\int_a^b \left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} v_i\left(s,t,\sum_{l=0}^{\infty} f_{1l}\lambda^l,\dots,\sum_{l=0}^{\infty} f_{nl}\lambda^l\right)\right]_{\lambda=0} \mathrm{d}s. \quad (14)$$

So, for the nonlinear systems of Fredholm integral equations, we use the following Adomian scheme:

$$\begin{cases} f_{i0}(t) = g_i(t) \\ f_{i,m+1}(t) = \dfrac{1}{m!}\displaystyle\int_a^b \left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} v_i\left(s,t,\sum_{l=0}^{\infty} f_{1l}\lambda^l,\dots,\sum_{l=0}^{\infty} f_{nl}\lambda^l\right)\right]_{\lambda=0} \mathrm{d}s. \end{cases} \quad (15)$$

To illustrate the method, considering the special case $n = 2$, we derive:

$$A_{im}(f_{10},\dots,f_{1m},f_{20},\dots,f_{2m}) = \frac{1}{m!}\int_a^b \left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} v_i\left(s,t,\sum_{l=0}^{\infty} f_{1l}\lambda^l,\sum_{l=0}^{\infty} f_{2l}\lambda^l\right)\right]_{\lambda=0} \mathrm{d}s. \quad (16)$$

By introducing the notation:

$$v_{ikl}(s,t,f_{10},f_{20}) = \frac{\partial^{k+l}}{\partial f_1^k \partial f_2^l} v_i(s,t,f_{1\lambda}(s),f_{2\lambda}(s)) \mid_{\lambda=0}, \quad (17)$$

we can list the polynomials A_{im}, for $m = 0,1,\dots,4$, as the following:

$$A_{i0}(f_{10},f_{20}) = \int_a^b v_i(s,t,f_{10},f_{20})\,\mathrm{d}s,$$

$$A_{i1}(f_{10},f_{11},f_{20},f_{21}) = \int_a^b [f_{11}v_{i10}(s,t,f_{10},f_{20}) + f_{21}v_{i01}(s,t,f_{10},f_{20})]\,\mathrm{d}s,$$

$$\begin{aligned} A_{i2}(f_{10},f_{11},f_{12},f_{20},f_{21},f_{22}) = \int_a^b \Big[& f_{12}v_{i10}(s,t,f_{10},f_{20}) + f_{22}v_{i01}(s,t,f_{10},f_{20}) \\ & + \frac{1}{2!}f_{11}^2 v_{i20}(s,t,f_{10},f_{20}) + \frac{1}{2!}f_{21}^2 v_{i02}(s,t,f_{10},f_{20}) \\ & + f_{11}f_{21}v_{i11}(s,t,f_{10},f_{20})\Big]\,\mathrm{d}s, \end{aligned}$$

$$
\begin{aligned}
&A_{i3}(f_{10},f_{11},f_{12},f_{13},f_{20},f_{21},f_{22},f_{23})\\
&\quad=\int_a^b\Bigg[f_{13}v_{i10}(s,t,f_{10},f_{20})+f_{23}v_{i01}(s,t,f_{10},f_{20})+f_{11}f_{12}v_{i20}(s,t,f_{10},f_{20})\\
&\qquad+(f_{11}f_{22}+f_{12}f_{21})v_{i11}(s,t,f_{10},f_{20})+f_{21}f_{22}v_{i30}(s,t,f_{10},f_{20})\\
&\qquad+\frac{1}{3!}f_{11}^3v_{i30}(s,t,f_{10},f_{20})+\frac{1}{3!}f_{21}^3v_{i03}(s,t,f_{10},f_{20})\\
&\qquad+\frac{1}{2!}f_{11}^2f_{21}v_{i21}(s,t,f_{10},f_{20})+\frac{1}{2!}f_{11}f_{21}^2v_{i12}(s,t,f_{10},f_{20})\Bigg]\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
&A_{i4}(f_{10},\ldots,f_{14},f_{20},\ldots,f_{24})\\
&\quad=\int_a^b\Bigg[f_{14}v_{i10}(s,t,f_{10},f_{20})+f_{24}v_{i01}(s,t,f_{10},f_{20})\\
&\quad+\left(\frac{1}{2!}f_{12}^2+f_{11}f_{23}\right)v_{i20}(s,t,f_{10},f_{20})+\frac{1}{2!}f_{11}^2f_{12}v_{i30}(s,t,f_{10},f_{20})\\
&\quad+\left(\frac{1}{2!}f_{22}^2+f_{21}f_{23}\right)v_{i02}(s,t,f_{10},f_{20})+(f_{11}f_{23}+f_{12}f_{22}\\
&\qquad+f_{13}f_{21})v_{i11}(s,t,f_{10},f_{20})+\left(\frac{1}{2!}f_{11}^2f_{22}+f_{11}f_{12}f_{21}\right)v_{i21}(s,t,f_{10},f_{20})\\
&\quad+\left(f_{11}f_{21}f_{22}+\frac{1}{2!}f_{12}f_{21}^2\right)v_{i12}(s,t,f_{10},f_{20})+\frac{1}{2!}f_{21}^2f_{22}v_{i03}(s,t,f_{10},f_{20})\\
&\quad+\frac{1}{4!}f_{11}^4v_{i40}(s,t,f_{10},f_{20})+\frac{1}{3!}f_{11}^3f_{21}v_{i31}(s,t,f_{10},f_{20})\\
&\quad+\frac{1}{2!}\frac{1}{2!}f_{11}^2f_{21}^2v_{i22}(s,t,f_{10},f_{20})+\frac{1}{3!}f_{11}f_{21}^3v_{i13}(s,t,f_{10},f_{20})\\
&\quad+\frac{1}{4!}f_{21}^4v_{i04}(s,t,f_{10},f_{20})\Bigg]\,\mathrm{d}s.
\end{aligned}
$$

