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Numerical treatment of second kind Fredholm integral equations systems on bounded intervals☆

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Abstract

In this paper the authors propose numerical methods to approximate the solutions of systems of second kind Fredholm integral equations. They prove that such methods are stable and convergent. Error estimates in weighted L^p norm, $1 \leqslant p \leqslant +\infty$, are given and some numerical tests are shown.

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Keywords: Fredholm integral equations; Projection method; Nyström method; Lagrange interpolation; Condition number

1. Introduction

Systems of linear integral equations appear in different contexts. For instance, they can represent integral equations defined on the union of curves in the plane whose solutions sometimes give the integral representations of the solutions of original boundary value problems. For the details the reader can consult [2,10,15,17]. On the other hand, in order to approximate the solution of an integral equation having singular kernel and/or known term, it should be convenient to represent it by means of a system of integral equations whose solutions have to be regularized.

In this paper, extending an idea in [3,5], we propose a projection method and an equivalent Nyström-type method in different weighted L^p spaces, $1 \leqslant p \leqslant +\infty$. We show that such procedures are stable and convergent. Moreover the error estimates we prove seem to be optimal and cover the ones available in literature (see, for instance, [1,6,7]). We point out that the systems of linear equations we come to solve are well-conditioned, i.e., the condition numbers of their matrices of coefficients are uniformly bounded except for a possible logarithmic factor. Here, for the sake of simplicity, we consider systems of two integral equations, but what we describe and prove can be generalized, mutatis mutandis, to systems of n second kind Fredholm integral equations.

The paper is organized as follows. In Section 2 we introduce some preliminary tools. In Section 3 we describe the proposed numerical procedures. Section 4 is dedicated to the analysis of their convergence and stability. In Section 5

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we give the proofs of the main results and, in the closing part of the paper, we provide numerical examples in order to illustrate the accuracy of the methods.

2. Preliminaries

We consider systems of the following type

$$\begin{cases} f_1(x) - \lambda \int_{-1}^{1} k^{1,1}(x,y) f_1(y) w_1(y)\,\mathrm{d}y - \lambda \int_{-1}^{1} k^{1,2}(x,y) f_2(y) w_2(y)\,\mathrm{d}y = g_1(x), \\ f_2(x) - \lambda \int_{-1}^{1} k^{2,1}(x,y) f_1(y) w_1(y)\,\mathrm{d}y - \lambda \int_{-1}^{1} k^{2,2}(x,y) f_2(y) w_2(y)\,\mathrm{d}y = g_2(x), \end{cases} \tag{1}$$

where w_i, $i=1,2$, are Jacobi weights, $\lambda \in \mathbb{R}$, $g_i, k^{i,j}$, $i, j = 1, 2$, are given functions and f_i, $i = 1, 2$ are unknown. In order to write the above system in a more compact form, we define

$$\mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \quad \mathbf{k}(x,y) = \begin{pmatrix} k^{1,1}(x,y) & k^{1,2}(x,y) \\ k^{2,1}(x,y) & k^{2,2}(x,y) \end{pmatrix}, \quad \mathbf{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

and

$$\mathbf{w}(x) = \begin{pmatrix} w_1(x) & 0 \\ 0 & w_2(x) \end{pmatrix}.$$

Then, (1) becomes

$$\mathbf{f}(x) - \lambda \int_{-1}^{1} \mathbf{k}(x,y)\mathbf{w}(y)\mathbf{f}(y)\,\mathrm{d}y = \mathbf{g}(x). \tag{2}$$

Moreover, defining the matrices

$$\mathbf{K} = \begin{pmatrix} K^{1,1} & K^{1,2} \\ K^{2,1} & K^{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where the operators $K^{i,j}$, $i, j = 1, 2$, are given by

$$K^{i,j} f_j(x) = \int_{-1}^{1} k^{i,j}(x,y) f_j(y) w_j(y)\,\mathrm{d}y$$

and I denotes the identity operator, it results

$$\mathbf{K}\mathbf{f}(x) = \int_{-1}^{1} \mathbf{k}(x,y)\mathbf{w}(y)\mathbf{f}(y)\,\mathrm{d}y$$

and (1) can be also written as

$$(\mathbf{I} - \lambda\mathbf{K})\mathbf{f} = \mathbf{g}. \tag{3}$$

Now we define the spaces in which we are going to study the above system. Let L^p be the space of all measurable functions f such that

$$\|f\|_{L^p} = \left(\int_{-1}^{1} |f(x)|^p\,\mathrm{d}x \right)^{1/p} < +\infty, \quad 1 \leqslant p < +\infty.$$

With $w(x) = v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1/p$, a Jacobi weight, we set $f \in L^p_w$ if and only if $fw \in L^p$, $1 \leqslant p < +\infty$. We equip the space L^p_w with the norm

$$\|f\|_{L^p_w} := \|fw\|_p = \left(\int_{-1}^{1} |f(x)w(x)|^p\,\mathrm{d}x \right)^{1/p} < +\infty, \quad 1 \leqslant p < +\infty.$$

When $p=+\infty$ we define, for $\alpha, \beta > 0$,

$$L^\infty_w := C_w = \left\{ f \in C^0((-1,1)) : \lim_{|x|\to 1} (fw)(x) = 0 \right\},$$

where $C^0(A)$ is the collection of the continuous functions in $A \subset [-1,1]$. In the case $\alpha=0$ (respectively, $\beta=0$) C_w consists of all continuous functions on $(-1,1]$ (respectively, $[-1,1)$) such that

$$\lim_{x\to -1} (fw)(x) = 0 \quad \left(\text{respectively, } \lim_{x\to 1} (fw)(x) = 0\right).$$

In the case $\alpha=\beta=0$, we set $C_w = C^0([-1,1])$. We equip the space C_w with the norm

$$\|f\|_{C_w} := \|fw\|_\infty = \max_{|x|\leqslant 1} |(fw)(x)|.$$

Somewhere, for brevity, we will write $\|f\|_A = \max_{x\in A}|f(x)|,\ \ A \subseteq [-1,1]$.

For more regular functions, we consider the following Sobolev-type space

$$W^p_s(w) = \{ f \in L^p_w : \|f\|_{W^p_s(w)} := \|f\|_{L^p_w} + \|f^{(s)}\varphi^s w\|_p < +\infty \},$$

where s is a positive integer, $1 \leqslant p \leqslant \infty$ and $\varphi(x) = \sqrt{1-x^2}$. For brevity we will set $W_s = W^\infty_s$.

Moreover, let us consider the spaces $C_w \times C_w$, $L^p_w \times L^p_w$ and $W^p_s(w) \times W^p_s(w)$ equipped with the norms:

$$\|\mathbf{f}\|_{C_w\times C_w} = \max\{\|f_1\|_{C_w}, \|f_2\|_{C_w}\},$$

$$\|\mathbf{f}\|_{L^p_w\times L^p_w} = \max\{\|f_1\|_{L^p_w}, \|f_2\|_{L^p_w}\}$$

and

$$\|\mathbf{f}\|_{W^p_s(w)\times W^p_s(w)} = \max\{\|f_1\|_{W^p_s(w)}, \|f_2\|_{W^p_s(w)}\},$$

respectively.

In the following $\mathscr{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathscr{C} \neq \mathscr{C}(a,b,\ldots)$ to say that $\mathscr{C}$ is independent of the parameters $a, b, \ldots$. If $A, B \geqslant 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathscr{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathscr{C}} \leqslant A \leqslant \mathscr{C} B.$$

3. Numerical method for systems of integral equations

We are looking for an array of polynomials $\mathbf{p}^*$ approximating the solution $\mathbf{f}^*$ (if it exists) of (3). To this end, we consider the Lagrange projection $L_m(v^{\alpha,\beta})$ based on the zeros $t_1 < t_2 < \cdots < t_m$ of the orthonormal Jacobi polynomial $p_m(v^{\alpha,\beta})$, i.e., with $F \in C^0((-1,1))$,

$$L_m(v^{\alpha,\beta}, F, x) = \sum_{i=1}^m l_i(v^{\alpha,\beta}, x) F(t_i), \quad l_i(v^{\alpha,\beta}, x) = \frac{p_m(v^{\alpha,\beta}, x)}{p'_m(v^{\alpha,\beta}, t_i)(x-t_i)}.$$

Now, we can introduce the following matrix of operators

$$\mathbf{K}_m = \begin{pmatrix} K^{1,1}_m & K^{1,2}_m \\ K^{2,1}_m & K^{2,2}_m \end{pmatrix}, \quad m = 1, 2, \ldots,$$

where

$$(K^{i,j}_m f_j)(x) = L_m(w_i, \bar{K}^{i,j} f_j, x), \quad i, j = 1, 2, \tag{4}$$

with

$$(\bar{K}^{i,j} f_j)(x) = \int_{-1}^{1} L_m(w_j, k^{i,j}(x,\cdot), y) f_j(y) w_j(y)\,\mathrm{d}y, \quad i, j = 1, 2, \tag{5}$$

and the arrays of polynomials of degree $m-1$

$$\mathbf{f}_m = \begin{pmatrix} f_{m,1} \\ f_{m,2} \end{pmatrix},$$

$$\mathbf{g}_m = \begin{pmatrix} g_{m,1} \\ g_{m,2} \end{pmatrix} = \begin{pmatrix} L_m(w_1, g_1) \\ L_m(w_2, g_2) \end{pmatrix}.$$

Then, we consider the following system of polynomial equations

$$(\mathbf{I} - \lambda \mathbf{K}_m)\mathbf{f}_m = \mathbf{g}_m, \quad \forall m \in \mathbb{N} \text{ fixed.} \tag{6}$$

Our aim is to approximate the solution $\mathbf{f}^*$ of (3) with the solution $\mathbf{f}_m^*$ (if it exists) of (6). To this end, we look for a linear system whose solution allow us to compute $\mathbf{f}_m^*$. We are going to study (3) in the spaces $C_v \times C_v$ and $L_u^p \times L_u^p$, with v and u suitable Jacobi weights.

