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Reduced models for ferromagnetic nanowires

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In this paper, we consider the micromagnetic variational problem for soft ferromagnetic nanowires. We show that, as the diameter of the wire is small, the magnetization inside the wire depends only on the length variable of the wire. The micromagnetic energy of the wire, in this case, is greatly simplified and in order to find the optimal magnetization distribution, one has to solve a 1D local variational problem.

Keywords: micromagnetics; nanowire; Γ-convergence.

1. Introduction

Magnetic structures of reduced dimensions (thin films, nanowires and nanodots) attract a lot of attention because of their applications to magnetic storage and logic devices (see Skomski, 2003; Vaz *et al*., 2008). Continual miniaturization of magnetic devices raises questions that seemed unimportant before and phenomena, that previously seemed negligible, gain importance. Theoretical understanding of magnetic properties of nanostructures with reduced dimensions is of utter importance. The problem is very difficult from an analytical point of view due to its non-local character and the presence of multiple length scales. Relation between the material properties and the geometry of the ferromagnetic nanostructures create a variety of different regimes. Some of these regimes have been investigated in mathematical literature in the context of thin films (see Gioia & James, 1997; DeSimone *et al*., 2002; Kohn & Slastikov, 2005; Kurzke, 2006; Slastikov, 2005), multilayers (see García-Cervera, 2005), nanowires (see Kühn, 2007; Sanchez, 2009) and nanodots (see DeSimone, 1995; Slastikov, 2010).

We focus our attention on soft ferromagnetic nanowires. These nanostructures are widely used in new technological applications related to magnetic memory devices and are therefore of major interest to both physical and mathematical communities. In the last few years, there were several mathematical studies of static and dynamic phenomena in straight nanowires. The optimal profile problem, for straight wires with a circular cross-section, was studied by Kühn (2007) using Fourier transformations. The Landau–Lifshitz–Gilbert equations, for straight wires in the regime when the exchange coefficient and diameter of the wire tend to zero, were studied by Sanchez (2009) using quite involved asymptotic analysis. In many cases, nanowires are fabricated with cross-sections that are different from a disk (rectangular, elliptic, etc.), have a curvature and exhibit some surface roughness. These effects significantly influence the magnetic properties of a nanowire.

In this paper, we study the properties of cylindrical nanowires with an arbitrary cross-section and non-zero curvature. Using the micromagnetic variational principle, we rigorously derive a 1D reduced micromagnetic model for ferromagnetic nanowires. In this reduced model, the non-local magnetostatic energy term becomes local and plays a role of additional anisotropy. This has been rigorously shown for straight cylindrical wires with a circular cross-section (see Kühn, 2007). In such wires, the additional anisotropy is uniaxial (directed along the wire) and isotropic in the transverse directions. Our results indicate that this, in general, is not true and additional anisotropy: (a) strongly depends on the shape of the cross-section and (b) favours a preferred plane rather than a preferred direction.

The paper is organized as follows. Below we briefly discuss the micromagnetic variational principle. In Section 2, we set up the variational problem for a straight cylindrical wire with a general cross-section. In Section 3, we simplify the magnetostatic energy, which allows us to prove a Γ-convergence result in Section 4. Section 5 is devoted to the specific example of a straight wire with an elliptical cross-section: we explicitly derive the reduced energy and calculate the optimal profile. In Section 6, we prove a Γ-convergence result for a curved cylindrical wire with a general cross-section, based on the previously obtained results.

1.1 *The micromagnetic variational principle*

The micromagnetic variational principle captures the remarkable multiscale complexity of the magnetization behaviour inside of ferromagnets. The local minima of the micromagnetic energy correspond to the stable, and therefore observable, magnetization distributions (see Aharoni, 1996; Hubert & Schäfer, 1998).

The normalized form of the micromagnetic energy is given by

$$\mathcal{E}(m) = w^2 \int_{\Omega} |\nabla m|^2 + Q \int_{\Omega} \phi(m) + \int_{\mathbf{R}^3} |\nabla u|^2 - 2 \int_{\Omega} h_{\text{ext}} \cdot m, \tag{1.1}$$

where the four terms of the energy (1.1) are the exchange, anisotropy, magnetostatic and Zeeman energies, respectively.

The ferromagnet being investigated is defined by the domain $\Omega \subset \mathbf{R}^3$, with the magnetization, $m\colon \Omega \to \mathbf{R}^3$, given such that

$$|m(x)| = \chi_{\Omega}. \tag{1.2}$$

Using Maxwell's equation, we have that u satisfies

$$\operatorname{div}(\nabla u + m\chi_{\Omega}) = 0 \quad \text{in } \mathbf{R}^3, \tag{1.3}$$

in the sense of distributions. It is clear that the magnetostatic energy term is non-local in m. Using integration by parts, we obtain

$$\int_{\mathbf{R}^3} |\nabla u|^2 = -\int_{\Omega} m \cdot \nabla u. \tag{1.4}$$

This form of the magnetostatic energy will be useful for analysis done in Section 3.

We are seeking a Γ-convergence result, and since Γ-convergence is insensitive to compact perturbations of the functional, we can disregard the anisotropy and Zeeman terms to simplify the presentation.

