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Convergence of approximate solution of system of Fredholm integral equations

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Abstract

In this paper numerical solution of system of linear Fredholm integral equations by means of the Sinc-collocation method is considered. This approximation reduces the system of integral equations to an explicit system of algebraic equations. The exponential convergence rate of the method $O(e^{-k\sqrt{N}})$ is proved. The method is applied to a few test examples with continuous kernels to illustrate the accuracy and the implementation of the method.

Keywords: System of Fredholm integral equations; Sinc method; Exponential convergence

1. Introduction

We consider the system of linear Fredholm integral equations of the form:

$$\mathbf{F}(x) = \mathbf{G}(x) + \int_{\Gamma} \mathbf{K}(x,t)\mathbf{F}(t)\,dt, \quad x \in \Gamma = [0,1], \tag{1}$$

where

$$\mathbf{F}(x) = \left[f_1(x), f_2(x), \ldots, f_n(x)\right]^T, \qquad \mathbf{G}(x) = \left[g_1(x), g_2(x), \ldots, g_n(x)\right]^T,$$
$$\mathbf{K}(x,t) = \left[K_{ij}(x,t)\right], \quad i,j = 1,2,\ldots,n.$$

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In system (1) the known kernel $\mathbf{K}(x,t)$ is continuous, the function $\mathbf{G}(x)$ is given, and $\mathbf{F}(x)$ is the solution to be determined [1,2]. There have been considerable interest in solving integral equation (1). E. Babolian et al. [3] applied an Adomian decomposition method for solving system of linear Fredholm integral equations of the second kind. Numerical solution of the system of linear Fredholm integral equations (1) in the case that $\Gamma = [0,1)$, has been proposed by Maleknejad et al. in [4,5]. Rationalized Haar wavelet has been used for direct numerical solution in [4] and also Block-Pulse functions to propose solutions for the system of Fredholm integral equations have been developed in [5].

Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied physics and engineering [6]. The books [6,7] provide excellent overviews of methods based on Sinc functions for solving ordinary and partial differential equations and integral equations. Sinc methods have also been employed as forward solvers in the solution of inverse problems [8,9]. In [10,11] the Sinc-collocation procedures for the eigenvalue problems are presented. Sinc-Nyström method for numerical solution of one-dimensional Cauchy singular integral equations given on a smooth arc in the complex plane has been developed in [12]. In [13], we used a Sinc-collocation procedure for numerical solution of linear Fredholm integral equations of the second kind. Approximation by Sinc functions are typified by errors of the form $o(\exp(-k/h))$, where $k > 0$ is a constant and h is a step size. Finally, these kinds of approximation yield both an effective and rapidly convergent scheme for solving the problems.

The main purpose of the present paper is to develop methods for numerical solution of the system of Fredholm integral equations (1). Our method consists of reducing the solution of (1) to a set of algebraic equations. The properties of Sinc function are then utilized to evaluate the unknown coefficients.

The paper is organized into five section. Section 2 outlines some of the main properties of Sinc function and Sinc method that are necessary for the formulation of the discrete linear system. In Section 3, we illustrate how the Sinc method may be used to replace system (1) by an explicit system of linear algebraic equations. In Section 4, the convergence analysis of the method has been discussed. It is shown that the Sinc procedure converges to the solution at an exponential rate. Finally we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in Section 5.

2. Sinc function properties and Sinc interpolation

In this section, we will review Sinc function properties, Sinc quadrature rule, and the Sinc method. These are discussed thoroughly in [6,7]. For solving integral equations system (1), these properties will be used later in Section 3.