3. Convergence of the method

Consider the system of equations (1). We are looking for a solution in the family:

$$
f_{i\lambda}=\sum_{j=0}^{\infty}f_{ij}\lambda^j,\quad i=1,2,\ldots,n. \tag{18}
$$

Let $\rho=\min\{\rho_1,\rho_2,\ldots,\rho_n\}$, where ρ_i is the convergence radius of the series (18), assuming $\rho>1$. Following [3] and extending to n-dimensional space, thus (18) converges for $|\lambda|\leqslant\rho$, with $\rho>1$. Let us now suppose that $N_{i\lambda}(f_1,f_2,\ldots,f_n)$ can be expanded in an entire series:

$$N_{i\lambda}(f_1,f_2,\ldots,f_n)=\sum_{m=0}^{\infty}\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1,\ldots,k_n\in W=\{0,1,\ldots\}}} a_{k_1k_2\ldots k_n}f_1^{k_1}f_2^{k_2}\ldots f_n^{k_n}, \tag{19}$$

with convergence radius $\rho^*>1$. This implies that the series (18) converges for $\|\vec{F}\|<\rho^*$ with $\rho^*>1$. Using the extension of a classical result given in [8], substituting (18) into (19) we obtain a new series $\sum_{m=0}^{\infty}c_m\lambda^m$, which its convergence radius is strictly greater than 1:

$$N_{i\lambda}(f_1,f_2,\ldots,f_n)=\sum_{m=0}^{\infty}\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1,\ldots,k_n\in W=\{0,1,\ldots\}}} a_{k_1k_2\ldots k_n}\prod_{i=1}^{n}\left(\sum_{j=0}^{\infty}f_{ij}\lambda^j\right)^{k_i}. \tag{20}$$

The m-row of this array converges to A_{im} defined in (10), when we set $\lambda=1$, because $N_{i\lambda}(f_1,f_2,\ldots,f_n)$ can be developed in a Taylor series. Our problem is now to prove the convergence of the double series in the array (20) for $\lambda=1$.

Theorem. *If $N(\vec{F})$ is an analytical function of n variables $f_1,f_2,\ldots,f_n$, in $\|\vec{F}\|<R$, $f_i(t)$, $i=1,2,\ldots,n$, can be decomposed as an infinite series $f_i=\sum f_{im}$, the parameterization $f_{i\lambda}=\sum f_{im}\lambda^m$ is absolutely convergent for $\lambda\in[-1,1]$ and the series f_i can be majored by*:

$$\frac{f'}{n(1+\varepsilon)}\left(1+\frac{1}{(1+\varepsilon)}\left(\frac{\lambda}{\rho}\right)+\cdots+\frac{1}{(1+\varepsilon)^n}\left(\frac{\lambda}{\rho}\right)^n+\cdots\right), \tag{21}$$

where $f'\geqslant f$, $f=\max\{\tilde{f}_1,\tilde{f}_2,\ldots,\tilde{f}_n\}$, $\tilde{f}_i$ is the upper limit of the series elements in (18), *$\varepsilon>f/R$ and $\rho\geqslant 1$, then the double series converges for $\lambda=1$.*