2. Mathematical formulation: straight wire

In this section, we are going to present a mathematically precise formulation of the problem for a straight generalized wire (see Fig. 1). We define the following reference domain:

$$\Omega = \{(x, y, z)\colon x \in [-L, L], (y, z) \in \omega \subset \mathbf{R}^2\},$$

FIG. 1. Straight generalized cylindrical wire.

where ω has C^1 boundary. The ferromagnetic wire is represented by

$$\Omega_h = \{(x, y, z)\colon (x, y/h, z/h) \in \Omega\},$$

where $h \ll 1$ is a small parameter corresponding to the thickness of the wire and L corresponds to the length of the wire. For simplicity of the presentation, we concentrate here on the case when L is finite, however, one can modify the proofs to include $L = \infty$.

We will study the following one parameter family of micromagnetic energy functionals:

$$E_h(m_h) = d^2 \int_{\Omega_h} |\nabla m_h|^2 + \int_{\mathbf{R}^3} |\nabla u_h|^2, \tag{2.1}$$

where $|m_h| = 1$ in Ω_h and u_h satisfies the following equation:

$$-\Delta u_h = \operatorname{div}(m_h \chi_{\Omega_h}) \quad \text{in } \mathbf{R}^3. \tag{2.2}$$

Rescaling the cross-section variables and the energy, we obtain the following problem:

$$E_h(\tilde{m}_h) = d^2 \int_{\Omega} \left(\frac{\partial \tilde{m}_h}{\partial x}\right)^2 + \frac{1}{h^2}|\nabla' \tilde{m}_h|^2 + \frac{1}{h^2}\int_{\mathbf{R}^3} |\nabla u_h|^2, \tag{2.3}$$

where

$$\tilde{m}_h(x, y, z) = m_h(x, hy, hz) \quad \text{for } (x, y, z) \in \Omega.$$

and $\nabla' = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ (note that the magnetostatic energy is written as in (2.1), (2.2)). We will assume that d is a fixed constant that corresponds to material parameters of the wire.

We describe the behaviour of the stable equilibrium magnetization distributions of the energy (2.3) as $h \to 0$. In order to do this, we will show that E_h Γ-converges to the 1D energy E_0, whose minimizers are much easier to study.

THEOREM 2.1 Assume that d is a constant then

- if $E_h(\tilde{m}_h) \leqslant C$, then $\tilde{m}_h \to m$ weakly in $H^1(\Omega; S^2)$ (maybe for a subsequence), $m = m(x)$ depends only one variable x;
- A sequence E_h Γ-converges to the energy E_0 in $H^1_w(\Omega; S^2)$, where

$$E_0(m) = \begin{cases} \displaystyle\int_{-L}^{L} d^2 |m'(x)|^2 + \int_{-L}^{L} (Mm(x), m(x)) & \text{if } m = m(x), \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Here, M is a constant symmetric matrix defined as

$$M = -\frac{1}{2\pi} \int_{\partial\omega} \int_{\partial\omega} n(\mathbf{x}) \otimes n(\mathbf{y}) \ln |\mathbf{x} - \mathbf{y}|, \tag{2.5}$$

where $n(\mathbf{x}) = (0, n_2, n_3)$ is a normal vector to $\partial\omega$.

3. Calculation of the magnetostatic energy

In this section, we simplify the magnetostatic energy for the straight generalized wire. In order to do this, we follow the arguments of Kohn & Slastikov (2005) used to study thin film behaviour. We first show that one can replace $m_h(x, y, z)$ by its average over the cross-section ω_h.

LEMMA 3.1 Define $\bar{m}_h(x) = \frac{1}{|\omega_h|} \int_{\omega_h} m_h(x, y, z)$ and let $\bar{u}_h$ be a solution of (2.2) with m_h replaced by $\bar{m}_h$. Then the following estimate is true:

$$\frac{1}{h^2} \left| \int_{\mathbf{R}^3} |\nabla u_h|^2 - \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right| \leqslant Ch \left(\frac{1}{h^2} \|\nabla' \tilde{m}_h\|^2_{L^2(\Omega)} + 1 \right). \tag{3.1}$$

Proof of Lemma 3.1. Applying formula (1.4) and using the definitions of u_h and $\bar{u}_h$, we obtain

$$\int_{\mathbf{R}^3} |\nabla u_h - \nabla \bar{u}_h|^2 \leqslant \int_{\Omega_h} |m_h - \bar{m}_h|^2.$$

Poincaré's inequality applied with respect to the (y, z) variables yields

$$\int_{\Omega_h} |m_h - \bar{m}_h|^2 = h^2 \int_{\Omega} |\widetilde{m}_h - \bar{m}_h|^2 \leqslant Ch^2 \int_{\Omega} |\nabla' \widetilde{m}_h|^2.$$

The last two inequalities, together with the triangle inequality, imply

$$\left| \left(\int_{\mathbf{R}^3} |\nabla u_h|^2 \right)^{\frac{1}{2}} - \left(\int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right)^{\frac{1}{2}} \right| \leqslant Ch \|\nabla' \widetilde{m}_h\|_{L^2(\Omega)}. \tag{3.2}$$

Using (1.4), it is clear that

$$\left(\int_{\mathbf{R}^3} |\nabla u_h|^2 \right)^{\frac{1}{2}} \leqslant \|m_h\|_{L^2(\Omega_h)} \quad \text{and} \quad \left(\int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right)^{\frac{1}{2}} \leqslant \|\bar{m}_h\|_{L^2(\Omega_h)}. \tag{3.3}$$

Finally, since $|\bar{m}_h| \leqslant |m_h| = 1$, we can combine (3.2) and (3.3) to obtain

$$\left| \int_{\mathbf{R}^3} |\nabla u_h|^2 - \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 \right| \leqslant Ch^2 \|\nabla' \widetilde{m}_h\|_{L^2(\Omega)} \leqslant Ch^3 \left(\frac{1}{h^2} \|\nabla' \tilde{m}_h\|^2_{L^2(\Omega)} + 1 \right).$$

The lemma is proved.