The Sinc function. The Sinc function is defined on the whole real line, $-\infty < z < \infty$, by

$$\mathrm{Sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & x \neq 0; \\ 1, & x = 0. \end{cases} \tag{2}$$

For any $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(j,h)(z) = \mathrm{Sinc}\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots. \tag{3}$$

The Sinc function for the interpolating points $z_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases} \tag{4}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h)(z) = \sum_{j=-\infty}^{\infty} f(jh)\,\mathrm{Sinc}\left(\frac{z - jh}{h}\right) \tag{5}$$

is called the Whittaker cardinal expansion of f whenever this series converges. They are based in the infinite strip D_d in the complex plane

$$D_d = \left\{ w = u + iv\colon\ |v| < d \leqslant \frac{\pi}{2} \right\}. \tag{6}$$

To construct approximation on the interval $\Gamma = [0, 1]$, we consider the conformal maps

$$\phi(z) = \ln\left(\frac{z}{1-z}\right). \tag{7}$$

The map ϕ carries the eye-shaped region

$$D = \left\{ z = x + iy\colon\ \left|\arg\left(\frac{z}{1-z}\right)\right| < d \leqslant \frac{\pi}{2} \right\}. \tag{8}$$

For the Sinc method, the basis functions on the interval $\Gamma = [0, 1]$ for $z \in D$ are derived from the composite translated Sinc functions,

$$S_j(z) = S(j,h) \circ \phi(z) = \mathrm{Sinc}\left(\frac{\phi(z) - jh}{h}\right). \tag{9}$$

The function

$$z = \phi^{-1}(w) = \frac{e^w}{1 + e^w} \tag{10}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{ \psi(u) = \phi^{-1}(u) \in D\colon\ -\infty < u < \infty \right\}. \tag{11}$$

The Sinc grid points $z_k \in \Gamma$ in D will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \quad k = \pm 1, \pm 2, \ldots. \tag{12}$$

Sinc interpolation. For further explanation of the procedure, the important class of functions is denoted by $L_\alpha(D)$. The properties of functions in $L_\alpha(D)$ and detailed discussions are given in [6]. We recall the following definition and theorem for our purpose.

Definition 1. Let $L_\alpha(D)$ be the set of all analytic functions u in D, for which there exists a constant C such that

$$\left|u(z)\right| \leqslant C \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D,\ 0 < \alpha \leqslant 1, \tag{13}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 1. *Let* $\frac{u}{\phi'} \in L_\alpha(D)$*, let* N *be a positive integer, and let* h *be selected by the formula*

$$h = \left(\frac{2\pi d}{\alpha N}\right)^{\frac{1}{2}}, \tag{14}$$

then there exists positive constant C_1*, independent of* N*, such that*

$$\left| \int_\Gamma u(z)\,dz - h \sum_{k=-N}^{N} \frac{u(z_k)}{\phi'(z_k)} \right| \leqslant C_1 e^{(-2\pi d\alpha N)^{\frac{1}{2}}}. \tag{15}$$

3. The approximate solution of system of Fredholm integral equations

Let us consider the system of linear integral equations (1). For convenience, we consider the ith equation of (1):

$$f_i(x) = g_i(x) + \sum_{j=1}^{n} \int_\Gamma K_{ij}(x,t) f_j(t)\,dt, \quad i = 1, 2, \ldots, n. \tag{16}$$

For the second term on the right-hand side of (16), we suppose that $\frac{K_{ij}(x,.)}{\phi'} \in L_\alpha(D)$, then by using Theorem 1, we obtain

$$\int_\Gamma K_{ij}(x,t) f_j(t)\,dt \approx h \sum_{l=-N}^{N} \frac{K_{ij}(x,t_l)}{\phi'(t_l)} f_{jl} \tag{17}$$

where f_{jl} denotes an approximate value of $f_j(x_l)$, and

$$\phi(x) = \ln\left(\frac{x}{1-x}\right), \quad \phi(0) = -\infty,\ \phi(1) = +\infty,$$
$$\phi'(x) = \frac{1}{x(1-x)}.$$

