Demagnetization Fields

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Summary

Generaly in the study of magnetism, and specifically for micromagnetics simulations, one needs to know the magnetic field $\vec{H}_M$ inside a macroscopic magnet, that is caused by the magnetization $\vec{M}$ (the dipole moment per unit volume) of that magnet itself. Some aspects of how that demagnetization field can be found are discussed. There are two basic cases: (1) The demagnetization field $\vec{H}_M$ inside a finite element, that is caused by $\vec{M}$ of that particular element; (2) The demagnetization field caused by one finite element, but measured at the position of another element. The discussion here is based on continuum description of the magnet and the field, although it can be connected to an alternative analysis that considers the superposition of many fields from a multitude of individual magnetic dipoles.

1 The magnetic field inside a magnet: Basic theory

In solving magnetostatics, and even electrodynamics, there there are no magnetic monopoles. So the magnetic induction $\vec{B}$ obeys a Gauss' Law where there is no fundamental source charge:

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{1.1}$$

Here $\vec{M}$ is the dipole moment per unit volume (magnetization) and $\vec{H}$ is called the magnetic field, or really, for the situation considered here, the demagnetization field. The magnetic field is important in that it determines part of the magnetic energy in the system, according to a volume integral,

$$U_M = -\frac{1}{2} \int dV \ \vec{M} \cdot \vec{H}. \tag{1.2}$$

The equation for the divergence-free $\vec{B}$ can be rearranged as

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}. \tag{1.3}$$

This suggests the idea that the magnetic field $\vec{H}$ is generated by an effective magnetic charge density, given by

$$\rho = -\vec{\nabla} \cdot \vec{M}. \tag{1.4}$$

This is not a monopole density! In a situation where there are no free currents (current density of free charges, $\vec{J} = 0$), the magnetic field can be found from a magnetic potential,

$$\vec{H} = -\vec{\nabla}\Phi \tag{1.5}$$

This leads to the Poisson equation to be solved to get the demagnetization field inside the magnet:

$$\nabla^2 \Phi = -\rho. \tag{1.6}$$

For three dimensions, this is solved using the potential of a unit point charge as the Green's function:

$$G(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}|}. \tag{1.7}$$

Then the solution of the Poisson equation for the potential is:

$$\Phi(\mathbf{r}) = \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') = \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int d\mathbf{r}' \frac{-\vec{\nabla}'\cdot\vec{M}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{1.8}$$

This gives the solution either directly from ρ, or from the divergence of $\vec{M}$. But in some situations, these are not as convenient as obtaining Φ directly from $\vec{M}$. So one can do an integration by parts here, using some vector calculus manipulations, letting the gradient act instead on the Green's function:

$$\Phi(\mathbf{r}) = \int d\mathbf{r}'\; \vec{\nabla}' \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) \cdot \vec{M}(\mathbf{r}') \tag{1.9}$$

There would have also been a surface term, but by taking that surface outside of the magnet, its contribution is zero. So another way to write this is seen to be

$$\Phi(\mathbf{r}) = \int d\mathbf{r}' \left(\frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3}\right) \cdot \vec{M}(\mathbf{r}'). \tag{1.10}$$

This defines another Green's function, a radial vector to be used acting directly on $\vec{M}$,

$$\vec{K}(\mathbf{r}) = \frac{\mathbf{r}}{4\pi|\mathbf{r}|^3} \quad\rightarrow\quad \Phi(\mathbf{r}) = \int d\mathbf{r}'\, \vec{K}(\mathbf{r}-\mathbf{r}')\cdot\vec{M}(\mathbf{r}'). \tag{1.11}$$

Also note the simple relation between the G and the $\vec{K}$ (radial component only) Green's operators:

$$K_r(r) = -\frac{d}{dr}G(r) = \frac{1}{4\pi r^2}. \tag{1.12}$$

This latter form using $\vec{K}$ is preferred if we want to calculate the field without going through the intermediate step of getting the charge density, which be a confusing physical concept anyway (at least, if you think there should be some physical experiment to detect ρ, on which its reality could be based).

In this last form, the Green's operator $\vec{K}$ acts directly on the magnetization. We could finally take the gradient w.r.t. r to get $\vec{H}$, however, without some special averaging procedures, that can lead to an undefined integral. So it is better to wait to do that. It is interesting to realize that equation (1.10) is simply a representation of the effective magnetostatic potential around a dipole (then summed over dipoles). This is because the well-known formula for the potential of a point dipole $\vec{p}$ (could be magnetic or electric) at the origin, is

$$\Phi(\mathbf{r}) = \frac{\mathbf{r}\cdot\mathbf{p}}{4\pi r^3}. \tag{1.13}$$

In (1.10), each dipole is $dV\,\vec{M}(\mathbf{r}')$, and thus one needs the displacement from its position, $\mathbf{r}-\mathbf{r}'$. Now, if the gradient of the potential of a point dipole is performed, it leads to the other well-known expression for the field caused by that point dipole.

$$\vec{H} = -\vec{\nabla}\Phi = \frac{1}{4\pi r^3}\left[3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}\right] \tag{1.14}$$

It is good to point out that in a magnet that is "uniformly magnetized," the internal charge density ρ is zero within the magnet. So how can there be any $\vec{H}$? The answer is that at the surface of the magnet, there is a discontinuous change in the magnetization; it suddenly goes from some nonzero value to zero. This change corresponds to a delta-function charge density. Stated otherwise, Gauss' Law used on $\vec{H}$ (i.e., the divergence theorem applied to a pillbox at the surface) will tell us that there is a local surface charge density, given by

$$\sigma = \vec{M}\cdot\hat{n}, \tag{1.15}$$

where $\hat{n}$ is the outward normal vector from the surface. Mostly, I will use this surface charge density as generating the field, because this is the most consistent way to think of doing the continuum field mechanics, that does not have singularities. (One could imagine trying to sum over fields of individual dipoles. It does not work out well in all cases.)

Usually the demagnetization field, the charge density, and the surface charge density, are given the subscript M to show they are those associated with or caused by $\vec{M}$. Here I suppress this subscript; I only discuss these demagnetization quantities. There is no external field being considered.

2 The magnetic field inside a cylindrical magnet

First, consider a magnet of length L along the z axis, which is the longitudinal axis of a cylinder. The upper end lies at $z = +\delta$, the lower end at $z = -\delta$, so that the length is $L = 2\delta$, and $z = 0$ is at the middle of the cylinder. The cross-section could be a circle of radius R, for most simplicity, but it doesn't absolutely have to be. For the circular cross section, there is no need yet to make any special assumption about the radius R compared to the cylinder length L.

2.1 Longitudinal magnetization M_z

Initially, suppose the cylinder is magnetized in the z direction, that is, along its axis of symmetry. Then $\vec{M} = M_z\hat{z}$. This places surface charge densities of $\sigma = \pm M_z$ at $z = \pm\delta$, respectively. So the top end has positive charge, the bottom end has negative charge.

To find the potential at an observer point $\mathbf{r} = (x, y, z)$, inside the magnet, consider the positive source charges at $\mathbf{r}' = (x', y', \delta)$, and the negative source charges at $\mathbf{r}' = (x', y', -\delta)$. From the Green function integral over charge density, one has now only surface integrals on the ends,

$$\Phi(x,y,z) = \frac{M_z}{4\pi}\int dx'dy' \left\{\frac{1}{\sqrt{\tilde{r}^2+(z-\delta)^2}} - \frac{1}{\sqrt{\tilde{r}^2+(z+\delta)^2}}\right\}. \tag{2.1}$$

To save space, I wrote $\tilde{r}^2 = (x-x')^2 + (y-y')^2$ here, and in what follows this may be used again. The integral is over the cross-section of the cylinder.