Using the above lemma, we may focus on estimating $\int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2$. We know that

$$\int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 = -\int_{\Omega_h} \nabla \bar{u}_h \cdot \bar{m}_h = \int_{\Omega_h} \bar{u}_h \operatorname{div} \bar{m}_h - \int_{\partial\Omega_h} \bar{u}_h (\bar{m}_h \cdot n). \tag{3.4}$$

Explicitly solving (1.3) for u_h, we obtain

$$4\pi \bar{u}_h(\mathbf{x}) = \int_{\Omega_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} \operatorname{div} \bar{m}_h(\mathbf{y}) - \int_{\partial\Omega_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} (\bar{m}_h \cdot n)(\mathbf{y}). \tag{3.5}$$

Plugging the expression (3.5) for $\bar{u}_h$ into formula (3.4) and recalling that $\bar{m}_h = (\bar{m}_{1,h}, \bar{m}_{2,h}, \bar{m}_{3,h})$ depends only on one variable x, we have

$$\begin{aligned}
4\pi \int_{\mathbf{R}^3} |\nabla \bar{u}_h|^2 = &\int_{\Omega_h}\int_{\Omega_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} \bar{m}'_{1,h}(y)\bar{m}'_{1,h}(x) \\
&-2\int_{\partial\Omega_h}\int_{\Omega_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} \bar{m}'_{1,h}(y)(\bar{m}_h \cdot n)(\mathbf{x}) \\
&+\int_{\partial\Omega_h}\int_{\partial\Omega_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} (\bar{m}_h \cdot n)(\mathbf{y})(\bar{m}_h \cdot n)(\mathbf{x}).
\end{aligned} \tag{3.6}$$

We will refer to the three terms of (3.6) as the 'bulk–bulk term', 'bulk–boundary term' and 'boundary–boundary term', respectively. Using the fact that $\bar{m}_h$ is dependent only on the length of the wire, we will expand these terms and estimate them. At this point, we will change variables, for added clarity, so that we have

$$\mathbf{x}, \mathbf{y} \in \omega_h \text{ are cross-section variables}, \quad s, t \in [-L, L] \text{ are length variables}.$$

With these new variables, we redefine the three terms of (3.6).

Bulk–bulk term

$$A_1 = \int_{-L}^{L}\int_{-L}^{L}\int_{\omega_h}\int_{\omega_h} \frac{\bar{m}'_{1,h}(s)\bar{m}'_{1,h}(t)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}}. \tag{3.7}$$

Bulk–boundary term

$$\begin{aligned}
A_2 = &-2\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\omega_h} \frac{\bar{m}'_{1,h}(s)(\bar{m}_h \cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}} - 2\int_{-L}^{L}\int_{\omega_h}\int_{\omega_h} \frac{\bar{m}'_{1,h}(s)\bar{m}_{1,h}(L)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-L)^2}} \\
&+2\int_{-L}^{L}\int_{\omega_h}\int_{\omega_h} \frac{\bar{m}'_{1,h}(s)\bar{m}_{1,h}(-L)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s+L)^2}}.
\end{aligned} \tag{3.8}$$

Boundary–boundary term

$$\begin{aligned}
A_3 = &\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})(\bar{m}_h \cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}} + \int_{\omega_h}\int_{\omega_h} \frac{|\bar{m}_{1,h}(-L)|^2 + |\bar{m}_{1,h}(L)|^2}{|\mathbf{x}-\mathbf{y}|} \\
&-2\int_{\omega_h}\int_{\omega_h} \frac{\bar{m}_{1,h}(-L)\bar{m}_{1,h}(L)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + 4L^2}} - 2\int_{-L}^{L}\int_{\partial\omega_h}\int_{\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})\bar{m}_{1,h}(-L)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s+L)^2}} \\
&+2\int_{-L}^{L}\int_{\partial\omega_h}\int_{\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})\bar{m}_{1,h}(L)}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-L)^2}}.
\end{aligned} \tag{3.9}$$

Next, we seek to estimate $|A_1|$, $|A_2|$ and $|A_3|$. We first state the following simple lemma (for proof see Kohn & Slastikov 2005).

LEMMA 3.2 (Generalized Young's inequality). Assume $\Omega \subset \mathbf{R}^n$ is a bounded set, $f, g \in L^2(\Omega)$ and $K \in L^1_{loc}(\mathbf{R}^n)$. Then

$$\int_\Omega \int_\Omega f(x)g(y)K(x-y) \leqslant \|K\|_{L^1(B)}\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}, \tag{3.10}$$

for some ball $B \subset \mathbf{R}^n$, depending only on Ω.

Using lemma 3.2, it is not difficult to obtain the following estimate on $|A_1|$:

$$|A_1| \leqslant Ch^4|\ln h|\|\bar{m}'_{1,h}\|^2. \tag{3.11}$$

It is straightforward to estimate the first term in A_2 by $Ch^3|\ln h|(\|\bar{m}'_{1,h}\|^2 + 1)$ and the last two terms by $Ch^4|\ln h|$. Therefore, we obtain

$$|A_2| \leqslant Ch^3|\ln h|(\|\bar{m}'_{1,h}\|^2 + 1). \tag{3.12}$$

The calculation of the first term in A_3 will be provided later but we see that this is the dominating term here since the rest of the terms in A_3 can be estimated by Ch^3, Ch^4 and $Ch^3|\ln h|$, respectively.