Having replaced the second term on the right-hand side of (16) with Eq. (17), we have

$$f_i(x) - h \sum_{j=1}^{n} \left[\sum_{l=-N}^{N} \frac{K_{i,j}(x,t_l)}{\phi'(t_l)} f_{jl} \right] \approx g_i(x), \quad i = 1, 2, \ldots, n. \tag{18}$$

There are $n \times (2N+1)$ unknowns f_{jl}, $l = -N, -N+1, \ldots, N-1, N$, $j = 1, \ldots, n$, to be determined in (18). In order to determine these $n \times (2N+1)$ unknowns, we apply the collocation method and as the collocation points, thus by setting $x = x_k$, $k = -N, \ldots, N$, in (18) that x_k are Sinc grid points

$$x_k = \psi(kh) = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}},$$

and applying the collocation to it, we obtain the following system of $n \times (2N+1)$ linear equations with $n \times (2N+1)$ unknowns f_{jl}, $l = -N, -N+1, \ldots, N-1, N$, $j = 1, \ldots, n$:

$$f_{ik} - h \sum_{j=1}^{n} \left[\sum_{l=-N}^{N} \frac{K_{i,j}(x_k,t_l)}{\phi'(t_l)} f_{jl} \right] = g_i(x_k), \quad i = 1, 2, \ldots, n,\ k = -N, \ldots, N. \tag{19}$$

We denote $\widetilde{K}_{ij}=[\frac{K_{ij}(x_k,t_l)}{\phi'(t_l)}]$ and

$$A_{ij}=\begin{cases} I-h\widetilde{K}_{ij}, & i=j; \\ -h\widetilde{K}_{ij}, & i\neq j, \end{cases}$$

which are the square matrices of order $(2N+1)\times(2N+1)$, then the system of linear equations (19) for $n\times(2N+1)$ unknown coefficients f_{jl}, $j=1,\dots,n$, $l=-N,\dots,N$, can be expressed in a matrix form

$$\mathbf{A}\widetilde{\mathbf{F}}=\mathbf{P} \tag{20}$$

where

$$\mathbf{A}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{2n} & \cdots & A_{nn} \end{pmatrix},$$

$$\mathbf{P}=\big[g_1(x_{-N}),\dots,g_1(x_N),\dots,g_n(x_{-N}),\dots,g_n(x_N)\big]^T,$$
$$\widetilde{\mathbf{F}}=[f_{1l},\dots,f_{nl}]^T,\quad l=-N,\dots,N.$$

By solving the linear system (20), we obtain an approximate solution f_{jl}, $j=1,2,\dots,n$, $l=-N,-N+1,\dots,N$, with correspond to the exact solution $f_j(x_l)$, $j=1,2,\dots,n$, $l=-N,-N+1,\dots,N$, of the system of integral equations (1) at the Sinc points. Having used the approximate solution f_{jl}, $j=1,2,\dots,n$, $l=-N,-N+1,\dots,N$, in the system (20), we employ a method similar to the Nyström's idea for the system of linear Fredholm integral equations [6], i.e., we use

$$\mathbf{F}_N(x)=h\sum_{l=-N}^{N}\frac{\mathbf{K}(x,t_l)}{\phi'(t_l)}\mathbf{F}_l+\mathbf{G}(x), \tag{21}$$

where

$$\mathbf{F}_l=[f_{1l},f_{2l},\dots,f_{nl}]^T,$$
$$\mathbf{G}(x)=\big[g_1(x),g_2(x),\dots,g_n(x)\big]^T,$$
$$\mathbf{K}(x,t_l)=\big[K_{ij}(x,t_l)\big],\quad i,j=1,2,\dots,n.$$

4. Convergence analysis

Now we discuss the convergence of the Sinc method for the system of linear Fredholm integral equations (1). For each N, f_{jl} is the solution of the system given in (20) and consequently $\mathbf{F}_N(x)$ is approximate solution. In order to derive a bound for $|\mathbf{F}(x)-\mathbf{F}_N(x)|$ we need to estimate the bound of the vector $A\mathbf{F}-\mathbf{P}$ where $\mathbf{F}$ is a vector defined by

$$\mathbf{F}=\big(f_1(x_{-N}),\dots,f_1(x_N),\dots,f_n(x_{-N}),\dots,f_n(x_N)\big)^T.$$

Let $f_i(x_k)$ be the value of the exact solution of integral equation at the Sinc points x_k.