If we want to just find the field in the center of the cylinder, it is not so difficult, putting here $x = y = 0$. Then $\vec{H}_z$ can be found as a function of z. In this case there is dependence only on $r' = \sqrt{x'^2 + y'^2}$ and only a radial integration is needed ($dx'dy' \to d\theta' r' dr'$),

$$\Phi(z) = \frac{M_z}{4\pi}\int d\theta' \int r'dr' \left\{\frac{1}{\sqrt{r'^2+(z-\delta)^2}} - \frac{1}{\sqrt{r'^2+(z+\delta)^2}}\right\}. \tag{2.2}$$

The integration is quite simple, if there is circular symmetry. For a circular cross-section, it gives

$$\begin{aligned}\Phi(z) &= \frac{M_z}{2}\left[\sqrt{r'^2+(z-\delta)^2} - \sqrt{r'^2+(z+\delta)^2}\right]_0^R \\ &= \frac{M_z}{2}\left[\sqrt{R^2+(z-\delta)^2} - \sqrt{R^2+(z+\delta)^2} - |z-\delta| + |z+\delta|\right]\end{aligned} \tag{2.3}$$

The resulting field has to be an even function of z. Thus it can be calculated for $z > 0$; the result for $z < 0$ will be symmetrical. Indeed, for any z between $\pm\delta$, this is

$$\Phi(z) = \frac{M_z}{2}\left[\sqrt{R^2+(z-\delta)^2} - \sqrt{R^2+(z+\delta)^2} + 2z\right] \tag{2.4}$$

Then the field on the axis of the cylinder is found quickly,

$$H_z = -\frac{d\Phi}{dz} = -M_z\left[1 + \frac{1}{2}\frac{z-\delta}{\sqrt{R^2+(z-\delta)^2}} - \frac{1}{2}\frac{z+\delta}{\sqrt{R^2+(z+\delta)^2}}\right]. \tag{2.5}$$

Note the somewhat surprizing result. At $z=0$, the last terms equal each other and combine, to give

$$H_z(0) = -M_z\left(1-\frac{\delta}{\sqrt{R^2+\delta^2}}\right) \approx \begin{cases} -M_z \frac{R^2}{2\delta^2} & \text{for } \delta \gg R \\ -M_z\left(1-\frac{\delta}{R}\right) & \text{for } \delta \ll R \end{cases} \tag{2.6}$$

The field points *opposite* to $\vec{M}$, which is why it is *demagnetization*. Further, its strength depends on the aspect ratio of the cylinder. Note the limiting behaviors. When the cylinder is long and thin, the longitudinal field at its center gets very small. On the other hand, if the cylinder is short and wide, the longitudinal field at its center is maximized, nearly equal to the strength of its magnetization. This latter case corresponds to the strongest demagnetization that can take place.

Note the reason for the name, demagnetization. In some situations, $\vec{M}$ could be generated by the action of an externally applied field, according to $\vec{M} = \chi\vec{H}_{\text{ext}}$, where $\chi > 0$ is a paramagnetic susceptibility. Then the total magnetic field in the sample will be the conbination of applied field and this demagnetization field. They oppose each other, hence, the demagnetization field tends to reduce the internal effect of the applied field. It seems to prevent the applied field from entering the sample.

2.1.1 Average of H_z

It is common to want to know the average of the magnetic field in the sample. This can be done easily for the field on the axis of the circular cylinder. The average over z is simple:

$$\begin{aligned} \overline{H}_z &= \frac{1}{L}\int_{-\delta}^{\delta} dz\ H_z(z) = -M_z - \frac{M_z}{2L}\int_{-\delta}^{\delta} dz \left[\frac{z-\delta}{\sqrt{R^2+(z-\delta)^2}} - \frac{z+\delta}{\sqrt{R^2+(z+\delta)^2}}\right] \\ &= -M_z - \frac{M_z}{2L}\left[\sqrt{R^2+(z-\delta)^2} - \sqrt{R^2+(z+\delta)^2}\right]_{-\delta}^{\delta} \\ &= -\frac{M_z}{L}\left(L+R-\sqrt{R^2+L^2}\right) \approx \begin{cases} -M_z\frac{R}{L} & \text{for } L \gg R \\ -M_z\left(1-\frac{L}{2R}\right) & \text{for } L \ll R \end{cases} \end{aligned} \tag{2.7}$$

Again, the average has a physical behavior similar to that for the value at $z=0$. This is summarized by saying that the longitudinal demagnetization factor N_z is

$$N_z = \frac{1}{L}\left(L+R-\sqrt{L^2+R^2}\right), \qquad \overline{H}_z = -N_z M_z. \tag{2.8}$$

One can note that there is a theorem which says that the sum of the demagnetization factors for x, y, z, call them N_x, N_y, and N_z, should add up to 1. Although we haven't yet solved the case of M_x or M_y, if we apply this theorem, and using the symmetry that $N_x = N_y$ for the cylinder, then there also results

$$N_x = N_y = \frac{1}{2}(1-N_z) = \frac{1}{2L}\left(\sqrt{L^2+R^2}-R\right). \tag{2.9}$$

These results are plotted in Figure 2.8. Notably, for very skinny cylinders with $R \ll L$, the longitudinal demagnetization factor is $N_z \to 0$ while the transverse factors are $N_x \to \frac{1}{2}$. At the other limit, for a flat cylinder, $R \gg L$, we have $N_z \to 1$ and $N_x \to 0$. The flat cylinder has no demagnetization effect within the xy-plane. Generally, the greatest demagnetization effects will always take place through the shortest dimension of a object.

2.2 Transverse magnetization M_x

Next suppose that the magnet is magnetized only along the x direction. Again, it is simplest to look at the case of a circular cylinder. This is a magnetization along a radius of the cross-section. This is not really a line of symmetry, so the mathematics is more complicated. Taking $\vec{M} = M_x\hat{x}$, this will generate a surface charge distribution, on the curved surface

$$\sigma(\theta) = M_x \cos\theta, \tag{2.10}$$

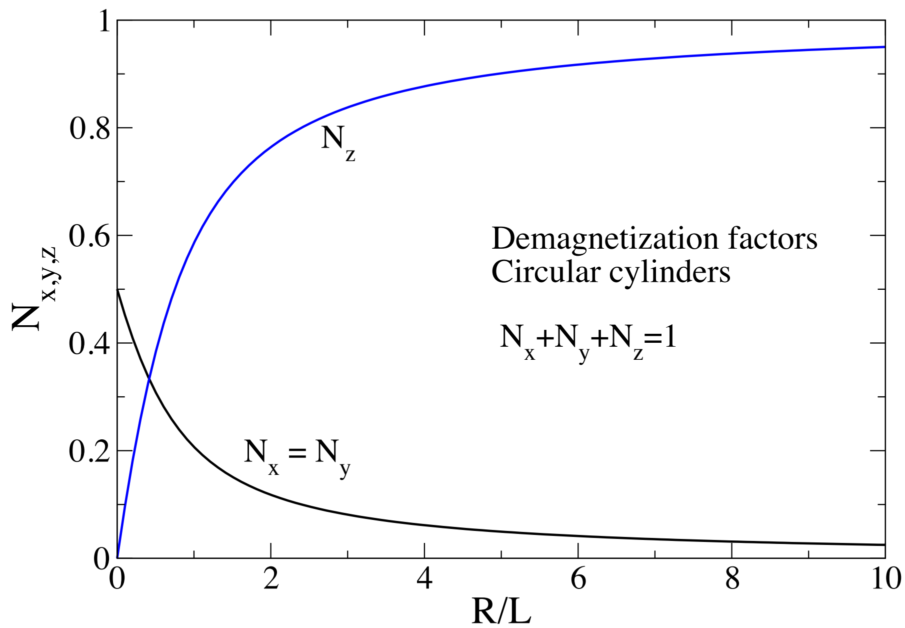

Figure 1: The behavior of the demagnetization factors as a function of aspect ratio for right circular cylinders, based on Equations (2.8) and (2.9), for the averaged field along the cylinder axis. The point where $N_x = N_y = N_z = \frac{1}{3}$ is close to the radius such that $2R = L$.