Therefore, the magnetostatic energy can be rewritten as

$$\begin{aligned} 4\pi \int_{\mathbf{R}^3} |\nabla u_h|^2 &= \int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})(\bar{m}_h \cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}} \\ &\quad + \mathrm{O}(h^3)\left(\frac{1}{h^2}\|\nabla'\tilde{m}_h\|^2_{L^2(\Omega)} + 1\right) + \mathrm{O}(h^3|\ln h|)\left(\|\bar{m}'_{1,h}\|^2_{L^2(\Omega)} + 1\right). \end{aligned} \tag{3.13}$$

4. Reduced energy: Γ-convergence result

In this section, we are going to derive the reduced energy for a straight ferromagnetic nanowire using Γ-convergence techniques. From Section 3, we see that in order to do this, it is necessary to understand the asymptotic behaviour, as $h \to 0$, of the following term:

$$\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})(\bar{m}_h \cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}}.$$

Before proving the actual Γ-convergence result, we will prove the following lemma, which identifies its limiting behaviour.

LEMMA 4.1 Assume $\bar{m}_h \to m$ weakly in $H^1(-L, L)$, then we have

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h^2} \int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h} \frac{(\bar{m}_h \cdot n)(s,\mathbf{x})(\bar{m}_h \cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + (s-t)^2}} &= \\ -2\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} (m \cdot n)(t,\mathbf{x})(m \cdot n)(t,\mathbf{y}) \ln|\mathbf{x}-\mathbf{y}|. \end{aligned} \tag{4.1}$$

Proof of Lemma 4.1. Since $n(\mathbf{x})$ depends only on the cross-section variable $\mathbf{x}$ and $\bar{m}_h(s)$ depends only on the length variable s, it is clear that in this case (not relabelling n)

$$\frac{1}{h^2}\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2+(s-t)^2}} = \int_{\partial\omega}\int_{\partial\omega}\int_{-L}^{L}\int_{-L}^{L}\frac{(\bar{m}_h(s)\cdot n(\mathbf{x}))(\bar{m}_h(t)\cdot n(\mathbf{y}))}{\sqrt{h^2|\mathbf{x}-\mathbf{y}|^2+(s-t)^2}}. \tag{4.2}$$

Therefore, we essentially have to understand how to evaluate the following expression:

$$G_h = \int_{-L}^{L}\int_{-L}^{L}\frac{f_h(s,x)f_h(t,y)}{\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}},$$

where $f_h(t,\mathbf{y}) = (\bar{m}_h(t)\cdot n(\mathbf{y}))$ and $f_h(s,\mathbf{x}) = (\bar{m}_h(s)\cdot n(\mathbf{x}))$. Since we integrate over the s and t variables, we may treat $\mathbf{x}$ and $\mathbf{y}$ as parameters here. Let's rewrite G_h as

$$\int_{-L}^{L} f_h(t,\mathbf{y})\int_{-L}^{t}\frac{f_h(s,\mathbf{x})\mathrm{d}s\,\mathrm{d}t}{\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}} + \int_{-L}^{L} f_h(t,\mathbf{y})\int_{t}^{L}\frac{f_h(s,\mathbf{x})\mathrm{d}s\,\mathrm{d}t}{\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}}.$$

Integration by parts yields

$$\begin{aligned}\int_{-L}^{t}\frac{f_h(s,\mathbf{x})\mathrm{d}s}{\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}} &= \int_{-L}^{t} f_h'(s,\mathbf{x})\ln\left(t-s+\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\mathrm{d}s\\ &\quad + f_h(-L,\mathbf{x})\ln\left(t+L+\sqrt{(t+L)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\\ &\quad - f_h(t,\mathbf{x})\ln(h|\mathbf{x}-\mathbf{y}|),\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_{t}^{L}\frac{f_h(s,\mathbf{x})\mathrm{d}s}{\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}} &= -\int_{t}^{L} f_h'(s,\mathbf{x})\ln\left(s-t+\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\mathrm{d}s\\ &\quad + f_h(L,\mathbf{x})\ln\left(L-t+\sqrt{(L-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\\ &\quad - f_h(t,\mathbf{x})\ln(h|\mathbf{x}-\mathbf{y}|).\end{aligned} \tag{4.4}$$

Therefore, we can explicitly obtain

$$G_h = -2\int_{-L}^{L} f_h(t,\mathbf{x})f_h(t,\mathbf{y})\ln(h|\mathbf{x}-\mathbf{y}|) \tag{4.5}$$

$$\begin{aligned}&+f_h(-L,\mathbf{x})\int_{-L}^{L} f_h(t,\mathbf{y})\ln\left(t+L+\sqrt{(t+L)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\\ &+f_h(L,\mathbf{x})\int_{-L}^{L} f_h(t,\mathbf{y})\ln\left(L-t+\sqrt{(L-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\\ &+\int_{-L}^{L} f_h(t,\mathbf{y})\int_{-L}^{t} f_h'(s,\mathbf{x})\ln\left(t-s+\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\\ &-\int_{-L}^{L} f_h(t,\mathbf{y})\int_{t}^{L} f_h'(s,\mathbf{x})\ln\left(s-t+\sqrt{(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right).\end{aligned} \tag{4.6}$$