Proof. $N_{i\lambda}(f_1,\ldots,f_n)$ is analytical in $\|\vec{F}\|<R$, then we can write:

$$N_{i\lambda}(\vec{F})\leqslant M\left(1+n\left(\frac{f(t)}{R'}\right)+\cdots+n^n\left(\frac{f(t)}{R'}\right)^n+\cdots\right), \tag{22}$$

where $\|N_{i\lambda}(\vec{F})\|^*\leqslant M$ ($\|\,\|^*$ is the dual norm), and $f(t)=\max\{f_1,\ldots,f_n\}$ and $R'\in[f,R]$. We now employ the hypothesis (21) to write:

$$\begin{aligned} f(t)&\leqslant\frac{f'}{n(1+\varepsilon)}\left(1+\frac{1}{(1+\varepsilon)}\left(\frac{\lambda}{\rho}\right)+\cdots+\frac{1}{(1+\varepsilon)^n}\left(\frac{\lambda}{\rho}\right)^n+\cdots\right)\\ &=\frac{f'}{n(1+\varepsilon)}\left(\frac{1}{1-\frac{1}{(1+\varepsilon)}\left(\frac{\lambda}{\rho}\right)}\right)=\frac{f'}{n\left(1+\varepsilon-\frac{\lambda}{\rho}\right)}. \end{aligned} \tag{23}$$

Substituting (23) into (22) we obtain:

$$N_{i\lambda}(\vec{F}) \leqslant M\left[1 + \frac{f'}{R'\left[1+\varepsilon-\left(\frac{\lambda}{\rho}\right)\right]} + \left(\frac{f'}{R'\left[1+\varepsilon-\left(\frac{\lambda}{\rho}\right)\right]}\right)^2 + \cdots + \left(\frac{f'}{R'\left[1+\varepsilon-\left(\frac{\lambda}{\rho}\right)\right]}\right)^n + \cdots\right]. \tag{24}$$

Obviously for having the convergence of (22) we must have:

$$\frac{f'}{R'\left[1+\varepsilon-\left(\frac{\lambda}{\rho}\right)\right]} < 1, \tag{25}$$

that is to say that:

$$\lambda < \rho\left(1+\varepsilon-\frac{f}{R'}\right). \tag{26}$$

We choose $\rho = 1$, then (22) converges for $\lambda = 1$, if $\varepsilon > f/R'$. □

4. Numerical results

In this section two examples are presented. In the first example, we use the proposed method to solve a linear system of Fredholm integral equations of the second kind and in the second example a nonlinear system of Fredholm integral equations of the second kind is solved by Adomian decomposition method.

Example 1. Consider the following system of linear Fredholm integral equations with the exact solutions $f_1(t) = t + 1$ and $f_2(t) = t^2 + 1$.

$$\begin{cases} f_1(t) = \dfrac{t}{18} + \dfrac{17}{36} + \displaystyle\int_0^1 \frac{s+t}{3}(f_1(s) + f_2(s))\,\mathrm{d}s, \\ f_2(t) = t^2 - \dfrac{19}{12}t + 1 + \displaystyle\int_0^1 st(f_1(s) + f_2(s))\,\mathrm{d}s. \end{cases}$$

To derive the solutions by using the decomposition method, we can use the following Adomian scheme:

$$\begin{cases} f_{10}(t) = \dfrac{t}{18} + \dfrac{17}{36} \simeq 0.0556t + 0.4722, \\ f_{20}(t) = t^2 - \dfrac{19}{12}t + 1 \simeq t^2 - 1.5833t + 1 \end{cases}$$

and

$$\begin{cases} f_{1,m+1}(t) = \int_0^1 \frac{(s+t)}{3}(f_{1m}(s) + f_{2m}(s))\,\mathrm{d}s, \\ f_{2,m+1}(t) = \int_0^1 st(f_{1m}(s) + f_{2m}(s))\,\mathrm{d}s, \quad m = 0, 1, 2, \ldots \end{cases}$$

For the first iteration, we have:

$$\begin{cases} f_{11}(t) = \int_0^1 \frac{(s+t)}{3}(f_{10}(s) + f_{20}(s))\,\mathrm{d}s = \frac{25}{72}t + \frac{103}{648} \simeq 0.3472t + 0.1590, \\ f_{21}(t) = \int_0^1 st(f_{10}(s) + f_{20}(s))\,\mathrm{d}s = \frac{103}{216}t \simeq 0.4769t. \end{cases}$$