where θ is the angular position of a point on the surface, measured from the x-axis. This produces positive charges on one side ($x > 0$) and negative charges on the other side ($x < 0$), hence, it is easy to see that the field $\vec{H}$ will point generally towards $-\hat{x}$. This now gives the integral expression for the potential, using cylindrical coordinates, with $r' = R$,

$$\Phi(\mathbf{r}) = \frac{M_x}{4\pi} \int_0^{2\pi} R\, d\theta' \int_{-\delta}^{\delta} dz' \, \frac{\cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos(\theta - \theta') + (z - z')^2}} \tag{2.11}$$

Let's already do the derivative to get H_x, using points along the x-axis, $\theta = 0$.

$$H_x(z) = -\frac{d\Phi}{dx} = \frac{M_x}{4\pi} \int_0^{2\pi} R\, d\theta' \int_{-\delta}^{\delta} dz' \, \frac{(x - R\cos\theta')\cos\theta'}{[r^2 + R^2 - 2rR\cos\theta' + (z - z')^2]^{3/2}} \tag{2.12}$$

Now evaluating on the axis of the cylinder, $x = y = r = 0$, leaves only one term in the numerator and simplifies the denominator:

$$H_x(z) = \frac{M_x}{4\pi} \int_0^{2\pi} R\, d\theta' \int_{-\delta}^{\delta} dz' \, \frac{-R\cos^2\theta'}{[R^2 + (z - z')^2]^{3/2}} \tag{2.13}$$

The angular integration of $\cos^2\theta'$ leads to a factor of $\frac{1}{2}(2\pi)$. So now we have

$$H_x(z) = \frac{-M_x}{4} \int_{-\delta}^{\delta} dz' \, \frac{R^2}{[R^2 + (z - z')^2]^{3/2}} \tag{2.14}$$

To do this integral, it helps to let $z' - z = R\tan\phi$, then $dz' = R\sec^2\phi$, and we will have some algebra like

$$\int \frac{R^2 dz'}{[R^2 + (z - z')^2]^{3/2}} = \int \frac{d\phi\, \sec^2\phi}{\sec^3\phi} = \int d\phi\, \cos\phi = \sin\phi = \frac{z' - z}{\sqrt{R^2 + (z' - z)^2}}. \tag{2.15}$$

This results in

$$H_x(z) = \frac{-M_x}{4}\left[\frac{\delta - z}{\sqrt{R^2+(\delta-z)^2}} + \frac{\delta+z}{\sqrt{R^2+(\delta+z)^2}}\right] \tag{2.16}$$

Check that is gives a correct order of magnitude by looking at the cylinder center, $z=0$. The result is

$$H_x(0) = \frac{-M_x}{2}\frac{\delta}{\sqrt{R^2+\delta^2}} \approx \begin{cases} \frac{-M_x}{2} & \text{for } \delta \gg R \\ \frac{-M_x}{2}\frac{\delta}{R} & \text{for } \delta \ll R \end{cases} \tag{2.17}$$

One can note that compared to $H_z(0) = -N_z M_z$ calculated for longitudinal magnetization, with $N_z = 1 - \delta/\sqrt{R^2+\delta^2}$, this demagnetization factor is

$$N_x = N_y = \frac{1}{2}\frac{\delta}{\sqrt{R^2+\delta^2}}. \tag{2.18}$$

Then at least for the central point of the cylinder, there results

$$N_x + N_y + N_z = 1. \tag{2.19}$$

One can also check the average of $H_x(z)$ over the position z. That result is

$$\begin{aligned}
\overline{H}_x &= \frac{1}{L}\int_{-\delta}^{\delta} dz \left\{\frac{-M_x}{4}\left[\frac{\delta - z}{\sqrt{R^2+(\delta-z)^2}} + \frac{\delta+z}{\sqrt{R^2+(\delta+z)^2}}\right]\right\} \\
&= \frac{-M_x}{4L}\left[-\sqrt{R^2+(\delta-z)^2} + \sqrt{R^2+(\delta+z)^2}\right]_{-\delta}^{\delta} \\
&= \frac{-M_x}{2L}\left[\sqrt{R^2+L^2} - R\right]
\end{aligned} \tag{2.20}$$

Here we can read off the transverse demagnetization factor (although not averaged over x),

$$N_x = N_y = \frac{1}{2L}\left[\sqrt{L^2+R^2} - R\right] \tag{2.21}$$

That is completely the same as we expected to get, based on the earlier result for N_z from $\overline{H}_z$, and the symmetry relation, $N_x + N_y + N_z = 1$. It should be stressed again, that in the limit of a long skinny cylinder (like a pencil, $L \gg R$), this demag factor becoms $N_x \approx \frac{1}{2}$.

3 A cylindrical magnet with a square cross-section

The demagnetization factors depend on geometry. Here consider a cylinder with a square cross-section of size $a \times a$; the height is $L = 2\delta$. We should still expect to get demagnetization factors that have the same limiting values as for the circular cross section. So only a few minor details otherwise should change.

3.1 Longitudinal magnetization M_z

The first step is the same as before, leaving an integration over the ends with the square area. Go ahead and evaluate on the axis, $x = y = 0$:

$$\Phi(z) = \frac{M_z}{4\pi}\int_{-\Delta}^{\Delta} dx' \int_{-\Delta}^{\Delta} dy' \left\{\frac{1}{\sqrt{x'^2+y'^2+(z-\delta)^2}} - \frac{1}{\sqrt{x'^2+y'^2+(z+\delta)^2}}\right\}. \tag{3.1}$$

The end edge is $a = 2\Delta$. This time go to the average over z directly, for the field:

$$\overline{H}_z = \frac{1}{L}\int_{-\delta}^{\delta} dz\, H_z(z) = \frac{1}{L}\int_{-\delta}^{\delta} dz \left(-\frac{d\Phi}{dz}\right) = \frac{-1}{L}\left[\Phi(\delta) - \Phi(-\delta)\right] \tag{3.2}$$

$$\overline{H}_z = \frac{-M_z}{2\pi L}\int_{-\Delta}^{\Delta} dx' \int_{-\Delta}^{\Delta} dy' \; \left\{ \frac{1}{\sqrt{x'^2+y'^2}} - \frac{1}{\sqrt{x'^2+y'^2+L^2}} \right\} \tag{3.3}$$

Consider the integration over y'. We know the basic indefinite integral,

$$\int \frac{dy'}{\sqrt{b^2+y'^2}} = \sinh^{-1}\frac{y'}{|b|}. \tag{3.4}$$

The absolute value is needed on b, because the result is independent of its sign. The integrals here are over a symmetric interval, and the integrands are even. So change the integration to twice that from 0 to Δ. It means we needs something like

$$\int_{-\Delta}^{\Delta} \frac{dy'}{\sqrt{b^2+y'^2}} = 2\sinh^{-1}\frac{\Delta}{|b|}. \tag{3.5}$$