It is clear that as $h \to 0$, $f_h \to f$ ($f(t,\mathbf{x}) = m(t)\cdot n(\mathbf{x})$) and we can pass to the limit to get

$$\begin{aligned} G_h + 2\ln h \int_{-L}^{L} f_h(t,\mathbf{x}) f_h(t,\mathbf{y}) \to & -2\int_{-L}^{L} f(t,\mathbf{x}) f(t,\mathbf{y}) \ln|\mathbf{x}-\mathbf{y}| \\ & + f(-L,\mathbf{x}) \int_{-L}^{L} f(t,\mathbf{y}) \ln(2(t+L)) \\ & + f(L,\mathbf{x}) \int_{-L}^{L} f(t,\mathbf{y}) \ln(2(L-t)) \\ & + \int_{-L}^{L} f(t,\mathbf{y}) \int_{-L}^{t} f'(s,\mathbf{x}) \ln(2(t-s)) \\ & - \int_{-L}^{L} f(t,\mathbf{y}) \int_{t}^{L} f'(s,\mathbf{x}) \ln(2(s-t)) \end{aligned} \tag{4.7}$$

uniformly in $\mathbf{x}, \mathbf{y}$. Recalling the definition of $f_h(t,\mathbf{x})$ and $f(t,\mathbf{x})$, we can deduce, using Stokes' theorem, that

$$\int_{\partial\omega} f_h(t,\mathbf{x}) = \int_{\partial\omega} (\bar{m}_h(t)\cdot n(\mathbf{x})) = 0 \quad \text{and} \quad \int_{\partial\omega} f(t,\mathbf{x}) = \int_{\partial\omega} (m(t)\cdot n(\mathbf{x})) = 0. \tag{4.8}$$

Integrating (4.7) over $\partial\omega$ with respect to $\mathbf{x}$ and $\mathbf{y}$, having in mind (4.8) and recalling the definition of G_h, we obtain

$$\begin{aligned} &\lim_{h\to 0} \int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} \frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})}{\sqrt{|\mathbf{x}-\mathbf{y}|^2+(s-t)^2}} \\ &\quad = -2\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} (m(t)\cdot n(\mathbf{x}))(m(t)\cdot n(\mathbf{y})) \ln|\mathbf{x}-\mathbf{y}|. \end{aligned} \tag{4.9}$$

The lemma is proved.

Now we are ready to prove the Γ-convergence result. Let us recall the definition of the micromagnetic energy we are considering here (for simplicity of notation we drop all tildes)

$$E_h(m_h) = d^2 \int_{\Omega} \left(\frac{\partial m_h}{\partial x}\right)^2 + \frac{1}{h^2}|\nabla' m_h|^2 + \frac{1}{h^2}\int_{\mathbf{R}^3} |\nabla u_h|^2.$$

Proof of Theorem 2.1. Using bounds on the energy $E_h(m_h) \leqslant C$, it is straightforward to deduce that

- $m_h \to m$ weakly in $H^1(\Omega; S^2)$,
- $m = m(x)$ is independent of (y, z) variables,
- $\bar{m}_h \to m$ weakly in $H^1(-L, L)$.

Using (2.3), (3.13) and Lemma 4.1, we obviously have

$$\liminf E_h(m_h) \geqslant E_0(m),$$

for any $m_h \to m$ weakly in $H^1(\Omega; S^2)$.

FIG. 2. Elliptical wire.

Now take any $m \in H^1(\Omega; S^2)$. We can construct the recovery sequence by defining $m_h = m$. Again, using (2.3), (3.13) and Lemma 4.1, it is straightforward to see that

$$E_h(m_h) = d^2 \int_\Omega \left(\frac{\partial m}{\partial x}\right)^2 + \frac{1}{h^2}|\nabla' m|^2 - \frac{1}{2\pi}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} (\bar{m}\cdot n)(t,\mathbf{x})(\bar{m}\cdot n)(t,\mathbf{y}) \ln|\mathbf{x}-\mathbf{y}| + o(1). \tag{4.10}$$

Therefore, it is clear that $E_h(m_h)$ Γ-converges to $E_0(m)$, hence Theorem 2.1 is proved.

5. Example of a wire with an elliptic cross-section

In this section, we will calculate the reduced energy and equilibrium magnetization distribution for a specific example of ferromagnetic generalized cylindrical wire with an elliptic cross section (see Fig. 2). The result turns out to be quite different from the well-known energy for a wire with a circular cross-section (see Kühn, 2007).

Looking back at (2.4), we just need to find the symmetric matrix $M = \{M_{ij}\}$ to get the magnetostatic energy. Parameterizing the boundary of an ellipse in $\mathbf{R}^2$ by $(a\cos\theta, b\sin\theta)$, with $\theta \in [0, 2\pi)$ and $a, b > 0$, we obtain

$$M_{11} = -\frac{b^2}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \cos\theta\cos\phi \ln(a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2), \tag{5.1}$$

$$M_{12} = -\frac{ab}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \cos\theta\sin\phi \ln(a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2), \tag{5.2}$$

$$M_{22} = -\frac{a^2}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \sin\theta\sin\phi \ln(a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2). \tag{5.3}$$

Using the change of variables $\theta \to 2\pi - \theta$ and $\phi \to 2\pi - \phi$, it is clear that $M_{12} = 0$. It is then straightforward to obtain

$$M_{11} = \frac{\pi ab^2}{a+b}, \quad M_{22} = \frac{\pi a^2 b}{a+b}.$$

Hence, the reduced magnetostatic energy is

$$\frac{\pi abh^2}{a+b}\int_{-L}^{L} bm_2^2(x) + am_3^2(x)\mathrm{d}x. \tag{5.4}$$

Putting everything together, we now have

$$E_0(m) = d^2\int_{-L}^{L} |m'(x)|^2\,\mathrm{d}x + \frac{\pi ab}{a+b}\int_{-L}^{L} bm_2^2(x) + am_3^2(x)\mathrm{d}x. \tag{5.5}$$

In order to obtain the magnetization distribution of this energy, we will now need to find a minimizer of the energy for a set of Dirichlet boundary conditions. We will choose our boundary conditions such that the magnetization vector is tangent to the wire at each end, i.e $m(-L) = (-1, 0, 0)$, $m(L) = (1, 0, 0)$.