3.1. Numerical method in $C_v \times C_v$

Let us study (3) in the space $C_v \times C_v$. In order to obtain a linear system in some sense equivalent to (6), we expand $\mathbf{f}_m$, $\mathbf{g}_m$ and $\mathbf{K}_m\mathbf{f}_m$ in a suitable basis $\{(\varphi_i, 0), (0, \psi_i)\}_{i=1,\dots,m}$ of $\mathbb{P}_{m-1} \times \mathbb{P}_{m-1}$, where $\mathbb{P}_{m-1}$ denotes the set of all polynomials of degree at most $m-1$. We choose $\varphi_i = l_i(w_1)/v(x_i)$, x_i zeros of $p_m(w_1)$, and $\psi_i = l_i(w_2)/v(y_i)$, y_i zeros of $p_m(w_2)$, i.e., we write

$$\mathbf{f}_m = \sum_{i=1}^{m} \left[a_i \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} + b_i \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \right], \tag{7}$$

$$\mathbf{g}_m = \sum_{i=1}^{m} \left[(g_1 v)(x_i) \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} + (g_2 v)(y_i) \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \right] \tag{8}$$

and

$$\begin{aligned} \mathbf{K}_m\mathbf{f}_m = \sum_{i=1}^{m} \Bigg[& v(x_i)(\bar{K}^{1,1} f_{m,1} + \bar{K}^{1,2} f_{m,2})(x_i) \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} \\ & + v(y_i)(\bar{K}^{2,1} f_{m,1} + \bar{K}^{2,2} f_{m,2})(y_i) \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \Bigg], \end{aligned} \tag{9}$$

where, recalling definition (5) and applying a Gaussian rule w.r.t. the weight w_1 and w_2, respectively,

$$(\bar{K}^{i,1} f_{m,1})(x) = \sum_{k=1}^{m} k^{i,1}(x, x_k) \frac{\lambda_k(w_1)}{v(x_k)} a_k, \quad i = 1, 2, \tag{10}$$

and

$$(\bar{K}^{i,2} f_{m,2})(x) = \sum_{k=1}^{m} k^{i,2}(x, y_k) \frac{\lambda_k(w_2)}{v(y_k)} b_k, \quad i = 1, 2. \tag{11}$$

Here $\lambda_k(w)$, $k=1,\dots,m$, are the Christoffel numbers w.r.t. the weight w. Moreover, substituting (7)–(9) into (6) and comparing the coefficients of both sides, we get the system

$$\begin{cases} \sum_{k=1}^{m}\left[\delta_{i,k}-\lambda k^{1,1}(x_i,x_k)\dfrac{v(x_i)}{v(x_k)}\lambda_k(w_1)\right]a_k \\ \quad -\lambda\sum_{k=1}^{m}k^{1,2}(x_i,y_k)\dfrac{v(x_i)}{v(y_k)}\lambda_k(w_2)b_k=g_1(x_i)v(x_i), \\ -\lambda\sum_{k=1}^{m}k^{2,1}(y_i,x_k)\dfrac{v(y_i)}{v(x_k)}\lambda_k(w_1)a_k+\sum_{k=1}^{m}\left[\delta_{i,k}-\lambda k^{2,2}(y_i,y_k)\dfrac{v(y_i)}{v(y_k)}\lambda_k(w_2)\right]b_k \\ \quad =g_2(y_i)v(y_i), \quad i=1,\dots,m. \end{cases} \tag{12}$$

The above system is equivalent to (6) in the following sense: the array $(a_1^*,\dots,a_m^*,b_1^*,\dots,b_m^*)$ is solution of (12) if and only if

$$\mathbf{f}_m^*=\sum_{i=1}^{m}\left[a_i^*\begin{pmatrix}\varphi_i\\0\end{pmatrix}+b_i^*\begin{pmatrix}0\\\psi_i\end{pmatrix}\right] \tag{13}$$

is solution of (6).

3.1.1. Nyström-type method

Now we compare the previous procedure with a Nyström method. In order to apply the Nyström method in $C_v\times C_v$, we multiply both the equations of (1) by the weight v and approximate the integrals by suitable Gaussian quadrature rules. Then we get

$$\begin{cases} v(x)\tilde f_{m,1}(x)-\lambda\sum_{k=1}^{m}k^{1,1}(x,x_k)\dfrac{v(x)}{v(x_k)}a_k\lambda_k(w_1) \\ \quad -\lambda\sum_{k=1}^{m}k^{1,2}(x,y_k)\dfrac{v(x)}{v(y_k)}b_k\lambda_k(w_2)=g_1(x)v(x), \\ v(x)\tilde f_{m,2}(x)-\lambda\sum_{k=1}^{m}k^{2,1}(x,x_k)\dfrac{v(x)}{v(x_k)}a_k\lambda_k(w_1) \\ \quad -\lambda\sum_{k=1}^{m}k^{2,2}(x,y_k)\dfrac{v(x)}{v(y_k)}b_k\lambda_k(w_2)=g_2(x)v(x), \end{cases} \tag{14}$$

where $a_k=\tilde f_{m,1}(x_k)v(x_k)$ and $b_k=\tilde f_{m,2}(y_k)v(y_k)$, $k=1,\dots,m$. Moreover, collocating the first equation on the knots x_i, $i=1,\dots,m$, and the second equation on the knots y_i, $i=1,\dots,m$, we obtain linear system (12). Thus, the solution $(a_1^*,\dots,a_m^*,b_1^*,\dots,b_m^*)$ of system (12) permit us to construct both $\mathbf{f}_m^*$ and the array of the Nyström interpolating functions

$$\tilde{\mathbf{f}}_m=\begin{pmatrix}\tilde f_{m,1}\\\tilde f_{m,2}\end{pmatrix}$$

with

$$\tilde f_{m,1}(x)=\lambda\sum_{k=1}^{m}\left[k^{1,1}(x,x_k)\frac{\lambda_k(w_1)}{v(x_k)}a_k^*+k^{1,2}(x,y_k)\frac{\lambda_k(w_2)}{v(y_k)}b_k^*\right]+g_1(x)$$

and

$$\tilde f_{m,2}(x)=\lambda\sum_{k=1}^{m}\left[k^{2,1}(x,x_k)\frac{\lambda_k(w_1)}{v(x_k)}a_k^*+k^{2,2}(x,y_k)\frac{\lambda_k(w_2)}{v(y_k)}b_k^*\right]+g_2(x).$$

Let us note that each solution of (12) furnishes a solution to (14): merely gives the values of $\tilde f_{m,1}v$ at the nodes x_i, $i=1,\dots,m$, and of $\tilde f_{m,2}v$ at the nodes y_i, $i=1,\dots,m$. The converse is also true.

3.2. Numerical method in $L^p_u \times L^p_u$

If we study system of integral equations (3) in the space $L^p_u \times L^p_u$, where u is a Jacobi weight, we can proceed analogously to the previous case. In order to obtain a linear system equivalent to (6) we expand $\mathbf{f}_m$, $\mathbf{g}_m$ and $\mathbf{K}_m\mathbf{f}_m$ in the basis $\{(\bar\varphi_i, 0), (0, \bar\psi_i)\}_{i=1,\ldots,m}$ of $\mathbb{P}_{m-1} \times \mathbb{P}_{m-1}$, with $\bar\varphi_i = \lambda_m^{-1/p}(u^p, x_i)l_i(w_1)$, $\bar\psi_i = \lambda_m^{-1/p}(u^p, y_i)l_i(w_2)$, x_i zeros of $p_m(w_1)$, y_i zeros of $p_m(w_2)$ and $\lambda_m(u^p, x) = [\sum_{k=0}^{m-1} p_k^2(u^p, x)]^{-1}$ mth Christoffel function related to the weight u^p. Thus we write

$$\mathbf{f}_m = \sum_{i=1}^m \left[a_i \begin{pmatrix} \bar\varphi_i \\ 0 \end{pmatrix} + b_i \begin{pmatrix} 0 \\ \bar\psi_i \end{pmatrix} \right], \tag{15}$$

$$\mathbf{g}_m = \sum_{i=1}^m \left[g_1(x_i)\lambda_m^{1/p}(u^p, x_i) \begin{pmatrix} \bar\varphi_i \\ 0 \end{pmatrix} + g_2(y_i)\lambda_m^{1/p}(u^p, y_i) \begin{pmatrix} 0 \\ \bar\psi_i \end{pmatrix} \right] \tag{16}$$

and

$$\begin{aligned}\mathbf{K}_m\mathbf{f}_m = \sum_{i=1}^m \Bigg[& \lambda_m^{1/p}(u^p, x_i)(\bar K^{1,1} f_{m,1} + \bar K^{1,2} f_{m,2})(x_i) \begin{pmatrix} \bar\varphi_i \\ 0 \end{pmatrix} \\ & + \lambda_m^{1/p}(u^p, y_i)(\bar K^{2,1} f_{m,1} + \bar K^{2,2} f_{m,2})(y_i) \begin{pmatrix} 0 \\ \bar\psi_i \end{pmatrix} \Bigg],\end{aligned} \tag{17}$$

where

$$(\bar K^{i,1} f_{m,1})(x) = \sum_{k=1}^m k^{i,1}(x, x_k) \frac{\lambda_k(w_1)}{\lambda_m^{1/p}(u^p, x_k)} a_k, \quad i = 1, 2, \tag{18}$$

and

$$(\bar K^{i,2} f_{m,2})(x) = \sum_{k=1}^m k^{i,2}(x, y_k) \frac{\lambda_k(w_2)}{\lambda_m^{1/p}(u^p, y_k)} b_k, \quad i = 1, 2. \tag{19}$$

Replacing (15)–(17) into (6) we get the system

$$\begin{cases} \displaystyle\sum_{k=1}^m \left[\delta_{i,k} - \lambda k^{1,1}(x_i, x_k) \frac{\lambda_m^{1/p}(u^p, x_i)}{\lambda_m^{1/p}(u^p, x_k)} \lambda_k(w_1) \right] a_k \\ \displaystyle\quad -\lambda \sum_{k=1}^m k^{1,2}(x_i, y_k) \frac{\lambda_m^{1/p}(u^p, x_i)}{\lambda_m^{1/p}(u^p, y_k)} \lambda_k(w_2) b_k = g_1(x_i)\lambda_m^{1/p}(u^p, x_i), \\ \displaystyle -\lambda \sum_{k=1}^m k^{2,1}(y_i, x_k) \frac{\lambda_m^{1/p}(u^p, y_i)}{\lambda_m^{1/p}(u^p, x_k)} \lambda_k(w_1) a_k \\ \displaystyle\quad + \sum_{k=1}^m \left[\delta_{i,k} - \lambda k^{2,2}(y_i, y_k) \frac{\lambda_m^{1/p}(u^p, y_i)}{\lambda_m^{1/p}(u^p, y_k)} \lambda_k(w_2) \right] b_k = g_2(y_i)\lambda_m^{1/p}(u^p, y_i). \\ \quad i = 1, \ldots, m. \end{cases} \tag{20}$$

Analogously to the previous case, the array $(a_1^*, \ldots, a_m^*, b_1^*, \ldots, b_m^*)$ is solution of (20) if and only if

$$\mathbf{f}_m^* = \sum_{i=1}^m \left[a_i^* \begin{pmatrix} \bar\varphi_i \\ 0 \end{pmatrix} + b_i^* \begin{pmatrix} 0 \\ \bar\psi_i \end{pmatrix} \right] \tag{21}$$

is solution of (6).

Remarks.

(1) When $p=2$, if $w_1=w_2$ and $u=\sqrt{w_1}=\sqrt{w_2}$ then the mth Christoffel function computed on x_k, $\lambda_m(u^2,x_k)$, is equal to the kth Christoffel number $\lambda_k(w_1)=\lambda_k(w_2)$.

(2) When $p\neq 2$ or $p=2$ and $w_1\neq w_2$, the computation of the entries of the matrix of system (20) requires to evaluate the orthonormal polynomials $p_k(u^p,x)$ and it is expensive from the computational point of view. In order to reduce such computational effort, taking into account that (see [14])

$$\lambda_m(u^p,x_i)\sim(\Delta x_i)u^p(x_i),\quad \lambda_m(u^p,y_i)\sim(\Delta y_i)u^p(y_i),\quad i=1,\dots,m,$$

with $\Delta x_i=x_{i+1}-x_i$ and $\Delta y_i=y_{i+1}-y_i$, we consider the new system obtained replacing into (20) $\lambda_m(u^p,x_i)$ by $(\Delta x_i)u^p(x_i)$ and $\lambda_m(u^p,y_i)$ by $(\Delta y_i)u^p(y_i)$, $i=1,\dots,m$. Consequently, we construct the solution $\mathbf{f}_m^*$ by using (21) with $\bar{\varphi}_i$ replaced by $l_i(w_1)/(\Delta x_i)^{1/p}u(x_i)$ and $\bar{\psi}_i$ by $l_i(w_2)/[(\Delta y_i)^{1/p}u(y_i)]$.