Now we have

$$\overline{H}_z = \frac{-M_z}{\pi L}\int_0^{\Delta} 2dx' \left\{ \sinh^{-1}\frac{\Delta}{|x'|} - \sinh^{-1}\frac{\Delta}{\sqrt{x'^2+L^2}} \right\} \tag{3.6}$$

Really, absolute value is unnecessary now, because $x' > 0$ only. To do this type of integral, note the other way to write the inverse sinh function,

$$\sinh^{-1} x = \ln\left[x+\sqrt{1+x^2}\right] \tag{3.7}$$

So now this gives for one part, the indefinite integral,

$$\int dx' \; \sinh^{-1}\frac{\Delta}{x'} = \int dx' \; \ln\left[\frac{\Delta}{x'} + \sqrt{1+\frac{\Delta^2}{x'^2}}\right] \tag{3.8}$$

Here, try the transformation, $x' = \Delta \operatorname{csch}\phi$, then $dx' = -\Delta \operatorname{csch}^2\phi \cosh\phi d\phi$, and $1+\Delta^2/x'^2 = \cosh^2\phi$. The changes the integral into

$$\int -\Delta\frac{\cosh\phi}{\sinh^2\phi}d\phi \; \ln\left[\sinh\phi+\cosh\phi\right] = -\Delta\int d\phi\,\frac{\phi\;\cosh\phi}{\sinh^2\phi} = -\Delta\int \phi\frac{d(\sinh\phi)}{\sinh^2\phi}. \tag{3.9}$$

That is set up for an integration by parts,

$$\text{integral} \longrightarrow -\Delta\left\{\phi\left(\frac{-1}{\sinh\phi}\right) + \int d\phi\,\left(\frac{1}{\sinh\phi}\right)\right\} = \Delta\left\{\frac{\phi}{\sinh\phi} - \int\frac{d(\cosh\phi)}{\cosh^2\phi - 1}\right\} \tag{3.10}$$

As hyperbolic cosine is always greater than 1, the last integral is aided by doing $\cosh\phi = \coth s$ with $\sinh\phi\, d\phi = -\operatorname{csch}^2 s\, ds$, then,

$$\text{integral} = \Delta\left\{\frac{\phi}{\sinh\phi} + \int ds\right\} = \Delta\left\{\frac{\phi}{\sinh\phi} + \coth^{-1}\left(\cosh\phi\right)\right\} \tag{3.11}$$

But the angle ϕ was defined with $\sinh\phi = \Delta/x'$ and $\cosh\phi = \sqrt{1+\Delta^2/x'^2}$ So this demonstrates the basic integral,

$$\int dx' \; \sinh^{-1}\frac{\Delta}{x'} = \Delta\left\{\frac{x'}{\Delta}\sinh^{-1}\frac{\Delta}{x'} + \coth^{-1}\sqrt{1+\frac{\Delta^2}{x'^2}}\right\} \tag{3.12}$$

For the limits from 0 to Δ, this definite integral becomes

$$\int_0^{\Delta} dx' \; \sinh^{-1}\frac{\Delta}{x'} = \Delta\left\{\sinh^{-1}(1) - 1 + \coth^{-1}\left(\sqrt{2}\right)\right\} = \Delta\left(2\sinh^{-1}(1) - 1\right) \approx 0.7627\Delta. \tag{3.13}$$

There is still the other integral, ugh.

$$I_2 = \int dx' \, \sinh^{-1} \frac{\Delta}{\sqrt{x'^2+L^2}} = \int dx' \, \ln\left[\frac{\Delta}{\sqrt{x'^2+L^2}} + \sqrt{1+\frac{\Delta^2}{x'^2+L^2}}\right] \tag{3.14}$$

One can try a similar kind of tranformation. It might help to choose the transformation as

$$\sinh\phi = \frac{\Delta}{\sqrt{x'^2+L^2}}, \quad \cosh^2\phi = 1+\sinh^2\phi = 1+\frac{\Delta^2}{x'^2+L^2}, \tag{3.15}$$

The forces the argument of the logarithmm to be $\sinh\phi+\cosh\phi = e^{\phi}$. Then get the derivative, based on a rearrangment,

$$x'^2+L^2 = \frac{\Delta^2}{\sinh^2\phi}, \quad 2x'\,dx' = \frac{-2\Delta^2\cosh\phi}{\sinh^3\phi}d\phi. \tag{3.16}$$

But also need

$$x' = \sqrt{\frac{\Delta^2}{\sinh^2\phi} - L^2} = L\sqrt{\frac{\Delta^2}{L^2\sinh^2\phi} - 1}. \tag{3.17}$$

So this gives

$$I_2 = \int dx' \, \phi = \frac{-\Delta^2}{L}\int \frac{\phi\cosh\phi\,d\phi}{\sinh^3\phi\sqrt{\frac{\Delta^2}{L^2\sinh^2\phi}-1}} \tag{3.18}$$

This is aided by letting

$$s = \frac{\Delta^2}{L^2\sinh^2\phi}, \quad ds = \frac{-2\Delta^2\,\cosh\phi\,d\phi}{L^2\sinh^3\phi}. \tag{3.19}$$

Now the integral is

$$I_2 = \frac{L}{2}\int \phi\frac{ds}{\sqrt{s-1}} \tag{3.20}$$

Thus it is set up for an integration by parts, doing first the s-integral. That is now trivial, and gives

$$I_2 = L\left[\phi\sqrt{s-1} - \int d\phi\,\sqrt{s-1}\right] = L\left\{\phi\sqrt{\frac{\Delta^2}{L^2\sinh^2\phi}-1} - \int d\phi\,\sqrt{\frac{\Delta^2}{L^2\sinh^2\phi}-1}\right\} \tag{3.21}$$

Now who knows if this last integral is tractable! We can also write for the differentials, let's see if this helps,

$$ds = -2s\coth\phi\,d\phi, \quad \text{but} \quad \frac{\cosh^2\phi}{\sinh^2\phi} = 1+\frac{1}{\sinh^2\phi} = 1+\frac{L^2}{\Delta^2}s. \tag{3.22}$$

$$ds = -2s\sqrt{1+\frac{L^2}{\Delta^2}s}\;\;d\phi \quad \longrightarrow \quad d\phi = \frac{-ds}{2s\sqrt{1+\frac{L^2}{\Delta^2}s}}. \tag{3.23}$$

So in terms of the s-variable, one needs now

$$I_3 = -\int d\phi\,\sqrt{s-1} = \frac{1}{2}\int ds\,\frac{\sqrt{s-1}}{s\sqrt{\frac{L^2}{\Delta^2}s+1}} \tag{3.24}$$

Ehh, not sure if that helped! This is a lot of work for one small integral. And it looked simpler in terms of hyperbolic functions. Abandon the exact evaluation for now.