In the following discussion all norms ($\|.\|$) are norm two, otherwise stated. For this purpose first we will need the following lemma.

Lemma 1. *Let* $\mathbf{F}(x)$ *be the exact solution of the given integral equation* (1) *and let* $h = (\frac{2\pi d}{\alpha N})^{\frac{1}{2}}$, $\frac{K_{ij}(x,.)}{\phi'} \in L_\alpha(D)$ *for all* $x \in \Gamma$, *then there exists a constant* c_2 *independent of* N *such that*

$$\|A\mathbf{F} - \mathbf{P}\| \leqslant c_2 N^{\frac{1}{2}} \exp\bigl\{-(2\pi d\alpha N)^{\frac{1}{2}}\bigr\}. \tag{22}$$

Proof. We derive a bound of the kth component v_k of the vector $v = A\mathbf{F} - \mathbf{P}$. Using Theorem 1 with the optimal mesh size $h = (\frac{2\pi d}{\alpha N})^{\frac{1}{2}}$ and by the assumption on the kernel, we have the following bound on v_k:

$$\begin{aligned} |v_k| &= \bigl|(A\mathbf{F} - \mathbf{P})_k\bigr| \\ &= \left|\mathbf{F}(x_k) - h\sum_{l=-N}^{N} \frac{\mathbf{K}(x_k, t_l)}{\phi'(t_l)}\mathbf{F}(t_l) - \mathbf{G}(x_k)\right| \\ &= \left|f_i(x_k) - h\sum_{j=1}^{n}\sum_{l=-N}^{N} \frac{K_{i,j}(x_k, t_l)}{\phi'(t_l)} f_j(t_l) - g_i(x_k)\right|, \quad i = 1, 2, \ldots, n \\ &\leqslant c_1 \exp\bigl\{-(2\pi d\alpha N)^{\frac{1}{2}}\bigr\}. \end{aligned}$$

Therefore, we have

$$\|A\mathbf{F} - \mathbf{P}\| = \left(\sum_{k=-N}^{N} |v_k|^2\right)^{\frac{1}{2}} \leqslant c_2 N^{\frac{1}{2}} \exp\bigl\{-(2\pi d\alpha N)^{\frac{1}{2}}\bigr\}. \qquad \square$$

We can also obtain a bound on the errors $\mathbf{F}(x) - \mathbf{F}_N(x)$ in the maximum norm, where $\mathbf{F}(x)$ is the exact solution and $\mathbf{F}_N(x)$ is the approximate solution (21). From Lemma 1 we can show that the Sinc method converges at rate of $O(e^{-k\sqrt{N}})$, where $k > 0$.

Theorem 2. *Let us consider all assumptions of Lemma* 1 *and let* $\mathbf{F}_N(x)$ *be the approximate solution of integral equation* (1) *given by* (21)*, then there exists a constant* c_3 *independent of* N *such that*

$$\sup_{x\in(a,b)} \bigl|\mathbf{F}(x) - \mathbf{F}_N(x)\bigr| \leqslant c_3 \mu N^{\frac{1}{2}} \exp\bigl\{-(2\pi d\alpha N)^{\frac{1}{2}}\bigr\} \tag{23}$$

where $\mu = \|A^{-1}\|$.