(3) Analogously to the case in which we study (1) in the space $C_v\times C_v$, the solution $(a_1^*,\dots,a_m^*,b_1^*,\dots,b_m^*)$ of the linear system (20) can also be used to construct the array of the Nyström interpolating functions

$$\tilde{\mathbf{f}}_m=\begin{pmatrix}\tilde{f}_{m,1}\\ \tilde{f}_{m,2}\end{pmatrix} \tag{22}$$

with

$$\tilde{f}_{m,1}(x)=\lambda\sum_{k=1}^{m}\left[k^{1,1}(x,x_k)\frac{\lambda_k(w_1)}{\lambda_m^{1/p}(u^p,x_k)}a_k^*+k^{1,2}(x,y_k)\frac{\lambda_k(w_2)}{\lambda_m^{1/p}(u^p,y_k)}b_k^*\right]+g_1(x)$$

and

$$\tilde{f}_{m,2}(x)=\lambda\sum_{k=1}^{m}\left[k^{2,1}(x,x_k)\frac{\lambda_k(w_1)}{\lambda_m^{1/p}(u^p,x_k)}a_k^*+k^{2,2}(x,y_k)\frac{\lambda_k(w_2)}{\lambda_m^{1/p}(u^p,y_k)}b_k^*\right]+g_2(x).$$

In next section, according to the choice of the space $C_v\times C_v$ or $L_u^p\times L_u^p$, we establish the hypotheses on the weight v or u, on the kernels $k^{i,j}$, $i,j=1,2$, and on the known terms g_i, $i=1,2$, such that the numerical methods previously introduced are stable and convergent. In other words, we state the assumptions under which systems (12) and (20) are unisolvent and well-conditioned and the corresponding arrays of polynomials $\mathbf{f}_m^*$ converge to the unique solution $\mathbf{f}^*$ of system of integral equations (1).

3.3. Stability and convergence analysis

Assume $w_1(x)=(1-x)^{\alpha_1}(1+x)^{\beta_1}$ with $-1<\alpha_1,\beta_1<1$, and $w_2(x)=(1-x)^{\alpha_2}(1+x)^{\beta_2}$ with $-1<\alpha_2,\beta_2<1$. Let us observe that we can suppose $\alpha_i,\beta_i<1$, $i=1,2$ without loss of generality. In fact, if $v^{\alpha,\beta}(x)$ is a Jacobi weight with $\alpha\geqslant 1$ and/or $\beta\geqslant 1$, one can factorize it as follows $v^{\alpha,\beta}(x)=(1-x)^{[\alpha]}(1+x)^{[\beta]}v^{\alpha-[\alpha],\beta-[\beta]}(x)$ and consider as new weight the function $v^{\alpha-[\alpha],\beta-[\beta]}(x)$ ($[a]$ denotes the integer part of $a\in\mathbb{R}$).

If we consider (3) in the space $C_v\times C_v$, $v:=v^{\gamma,\delta}$, we choose γ,δ according to

$$\begin{cases}\max\left\{0,\dfrac{\alpha_1}{2}+\dfrac{1}{4},\dfrac{\alpha_2}{2}+\dfrac{1}{4}\right\}\leqslant\gamma<\min\left\{\dfrac{\alpha_1}{2}+\dfrac{3}{4},\alpha_1+1,\dfrac{\alpha_2}{2}+\dfrac{3}{4},\alpha_2+1\right\},\\[2ex] \max\left\{0,\dfrac{\beta_1}{2}+\dfrac{1}{4},\dfrac{\beta_2}{2}+\dfrac{1}{4}\right\}\leqslant\delta<\min\left\{\dfrac{\beta_1}{2}+\dfrac{3}{4},\beta_1+1,\dfrac{\beta_2}{2}+\dfrac{3}{4},\beta_2+1\right\}\end{cases} \tag{23}$$

and establish the following assumptions, for some positive integer s,

$$M_s^{i,j}:=\sup_{|y|\leqslant 1}\|k_y^{i,j}\|_{W_s(v)}<+\infty,\quad i,j=1,2, \tag{24}$$

$$N_s^{i,j}:=\sup_{|x|\leqslant 1}v(x)\left\|k_x^{i,j}\right\|_{W_s}<+\infty,\quad i,j=1,2, \tag{25}$$

where $k^{i,j}(x,y)=k_x^{i,j}(y)=k_y^{i,j}(x)$, and

$$g_i \in W_s(v), \quad i=1,2. \tag{26}$$

Let us observe that, under our hypotheses on the weights w_1 and w_2, there always exist γ and δ satisfying (23).

The following proposition holds true.

Proposition 3.1. *If $v := v^{\gamma,\delta}$ is a Jacobi weight with γ, δ such that*

$$0 \leqslant \gamma < \min\{\alpha_1+1, \alpha_2+1\}, \quad 0 \leqslant \delta < \min\{\beta_1+1, \beta_2+1\}$$

and the kernels $k^{i,j}$, $i, j = 1, 2$, satisfy (24), *then the operator $\mathbf{K} : C_v \times C_v \to C_v \times C_v$ is compact and for* (3) *the Fredholm alternative is true in $C_v \times C_v$.*

Then we can state the next theorem.

Theorem 3.1. *Let us assume* (23)–(26) *and* $\mathrm{Ker}(\mathbf{I}-\lambda\mathbf{K})=\{\mathbf{0}\}$ *in $C_v \times C_v$. Then, for any sufficiently large m,* (*say $m > m_0$*), *system* (12) *has a unique solution $(a_1^*, \ldots, a_m^*, b_1^*, \ldots, b_m^*)$ and the corresponding*

$$f_m^* = \sum_{i=1}^{m} \left[a_i^* \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} + b_i^* \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \right]$$

is the unique solution of (6) *in $\mathbb{P}_{m-1} \times \mathbb{P}_{m-1}$. If A_m is the matrix of the coefficients of* (12) *and* $\mathrm{cond}(A_m) = \|A_m\|_\infty \|A_m^{-1}\|_\infty$ *denotes its condition number in uniform norm* (*the so-called* "*row sum norm*"), *then we have*

$$\sup_m \frac{\mathrm{cond}(A_m)}{\log m} < +\infty. \tag{27}$$

Moreover, $\mathbf{f}_m^*$ converges to the unique solution $\mathbf{f}^*$ of (3) *in $C_v \times C_v$ with the error*

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{C_v \times C_v} \leqslant \mathscr{C} \frac{\log m}{m^s} \|\mathbf{g}\|_{W_s(v) \times W_s(v)}, \tag{28}$$

where $\mathscr{C} \neq \mathscr{C}(m, \mathbf{f}^*)$.

From the previous theorem we can also deduce the stability of the Nyström method, while its convergence is stated by the following.

Theorem 3.2. *Under assumptions* (23)–(26) *with $v = v^{\gamma,\delta}$, the unique solution $\tilde{\mathbf{f}}_m$ of* (14) *satisfies*

$$\|\mathbf{f}^* - \tilde{\mathbf{f}}_m\|_{C_v \times C_v} \leqslant \frac{\mathscr{C}}{m^s} \|\mathbf{f}^*\|_{W_s(v) \times W_s(v)}, \tag{29}$$

where $\mathscr{C} \neq \mathscr{C}(m, \mathbf{f}^*)$.

If we consider (3) in the space $L_u^p \times L_u^p$, $u = v^{\rho,\theta}$, we choose ρ, θ according to

$$\rho_1 < \rho < \rho_2, \quad \theta_1 < \theta < \theta_2, \tag{30}$$

with

$$\rho_1 = \max\left\{ -\frac{1}{p}, \frac{\alpha_1}{2} - \frac{1}{p} + \frac{1}{4}, \frac{\alpha_2}{2} - \frac{1}{p} + \frac{1}{4} \right\},$$

$$\rho_2 = \min\left\{ \frac{\alpha_1}{2} + \frac{1}{q} + \frac{1}{4}, \alpha_1 + \frac{1}{q}, \frac{\alpha_2}{2} + \frac{1}{q} + \frac{1}{4}, \alpha_2 + \frac{1}{q} \right\},$$

$$\theta_1 = \max\left\{-\frac{1}{p}, \frac{\beta_1}{2} - \frac{1}{p} + \frac{1}{4}, \frac{\beta_2}{2} - \frac{1}{p} + \frac{1}{4}\right\},$$

$$\theta_2 = \min\left\{\frac{\beta_1}{2} + \frac{1}{q} + \frac{1}{4}, \beta_1 + \frac{1}{q}, \frac{\beta_2}{2} + \frac{1}{q} + \frac{1}{4}, \beta_2 + \frac{1}{q}\right\}$$

and $(1/p) + (1/q) = 1$. Moreover, we assume that

$$\bar{M}_s^{i,j} := \sup_{|y| \leqslant 1} \|k_y^{i,j}\|_{W_s^p(u)} < +\infty, \quad i, j = 1, 2, \tag{31}$$

$$\bar{N}_s^{i,j} := \sup_{|x| \leqslant 1} \|k_x^{i,j}\|_{W_s^q(w_j/u)} < +\infty, \quad q = \frac{p}{p-1}, \quad i, j = 1, 2 \tag{32}$$

and

$$g_i \in W_s^p(u), \quad i = 1, 2. \tag{33}$$

We point out that, under our assumptions, there always exist ρ and θ satisfying (30). We prove the following proposition.

Proposition 3.2. *Let $u := v^{\rho,\theta}$ be a Jacobi weight such that*

$$-\frac{1}{p} < \rho < \min\left\{\alpha_1 + 1 - \frac{1}{p}, \alpha_2 + 1 - \frac{1}{p}\right\}$$

and

$$-\frac{1}{p} < \theta < \min\left\{\beta_1 + 1 - \frac{1}{p}, \beta_2 + 1 - \frac{1}{p}\right\}.$$

If the kernels $k^{i,j}$, $i, j = 1, 2$, satisfy (31) then the operator $\mathbf{K} : L_u^p \times L_u^p \to L_u^p \times L_u^p$ is compact and for (3) the Fredholm alternative is true in $L_u^p \times L_u^p$.

Moreover the following theorem holds true.