Long thin limit, $L \gg a$. In this case the second integral can be done, going back to its original form. A transformation that uses the circular symmetry, but integrates to the square edge, is to do $r'(\theta') = \frac{\Delta}{\cos\theta'} = \Delta\sec\theta'$, applied in each octant of the plane. So this integral is

$$\begin{aligned} I_2 &= \int_{-\Delta}^{\Delta} dx' \int_{-\Delta}^{\Delta} dy' \, \frac{1}{\sqrt{r'^2+L^2}} = 8\int_0^{\pi/4} d\theta' \int_0^{r'(\theta')} r'\,dr'\,\frac{1}{\sqrt{r'^2+L^2}} \\ &= 8\int_0^{\pi/4} d\theta' \left[\sqrt{r'^2(\theta')+L^2} - L\right] = 8\int_0^{\pi/4} d\theta' \left[\sqrt{\Delta^2\sec^2\theta'+L^2} - L\right] \end{aligned} \tag{3.25}$$

That is still exact. But I don't see how to integrate it. So go to the large L limit. This is

$$I_2 \approx 8\int_0^{\pi/4} d\theta' \, \frac{\Delta^2}{2L}\sec^2\theta' = \frac{4\Delta^2}{L}\left[\tan\theta'\right]_0^{\pi/4} = \frac{4\Delta^2}{L} = \frac{a^2}{L}. \tag{3.26}$$

Note what is obtained for the first integral by this procedure (a simpler alternative to the mess I did above):

$$\begin{aligned} I_1 &= \int_{-\Delta}^{\Delta} dx' \int_{-\Delta}^{\Delta} dy' \, \frac{1}{\sqrt{r'^2}} = 8\int_0^{\pi/4} d\theta' \int_0^{r'(\theta')} r' \, dr' \, \frac{1}{r'} = 8\int_0^{\pi/4} d\theta' \; r'(\theta') \\ &= 8\int_0^{\pi/4} d\theta' \; \Delta\sec\theta' = 8\Delta\int_0^{\pi/4} d\theta' \, \frac{\cos\theta'}{1-\sin^2\theta'} = 8\Delta\int_0^{1/\sqrt{2}} \frac{du}{1-u^2} \end{aligned} \tag{3.27}$$

In the last step I used $u = \sin\theta'$. But that is the derivative of inverse hyperbolic tangent. So this gives

$$I_1 = 8\Delta\left[\tanh^{-1} u\right]_0^{1/\sqrt{2}} = 8\Delta\tanh^{-1}\frac{1}{\sqrt{2}} = 8\Delta\cosh^{-1}\sqrt{2} = 8\Delta\sinh^{-1}(1) \approx 7.0\Delta \approx 3.5a. \tag{3.28}$$

Putting the parts together, this gives in this limit,

$$\overline{H}_z = \frac{-M_z}{2\pi L}(I_1 - I_2) \approx \frac{-M_z}{2\pi L}\left(4a\sinh^{-1}(1) - \frac{a^2}{L}\right) = \frac{-M_z a}{2\pi L}\left(4\sinh^{-1}(1) - \frac{a}{L}\right) \tag{3.29}$$

Here in this limit the I_2 integral is just the correction term. This goes to zero when $L \to \infty$, the correct result for a long thin cylinder, even with a square cross-section. In place of R/L for circular symmetry, the leading factor here is $\frac{4\Delta}{\pi L}\sinh^{-1}(1) \approx 1.122\frac{\Delta}{L}$. So with Δ playing the role of the "radius," it makes little difference if the cross-section is circular or square.

Short wide limit, $L \ll a$. The I_1 integral is unchanged, it does not depend on L. The I_2 integral is expanded now as

$$I_2 \approx 8\int_0^{\pi/4} d\theta' \left[\Delta\sec\theta'\left(1 + \frac{1}{2}\frac{L^2}{\Delta^2\sec^2\theta'}\right) - L\right] \tag{3.30}$$

The first term is a copy of I_1; it will cancel out. That leaves terms linear and quadratic in L.

$$I_2 = I_1 + 8\int_0^{\pi/4} d\theta' \left(\frac{L^2}{2\Delta}\cos\theta' - L\right) = I_1 + 8\left(\frac{L^2}{2\Delta}\sin\frac{\pi}{4} - L\frac{\pi}{4}\right). \tag{3.31}$$

Putting all together, the result is

$$\overline{H}_z = \frac{-M_z}{2\pi L}(I_1 - I_2) \approx \frac{-M_z}{2\pi L} \times (-8)\left(\frac{L^2}{2\Delta}\sin\frac{\pi}{4} - L\frac{\pi}{4}\right) = -M_z\left(1 - \frac{2L}{\pi\Delta}\sin\frac{\pi}{4}\right). \tag{3.32}$$

The last term is $\frac{2L}{\pi\Delta}\sin\frac{\pi}{4} \approx 0.45\frac{L}{\Delta}$. In the circular case, this factor comes in as $\frac{L}{2R}$, hence, they are about equivalent, considering Δ like the radius.

3.2 Transverse magnetization, M_x

Based on the previous case, this is not easy to do exactly analytically. But it is the more important case, so see how much can be worked out. The potential at an observer point $\mathbf{r}$, due to only the surface charge density $\sigma = +M_x$ on the face at $x = +\Delta$, is

$$\Phi^+(\mathbf{r}) = \frac{M_x}{4\pi}\int_{-\Delta}^{\Delta} dy' \int_{-\delta}^{\delta} dz' \, \frac{1}{\sqrt{(x-\Delta)^2 + (y-y')^2 + (z-z')^2}} \tag{3.33}$$

There is another term coming from the negative charge on the left face at $x=-\Delta$, to be included now. Now find the field averaged over the x axis, including both plus and minus charges, is:

$$\overline{H}_x = \frac{-2M_x}{4\pi a}\int_{-\Delta}^{\Delta} dy' \int_{-\delta}^{\delta} dz' \left\{ \frac{1}{\sqrt{(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{a^2+(y-y')^2+(z-z')^2}} \right\} \tag{3.34}$$

This is just the double of the contribution from one face's charge. This is the same form of integrals already discussed, but I didn't get the exact result. I am most interested in the case where $\Delta \ll \delta$, the long thin cylinder.

Let me instead do a slightly different approach. First, find the field. without any averaging. From the positive charges, there is

$$H_x^+(\mathbf{r}) = -\frac{d\Phi}{dx} = \frac{M_x}{4\pi}\int_{-\Delta}^{\Delta} dy' \int_{-\delta}^{\delta} dz' \frac{x-\Delta}{[(x-\Delta)^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{3.35}$$

The integration over z' was done in equation (2.15). Using that result here, with the effective $R^2 \equiv (x-\Delta)^2+(y-y')^2$, there results

$$\begin{aligned} H_x^+(\mathbf{r}) &= \frac{M_x\cdot(x-\Delta)}{4\pi}\int_{-\Delta}^{\Delta} dy' \left[\frac{z'-z}{R^2\sqrt{R^2+(z'-z)^2}}\right]_{z'=-\delta}^{z'=\delta} \\ &= \frac{M_x\cdot(x-\Delta)}{4\pi}\int_{-\Delta}^{\Delta} \frac{dy'}{R^2}\left[\frac{\delta-z}{\sqrt{R^2+(\delta-z)^2}} + \frac{\delta+z}{\sqrt{R^2+(\delta+z)^2}}\right] \end{aligned} \tag{3.36}$$

For now, just find the field at the middle of the cylinder, $z=0$. Further, consider the long thin limit, $L \gg a$, which means also $\delta \gg R$. Then this contribution is

$$H_x^+(x,y) = \frac{2M_x(x-\Delta)}{4\pi}\int_{-\Delta}^{\Delta} \frac{dy'}{(x-\Delta)^2+(y'-y)^2} \tag{3.37}$$

The integral is an inverse tangent. Use the fact that $x<\Delta$ for any point inside the system:

$$H_x^+(x,y) = \frac{2M_x}{4\pi}\frac{x-\Delta}{|x-\Delta|}\left[\tan^{-1}\frac{y'-y}{|x-\Delta|}\right]_{-\Delta}^{\Delta} = \frac{-M_x}{2\pi}\left[\tan^{-1}\frac{\Delta-y}{\Delta-x} + \tan^{-1}\frac{\Delta+y}{\Delta-x}\right] \tag{3.38}$$