Considering (7), the approximated solutions with two terms are:

$$\begin{cases} \varphi_{12}(t) = f_{10}(t) + f_{11}(t) \simeq 0.4028t + 0.6312, \\ \varphi_{22}(t) = f_{20}(t) + f_{21}(t) \simeq t^2 - 1.1065t + 1. \end{cases}$$

Next terms are:

$$\begin{cases} f_{12}(t) = \int_0^1 \frac{(s+t)}{3}(f_{11}(s) + f_{21}(s))\,\mathrm{d}s = \frac{185}{972}t + \frac{17}{144} \simeq 0.1903t + 0.1181, \\ f_{22}(t) = \int_0^1 st(f_{11}(s) + f_{21}(s))\,\mathrm{d}s = \frac{17}{48}t \simeq 0.3542t. \end{cases}$$

Solutions with three terms are:

$$\begin{cases} \varphi_{13}(t) = f_{10}(t) + f_{11}(t) + f_{12}(t) \simeq 0.5931t + 0.7492, \\ \varphi_{23}(t) = f_{20}(t) + f_{21}(t) + f_{22}(t) \simeq t^2 - 0.7523t + 1. \end{cases}$$

Table 1
The results of example 1

t	$f_1(t)$	$\varphi_{1,11}(t)$	$e(\varphi_{1,11}(t))$	$f_2(t)$	$\varphi_{2,11}(t)$	$e(\varphi_{2,11}(t))$
0	1	0.988498	1.15×10^{-2}	1	1	0
0.1	1.1	1.086632	1.33×10^{-2}	1.01	1.006549	3.45×10^{-3}
0.2	1.2	1.184766	1.52×10^{-2}	1.04	1.033099	6.90×10^{-3}
0.3	1.3	1.282899	1.71×10^{-2}	1.09	1.079648	1.03×10^{-2}
0.4	1.4	1.381033	1.89×10^{-2}	1.16	1.146198	1.38×10^{-2}
0.5	1.5	1.479167	2.08×10^{-2}	1.25	1.232747	1.72×10^{-2}
0.6	1.6	1.577301	2.26×10^{-2}	1.36	1.339296	2.07×10^{-2}
0.7	1.7	1.675435	2.45×10^{-2}	1.49	1.465846	2.41×10^{-2}
0.8	1.8	1.773569	2.64×10^{-2}	1.64	1.612695	2.76×10^{-2}
0.9	1.9	1.871702	2.82×10^{-2}	1.81	1.778945	3.10×10^{-2}
1	2	1.969836	3.02×10^{-1}	2	1.965494	3.45×10^{-2}

In the same way, the components $\varphi_{1k}(t)$ and $\varphi_{2k}(t)$ can be calculated for $k = 3, 4, \ldots$ The solutions with eleven terms are given as:

$$\begin{cases} \varphi_{1,11}(t) = f_{10}(t) + f_{11}(t) + \cdots + f_{1,10}(t) \simeq 0.9813t + 0.9885, \\ \varphi_{2,11}(t) = f_{20}(t) + f_{21}(t) + \cdots + f_{2,10}(t) \simeq t^2 - 0.0345t + 1. \end{cases}$$

Approximated solutions for some values of t and the corresponding absolute errors are presented in Table 1.

Example 2. Consider the following system of nonlinear Fredholm integral equations with the exact solutions $f_1(t) = t$ and $f_2(t) = t^2$.

$$\begin{cases} f_1(t) = t - \dfrac{5}{18} + \displaystyle\int_0^1 \frac{1}{3}(f_1(s) + f_2(s))\,\mathrm{d}s, \\ f_2(t) = t^2 - \dfrac{2}{9} + \displaystyle\int_0^1 \frac{1}{3}(f_1^2(s) + f_2(s))\,\mathrm{d}s. \end{cases}$$

The following Adomian scheme can be used to solve this system:

$$\begin{cases} f_{10}(t) = t - \dfrac{5}{18} \simeq t - 0.2778, \\ f_{20}(t) = t^2 - \dfrac{2}{9} \simeq t^2 - 0.2222, \end{cases}$$

and

$$\begin{cases} f_{1,m+1}(t) = A_{1m}(f_{10}, \ldots, f_{1m}, f_{20}, \ldots, f_{2m}), \\ f_{2,m+1}(t) = A_{2m}(f_{10}, \ldots, f_{1m}, f_{20}, \ldots, f_{2m}), \quad m = 0, 1, 2, \ldots \end{cases}$$