Theorem 3.3. *Let us assume (30)–(33) and $\mathrm{Ker}(\mathbf{I} - \lambda\mathbf{K}) = \{\mathbf{0}\}$ in $L_u^p \times L_u^p$. Then, for any sufficiently large m, (say $m > m_0$), system (20) has a unique solution $(a_1^*, \ldots, a_m^*, b_1^*, \ldots, b_m^*)$ and the corresponding*

$$\mathbf{f}_m^* = \sum_{i=1}^{m}\left[a_i^*\begin{pmatrix}\bar{\varphi}_i \\ 0\end{pmatrix} + b_i^*\begin{pmatrix}0 \\ \bar{\psi}_i\end{pmatrix}\right]$$

is the unique solution of (6) in $\mathbb{P}_{m-1} \times \mathbb{P}_{m-1}$. If $\bar{A}_m$ is the matrix of the coefficients of (20) and $\mathrm{cond}(\bar{A}_m)_p = \|\bar{A}_m\|_p\|\bar{A}_m^{-1}\|_p$ denotes its condition number in the matrix p-norm, then we have

$$\mathrm{cond}(\bar{A}_m)_p \leqslant \mathscr{C}\mathrm{cond}(\mathbf{I} - \lambda\mathbf{K}) + \mathcal{O}(m^{-s}), \quad \mathscr{C} \neq \mathscr{C}(m). \tag{34}$$

Moreover, $\mathbf{f}_m^*$ converges to the unique solution $\mathbf{f}^*$ of (3) in $L_u^p \times L_u^p$ with the error

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{L_u^p \times L_u^p} \leqslant \frac{\mathscr{C}}{m^s}\|g\|_{W_s^p(u) \times W_s^p(u)}, \tag{35}$$

where $\mathscr{C} \neq \mathscr{C}(m, \mathbf{f}^*)$.

Theorem 3.4. *Under assumptions (30)–(33) with $u = u^{\rho,\theta}$, the array $\tilde{\mathbf{f}}_m$ given by (22) satisfies*

$$\|\mathbf{f}^* - \tilde{\mathbf{f}}_m\|_{L_u^p \times L_u^p} \leqslant \frac{\mathscr{C}}{m^s}\|\mathbf{f}^*\|_{W_s^p(u) \times W_s^p(u)}, \tag{36}$$

where $\mathscr{C} \neq \mathscr{C}(m, \mathbf{f}^*)$.

Let us make a remark. Till now, we assumed the known functions $k^{i,j}(x,y)$, $i,j=1,2$, and g_i, $i=1,2$, belonging to Sobolev-type spaces. These conditions can be relaxed if we replace such spaces by the Zygmund-type ones defined as follows:

$$Z_s^p(u)=\left\{f\in L_u^p(-1,1):\|f\|_{Z_s^p(u)}:=\|fu\|_p+\sup_{t>0}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}<\infty\right\},$$

where s is a positive real number, $r>s$ is integer, $1\leqslant p\leqslant+\infty$,

$$\Omega_\varphi^r(f,t)_{u,p}=\sup_{0<h\leqslant t}\|(\vec{\Delta}_{h\varphi}^r f)u\|_{L^p(I_{rh})},\quad I_{rh}=[-1+(2hr)^2,1-(2hr)^2],$$

$$\vec{\Delta}_{h\varphi}^r f(x)=\sum_{i=0}^r(-1)^i\binom{r}{i}f\left(x+\left(\frac{r}{2}-i\right)h\varphi(x)\right),\quad \varphi(x)=\sqrt{1-x^2}.$$

All the above results are still true in Zygmund-type spaces. We used Sobolev-type spaces only to simplify the proofs.

4. Proofs

We need some notations and preliminary results. We denote by

$$E_m(f)_{w,p}=\inf_{P\in\mathbb{P}_m}\|(f-P)w\|_p,\quad 1\leqslant p\leqslant+\infty,$$

the error of best approximation of a function $f\in L_w^p$ by means of polynomials of degree at most m. We set $E_m(f)_w=E_m(f)_{w,\infty}$. For all functions $f\in W_s^p(w)$, we have [4]

$$E_m(f)_{w,p}\leqslant\frac{\mathscr{C}}{m^s}\|f^{(s)}\varphi^s w\|_p\leqslant\frac{\mathscr{C}}{m^s}\|f\|_{W_s^p(w)}. \tag{37}$$

Proof of Proposition 3.1. Since the operator $\mathbf{K}:C_v\times C_v\to C_v\times C_v$ is compact if and only if all the operators $K^{i,j}:C_v\to C_v$, $i,j=1,2$, are compact, we prove the compactness of the operators $K^{i,j}$. For $i,j=1,2$ we have

$$\begin{aligned}|(K^{i,j}f_j)(x)v(x)|&\leqslant\|f_jv\|_\infty\int_{-1}^1|k^{i,j}(x,y)v(x)v^{\alpha_j-\gamma,\beta_j-\delta}(y)|\,\mathrm{d}y\\&\leqslant\|f_jv\|_\infty\|vk_y^{i,j}\|_\infty\int_{-1}^1 v^{\alpha_j-\gamma,\beta_j-\delta}(y)\,\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}|(K^{i,j}f_j)^{(s)}(x)\varphi^s(x)v(x)|&\leqslant\|f_jv\|_\infty\int_{-1}^1\left|\frac{\partial^s}{\partial x^s}k_y^{i,j}(x)\varphi^s(x)v(x)\right|v^{\alpha_j-\gamma,\beta_j-\delta}(y)\,\mathrm{d}y\\&\leqslant\|f_jv\|_\infty\left\|\frac{\partial^s}{\partial x^s}k_y^{i,j}\varphi^s v\right\|_\infty\int_{-1}^1 v^{\alpha_j-\gamma,\beta_j-\delta}(y)\,\mathrm{d}y.\end{aligned}$$

Then, under the assumptions, it results

$$\|K^{i,j}f_j\|_{W_s(v)}\leqslant\|f_jv\|_\infty M_s^{i,j}\int_{-1}^1 v^{\alpha_j-\gamma,\beta_j-\delta}(y)\,\mathrm{d}y\leqslant\mathscr{C}\|f_jv\|_\infty.$$

Now, by (37), we get

$$E_m(K^{i,j}f_j)_v\leqslant\frac{\mathscr{C}}{m^s}\|K^{i,j}f_j\|_{W_s(v)}\leqslant\frac{\mathscr{C}}{m^s}\|f_jv\|_\infty,$$

i.e.,

$$\lim_m \left(\sup_{\|f_j v\|_\infty = 1} E_m(K^{i,j} f_j)_v \right) = 0$$

and then the operators $K^{i,j}$, $i, j = 1, 2$, are compact (see, for example, [16, p. 44]). $\square$

The following result will be useful in the sequel [11].

Lemma 4.1. *For* α, β, γ *and* δ *satisfying*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \leqslant \gamma < \min\left\{\frac{\alpha}{2} + \frac{3}{4}, 1 + \alpha\right\},$$

$$\max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\} \leqslant \delta < \min\left\{\frac{\beta}{2} + \frac{3}{4}, 1 + \beta\right\} \tag{38}$$

and for every $f \in C_{v^{\gamma,\delta}}$, *we have*

$$\|L_m(v^{\alpha,\beta}, f)v^{\gamma,\delta}\|_\infty \leqslant \mathscr{C}(\log m)\|f v^{\gamma,\delta}\|_\infty \tag{39}$$

or, equivalently,

$$\|[f - L_m(v^{\alpha,\beta}, f)]v^{\gamma,\delta}\|_\infty \leqslant \mathscr{C}(\log m) E_{m-1}(f)_{v^{\gamma,\delta}}, \tag{40}$$

where the constant $\mathscr{C}$ *is independent of* m *and* f.

In order to prove Theorem 3.1 we need the following result.

Proposition 4.1. *Under assumptions* (23)–(25), *we have*

$$\|\mathbf{K} - \mathbf{K}_m\|_{C_v \times C_v \to C_v \times C_v} \leqslant \mathscr{C} \frac{\log m}{m^s},$$

where $\mathscr{C} \neq \mathscr{C}(m)$.

Proof. We have

$$\begin{aligned}
\|(\mathbf{K} - \mathbf{K}_m)\mathbf{f}\|_{C_v \times C_v} &= \max\{\|[(K^{1,1} - K_m^{1,1})f_1 + (K^{1,2} - K_m^{1,2})f_2]v\|_\infty, \\
&\quad \|[(K^{2,1} - K_m^{2,1})f_1 + (K^{2,2} - K_m^{2,2})f_2]v\|_\infty\} \\
&\leqslant \max\{\|K^{1,1} - K_m^{1,1}\|_{C_v \to C_v}\|f_1 v\|_\infty + \|K^{1,2} - K_m^{1,2}\|_{C_v \to C_v}\|f_2 v\|_\infty, \\
&\quad \|K^{2,1} - K_m^{2,1}\|_{C_v \to C_v}\|f_1 v\|_\infty + \|K^{2,2} - K_m^{2,2}\|_{C_v \to C_v}\|f_2 v\|_\infty\} \\
&\leqslant \|\mathbf{f}\|_{C_v \times C_v} \max\{\|K^{1,1} - K_m^{1,1}\|_{C_v \to C_v} + \|K^{1,2} - K_m^{1,2}\|_{C_v \to C_v}, \\
&\quad \|K^{2,1} - K_m^{2,1}\|_{C_v \to C_v} + \|K^{2,2} - K_m^{2,2}\|_{C_v \to C_v}\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{K} - \mathbf{K}_m\|_{C_v \times C_v \to C_v \times C_v} \leqslant \max\{&\|K^{1,1} - K_m^{1,1}\|_{C_v \to C_v} + \|K^{1,2} - K_m^{1,2}\|_{C_v \to C_v}, \\
&\|K^{2,1} - K_m^{2,1}\|_{C_v \to C_v} + \|K^{2,2} - K_m^{2,2}\|_{C_v \to C_v}\}.
\end{aligned}$$

Since, under assumptions (23)–(25), it results [3, Proof of Theorem 2.2]

$$\|K^{i,j} - K_m^{i,j}\|_{C_v \to C_v} \leqslant \mathscr{C}(M_s^{i,j} + N_s^{i,j}) \frac{\log m}{m^s}, \quad i, j = 1, 2,$$

we deduce the thesis. $\square$

Proof of Theorem 3.1. We first note that by well-known results (see for example [2]), Proposition 4.1 implies that, for sufficiently large m (say $m > m_0$), the inverse operators $(\mathbf{I} - \lambda\mathbf{K}_m)^{-1}$ exist and are uniformly bounded by

$$\|(\mathbf{I} - \lambda\mathbf{K}_m)^{-1}\|_{C_v\times C_v\to C_v\times C_v} \leqslant \frac{\|(\mathbf{I} - \lambda\mathbf{K})^{-1}\|_{C_v\times C_v\to C_v\times C_v}}{1 - \|(\mathbf{I} - \lambda\mathbf{K})^{-1}\|_{C_v\times C_v\to C_v\times C_v}\cdot\|\mathbf{K} - \mathbf{K}_m\|_{C_v\times C_v\to C_v\times C_v}}. \tag{41}$$

We start with the proof of (28). Using the identity

$$\mathbf{f}^* - \mathbf{f}_m^* = (\mathbf{I} - \lambda\mathbf{K}_m)^{-1}[(\mathbf{g} - \mathbf{g}_m) + (\mathbf{K} - \mathbf{K}_m)(\mathbf{I} - \lambda\mathbf{K})^{-1}\mathbf{g}], \tag{42}$$

by (41), we get

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{C_v\times C_v} \leqslant \mathscr{C}[\|\mathbf{g} - \mathbf{g}_m\|_{C_v\times C_v} + \|\mathbf{g}\|_{C_v\times C_v}\|\mathbf{K} - \mathbf{K}_m\|_{C_v\times C_v\to C_v\times C_v}]. \tag{43}$$

Thus, taking into account Proposition 4.1, we have to estimate only $\|\mathbf{g} - \mathbf{g}_m\|_{C_v\times C_v}$. We have

$$\|\mathbf{g} - \mathbf{g}_m\|_{C_v\times C_v} = \max\{\|[g_1 - L_m(w_1, g_1)]v\|_\infty, \|[g_2 - L_m(w_2, g_2)]v\|_\infty\},$$

thus, by (40) and (37) and the hypotheses (23) and (26), it results

$$\begin{aligned}\|\mathbf{g} - \mathbf{g}_m\|_{C_v\times C_v} &\leqslant \mathscr{C}\frac{\log m}{m^s}\max\{\|g_1\|_{W_s(v)}, \|g_2\|_{W_s(v)}\}\\ &\leqslant \mathscr{C}\frac{\log m}{m^s}\|\mathbf{g}\|_{W_s(v)\times W_s(v)}.\end{aligned} \tag{44}$$

Finally combining (44) and Proposition 4.1 with (43), (28) follows.