That was the contribution from the positive charge at $x=+\Delta$. The contribution from the negative charge at $x=-\Delta$ is similar, but with $\Delta \to -\Delta$ inside R, and the opposite sign:

$$H_x^-(x,y) = \frac{-2M_x}{4\pi}\frac{x+\Delta}{|x+\Delta|}\left[\tan^{-1}\frac{y'-y}{|x+\Delta|}\right]_{-\Delta}^{\Delta} = \frac{-M_x}{2\pi}\left[\tan^{-1}\frac{\Delta-y}{\Delta+x} + \tan^{-1}\frac{\Delta+y}{\Delta+x}\right] \tag{3.39}$$

So the total in the long thin cylinder limit is

$$H_x(x,y) = \frac{-M_x}{2\pi}\left(\tan^{-1}\frac{\Delta-y}{\Delta-x} + \tan^{-1}\frac{\Delta+y}{\Delta-x} + \tan^{-1}\frac{\Delta-y}{\Delta+x} + \tan^{-1}\frac{\Delta+y}{\Delta+x}\right) \tag{3.40}$$

At the center of the system, the inverse tangents are all $\frac{\pi}{4}$. Then the field at the center of the system is

$$H_x(0) = -\frac{1}{2}M_x. \tag{3.41}$$

So the demagnetization factor for this point is $N_x = \frac{1}{2}$, as we would expect. Near a corner, say, $x=y=\Delta-\epsilon$, or any of the other corners, by symmetry, there is, surprisingly, the same value:

$$H_x(\Delta,\Delta,0) = \frac{-M_x}{2\pi}\left(\tan^{-1}\frac{\epsilon}{\epsilon} + \tan^{-1}\frac{2\Delta}{\epsilon} + \tan^{-1}\frac{\epsilon}{2\Delta} + \tan^{-1}\frac{2\Delta}{2\Delta}\right) = -\frac{1}{2}M_x. \tag{3.42}$$

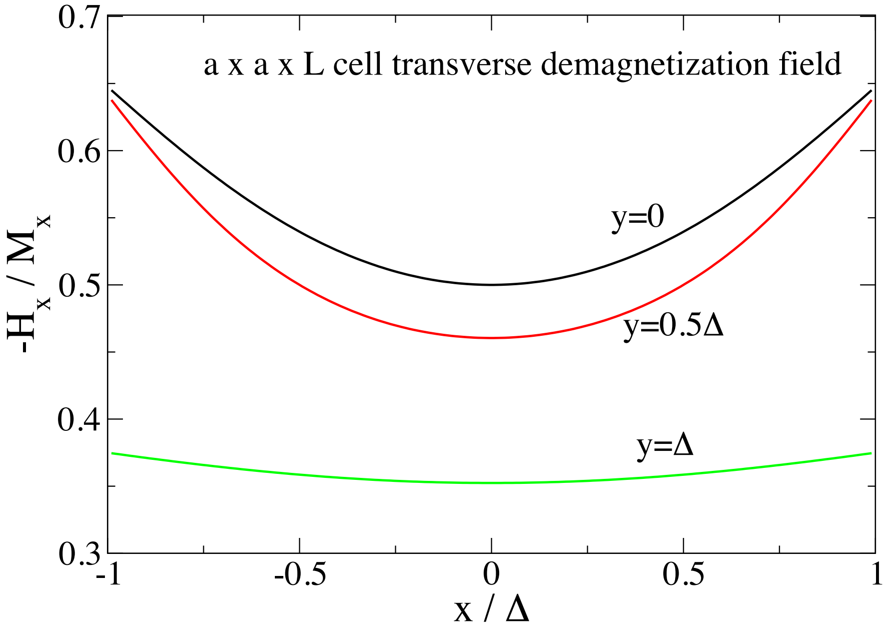

Figure 2: The behavior of the transverse demagnetization field for a rectangular cell of dimensions $a \times a \times L$, with $a \ll L$, as a function of position in the cross-section. Note that $a = 2\Delta$, and $z = 0$, i.e., the middle of a long thin cell.

(Note, this limit might be different if approached in a different direction. See Figure 2.) Trying instead a point along the x-axis, say, $x = \Delta - \epsilon$, $y = 0$, one has

$$H_x(\Delta, 0, 0) = \frac{-M_x}{2\pi}\left(\tan^{-1}\frac{\Delta}{\epsilon} + \tan^{-1}\frac{\Delta}{\epsilon} + \tan^{-1}\frac{\Delta}{2\Delta} + \tan^{-1}\frac{\Delta}{2\Delta}\right) = -\left(\frac{1}{2} + 0.148\right) M_x. \tag{3.43}$$

On the other hand, look at the point $x = 0$, $y = \Delta - \epsilon$:

$$H_x(0, \Delta, 0) = \frac{-M_x}{2\pi}\left(\tan^{-1}\frac{\epsilon}{\Delta} + \tan^{-1}\frac{2\Delta}{\Delta} + \tan^{-1}\frac{\epsilon}{\Delta} + \tan^{-1}\frac{2\Delta}{\Delta}\right) = -0.3524\, M_x. \tag{3.44}$$

This suggests that there is only little variation within the cross-section, in this limit. Then one can expect the demagnetization factor when averaging over different points, is to fair approximation,

$$N_x \approx \frac{1}{2}. \tag{3.45}$$

See Figure 2 for how the field varies within the cell, verses x, at different y (all with $z = 0$ in the long thin approximation).

3.3 About the transverse demagnetization for computations

In actual application in some calculations using computation cells of size $a \times a \times L$ for thin magnets, I would like to include the transverse demagnetization effect. However, I don't have that calculated exactly. We know that $N_x \approx \frac{1}{2}$ for long thin cells. But what if the cells are not so long and thin? As a slight improvement on that, one can use as a reasonable approximation, the transverse demagnetization factor found for the circular cylinder. So instead of simply using $N_x = \frac{1}{2}$, a reasonable improvement is to apply

$$N_x = \frac{1}{2L}\left(\sqrt{L^2 + R^2} - R\right). \tag{3.46}$$

To use this, one needs to choose R. That can be chosen by assuming a a circle with the same area as the square, that is,

$$A = a^2 = \pi R^2, \quad \Longrightarrow \quad R = \frac{a}{\sqrt{\pi}} \tag{3.47}$$

This is a type of approximation I have used for the smoothing of the longitudinal Green's function (see a following section). So it should be good also for the transverse demagnetization. It means we take

$$N_x = \frac{1}{2L}\left(\sqrt{L^2 + \frac{a^2}{\pi}} - \frac{a}{\sqrt{\pi}}\right). \tag{3.48}$$

I'd like to mention one other thing. I like to avoid subtraction in calculations, although it doesn't cause any real problme here. Nevertheless, another way to write the transverse demagnetization factor is by getting rid of the subtraction:

$$N_x = \frac{1}{2L}\left(\sqrt{L^2+R^2} - R\right) \times \frac{\sqrt{L^2+R^2}+R}{\sqrt{L^2+R^2}+R} = \frac{1}{2}\frac{L}{\sqrt{L^2+R^2}+R}. \tag{3.49}$$

Then inserting the effective radius $R = a/\sqrt{\pi}$ gives the formula I actually use in calculations:

$$N_x = \frac{\frac{1}{2}\sqrt{\pi}\,L}{\sqrt{\pi L^2 + a^2} + a}. \tag{3.50}$$

Similarly, the longitudinal factor can be written in alternative ways:

$$N_z = \frac{1}{L}\left(L + R - \sqrt{L^2+R^2}\right) = \frac{2R}{L + R + \sqrt{L^2+R^2}} \tag{3.51}$$

Then with the effective radius, the actual form in the calculations can be

$$N_z = \frac{2a}{\sqrt{\pi}\,L + a + \sqrt{\pi L^2 + a^2}} \tag{3.52}$$

4 Thin film magnets: Demagnetization fields outside of a source cell

This part summarizes some Green's functions that allow the calculation of the demagnetization field (numerically) in a model for a thin magnet (thickness L). The thin direction is the z-direction. The magnet coud have an arbitrary shape in the xy plane. We just do want to assume that $L \ll R$, where R is some transverse diameter of the magnet.