Proof. By considering the given system of linear Fredholm integral equations (1) and all assumptions we obtain

$$\begin{aligned} \bigl|\mathbf{F}(x) - \mathbf{F}_N(x)\bigr| &= \left|\int_\Gamma \mathbf{K}(x,t)\mathbf{F}(t)\,dt - h\sum_{l=-N}^{N} \frac{\mathbf{K}(x, t_l)}{\phi'(t_l)}\mathbf{F}_l\right| \\ &\leqslant h\sum_{l=-N}^{N} \left|\frac{\mathbf{K}(x, t_l)}{\phi'(t_l)}\right| \bigl|\mathbf{F}(t_l) - \mathbf{F}_l\bigr| = E_N \quad (Say). \end{aligned}$$

Note that

$$h\left(\sum_{l=-N}^{N}\left|\frac{\mathbf{K}(x,t_l)}{\phi'(t_l)}\right|^2\right)^{\frac{1}{2}} \leqslant M$$

holds for $x \in [0,1]$, then using the Schwarz inequality, we get

$$\begin{aligned} E_N &\leqslant h\left(\sum_{l=-N}^{N}\left|\frac{\mathbf{K}(x,t_l)}{\phi'(t_l)}\right|^2\right)^{\frac{1}{2}}\left(\left|\mathbf{F}(t_l)-\mathbf{F}_l\right|^2\right)^{\frac{1}{2}} \\ &\leqslant M\|\mathbf{F}-\widetilde{\mathbf{F}}\|. \end{aligned}$$

Since from (20)

$$\widetilde{\mathbf{F}} = \mathbf{A}^{-1}\mathbf{P},$$

then

$$\|\mathbf{F}-\widetilde{\mathbf{F}}\| = \left\|\mathbf{F}-\mathbf{A}^{-1}\mathbf{P}\right\| \leqslant \left\|\mathbf{A}^{-1}\right\|\|\mathbf{AF}-\mathbf{P}\| \tag{24}$$

holds, we have from Lemma 1

$$\begin{aligned} E_N &\leqslant M\left\|\mathbf{A}^{-1}\right\|\|\mathbf{AF}-\mathbf{P}\| \\ &= M\mu c_2 N^{\frac{1}{2}}\exp\left\{-(2\pi d\alpha N)^{\frac{1}{2}}\right\}. \end{aligned} \tag{25}$$

Therefore from the above relation we conclude that

$$\sup_{x\in(a,b)}\left|\mathbf{F}(x)-\mathbf{F}_N(x)\right| \leqslant c_3\mu N^{\frac{1}{2}}\exp\left\{-(2\pi d\alpha N)^{\frac{1}{2}}\right\}. \qquad \square$$

5. Numerical examples

In order to illustrate the performance of the Sinc method in solving system of linear Fredholm integral equations and justify the accuracy and efficiency of the method, we consider the following examples. The examples have been solved by presented method with different values of N and α, $0 < \alpha \leqslant 1$. In all examples we take $\alpha = 1$ and $d = \frac{\pi}{2}$, which yields $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. The errors are reported on the set of Sinc grid points

$$\begin{aligned} &S = \{x_{-N}, \ldots, x_0, \ldots, x_N\}, \\ &x_k = \frac{e^{kh}}{1+e^{kh}}, \quad k = -N, \ldots, N. \end{aligned} \tag{26}$$

The maximum error on the Sinc grid points is

$$\left\|E_F^S(h)\right\|_\infty = \max_{-N\leqslant j\leqslant N}\left|\mathbf{F}(x_j)-\mathbf{F}_N(x_j)\right|. \tag{27}$$

The numerical results are tabulated in Tables 1–3. We determine $\mu = \|A^{-1}\|$ that appears in the coefficient of the error term in (23) for several values of N numerically. We pursue the change of μ and judge whether the numerical solution of (20) is reliable or not, and also show the existence of A^{-1} practically.