Now we prove (27). The matrix A_m of system (12) can be written as follows:

$$A_m = \begin{pmatrix} A_m^{1,1} & A_m^{1,2} \\ A_m^{2,1} & A_m^{2,2}\end{pmatrix},$$

where

$$A_m^{1,1} = \left[\delta_{i,k} - \lambda k^{1,1}(x_i, x_k)\frac{v(x_i)}{v(x_k)}\lambda_k(w_1)\right]_{i,k=1,\dots,m} =: [a_{i,k}^{1,1}]_{i,k=1,\dots,m},$$

$$A_m^{1,2} = \left[-\lambda k^{1,2}(x_i, y_k)\frac{v(x_i)}{v(y_k)}\lambda_k(w_2)\right]_{i,k=1,\dots,m} =: [a_{i,k}^{1,2}]_{i,k=1,\dots,m},$$

$$A_m^{2,1} = \left[-\lambda k^{2,1}(y_i, x_k)\frac{v(y_i)}{v(x_k)}\lambda_k(w_1)\right]_{i,k=1,\dots,m} =: [a_{i,k}^{2,1}]_{i,k=1,\dots,m}$$

and

$$A_m^{2,2} = \left[\delta_{i,k} - \lambda k^{2,2}(y_i, y_k)\frac{v(y_i)}{v(y_k)}\lambda_k(w_2)\right]_{i,k=1,\dots,m} =: [a_{i,k}^{2,2}]_{i,k=1,\dots,m}.$$

Thus we have

$$\|A_m\|_\infty = \max\{\|(A_m^{1,1}\, A_m^{1,2})\|_\infty, \|(A_m^{2,1}\, A_m^{2,2})\|_\infty\},$$

with

$$\|(A_m^{1,1}A_m^{1,2})\|_\infty = \max_{1\leqslant i\leqslant m}\left(\sum_{k=1}^m |a_{i,k}^{1,1}| + \sum_{k=1}^m |a_{i,k}^{1,2}|\right)$$

and

$$\|(A_m^{2,1}A_m^{2,2})\|_\infty = \max_{1\leqslant i\leqslant m}\left(\sum_{k=1}^m |a_{i,k}^{2,1}| + \sum_{k=1}^m |a_{i,k}^{2,2}|\right).$$

We first estimate $\|(A_m^{1,1}A_m^{1,2})\|_\infty$. Recalling that, if t_k, $k=1,\dots,m$, are the zeros of the Jacobi polynomial $p_m(v^{\alpha,\beta})$ and $\Delta t_k = t_{k+1}-t_k$, then [13]

$$\lambda_k(v^{\alpha,\beta}) \sim \Delta t_k v^{\alpha,\beta}(t_k),$$

and

$$v^{\alpha,\beta}(t_k) \sim v^{\alpha,\beta}(x) \sim v^{\alpha,\beta}(t_{k+1}), \quad x\in[t_k,t_{k+1}],$$

under assumptions (23)–(24), we have

$$\begin{aligned}\sum_{k=1}^m |a_{i,k}^{1,1}| &\leqslant 1+|\lambda|\sum_{k=1}^m v^{\alpha_1-\gamma,\beta_1-\delta}(x_k)\Delta x_k |k^{1,1}(x_i,x_k)v^{\gamma,\delta}(x_i)|\\ &\leqslant 1+|\lambda|\sup_{|y|\leqslant 1}\|vk_y^{1,1}\|_\infty\int_{-1}^1 v^{\alpha_1-\gamma,\beta_1-\delta}(x)\,\mathrm{d}x\leqslant \mathscr{C}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^m |a_{i,k}^{1,2}| &\leqslant |\lambda|\sum_{k=1}^m v^{\alpha_2-\gamma,\beta_2-\delta}(y_k)\Delta y_k |k^{1,2}(x_i,y_k)v^{\gamma,\delta}(x_i)|\\ &\leqslant |\lambda|\sup_{|y|\leqslant 1}\|vk_y^{1,2}\|_\infty\int_{-1}^1 v^{\alpha_2-\gamma,\beta_2-\delta}(x)\,\mathrm{d}x\leqslant \mathscr{C}.\end{aligned}$$

Thus we have

$$\|(A_m^{1,1}A_m^{1,2})\|_\infty\leqslant\mathscr{C}.$$

Since, analogously it is possible to prove that

$$\|(A_m^{2,1}A_m^{2,2})\|_\infty\leqslant\mathscr{C},$$

we have

$$\|A_m\|_\infty\leqslant\mathscr{C}. \tag{45}$$

Now we estimate $\|A_m^{-1}\|_\infty$. In virtue of the equivalence between system (12) and Eq. (6), for all $\boldsymbol{\tau}=(\eta_1,\dots,\eta_m,\theta_1,\dots,\theta_m)\in\mathbb{R}^{2m}$ there exists a unique array $\mathbf{c}=(a_1,\dots,a_m,b_1,\dots,b_m)\in\mathbb{R}^{2m}$ such that $\mathbf{c}=A_m^{-1}\boldsymbol{\tau}$ if and only if

$$\mathbf{F}_m = (\mathbf{I}-\lambda\mathbf{K}_m)^{-1}\mathbf{G}_m,$$

with

$$\mathbf{F}_m = \begin{pmatrix} F_{m,1} \\ F_{m,2} \end{pmatrix} = \sum_{i=1}^{m} \left[a_i \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} + b_i \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \right],$$

$$a_i = F_{m,1}(x_i)v(x_i), \quad b_i = F_{m,2}(y_i)v(y_i)$$

and

$$\mathbf{G}_m = \begin{pmatrix} G_{m,1} \\ G_{m,2} \end{pmatrix} = \sum_{i=1}^{m} \left[\eta_i \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} + \theta_i \begin{pmatrix} 0 \\ \psi_i \end{pmatrix} \right],$$

$$\eta_i = G_{m,1}(x_i)v(x_i), \quad \theta_i = G_{m,2}(y_i)v(y_i).$$

Therefore, for all τ, we get

$$\begin{aligned} \|A_m^{-1}\boldsymbol{\tau}\|_{l^\infty} &= \|\mathbf{c}\|_{l^\infty} \leqslant \max\{\|F_{m,1}v\|_\infty, \|F_{m,2}v\|_\infty\} = \|\mathbf{F}_m\|_{C_v \times C_v} \\ &= \|(\mathbf{I} - \lambda \mathbf{K}_m)^{-1}\mathbf{G}_m\|_{C_v \times C_v} \\ &\leqslant \|(\mathbf{I} - \lambda \mathbf{K}_m)^{-1}\|_{C_v \times C_v \to C_v \times C_v} \|\mathbf{G}_m\|_{C_v \times C_v}. \end{aligned} \tag{46}$$

Moreover, by (39), we obtain

$$\begin{aligned} \|G_{m,1}v\|_\infty &\leqslant \left(\max_{|x| \leqslant 1} v(x) \sum_{k=1}^{m} \frac{|l_k(w_1, x)|}{v(x_k)} \right) \|(\eta_1, \dots, \eta_m)\|_{l^\infty} \\ &= \|L_m(w_1)\|_{C_v \to C_v} \|(\eta_1, \dots, \eta_m)\|_{l^\infty} \leqslant \mathscr{C} \|\boldsymbol{\tau}\|_{l^\infty} \log m \end{aligned}$$

and, analogously,

$$\|G_{m,2}v\|_\infty \leqslant \mathscr{C} \|\boldsymbol{\tau}\|_{l^\infty} \log m,$$

then

$$\|\mathbf{G}_m\|_{C_v \times C_v} = \max\{\|G_{m,1}v\|_\infty, \|G_{m,2}v\|_\infty\} \leqslant \mathscr{C} \|\boldsymbol{\tau}\|_{l^\infty} \log m. \tag{47}$$

Replacing (47) into (46) and taking into account (41) and Proposition 4.1, it results

$$\begin{aligned} \|A_m^{-1}\|_\infty &\leqslant \mathscr{C} \|(\mathbf{I} - \lambda \mathbf{K}_m)^{-1}\|_{C_v \times C_v \to C_v \times C_v} \log m \\ &\leqslant \mathscr{C} \|(\mathbf{I} - \lambda \mathbf{K})^{-1}\|_{C_v \times C_v \to C_v \times C_v} \log m \\ &\leqslant \mathscr{C} \log m. \end{aligned} \tag{48}$$

Combining (45) with (48), (27) follows. □

Proof of Theorem 3.2. Letting $\mathbf{f}^* = \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix}$, by (1) and (14), for $i = 1, 2$, it trivially follows:

$$\begin{aligned} &\|[f_i^* - \tilde{f}_{m,i}]v\|_\infty \\ &\quad \leqslant |\lambda| \sup_{|x| \leqslant 1} \sum_{j=1}^{2} \left| v(x) \int_{-1}^{1} [k_x^{i,j}(y) f_j^*(y) - L_m(w_j, (k_x^{i,j} f_j^*), y)] w_j(y) \, \mathrm{d}y \right|. \end{aligned} \tag{49}$$

Taking into account that $f_i \in W_s(v)$, being $K^{i,j} f_j \in W_s(v), i, j = 1, 2$, (see the proof of Proposition 3.1) and $g_i \in W_s(v)$, by estimating the error of the Gaussian quadrature rule [8,9,12], one can obtain

$$\begin{aligned}
&\left|\int_{-1}^{1} [k_x^{i,j}(y) f_j^*(y) - L_m(w_j, (k_x^{i,j} f_j^*), y)] w_j(y)\,\mathrm{d}y\right| \\
&\quad \leqslant \frac{\mathscr{C}}{m} E_{2m-2}((k_x^{i,j} f_j^*)')_{\varphi w_j,1} \\
&\quad \leqslant \frac{\mathscr{C}}{m} E_{2m-2}((k_x^{i,j} f_j^*)')_{\varphi v,\infty} \int_{-1}^{1} v^{\alpha_j-\gamma,\beta_j-\delta}(y)\,\mathrm{d}y \\
&\quad \leqslant \frac{\mathscr{C}}{m} E_{2m-2}((k_x^{i,j} f_j^*)')_{\varphi v,\infty} \\
&\quad \leqslant \frac{\mathscr{C}}{m} [E_{2m-2}((k_x^{i,j})' f_j^*)_{\varphi v,\infty} + E_{2m-2}(k_x^{i,j} (f_j^*)')_{\varphi v,\infty}] \\
&\quad =: A + B.
\end{aligned}$$