Clearly, minimizing the above is equivalent to finding a minimizer of

$$F(m) = \alpha \int_{-L}^{L} |m'(x)|^2 \, \mathrm{d}x + \int_{-L}^{L} bm_2^2(x) + am_3^2(x)\mathrm{d}x, \tag{5.6}$$

where α is a positive constant.

Since we have $|m(x)| = 1$, for $x \in [-L, L]$, we can write $m(x)$ as

$$m(x) = (\cos\theta(x), \sin\theta(x)\cos\phi(x), \sin\theta(x)\sin\phi(x)). \tag{5.7}$$

To achieve the boundary conditions, we have $\theta(0) = \pi$, $\theta(1) = 0$ and $\phi =$ constant. Rewriting (5.6), we now get

$$F(\theta, \phi) = \int_{-L}^{L} \alpha(\theta')^2 + (\alpha(\phi')^2 + (b-a)\cos^2\phi + a)\sin^2\theta \, \mathrm{d}x. \tag{5.8}$$

Clearly if $a = b$, then in order to minimize this functional, we see that we can choose ϕ to be any constant. However, if $a > b$, then $\phi = 0$ or $\phi = \pi$ will minimize the energy and if $b > a$, then $\phi = \frac{\pi}{2}$ or $\phi = \frac{3\pi}{2}$ will minimize the energy. In both cases, the energy is simplified to

$$F(\theta, \phi) = \int_{-L}^{L} \alpha(\theta')^2 + \min\{a, b\}\sin^2\theta \, \mathrm{d}x.$$

From this, it is not difficult to derive the Euler–Lagrange equation for θ

$$\theta'' = k\sin(2\theta), \tag{5.9}$$

where $k = \frac{\min\{a,b\}}{2\alpha}$ is a positive constant, $\theta(0) = \pi$, $\theta(1) = 0$.

We note that for a circular cross-section ($a = b$), the magnetization can lie in any plane (ϕ is any constant). On the other hand, if $a \neq b$, then there is a preferred plane in which magnetization lies $\left(\phi = 0 \text{ or } \phi = \frac{\pi}{2}\right)$.

REMARK 51 For a wire with an arbitrary cross-section, the matrix M defined in (2.5) is symmetric. Therefore, it is clear that there exists a frame in which M is a diagonal matrix. Using the above arguments for the elliptic wire, it is clear that the optimal magnetization distribution will always lie in some preferred plane (that will depend on the wire's cross-section).

6. Reduced energy for a curved wire

We now turn our attention to the problem of finding the reduced energy of a curved generalized cylindrical wire (see Fig. 3). For clarity of the presentation, we choose to parameterize the wire using the Frenet–Serret frame (see Dineen, 2001) .

We have that the tangent $T(s)$, normal $N(s)$ and binormal $B(s)$ to a 3D curve $\gamma(s)$ form the Frenet–Serret basis $\{T(s), N(s), B(s)\}$. When s is the arc length of the curve, the relation between the basis vectors is given by the following Frenet–Serret equations:

$$T'(s) = \kappa(s)N(s), \tag{6.1}$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \tag{6.2}$$

$$B'(s) = -\tau(s)N(s), \tag{6.3}$$

FIG. 3. Curved generalized cylindrical wire.

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of $\gamma(s)$, respectively.

We can define the following reference domain:

$$\Omega = \{(x, y, z) \in \mathbf{R}^3 \colon (x, y, z) = \gamma(s) + x_1 N(s) + x_2 B(s)\},$$

where $s \in [-L, L]$ is the arc length of the wire, $\mathbf{x} = (x_1, x_2) \in \omega$ (where ω is the cross-section) and $\gamma(s) \subset \mathbf{R}^3$ is the central curve of the wire with $N(s)$ and $B(s)$ being the normal and binormal to the curve, respectively. We will work with curves γ which are smooth, non-intersecting and regular. Without loss of generality, we assume the condition $\operatorname{diam}(\omega) < \frac{1}{\min_s \kappa(s)}$, where $\kappa(s)$ is the curvature. The ferromagnetic wire is represented by

$$\Omega_h = \{(x, y, z) \in \mathbf{R}^3 \colon (x, y, z) = \gamma(s) + h(x_1 N(s) + x_2 B(s))\},$$

where s is the arc length and $(x_1, x_2) \in \omega$.

As before we will study the following one parameter family of micromagnetic energy functionals:

$$E_h(m_h) = \frac{d^2}{h^2} \int_{\Omega_h} |\nabla m_h|^2 + \frac{1}{h^2} \int_{\mathbf{R}^3} |\nabla u_h|^2, \tag{6.4}$$

where $|m_h| = 1$ in Ω_h and u_h satisfies the following equation:

$$-\Delta u_h = \operatorname{div}(m_h \chi_{\Omega_h}) \quad \text{in } \mathbf{R}^3. \tag{6.5}$$

In this section, we will now prove the following theorem.