Table 1
Results for Example 1

N	h	$\Vert E^S_{f_1}(h)\Vert_\infty$	$\Vert E^S_{f_2}(h)\Vert_\infty$	$\Vert A^{-1}\Vert_2$
5	1.404963	1.45910×10^{-3}	1.57661×10^{-3}	6.88103
10	0.993459	8.67435×10^{-5}	9.78188×10^{-5}	8.73897
20	0.702481	1.65401×10^{-6}	1.87321×10^{-6}	11.39795
30	0.573574	7.55594×10^{-8}	8.55826×10^{-8}	13.43398
40	0.496729	5.50399×10^{-9}	6.23020×10^{-9}	15.14687
50	0.444288	5.42607×10^{-10}	6.14127×10^{-10}	16.65364
60	0.405578	6.64696×10^{-11}	7.52388×10^{-11}	18.01431
70	0.375492	9.60973×10^{-12}	1.08798×10^{-11}	19.26446
80	0.351241	1.58541×10^{-12}	1.79492×10^{-12}	20.42726
90	0.331153	2.91456×10^{-13}	3.29911×10^{-13}	21.51876
100	0.314159	5.86608×10^{-14}	6.63944×10^{-14}	22.55064

Table 2
Results for Example 2

N	h	$\Vert E^S_{f_1}(h)\Vert_\infty$	$\Vert E^S_{f_2}(h)\Vert_\infty$	$\Vert A^{-1}\Vert_2$
5	1.404963	4.83062×10^{-3}	5.77186×10^{-3}	6.27942
10	0.993459	3.06161×10^{-4}	3.56890×10^{-4}	8.03921
20	0.702481	5.80340×10^{-6}	6.74044×10^{-6}	10.58944
30	0.573574	2.64859×10^{-7}	3.07582×10^{-7}	12.56905
40	0.496729	1.92790×10^{-8}	2.23885×10^{-8}	14.24564
50	0.444288	1.90036×10^{-9}	2.20687×10^{-9}	15.72641
60	0.405578	2.32819×10^{-10}	2.70370×10^{-10}	17.06721
70	0.375492	3.36665×10^{-11}	3.90966×10^{-11}	18.30152
80	0.351241	5.55418×10^{-12}	6.45003×10^{-12}	19.45127
90	0.331153	1.02087×10^{-12}	1.18553×10^{-12}	20.53177
100	0.314159	2.05451×10^{-13}	2.38588×10^{-13}	21.55420

Table 3
Results for Example 3

N	h	$\Vert E^S_{f_1}(h)\Vert_\infty$	$\Vert E^S_{f_2}(h)\Vert_\infty$	$\Vert A^{-1}\Vert_2$
5	1.404963	4.42654×10^{-4}	8.36716×10^{-4}	4.72982
10	0.993459	2.30134×10^{-5}	6.17757×10^{-5}	6.55361
20	0.702481	4.21729×10^{-7}	1.21623×10^{-6}	9.15421
30	0.573574	1.92162×10^{-8}	5.55902×10^{-8}	11.15624
40	0.496729	1.39853×10^{-9}	4.04665×10^{-9}	12.84614
50	0.444288	1.37853×10^{-10}	3.98884×10^{-10}	14.33596
60	0.405578	1.68887×10^{-11}	4.88686×10^{-11}	15.68341
70	0.375492	2.44217×10^{-12}	7.06658×10^{-12}	16.92287
80	0.351241	4.02902×10^{-13}	1.16582×10^{-12}	18.07677
90	0.331153	7.40545×10^{-14}	9.51308×10^{-14}	19.68010
100	0.314159	1.49034×10^{-14}	4.31241×10^{-14}	20.18603

Example 1. We first consider the system of following Fredholm integral equations with exact solution $(f_1(x), f_2(x)) = (e^x, e^{-x})$ in [5].

$$\begin{cases} f_1(x) = 2e^x + \dfrac{e^{x+1}-1}{x+1} - \displaystyle\int_0^1 e^{x-t} f_1(t)\,dt - \int_0^1 e^{(x+2)t} f_2(t)\,dt, \\ f_2(x) = e^x + e^{-x} + \dfrac{e^{x+1}-1}{x+1} - \displaystyle\int_0^1 e^{xt} f_1(t)\,dt - \int_0^1 e^{x+t} f_2(t)\,dt. \end{cases} \tag{28}$$

We solved Example 1 for different values of N and the maximum of absolute errors on the Sinc grid S are tabulated in Table 1. This table indicates that as N increases the errors are decreasing more rapidly where excellent results are shown.