On the other hand, since, for a Jacobi weight w,

$$E_{2m-2}(fg)_{w,\infty} \leqslant \|fw\|_\infty E_{m-1}(g)_\infty + 2\|g\|_\infty E_{m-1}(f)_{w,\infty},$$

we deduce

$$A \leqslant \frac{\mathscr{C}}{m} [\|(k_x^{i,j})'\varphi\|_\infty E_{m-1}(f_j^*)_{v,\infty} + 2\|f_j^* v\|_\infty E_{m-1}((k_x^{i,j})')_{\varphi,\infty}]$$

and

$$B \leqslant \frac{\mathscr{C}}{m} [\|(f_j^*)'\varphi v\|_\infty E_{m-1}(k_x^{i,j})_\infty + 2\|k_x^{i,j}\|_\infty E_{m-1}((f_j^*)')_{\varphi v,\infty}].$$

Therefore, applying (37), we get

$$\left|\int_{-1}^{1} [k_x^{i,j}(y) f_j^*(y) - L_m(w_j, (k_x^{i,j} f_j^*), y)] w_j(y)\,\mathrm{d}y\right| \leqslant \frac{\mathscr{C}}{m^s} \|f_j^*\|_{W_s(v)} \|k_x^{i,j}\|_{W_s}$$

and, then,

$$\begin{aligned}
\|[f_i^* - \tilde{f}_{m,i}]v\|_\infty &\leqslant \frac{\mathscr{C}}{m^s} \sum_{j=1}^{2} \|f_j^*\|_{W_s(v)} \sup_{|x|\leqslant 1} v(x) \|k_x^{i,j}\|_{W_s} \\
&= \frac{\mathscr{C}}{m^s} \sum_{j=1}^{2} N_s^{i,j} \|f_j^*\|_{W_s(v)} \leqslant \frac{\mathscr{C}}{m^s} \|\mathbf{f}^*\|_{W_s(v)\times W_s(v)}, \quad i = 1, 2,
\end{aligned}$$

from which we deduce (29). □

Proof of Proposition 3.2. Using the same tools of the proof of Proposition 3.1, it is sufficient to prove

$$\|K^{i,j} f_j\|_{W_s^p(u)} \leqslant \mathscr{C} \|f_j u\|_p \tag{50}$$

for $i, j = 1, 2$ and $\mathscr{C} \neq \mathscr{C}(f_j)$. We start by estimating $\|(K^{i,j} f_j)u\|_p$. By applying The Minkowski and Hölder inequalities, we have

$$\begin{aligned}\|(K^{i,j} f_j)u\|_p &\leqslant \int_{-1}^{1} |f_j(y)|w_j(y)\left(\int_{-1}^{1} |k^{i,j}(x,y)u(x)|^p \, dx\right)^{1/p} dy \\ &\leqslant \sup_{|y|\leqslant 1} \|k_y^{i,j} u\|_p \|f_j u\|_p \left(\int_{-1}^{1} [v^{\alpha_j-\rho,\beta_j-\theta}(y)]^q \, dy\right)^{1/q}\end{aligned}$$

and

$$\begin{aligned}\|(K^{i,j} f_j)^{(s)}\varphi^s u\|_p &\leqslant \int_{-1}^{1} |f_j(y)|w_j(y)\left(\int_{-1}^{1} \left|\frac{\partial^s}{\partial x^s} k^{i,j}(x,y)\varphi^s(x)u(x)\right|^p dx\right)^{1/p} dy \\ &\leqslant \sup_{|y|\leqslant 1} \left\|\frac{\partial^s}{\partial x^s} k_y^{i,j} \varphi^s u\right\|_p \|f_j u\|_p \left(\int_{-1}^{1} [v^{\alpha_j-\rho,\beta_j-\theta}(y)]^q \, dy\right)^{1/q}.\end{aligned}$$

Then, under our assumptions, it results

$$\|K^{i,j} f_j\|_{W_s^p(u)} \leqslant \|f_j u\|_p \bar{M}_s^{i,j} \left(\int_{-1}^{1} [v^{\alpha_j-\rho,\beta_j-\theta}(y)]^q \, dy\right)^{1/q} \leqslant \mathscr{C}\|f_j u\|_p, \tag{51}$$

i.e., (50). □

In the sequel we shall use the following result [11].

Lemma 4.2. *For every $f \in W_s^p(v^{\rho,\theta})$, $s \geqslant 1$, the estimate*

$$\|[f - L_m(v^{\alpha,\beta}, f)]v^{\rho,\theta}\|_p \leqslant \frac{\mathscr{C}}{m^s}\|f^{(s)}\varphi^s v^{\rho,\theta}\|_p \tag{52}$$

holds, where $\varphi(x) = \sqrt{1-x^2}$ and $\mathscr{C} \neq \mathscr{C}(m, f)$, if and only if α, β, ρ and θ satisfy

$$\begin{aligned}&\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{p} < \rho < \frac{\alpha}{2} + \frac{5}{4} - \frac{1}{p}, \\ &\frac{\beta}{2} + \frac{1}{4} - \frac{1}{p} < \theta < \frac{\beta}{2} + \frac{5}{4} - \frac{1}{p}.\end{aligned} \tag{53}$$

In order to prove Theorem 3.3 we need the following proposition.

Proposition 4.2. *Under assumption* (30)–(32), *we have*

$$\|\mathbf{K} - \mathbf{K}_m\|_{L_u^p \times L_u^p \to L_u^p \times L_u^p} \leqslant \frac{\mathscr{C}}{m^s},$$

where $\mathscr{C} \neq \mathscr{C}(m)$.

Proof. By proceeding as in the proof of Proposition 4.1, it is sufficient to prove that, under our assumptions, for $i, j = 1, 2$, we have

$$\|[(K^{i,j} - K_m^{i,j})f_j]u\|_p \leqslant \frac{\mathscr{C}}{m^s}\|f_j u\|_p. \tag{54}$$

We first observe that

$$\|[(K^{i,j} - K_m^{i,j})f_j]u\|_p \leqslant \|[K^{i,j} f_j - L_m(w_i, K^{i,j} f_j)]u\|_p + \|L_m(w_i, (K^{i,j} - \bar{K}^{i,j})f_j)u\|_p. \tag{55}$$

Since, as shown in the proof of Proposition 3.2, for any $f \in L^p_u$, it results $K^{i,j} f_j \in W^p_s(u)$, by applying (52) and (51), we get

$$\|[K^{i,j} f_j - L_m(w_i, K^{i,j} f_j)]u\|_p \leqslant \frac{\mathscr{C}}{m^s} \|(K^{i,j} f_j)^{(s)} \varphi^s u\|_p$$
$$\leqslant \frac{\mathscr{C}}{m^s} \bar{M}^{i,j}_s \|f_j u\|_p. \tag{56}$$

For the second addendum of (55), by using the "inverse" Marcinkiewicz-type inequality (see [11, Theorem 2.7]), we can write

$$\|L_m(w_i, (K^{i,j} - \bar{K}^{i,j}) f_j)u\|_p \leqslant \mathscr{C} \left(\sum_{h=1}^{m} \lambda_m(u^p, t_h) |(K^{i,j} - \bar{K}^{i,j}) f_j(t_h)|^p \right)^{1/p},$$

where $t_h, h = 1, \ldots, m$, denotes the zeros of $p_m(w_i)$. Since, by the Hölder inequality and (52), one has

$$|(K^{i,j} - \bar{K}^{i,j}) f_j(x)| = \left| \int_{-1}^{1} [k^{i,j}(x, y) - L_m(w_j, k^{i,j}(x, \cdot), y)] f_j(y) w_j(y)\, \mathrm{d}y \right|$$
$$\leqslant \left\| [k^{i,j}_x - L_m(w_j, k^{i,j}_x)] \frac{w_j}{u} \right\|_q \|f_j u\|_p$$
$$\leqslant \frac{\mathscr{C}}{m^s} \|k^{i,j}_x\|_{W^q_s(\frac{w_j}{u})} \|f_j u\|_p$$
$$\leqslant \frac{\mathscr{C}}{m^s} \bar{N}^{i,j}_s \|f_j u\|_p,$$

using the Marcinkiewicz-type inequality (2.19) in [11], it follows:

$$\|L_m(w_i, (K^{i,j} - \bar{K}^{i,j}) f_j)u\|_p \leqslant \frac{\mathscr{C}}{m^s} \bar{N}^{i,j}_s \left(\sum_{h=1}^{m} \lambda_m(u^p, t_h) \right)^{1/p} \|f_j u\|_p$$
$$\leqslant \frac{\mathscr{C}}{m^s} \bar{N}^{i,j}_s \|f_j u\|_p. \tag{57}$$

Replacing (56) and (57) into (55), we deduce

$$\|[(K^{i,j} - K^{i,j}_m) f_j]u\|_p \leqslant \frac{\mathscr{C}}{m^s} (\bar{M}^{i,j}_s + \bar{N}^{i,j}_s) \|f_j u\|_p,$$

i.e., (54). □

Proof of Theorem 3.3. To obtain estimate (35), taking into account Lemma 4.2 and Proposition 4.2, one can repeat word by word the proof of Theorem 3.1. It remains to prove (34). In the sequel we will denote by $\|\mathbf{d}\|_{l_p(\mathbb{R}^n)} = (\sum_{i=1}^n |d_i|^p)^{1/p}$ the l_p-norm of an array $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n$. Now, let $\mathbf{c} = (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2m}$ be an arbitrary array with $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m$. Then, $\boldsymbol{\tau} = (\eta_1, \ldots, \eta_m, \theta_1, \ldots, \theta_m) \in \mathbb{R}^{2m}$ satisfies $\bar{A}_m \mathbf{c} = \boldsymbol{\tau}$ if and only if $(\mathbf{I} - \lambda \mathbf{K}_m)\mathbf{F}_m = \mathbf{G}_m$ with

$$\mathbf{F}_m = \begin{pmatrix} F_{m,1} \\ F_{m,2} \end{pmatrix} = \sum_{i=1}^{m} \left[a_i \begin{pmatrix} \bar{\varphi}_i \\ 0 \end{pmatrix} + b_i \begin{pmatrix} 0 \\ \bar{\psi}_i \end{pmatrix} \right],$$

$$a_i = F_{m,1}(x_i)\lambda_m(u^p, x_i), \quad b_i = F_{m,2}(y_i)\lambda_m(u^p, y_i),$$

and

$$\mathbf{G}_m = \begin{pmatrix} G_{m,1} \\ G_{m,2} \end{pmatrix} = \sum_{i=1}^{m} \left[\eta_i \begin{pmatrix} \bar{\varphi}_i \\ 0 \end{pmatrix} + \theta_i \begin{pmatrix} 0 \\ \bar{\psi}_i \end{pmatrix} \right],$$

$$\eta_i = G_{m,1}(x_i)\lambda_m(u^p, x_i), \quad \theta_i = G_{m,2}(y_i)\lambda_m(u^p, y_i).$$