THEOREM 6.1 Assume that d is a constant, then we have

- if $E_h(\tilde{m}_h) \leqslant C$, then $\tilde{m}_h \to m$ weakly in $H^1(\Omega; S^2)$ (maybe for a subsequence), $m = m(s)$;
- E_h Γ-converges to the following energy E_0 in $H^1_w(\Omega; S^2)$:

$$E_0(m) = \begin{cases} d^2 \int_{-L}^{L} |m'(s)|^2 + \int_{-L}^{L} (M(s)m(s), m(s)) & \text{if } m = m(s), \\ \infty & \text{otherwise.} \end{cases} \tag{6.6}$$

Here, $M(s)$ is a symmetric matrix defined as

$$M(s) = -\frac{1}{2\pi} \int_{\partial\omega} \int_{\partial\omega} n(s, \mathbf{x}) \otimes n(s, \mathbf{y}) \ln |\mathbf{x} - \mathbf{y}|,$$

where n is a normal vector to $\partial\Omega$ and parameter s corresponds to the arc length of the wire.

6.1 *Exchange energy*

We will begin by looking at the exchange energy and so we will need to change variables from $(x, y, z) \to (s, \mathbf{x})$. We have

$$x = \gamma_1(s) + x_1 N_1(s) + x_2 B_1(s), \tag{6.7}$$
$$y = \gamma_2(s) + x_1 N_2(s) + x_2 B_2(s), \tag{6.8}$$
$$z = \gamma_3(s) + x_1 N_3(s) + x_2 B_3(s), \tag{6.9}$$

where the Jacobian of this transformation is

$$J(s, \mathbf{x}) = |1 - x_1\kappa(s)|. \tag{6.10}$$

Next, by implicitly differentiating (6.7)–(6.9) by x, y and z, we can then solve a set of simultaneous equations to get all the partial derivatives needed for $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, respectively. Putting them together, we obtain

$$\begin{aligned}\frac{d^2}{h^2}\int_{\Omega_h} |\nabla m_h|^2 = {} & \frac{d^2}{h^2|1-x_1\kappa|}\int_{-L}^{L}\int_{\partial\omega_h} |m_{h,s} - x_1\tau m_{h,x_2} + x_2\tau m_{h,x_1}|^2 \\ & + \frac{d^2}{h^2}\int_{-L}^{L}\int_{\partial\omega_h} |1-x_1\kappa||m_{h,x_1}|^2 + \frac{d^2}{h^2}\int_{-L}^{L}\int_{\partial\omega_h} |1-x_1\kappa||m_{h,x_2}|^2. \end{aligned}\tag{6.11}$$

Rescaling ω_h to a fixed domain ω (not relabelling (x_1, x_2)) we finally get

$$\begin{aligned}\frac{d^2}{h^2}\int_{\Omega_h} |\nabla m_h|^2 = {} & \frac{d^2}{|1-hx_1\kappa|}\int_{-L}^{L}\int_{\partial\omega} |\tilde{m}_{h,s} - x_1\tau \tilde{m}_{h,x_2} + x_2\tau \tilde{m}_{h,x_1}|^2 \\ & + \frac{d^2}{h^2}\int_{-L}^{L}\int_{\partial\omega} |1-hx_1\kappa||\nabla'\tilde{m}_h|^2, \end{aligned}\tag{6.12}$$

where

$$\tilde{m}_h(s, x_1, x_2) = m_h(s, hx_1, hx_2) \quad \text{for } s \in [-L, L],\ (x_1, x_2) \in \omega$$

and $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$.

6.2 *Magnetostatic energy*

Now, we proceed to compute the magnetostatic energy. Before we begin we should note that when we change variables from $m_h(x, y, z)$ to $m_h(s, \mathbf{x})$, we can then replace $m_h(s, \mathbf{x})$ with $\bar{m}_h(s) = \frac{1}{|\omega_h|}\int_{\omega_h} m_h(s, \mathbf{x})$. The proof of this fact follows the same arguments as the proof of Lemma 3.1 and therefore, we omit it here.

Using expansion (3.6) of the magnetostatic energy and estimates for the bulk–bulk, bulk–boundary and boundary–boundary terms similar to (3.7)–(3.9), we can show as before that

$$\begin{aligned}4\pi\int_{\mathbf{R}^3} |\nabla u_h|^2 = {} & \mathrm{O}(h^3)\left(\frac{1}{h^2}\|\nabla'\tilde{m}_h\|^2_{L^2(\Omega)} + 1\right) + \mathrm{O}(h^3|\ln h|)\left(\|\bar{m}'_{1,h}\|^2_{L^2(\Omega)} + 1\right) \\ & + \int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h} \frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})J(s,\mathbf{x})J(t,\mathbf{y})}{|(\gamma(s) + x_1N(s) + x_2B(s)) - (\gamma(t) + y_1N(t) + y_2B(t))|}, \end{aligned}\tag{6.13}$$

where in this case $\mathbf{x}$ and $\mathbf{y}$ parameterize $\partial\omega_h$. We now proceed to prove the following lemma, analogous to Lemma 4.1.