Example 2. Consider the system of integral equations

$$\begin{cases} f_1(x) = \dfrac{x}{18} + \dfrac{17}{36} + \displaystyle\int_0^1 \frac{x+t}{3} f_1(t)\,dt + \int_0^1 \frac{x+t}{3} f_2(t)\,dt, \\ f_2(x) = x^2 - \dfrac{19}{12}x + 1 + \displaystyle\int_0^1 xt f_1(t)\,dt + \int_0^1 xt f_2(t)\,dt, \end{cases} \tag{29}$$

with exact solution $(f_1(x), f_2(x)) = (x+1, x^2+1)$ given in [3].

The approximate solution is calculated for different values of N and the optimal Sinc mesh size $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. The maximum absolute errors on the Sinc grid S are tabulated in Table 2. This table indicates that as N increases the errors are decreasing rapidly.

Example 3. We consider the following system of Fredholm integral equations of the second kind

$$\begin{cases} f_1(x) = x + \dfrac{\cos x}{3} + \dfrac{x\sin^2 1}{2} - \displaystyle\int_0^1 t\cos x f_1(t)\,dt - \int_0^1 x\sin t f_2(t)\,dt, \\ f_2(x) = \cos x + \dfrac{e^x - 1}{2x} + (x+1)\sin 1 + \cos 1 - 1 - \displaystyle\int_0^1 e^{xt^2} f_1(t)\,dt - \int_0^1 (x+t) f_2(t)\,dt, \end{cases} \tag{30}$$

with exact solution $(f_1(x), f_2(x)) = (x, \cos x)$ given in [4].

In Table 3, the approximate solution is calculated for different values of N, and the maximum absolute errors on the Sinc grid S are tabulated.

Appendix A

To prove the existence of a solution of the system (20), we recall the following theorems in [14].

Theorem I. (R. Kress [14, Theorem 2.8, p. 16]) *Let $A : X \to X$ be a bounded linear operator mapping a Banach space X into itself with $\|A\| < 1$ and let $I : X \to X$ denote the identity operator. Then $I - A$ has a bounded inverse operator on X which is given by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and which satisfies

$$\left\|(I - A)^{-1}\right\| \leqslant \frac{1}{1 - \|A\|}.$$

Theorem II. (R. Kress [14, Theorem 3.4, p. 29]) *Let X be a normed space, $A : X \to X$ a compact operator, and let $I - A$ be injective. Then the inverse operator $(I - A)^{-1} : X \to X$ exists and is bounded.*

Corollary III. (R. Kress [14, Corollary 10.7, p. 146]) *Let X be a Banach space and let $A_n : X \to X$ be a collectively compact and pointwise convergent sequence with limit operator $A : X \to X$. Then*

$$\left\|(A_n - A)A\right\| \to 0, \quad n \to \infty,$$

and

$$\left\|(A_n - A)A_n\right\| \to 0, \quad n \to \infty.$$

To show that the inverse operator A^{-1} exists and the norm of the A^{-1} is bounded, we consider the system of linear Fredholm integral equations given in Eq. (1):

$$\mathbf{F}(x) = \mathbf{G}(x) + \int_{\Gamma} \mathbf{K}(x, t)\mathbf{F}(t)\, dt, \quad x \in \Gamma = [0, 1],$$

we define the integral operator

$$(\mathcal{A}\mathbf{F})(x) := \int_{\Gamma} \mathbf{K}(x, t)\mathbf{F}(t)\, dt, \quad x \in \Gamma = [0, 1],$$