Since

$$\|\mathbf{c}\|_{l_p(\mathbb{R}^{2m})} \sim \max\{\|\mathbf{a}\|_{l_p(\mathbb{R}^m)}, \|\mathbf{b}\|_{l_p(\mathbb{R}^m)}\}$$

and, by the Marcinkiewicz inequality (see [11, Theorems 2.6 and 2.7]),

$$\|F_{m,1}u\|_p \sim \|\mathbf{a}\|_{l_p(\mathbb{R}^m)} \quad \text{and} \quad \|F_{m,2}u\|_p \sim \|\mathbf{b}\|_{l_p(\mathbb{R}^m)}$$

hold, one can deduce

$$\|\mathbf{F}_m\|_{L_u^p \times L_u^p} \sim \|\mathbf{c}\|_{l_p(\mathbb{R}^{2m})}.$$

Analogously,

$$\|\mathbf{G}_m\|_{L_u^p \times L_u^p} \sim \|\boldsymbol{\tau}\|_{l_p(\mathbb{R}^{2m})}$$

holds too. Taking into account the previous equivalences, we obtain

$$\begin{aligned} \|\bar{A}_m\|_p &= \sup_{\mathbf{c}\in\mathbb{R}^{2m},\mathbf{c}\neq\mathbf{0}} \frac{\|\bar{A}_m\mathbf{c}\|_{l_p(\mathbb{R}^{2m})}}{\|\mathbf{c}\|_{l_p(\mathbb{R}^{2m})}} \\ &\leqslant \mathscr{C} \sup_{\mathbf{F}_m\in\mathbb{P}_{m-1}\times\mathbb{P}_{m-1},\mathbf{F}_m\neq\mathbf{0}} \frac{\|(\mathbf{I}-\lambda\mathbf{K}_m)\mathbf{F}_m\|_{L_u^p\times L_u^p}}{\|\mathbf{F}_m\|_{L_u^p\times L_u^p}} \\ &\leqslant \mathscr{C}\|\mathbf{I}-\lambda\mathbf{K}_m)\|_{L_u^p\times L_u^p\to L_u^p\times L_u^p}, \end{aligned} \tag{58}$$

with $\mathscr{C} \neq \mathscr{C}(m)$. In the same way, for the inverse matrix we can write

$$\begin{aligned} \|\bar{A}_m^{-1}\|_p &= \sup_{\boldsymbol{\tau}\in\mathbb{R}^{2m},\boldsymbol{\tau}\neq\mathbf{0}} \frac{\|\bar{A}_m^{-1}\boldsymbol{\tau}\|_{l_p(\mathbb{R}^{2m})}}{\|\boldsymbol{\tau}\|_{l_p(\mathbb{R}^{2m})}} \\ &\leqslant \mathscr{C} \sup_{\mathbf{G}_m\in\mathbb{P}_{m-1}\times\mathbb{P}_{m-1},\mathbf{G}_m\neq\mathbf{0}} \frac{\|(\mathbf{I}-\lambda\mathbf{K}_m)^{-1}\mathbf{G}_m\|_{L_u^p\times L_u^p}}{\|\mathbf{G}_m\|_{L_u^p\times L_u^p}} \\ &\leqslant \mathscr{C}\|(\mathbf{I}-\lambda\mathbf{K}_m)^{-1}\|_{L_u^p\times L_u^p\to L_u^p\times L_u^p}, \end{aligned} \tag{59}$$

with $\mathscr{C} \neq \mathscr{C}(m)$. Combining (58) and (59) one obtains

$$\begin{aligned} \mathrm{cond}(\bar{A}_m)_p &\leqslant \mathscr{C}\|\mathbf{I}-\lambda\mathbf{K}_m\|_{L_u^p\times L_u^p\to L_u^p\times L_u^p}\|(\mathbf{I}-\lambda\mathbf{K}_m)^{-1}\|_{L_u^p\times L_u^p\to L_u^p\times L_u^p} \\ &= \mathscr{C}\mathrm{cond}(\mathbf{I}-\lambda\mathbf{K}_m). \end{aligned}$$

Since (see, for instance, [5])

$$|\mathrm{cond}(\mathbf{I}-\lambda\mathbf{K}_m) - \mathrm{cond}(\mathbf{I}-\lambda\mathbf{K})| = \mathcal{O}(\|\mathbf{K}-\mathbf{K}_m\|_{L_u^p\times L_u^p\to L_u^p\times L_u^p}),$$

by Proposition 4.2, (34) follows and the proof is complete. □

Proof of Theorem 3.4. Let us estimate $\|[f_i^* - \tilde{f}_{m,i}]u\|_p$, $i = 1, 2$. We have

$$\begin{aligned}\|[f_i^* - \tilde{f}_{m,i}]u\|_p &= |\lambda| \left(\int_{-1}^{1} |f_i^*(x) - \tilde{f}_{m,i}(x)|^p u^p(x)\,\mathrm{d}x \right)^{1/p} \\ &\leqslant \sum_{j=1}^{2} \left(\int_{-1}^{1} \left| \int_{-1}^{1} |k_x^{i,j}(y) f_j^*(y) - L_m(w_j, (k_x^{i,j} f_j^*), y)| w_j(y)\,\mathrm{d}y \right|^p u^p(x)\,\mathrm{d}x \right)^{1/p} \\ &=: \sum_{j=1}^{2} E_{i,j}.\end{aligned}$$

By proceeding as in the proof of Theorem 3.2, taking into account

$$E_{2m-2}(fg)_{w,1} \leqslant E_{m-1}(f)_{u,p} \left\| g \frac{w}{u} \right\|_q + 2\|fu\|_p E_{m-1}(g)_{w/u,q}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

with w, u Jacobi weights, and (37), we have

$$\begin{aligned}&\int_{-1}^{1} |k_x^{i,j}(y) f_j^*(y) - L_m(w_j, (k_x^{i,j} f_j^*), y)| w_j(y)\,\mathrm{d}y \\ &\leqslant \frac{\mathscr{C}}{m} E_{2m-2}((k_x^{i,j} f_j^*)')_{\varphi w_j,1} \\ &\leqslant \frac{\mathscr{C}}{m} [E_{2m-2}((k_x^{i,j})' f_j^*)_{\varphi w_j,1} + E_{2m-2}(k_x^{i,j} (f_j^*)')_{\varphi w_j,1}] \\ &\leqslant \frac{\mathscr{C}}{m^s} \|f_j^*\|_{W_s^p(u)} \|k_x^{i,j}\|_{W_s^q(w_j/u)},\end{aligned}$$

from which we deduce

$$\begin{aligned}E_{i,j} &\leqslant \frac{\mathscr{C}}{m^s} \|f_j^*\|_{W_s^p(u)} \left(\int_{-1}^{1} \|k_x^{i,j}\|^p_{W_s^q(w_j/u)} u^p(x)\,\mathrm{d}x \right)^{1/p} \\ &\leqslant \frac{\mathscr{C}}{m^s} \bar{N}_s^{i,j} \|f_j^*\|_{W_s^p(u)} \|u\|_p \leqslant \frac{\mathscr{C}}{m^s} \|\mathbf{f}^*\|_{W_s^p(u) \times W_s^p(u)}, \qquad i, j = 1, 2.\end{aligned}$$

Then, (36) trivially follows. $\square$

5. Numerical examples

In this section we show by some examples that our theoretical results are confirmed by the numerical tests. We recall that the convergence order of the proved estimates depends on the smoothness of the kernels and of the known terms.

When we dont know the exact solutions of the systems of integral equations, we will think as exact their approximate solutions obtained for $m = 512$, i.e., we assume $f_i^* = f_{512,i}^*$, $i = 1, 2$, and in all the tables we will report only the digits which are correct according to them.

Example 1. Consider the system

$$\mathbf{f}(x) - \frac{1}{3} \int_{-1}^{1} \mathbf{k}(x, y) \mathbf{w}(y) \mathbf{f}(y)\,\mathrm{d}y = \mathbf{g}(x)$$

with

$$\mathbf{k}(x, y) = \begin{pmatrix} x^2(y+2) & e^{x+y} \\ (x+1)\cos(y) & (y^2+1)\sin(x+1) \end{pmatrix}, \qquad \mathbf{w}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{g}(x) = \begin{pmatrix} \dfrac{e^x(3ex - e^2 + 5) - (e^2 - 1)x^2}{3e} \\ x^2 - \dfrac{16}{45}\sin(x+1) - \left(\dfrac{\cos(1) + e^2\cos(1) - 2\sin(1)}{6e}\right)(x+1) \end{pmatrix}.$$

The exact solution is

$$\mathbf{f}^*(x) = \begin{pmatrix} xe^x \\ x^2 \end{pmatrix}.$$

In Tables 1 and 2 we report the weighted absolute errors

$$e_{m,i}(x) = |(f^*_{m,i}(x) - f^*_i(x))v^{1/4,1/4}(x)|, \quad i = 1, 2$$

for $x = 0.5$ and $x = 0.9$. The weight function $v = v^{\frac{1}{4},\frac{1}{4}}$ is chosen according to (23) (with $\alpha = \beta = 0$) and the system is studied in $C_v \times C_v$. The approximate solution $\mathbf{f}^*_m$ defined in (13) is computed by solving system (12). Since both the kernels and the known functions are analytic, we get the machine precision in double arithmetics with small values of m. The condition number in uniform norm of the matrix of system (12) is less than 29.

Now we consider the system in the space $L^2_u \times L^2_u$, with chosen $u = v^{0,0}$ according to (30). The approximate solutions $f^*_{m,i}$, $i = 1, 2$, defined in (21) is computed by solving the linear system (20), taking into account Remark (1). In Tables 3 and 4, we show the absolute errors

$$\bar{e}_{m,i}(x) = |f^*_{m,i}(x) - f^*_i(x)|, \quad i = 1, 2,$$

in the points $x = 0.5$ and $x = 0.9$. Also in this case the numerical evidence agrees with the theoretical expectations.

Table 5 shows that the condition number cond_2 in the spectral norm of the matrix of system (20) is uniformly bounded with respect to m.