Then, the magnet is partitioned into cells of size $a \times a \times L$. Basically, one wants to estimate the field generated by $\vec{M}_i$ in cell i, but measured at some position $\mathbf{r}$ outside of that cell. This is the "external field" problem. This field will characterize the interaction between the cells of this magnet.

The source cell has some uniform magnetization $\vec{M}$. The cells are supposed to be somewhat "infinitesimal." So we avoid doing an extremely precise calculation. Let's see what comes out if we just treat the source cell as a column of dipole density (along the z'-axis), and get the potential that dipole density produces in its exterior. We will average that potential over altidtude z of an observer position, and use that averaged potential to get the averaged field $\vec{H}(\mathbf{r})$ in the observer position.

4.1 The longitudinal field H_z

The calculation of H_z is easy and comes only from M_z. This general case is nearly the same as what was calculated for a circular cylinder. Start from the potential at $\mathbf{r}$ due to the sum over sources at $\mathbf{r}'$, like we had earlier, with $\tilde{r}^2 = (x-x')^2 + (y-y')^2$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int dx'dy' \left\{ \frac{1}{\sqrt{\tilde{r}^2 + (z-\delta)^2}} - \frac{1}{\sqrt{\tilde{r}^2 + (z+\delta)^2}} \right\} M_z(x', y'). \tag{4.1}$$

There is no sum over z'; this just uses the charges at the surfaces $z' = \pm\delta$. But now we keep the (x', y') dependence of M_z present, as it will depend on the choice of position, i.e., the source cell. We already know how to get H_z and then average it over z:

$$\overline{H}_z(x,y) = \frac{1}{L}\int_{-\delta}^{\delta} dz\, \frac{-d\Phi}{dz} = \frac{-1}{L}\left(\Phi(\delta) - \Phi(-\delta)\right). \tag{4.2}$$

This gives a familiar-looking result,

$$\overline{H}_z(x,y) = \frac{-1}{2\pi L}\int dx'\, dy' \left\{\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + L^2}}\right\} M_z(x', y'). \tag{4.3}$$

Then it is seen that this involves the convolution of a (longitudinal) Green's function G_{zz} with the out-of-plane magnetization component. The Green's function is apparently,

$$G_{zz}(\tilde{r}) = \frac{-1}{2\pi L}\left\{\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + L^2}}\right\}. \tag{4.4}$$

Obviously this is written in terms of the difference of source and observer postions. Hence, it is applied as a convolution with the source magnetization. Note that it is always negative. Thus, it leads to the usual (negative) demagnetization effect.

There is only one small problem with it. I want to think of the computation cells as squares, yet, this Green's function has circular symmetry. Further, it is not defined (technically) if the observer cell is the same as the source cell. Of course, for that case, we already calculated the demagnetization field. But we want to apply this object in a computation using these finite element cells. So clearly the self-interaction at $\tilde{r} = 0$ needs to be corrected.

I imagine that correction to be done by averaging G_{zz} over a circle of radius r_0 whose area is the same as the cell area, $A = a^2$. This procedure was mentioned earlier. We need a radius $R = a/\sqrt{\pi}$. Look what happens for this averaged $G_{zz}(0)$, taking $x' = y' = 0$, but summing over (x, y) in this cirle:

$$\begin{aligned} G_{zz}^0 &\equiv \langle G_{zz}(0)\rangle = \frac{1}{a^2}\int_0^R 2\pi r dr\, \frac{-1}{2\pi L}\left\{\frac{1}{r} - \frac{1}{\sqrt{r^2 + L^2}}\right\} = \frac{-1}{a^2 L}\left\{R - \sqrt{R^2 + L^2} + L\right\} \\ G_{zz}^0 &= \frac{-1}{a^2 L}\left\{\frac{a}{\sqrt{\pi}} - \sqrt{\frac{a^2}{\pi} + L^2} + L\right\} \end{aligned} \tag{4.5}$$

Curiously (or not), G_{zz}^0 found this way is the same as $-N_z$ for a circular cylinder of length L and radius $R = a/\sqrt{\pi}$. This will get applied to an area element $dx'dy'$ of size a^2, hence that factor will cancel out. The rest is the longitudinal demagnetization factor already encountered. So it is nothing too new, but to get this, we needed to do this averaging procedure.

A similar averaging can be applied for cells at farther radii from the source, however, it is not essential. It does help to eliminate some roughness due to using square cross-section cells for a circularly symmetric function.

Then with this correction at the origin, the finite-element calculation of the demagnetization field proceeds from a sum over source cells, (discrete convolution)

$$\overline{H}_z(x,y) = \sum_i G_{zz}(x - x_i, y - y_i)\cdot M_z(x_i, y_i) \tag{4.6}$$

In actual practice, this is best to evaluate using a fast Fourier transform, to get the most speed.

4.2 The transverse field (H_x, H_y)

This could be developed in terms of charge density, but, it is mathematically easier to do in terms of the superposition of dipole fields from the source cells. Really, it is a superposition of a dipole field from each layer (at fixed z') in a source cell. One supposes that dipole is centered in the center of the source cell, at height z'. Thus, we apply the expression (1.10) for the potential, to only the xy components,

$$\Phi(\mathbf{r}) = \int d\mathbf{r}' \frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3} \cdot \vec{M}(\mathbf{r}') = \int dx'\, dy'\, dz'\, \frac{(x-x')M_x + (y-y')M_y}{4\pi[\tilde{r}^2 + (z-z')^2]^{3/2}} \tag{4.7}$$

The integration over (x', y') is left to become the sum over source cells. The source point (x', y') for this cell is set to the center of the cell; we still need to sum over z'. M_x or M_y are constant within a chosen cell. We do need to average over the observation height z.

So doing the integration over z' gives [see expression (2.15)]:

$$\begin{aligned} \Phi(x,y) &= \int dx'\, dy' \left[\frac{[(x-x')M_x + (y-y')M_y](z'-z)}{4\pi\tilde{r}^2\sqrt{\tilde{r}^2+(z-z')^2}} \right]_{-\delta}^{\delta} \\ &= \int dx'\, dy'\, \frac{[(x-x')M_x + (y-y')M_y]}{4\pi\tilde{r}^2} \left\{ \frac{\delta - z}{\sqrt{\tilde{r}^2+(\delta-z)^2}} + \frac{\delta+z}{\sqrt{\tilde{r}^2+(\delta+z)^2}} \right\} \end{aligned} \tag{4.8}$$

This has a very familiar look to it, as we had a similar expression for the transverse field within a cell. Now we can find the transverse field components and also do the averaging over z; the order in which this is done makes no difference.