with continuous kernel $\mathbf{K}$. We suppose that $\frac{\mathbf{K}}{\phi'} \in L_\alpha(D)$, then by using Theorem 1 (in manuscript), we obtain

$$(\mathcal{A}_N\mathbf{F})(x) := h \sum_{l=-N}^{N} \frac{\mathbf{K}(x, t_l)}{\phi'(t_l)}\mathbf{F}_l, \quad x \in \Gamma = [0, 1],$$

where $\mathbf{F}_l$ denotes an approximate value of $\mathbf{F}(t_l)$. The solution of the system of integral equations

$$\mathbf{F} - \mathcal{A}\mathbf{F} = \mathbf{G}$$

is replaced by the solution of

$$\mathbf{F}_N - \mathcal{A}_N\mathbf{F}_N = \mathbf{G}.$$

Let $\mathcal{A}$ be a compact linear operator, and also let $\mathcal{A}_N$ be the collectively compact operator and pointwise convergent $\mathcal{A}_N\mathbf{F} \to \mathcal{A}\mathbf{F}$, $N \to \infty$, for all $\mathbf{F} \in \mathbf{C}(\Gamma)$. Then we will prove the following theorem.

Theorem IV. *We assume that the kernel* $\mathbf{K}(x,t)$ *be continuous for* $x, t \in \Gamma$ *and* $I - \mathcal{A}$ *is invertible. For sufficiently large* N *provided*

$$\left\|(I-\mathcal{A})^{-1}(\mathcal{A}_N-\mathcal{A})\mathcal{A}_N\right\| < 1.$$

Then the approximate inverses $A^{-1} = (I - \mathcal{A}_N)^{-1}$ *exist and are uniformly bounded,*

$$\left\|A^{-1}\right\| = \left\|(I-\mathcal{A}_N)^{-1}\right\| \leqslant \frac{1 + \|(I-\mathcal{A})^{-1}\mathcal{A}_N\|}{1 - \|(I-\mathcal{A})^{-1}(\mathcal{A}_N-\mathcal{A})\mathcal{A}_N\|}. \tag{i}$$

Proof. Following Theorem II the inverse operator $(I-\mathcal{A})^{-1}$ exists and is bounded. We consider the identity

$$(I-\mathcal{A})^{-1} = I + (I-\mathcal{A})^{-1}\mathcal{A},$$

we denote

$$B_N \approx I + (I-\mathcal{A})^{-1}\mathcal{A}_N$$

as an approximate inverse for $A = I - \mathcal{A}_N$. Elementary calculations yield

$$B_N(I-\mathcal{A}_N) = I - S_N, \tag{ii}$$

where

$$S_N \approx (I-\mathcal{A})^{-1}(\mathcal{A}_N-\mathcal{A})\mathcal{A}_N.$$

From Corollary III we know that $\|S_N\| \to 0$, $N \to \infty$. For $\|S_N\| < 1$ the Neumann series Theorem I implies that $(I - S_N)^{-1}$ exists and is bounded by

$$\left\|(I-S_N)^{-1}\right\| \leqslant \frac{1}{1-\|S_N\|}.$$

Equation (ii) implies that $A = I - \mathcal{A}_N$ is injective and therefore, since $\mathcal{A}_N$ is compact, then by Theorem II the inverse $A^{-1} = (I - \mathcal{A}_N)^{-1}$ exists. By using (ii) we obtain

$$A^{-1} = (I-\mathcal{A}_N)^{-1} = (I-S_N)^{-1}B_N,$$

after simplification, we obtain

$$\left\|A^{-1}\right\| = \left\|(I-\mathcal{A}_N)^{-1}\right\| \leqslant \frac{1 + \|(I-\mathcal{A})^{-1}\mathcal{A}_N\|}{1 - \|(I-\mathcal{A})^{-1}(\mathcal{A}_N-\mathcal{A})\mathcal{A}_N\|}. \qquad \square$$

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