For the first iteration, we have:

$$\begin{cases} f_{11}(t) = A_{10}(f_{10}, f_{20}) = \displaystyle\int_0^1 \frac{1}{3}(f_{10}(s) + f_{20}(s))\,\mathrm{d}s = \frac{1}{9} \simeq 0.1111, \\ f_{21}(t) = A_{20}(f_{10}, f_{20}) = \displaystyle\int_0^1 \frac{1}{3}(f_{10}^2(s) + f_{20}(s))\,\mathrm{d}s = \frac{79}{972} \simeq 0.0813. \end{cases}$$

Considering (7), the solutions with two terms are:

$$\begin{cases} \varphi_{1,2}(t) = f_{10}(t) + f_{11}(t) \simeq t - 0.1667, \\ \varphi_{2,2}(t) = f_{20}(t) + f_{21}(t) \simeq t^2 - 0.1409. \end{cases}$$

For the second iteration, we have:

$$\begin{cases} f_{12}(t) = A_{11}(f_{10}, f_{11}, f_{20}, f_{21}) \\ \qquad = \displaystyle\int_0^1 \frac{1}{3}(f_{11}(s) + f_{21}(s))\,\mathrm{d}s = \frac{187}{2916} \simeq 0.0641, \\ f_{22}(t) = A_{21}(f_{10}, f_{11}, f_{20}, f_{21}) \\ \qquad = \displaystyle\int_0^1 \frac{1}{3}(2f_{10}(s)f_{11}(s) + f_{21}(s))\,\mathrm{d}s = \frac{127}{2916}t \simeq 0.0436. \end{cases}$$

Considering (7), the solutions with three terms are:

$$\begin{cases} \varphi_{1,3}(t) = f_{10}(t) + f_{11}(t) + f_{12}(t) \simeq t - 0.1025, \\ \varphi_{2,3}(t) = f_{20}(t) + f_{21}(t) + f_{22}(t) \simeq t^2 - 0.0974. \end{cases}$$

$$\vdots$$

The solutions after nine iteration and for the first 10 terms are given as:

$$\begin{cases} \varphi_{1,10}(t) = f_{10}(t) + f_{11}(t) + \cdots + f_{19}(t) \simeq t - 0.0230, \\ \varphi_{2,10}(t) = f_{20}(t) + f_{21}(t) + \cdots + f_{29}(t) \simeq t^2 - 0.0443. \end{cases}$$

More iteration will reduce the error. Obviously, the maximum absolute error for $t \in [0,1]$ is 0.0230 for $f_1(t)$ and 0.0443 for $f_2(t)$.

5. Conclusion

This paper presents the use of the Adomian decomposition method, both for systems of linear and nonlinear Fredholm integral equations of the second kind. As it can be seen in both examples, to derive a good approximation to the solution a large number of iteration should be done. But this method gives better approximations, in less iteration, when applied to solve linear and nonlinear systems of Volterra integral equations (see [5,6]). The reason could be found in the point that the system of Volterra integral equations of the second kind is basically more well posed than the system of Fredholm integral equations of the second kind [8]. Accelerating the convergence of the Adomian decomposition method when applied to a system of Fredholm integral equations of the second kind, is a good subject for further research.

References

[1] G. Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer, 1989.
[2] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994.
[3] Y. Cherruault, G. Saccomandi, New results for convergence of Adomian method applied to integral equations, Math. Comput. Modelling 16 (2) (1992) 83–93.
[4] Y. Cherruault, V. Seng, The resolution of nonlinear integral equations of the first kind using the decomposition method of Adomian, Kybernetes 26 (2) (1997) 198–206.
[5] E. Babolian, J. Biazar, Solution of a system of nonlinear Volterra equations by Adomian decomposition method, Far East J. Math. Sci. 2 (6) (2000) 935–945.
[6] E. Babolian, J. Biazar, Solution of a system of linear Volterra equations by Adomian decomposition method, Far East J. Math. Sci. 7 (1) (2001) 17–25.
[7] L.M. Delves, J.L. Mohamed, Computational Methods for Integral Equations, Cambridge University Press, 1985.
[8] L. Gabet, The theoretical foundation of Adomian method, Comput. Math. Appl. 27 (12) (1994) 41–52.