Do first the average over z. The integration is trivial, and after dividing by L gives

$$\begin{aligned} \overline{\Phi}(x,y) &= \int dx'\, dy'\, \frac{[(x-x')M_x + (y-y')M_y]}{4\pi L\, \tilde{r}^2} \left\{ -\sqrt{\tilde{r}^2+(z-\delta)^2} + \sqrt{\tilde{r}^2+(\delta+z)^2} \right\}_{z=-\delta}^{z=\delta} \\ &= -\int dx'\, dy'\, \frac{[(x-x')M_x + (y-y')M_y]}{2\pi L\, \tilde{r}^2} \left\{ \sqrt{\tilde{r}^2} - \sqrt{\tilde{r}^2+L^2} \right\} \end{aligned} \tag{4.9}$$

Note that this makes a type of Green's function to give the potential, based on the source $\vec{M}(x', y')$. It acts on a vector source, hence it is a vector Green's function, whose direction is radially outward from the source point. We see that this Green's function can be written:

$$\vec{K}(\tilde{\mathbf{r}}) = \frac{1}{2\pi L} \left(\sqrt{1+\frac{L^2}{\tilde{r}^2}} - 1 \right) \frac{\tilde{\mathbf{r}}}{|\tilde{r}|}. \tag{4.10}$$

With that definition, the expression for the potential it produces is

$$\overline{\Phi}(x,y) = \int dx'\, dy'\, \vec{K}(\mathbf{r}-\mathbf{r}') \cdot \vec{M}(\mathbf{r}') \tag{4.11}$$

The last factor in $\vec{K}$ is a radial unit vecctor. Then it is also interesting to realize that the radial component only of this Green's function is

$$K_r(\tilde{r}) = \frac{1}{2\pi L} \left(\sqrt{1+\frac{L^2}{\tilde{r}^2}} - 1 \right). \tag{4.12}$$

This must be the negative radial gradient of the Green's function $\overline{G}$ that produces Φ from the volume charge density ρ, see Equations (1.11) and (1.12). So it may be interesting to find the effective G associated with this $\vec{K}$. We get it from an indefinite integration,

$$\overline{G}(r) = -\int dr\, K_r(r) = \frac{-1}{2\pi L} \int dr \left[\sqrt{1+\frac{L^2}{r^2}} - 1 \right] \tag{4.13}$$

This is aided by using

$$\sinh\phi = \frac{L}{r}, \quad \sqrt{1+\frac{L^2}{r^2}} = \cosh\phi, \quad \cosh\phi\, d\phi = \frac{-L}{r^2} dr = \frac{-\sinh^2\phi}{L} dr. \tag{4.14}$$

Then the integral with the square root is

$$-\int dr\sqrt{1+\frac{L^2}{r^2}} \;=\; L\int d\phi\,\frac{\cosh^2\phi}{\sinh^2\phi}=L\int d\phi(1+\,\mathrm{csch}^2\phi)=L\,(\phi-\coth\phi)$$
$$= L\sinh^{-1}\frac{L}{r}-\sqrt{r^2+L^2}. \tag{4.15}$$

So the whole thing gives

$$\overline{G}(\tilde{r})=\frac{1}{2\pi L}\left(L\sinh^{-1}\frac{L}{\tilde{r}}-\sqrt{\tilde{r}^2+L^2}+\tilde{r}\right). \tag{4.16}$$

This is a well-known expression for the effective 2D in-plane Green function needed, to be applied on $\rho(\mathbf{r}')$, for finding the magnetic potential $\overline{\Phi}(\mathbf{r})$ for thin-film problems, as

$$\overline{\Phi}(\mathbf{r})=\int d\mathbf{r}'\;\overline{G}(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}'). \tag{4.17}$$

Now the gradient of $\overline{\Phi}$ in Equation (4.9) can be done to get the averaged field. To do this, recall that $\tilde{r}^2=(x-x')^2+(y-y')^2$. Find only $H_x=-\frac{d\Phi}{dx}$, then H_y will be obtained by switching some indices.

$$\overline{H}_x = \int\frac{dx'\,dy'}{2\pi L}\left\{\frac{M_x[\tilde{r}-\sqrt{\tilde{r}^2+L^2}]}{\tilde{r}^2}-[(x-x')M_x+(y-y')M_y]\frac{2(x-x')}{\tilde{r}^4}[\tilde{r}-\sqrt{\tilde{r}^2+L^2}]\right.$$
$$\left.+\frac{[(x-x')M_x+(y-y')M_y]}{\tilde{r}^2}\left[\frac{(x-x')}{\tilde{r}}-\frac{(x-x')}{\sqrt{\tilde{r}^2+L^2}}\right]\right\} \tag{4.18}$$

This has contributions from both M_x and M_y. There is nothing coming from M_z, obviously. The parts involving M_x contain the factors

$$M_x:\quad G_{xx}=\frac{\tilde{r}-\sqrt{\tilde{r}^2+L^2}}{2\pi L\,\tilde{r}^4}\left[\tilde{r}^2-2(x-x')^2-(x-x')^2\frac{\tilde{r}}{\sqrt{\tilde{r}^2+L^2}}\right] \tag{4.19}$$

This can be re-arranged as (and in various other ways...)

$$M_x:\quad G_{xx}=\frac{\sqrt{\tilde{r}^2+L^2}-\tilde{r}}{2\pi L\,\tilde{r}^4}\left\{(x-x')^2\left[1+\frac{\tilde{r}}{\sqrt{\tilde{r}^2+L^2}}\right]-(y-y')^2\right\} \tag{4.20}$$

Similarly, there is a term proportional to M_y:

$$M_y:\quad G_{xy}=\frac{\sqrt{\tilde{r}^2+L^2}-\tilde{r}}{2\pi L\,\tilde{r}^4}\left\{2(x-x')(y-y')\left[1+\frac{\tilde{r}}{\sqrt{\tilde{r}^2+L^2}}\right]\right\} \tag{4.21}$$

Thus these define some components of yet another Green function (a matrix) that produces H_α:

$$\overline{H}_\alpha(\mathbf{r})=\int dx'\,dy'\sum_{\beta=x,y}G_{\alpha\beta}(\mathbf{r}-\mathbf{r}')\cdot M_\beta(\mathbf{r}') \tag{4.22}$$

Indeed, this expression even applies to the full 3D field, when used with the 3D magnetization (and let $\alpha,\beta=x,y,x$). The other missing components clearly are obtained by swapping xy indices:

$$G_{yy} = \frac{\sqrt{\tilde{r}^2+L^2}-\tilde{r}}{2\pi L\,\tilde{r}^4}\left\{(y-y')^2\left[1+\frac{\tilde{r}}{\sqrt{\tilde{r}^2+L^2}}\right]-(x-x')^2\right\} \tag{4.23}$$

$$G_{yx}=G_{xy} = \frac{\sqrt{\tilde{r}^2+L^2}-\tilde{r}}{2\pi L\,\tilde{r}^4}\left\{2(x-x')(y-y')\left[1+\frac{\tilde{r}}{\sqrt{\tilde{r}^2+L^2}}\right]\right\} \tag{4.24}$$

Interestingly, we can note that these components of G reduce to those appropriate to give the usual far-field of a unit dipole. Some expansion for $\tilde{r} \gg L$ leads to

$$G_{\alpha\beta}(\tilde{\mathbf{r}}) = \frac{L}{4\pi\tilde{r}^5} \begin{pmatrix} 2\tilde{x}^2 - \tilde{y}^2 & 3\tilde{x}\tilde{y} \\ 3\tilde{x}\tilde{y} & 2\tilde{y}^2 - \tilde{x}^2 \end{pmatrix} \tag{4.25}$$

The tilde means evaluated with the difference of source and observer points, i.e., $\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{r}'$.

This last result is good for the field outside of a source cell. It must be remembered, however, to include the self-demagnetization, if the field within that same source cell is desired. Roughly, this is an extra field of

$$\vec{H}_{\text{local}} = -N_x(M_x\hat{x} + M_y\hat{y}). \tag{4.26}$$

Nominally N_x is a number near 1/2, but it could be smaller than this, taking, for example, the value expected for the transverse demagnetization of circular cylinders as discussed at the end of the previous section, Equations (3.50) and (3.52).