Table 1

m	$e_{m,1}(0.5)$	$e_{m,2}(0.5)$
4	2.9201e − 005	2.7176e − 005
6	4.9461e − 010	1.8161e − 010
8	7.4384e − 015	5.7731e − 015

Table 2

m	$e_{m,1}(0.9)$	$e_{m,2}(0.9)$
4	4.0397e − 005	2.0742e − 005
6	7.5501e − 010	1.3861e − 010
8	1.0436e − 014	4.3298e − 015
10	3.7747e − 015	2.1094e − 015

Table 3

m	$\bar{e}_{m,1}(0.5)$	$\bar{e}_{m,2}(0.5)$
4	3.1378e − 005	2.9202e − 005
6	5.3149e − 010	1.9516e − 010
8	8.4376e − 015	6.5503e − 015

Table 4

m	$\bar{e}_{m,1}(0.9)$	$\bar{e}_{m,2}(0.9)$
4	6.1188e − 005	3.1416e − 005
6	1.1435e − 009	2.0995e − 010
8	1.6431e − 014	7.1054e − 015
10	5.7731e − 015	3.1086e − 015

Table 5

m	cond_2
4	6.127319156004332
6	6.127621168085106
8	6.127621184605690

Table 6

m	$(v^{0.45,0.625} f^*_{m,1})(0.4)$	$(v^{0.45,0.625} f^*_{m,2})(0.4)$
8	0.57647	−0.48976
16	0.576479	−0.48976
32	0.5764792	−0.4897620
64	0.5764792	−0.48976202
128	0.576479219	−0.4897620226
256	0.57647921910	−0.48976202267

Example 2. The exact solution of the system

$$\mathbf{f}(x) - \frac{1}{2}\int_{-1}^{1} \mathbf{k}(x, y)\mathbf{w}(y)\mathbf{f}(y)\,\mathrm{d}y = \mathbf{g}(x)$$

with

$$\mathbf{k}(x, y) = \begin{pmatrix} \cos(x+y) & e^x(x+y)^8 \\ \sin((x-y)^2) & |x-y|^{9/2} \end{pmatrix},$$

$$\mathbf{w}(x) = \begin{pmatrix} (1-x)^{1/4}(1+x)^{1/2} & 0 \\ 0 & (1-x)^{2/5}(1+x)^{3/4} \end{pmatrix}, \quad \mathbf{g}(x) = \begin{pmatrix} e^x \cos(x) \\ |x|^{7/2} \end{pmatrix},$$

is unknown. We consider the system in the space $C_{v^{\gamma,\delta}} \times C_{v^{\gamma,\delta}}$ with $\gamma = 0.45$ and $\delta = 0.625$. In such a space the solution lives and the Lagrange operator is the projector having the smallest norm ($\log m$). From the smoothness of the kernels and the known terms it follows that the approximate solution $\{\mathbf{f}^*_m\}_m$, computed by (12)–(13), converges to the exact solution $\mathbf{f}^*$ with order at least $\log m/m^3$ (see estimate (28)). The numerical results, shown in Tables 6 and 7, confirm the theoretical ones.

In Table 8 cond_∞ denotes the condition number in uniform norm of the matrix of system (12).

In Tables 9 and 10 we show the weighted approximations $v^{\rho,\theta} f^*_{m,i}$ of the weighted solutions $v^{\rho,\theta} f^*_i$, $i = 1, 2$, obtained

Table 7

m	$(v^{0.45,0.625} f^*_{m,1})(0.8)$	$(v^{0.45,0.625} f^*_{m,2})(0.8)$
8	−3.65692	−0.260849
16	−3.65692	−0.260849
32	−3.656925	−0.2608499
64	−3.6569252	−0.26084996
128	−3.656925233	−0.260849964
256	−3.6569252339	−0.26084996421

Table 8

m	cond_∞
8	3.970276193357605e + 002
16	4.517934190048056e + 002
32	4.715268457817439e + 002
64	4.785627717233979e + 002
128	4.812383816476705e + 002
256	4.823439411745753e + 002
512	4.828348268823381e + 002

Table 9

m	$(v^{0.4,0.8} f^*_{m,1})(0.4)$	$(v^{0.4,0.8} f^*_{m,2})(0.4)$
8	0.6272	−0.5329
16	0.627261	−0.532905
32	0.6272613	−0.5329051
64	0.6272613	−0.53290518
128	0.627261309	−0.532905189
256	0.62726130928	−0.5329051896

Table 10

m	$(v^{0.45,0.625} f^*_{m,1})(0.8)$	$(v^{0.45,0.625} f^*_{m,2})(0.8)$
8	−4.39275	−0.31333
16	−4.39275	−0.313337
32	−4.392757	−0.313337
64	−4.3927579	−0.313337208
128	−4.39275791	−0.3133372085
256	−4.3927579141	−0.31333720855

by (20) and (21), with $p=3$, $\rho=0.4$ and $\theta=0.8$ (see condition (30)), and taking into account Remark (2). In this case the rate of convergence in $L^3_{v^{\rho,\theta}} \times L^3_{v^{\rho,\theta}}$ is m^{-3} as confirmed by the numerical results.

In Table 11 cond_2 denotes the condition number in the spectral norm of the matrix of system (20).

Note that the condition numbers in Tables 8 and 11 are essentially bounded.

Table 11

m	cond_2
8	1.361073458684526e + 002
16	1.534316804366401e + 002
32	1.609096274919522e + 002
64	1.645564012423444e + 002
128	1.665306270326582e + 002
256	1.676010025046390e + 002
512	1.681669455956384e + 002

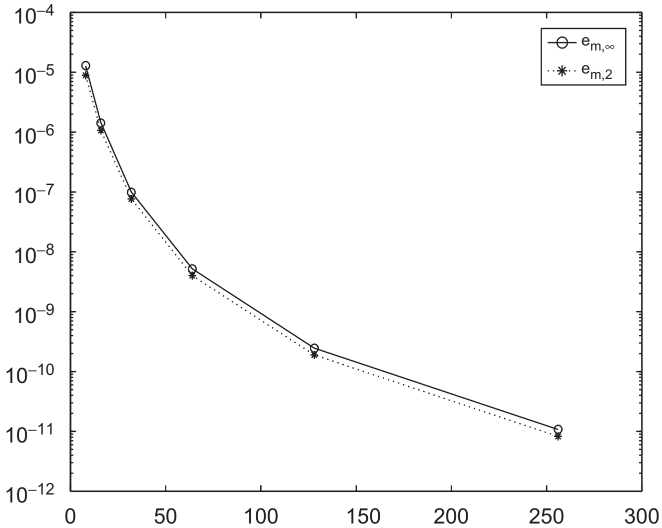

Fig. 1. Graph of the errors.

Finally, in order to emphasize that the numerical results are in accordance with the theoretical ones (see estimates (28) and (35)) we show in Fig. 1 the behavior of the errors $\|\mathbf{f}^*_{512} - \mathbf{f}^*_m\|_{C_{v^{\gamma,\delta}} \times C_{v^{\gamma,\delta}}}$ and $\|\mathbf{f}^*_{512} - \mathbf{f}^*_m\|_{L^2_{v^{\rho,\theta}} \times L^2_{v^{\rho,\theta}}}$. More precisely, we approximate the first norm by the quantity

$$e_{m,\infty} = \max\{e_{m,1,\infty}, e_{m,2,\infty}\},$$

where

$$e_{m,i,\infty} = \max_{k=1,\dots,800} |\mathbf{f}^*_{512}(y_k) - \mathbf{f}^*_m(y_k)| v^{\gamma,\delta}(y_k)$$

and y_k, $k = 1, \dots, 800$, are the equally spaced points in $[-1, 1]$, and we compute exactly

$$e_{m,2} = \|\mathbf{f}^*_{512} - \mathbf{f}^*_m\|_{L^2_{v^{\rho,\theta}} \times L^2_{v^{\rho,\theta}}}$$

by a suitable Gaussian quadrature rule.

Fig. 1 shows that the convergence of the approximations obtained by method (20)–(21) in the space $L^2_{v^{\rho,\theta}} \times L^2_{v^{\rho,\theta}}$ is a little bit faster than the convergence of the approximations computed by method (12)–(13) in the space $C_{v^{\gamma,\delta}} \times C_{v^{\gamma,\delta}}$.

6. Conclusions

In this paper we present a projection method, based on the Lagrange interpolation, for the numerical solution of second kind Fredholm integral equations systems on $[-1, 1]$.

The proposed numerical procedure essentially consists in the following steps:

- choose the weighted space $C_v \times C_v$ with $v = v^{\gamma,\delta}$ satisfying (23) ($L^p_u \times L^p_u$ with $u = v^{\rho,\theta}$ satisfying (30), respectively) in which to study the system;
- project equation $(\mathbf{I} - \lambda\mathbf{K})\mathbf{f} = \mathbf{g}$ by using suitable Lagrange interpolation operators and obtain the finite dimensional problem $(\mathbf{I} - \lambda\mathbf{K}_m)\mathbf{f}_m = \mathbf{g}_m$;
- solve the well-conditioned linear system (12) ((20), respectively) equivalent to $(\mathbf{I} - \lambda\mathbf{K}_m)\mathbf{f}_m = \mathbf{g}_m$ and construct the approximate solution $\mathbf{f}^*_m$ using (13) ((21), respectively);
- the approximate solution $\mathbf{f}^*_m$ converges to the exact solution $\mathbf{f}^*$ with the order of the best approximation as estimate (28) ((35), respectively) shows.

Note that the choice of the weight $v^{\gamma,\delta}$ ($v^{\rho,\theta}$, respectively) is crucial for the convergence of the considered interpolatory processes.

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References

[1] A. Akyüz-Daşcioğlu, Chebyshev polynomial solutions of systems of linear integral equations, Appl. Math. Comput. 151 (1) (2004) 221–232.
[2] K.E. Atkinsons, The Numerical Solution of Integral Equations of the Second Kind, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1997.
[3] M.C. De Bonis, G. Mastroianni, Projection Methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations, SIAM J. Numer. Anal. 44 (4) (2006) 1351–1374.
[4] Z. Ditzian, V. Totik, Moduli of Smoothness, SCMG Springer, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1987.
[5] C. Laurita, G. Mastroianni, Condition numbers in numerical methods for Fredholm integral equations of the second kind, J. Integral Equations Appl. 14 (3) (2002) 311–341.
[6] K. Maleknejad, N. Aghazadeh, M. Rabbani, Numerical solution of second kind Fredholm integral equations system by using a Taylor-series expansion method, Appl. Math. Comput. 175 (2) (2006) 1229–1234.
[7] K. Maleknejad, M. Shahrezaee, H. Khatami, Numerical solution of integral equations system of the second kind by block-pulse functions, Appl. Math. Comput. 166 (1) (2005) 15–24.
[8] G. Mastroianni, Generalized Christoffel function and error of positive quadrature rules, Special functions (Torino, 1993), Numer. Algorithms 10 (1–2) (1995) 113–126.
[9] G. Mastroianni, G.V. Milovanović, Interpolation Processes: Basic Theory and Applications, in progress.
[10] S. Mikhlin, S. Prössdorf, Singular Integral Operators, Springer, Berlin, 1986.
[11] G. Mastroianni, M.G. Russo, Lagrange interpolation in some weighted uniform spaces, Facta Univ. Ser. Math. Inform. 12 (1997) 185–201 (Dedicated to Professor Dragoslav S. Mitrinovic (1908–1995) (Nis, 1996).
[12] G. Mastroianni, P. Vértesi, Error estimates of product quadrature formulae, Numerical Integration, vol. IV, Oberwolfach, 1992, Internat. Ser. Numer. Math., 112 (1993) 241–252.
[13] P. Nevai, Orthogonal polynomials, Mem. Amer. Math. Soc. 213 (1979).
[14] P. Nevai, Mean convergence of Lagrange interpolation III, Trans. Amer. Math. Soc. 282 (1984) 669–698.
[15] S. Prössdorf, B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Birkhäuser, Basel, Switzerland, 1991.
[16] A.F. Timan, Theory of Approximation of Functions of a Real Variable, Dover, New York, 1994.
[17] N.P. Vekua, Systems of Singular Integral Equations, P. Noordhoff (LTD), Groningen, The Netherlands, 1967.