LEMMA 6.1 Assume $\bar{m}_h \to m$ weakly in $H^1(-L, L)$, then as $h \to 0$, we have

$$\frac{1}{h^2}\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega_h}\int_{\partial\omega_h}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})J(s,\mathbf{x})J(t,\mathbf{y})}{|(\gamma(s)+x_1N(s)+x_2B(s))-(\gamma(t)+y_1N(t)+y_2B(t))|}$$

$$\to -2\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}(m\cdot n)(t,\mathbf{x})(m\cdot n)(t,\mathbf{y})\ln|\mathbf{x}-\mathbf{y}|. \tag{6.14}$$

Proof of Lemma 6.1. We begin by rescaling (6.13), but not relabelling $(\bar{m}\cdot n)$, to get

$$\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})J(s,h\mathbf{x})J(t,h\mathbf{y})}{|(\gamma(s)+hx_1N(s)+hx_2B(s))-(\gamma(t)+hy_1N(t)+hy_2B(t))|}. \tag{6.15}$$

We will now show that as $h \to 0$ this integral can be approximated by the following one:

$$\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}} \tag{6.16}$$

with the error $O(h)$.

To show this, we will begin by expanding out the denominator of (6.15). Let $A_h = |(\gamma(s) + hx_1N(s)+hx_2B(s))-(\gamma(t)+hy_1N(t)+hy_2B(t))|$. Expanding this out, we get

$$\begin{aligned} A_h^2 = {} & |\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}|^2+h^2|\mathbf{y}|^2 \\ & +2hx_1N(s)\cdot(\gamma(s)-\gamma(t))+2hx_2B(s)\cdot(\gamma(s)-\gamma(t)) \\ & +2hy_1N(t)\cdot(\gamma(t)-\gamma(s))+2hy_2B(t)\cdot(\gamma(t)-\gamma(s)) \\ & -2h^2x_1y_1N(s)\cdot N(t)-2h^2x_1y_2N(s)\cdot B(t) \\ & -2h^2x_2y_1B(s)\cdot N(t)-2h^2x_2y_2B(s)\cdot B(t). \end{aligned} \tag{6.17}$$

Now, let $B_h = \sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}$. Using basic algebra and Taylor expansions, it is not difficult to see that

$$|A_h^2-B_h^2| \leqslant Ch^2|t-s|+Ch|t-s|^2, \tag{6.18}$$

$$B_h \geqslant \sqrt{\delta^2(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2}, \tag{6.19}$$

$$A_h \geqslant \sqrt{\delta^2(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2} \tag{6.20}$$

for some $\delta > 0$. So we now have

$$\begin{aligned} &\Bigg|\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})J(s,h\mathbf{x})J(t,h\mathbf{y})}{|(\gamma(s)+hx_1N(s)+hx_2B(s))-(\gamma(t)+hy_1N(t)+hy_2B(t))|} \\ &\qquad -\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}\frac{(\bar{m}_h\cdot n)(s,\mathbf{x})(\bar{m}_h\cdot n)(t,\mathbf{y})}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\Bigg| \\ &\qquad\qquad \leqslant C\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega}\frac{h^2|s-t|+h|s-t|^2}{(\delta^2(s-t)^2+h^2|\mathbf{x}-\mathbf{y}|^2)^{\frac{3}{2}}} \leqslant Ch. \end{aligned} \tag{6.21}$$

Hence, we have shown that we may replace (6.15) by (6.16). In order to tackle (6.16), we begin by investigating

$$\int_{-L}^{L} \frac{f_h(s,\mathbf{x})\mathrm{d}s}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}},$$

where $f_h = (\bar{m}_h \cdot n)$. This can be written as

$$\int_{-L}^{L} \frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)} \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}s, \tag{6.22}$$

where $\cos\theta(s,t) = \frac{\gamma'(s)\cdot(\gamma(s)-\gamma(t))}{|\gamma'(s)||\gamma(s)-\gamma(t)|} = \frac{\gamma'(s)\cdot(\gamma(s)-\gamma(t))}{|\gamma(s)-\gamma(t)|}$. It is clear that for $|s-t| \to 0$,

$$\left|\cos\theta(s,t) - \frac{s-t}{|s-t|}\right| \to 0. \tag{6.23}$$

We can do a similar integration by parts trick as for the straight wire case by rewriting (6.22) as

$$\begin{aligned}&\int_{-L}^{t} \frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)} \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}s \\ &\quad + \int_{t}^{L} \frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)} \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}s,\end{aligned} \tag{6.24}$$

and noting that

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln\left(|\gamma(s)-\gamma(t)| + \sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right) = \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}.$$

Integration by parts yields

$$\begin{aligned}&\int_{-L}^{t} \frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)} \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}s = -f_h(t,\mathbf{x})\ln(h|\mathbf{x}-\mathbf{y}|) \\ &- \frac{f_h(-L,\mathbf{x})}{\cos\theta(-L,t)}\ln\left(|\gamma(-L)-\gamma(t)| + \sqrt{|\gamma(-L)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right) \\ &- \int_{-L}^{t}\ln\left(|\gamma(s)-\gamma(t)| + \sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)}\right)\end{aligned} \tag{6.25}$$

and

$$\begin{aligned}&\int_{t}^{L} \frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)} \frac{\cos\theta(s,t)}{\sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}}\mathrm{d}s = -f_h(t,\mathbf{x})\ln(h|\mathbf{x}-\mathbf{y}|) \\ &+ \frac{f_h(L,\mathbf{x})}{\cos\theta(L,t)}\ln\left(|\gamma(L)-\gamma(t)| + \sqrt{|\gamma(L)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right) \\ &- \int_{t}^{L}\ln\left(|\gamma(s)-\gamma(t)| + \sqrt{|\gamma(s)-\gamma(t)|^2+h^2|\mathbf{x}-\mathbf{y}|^2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f_h(s,\mathbf{x})}{\cos\theta(s,t)}\right).\end{aligned} \tag{6.26}$$

Combining these, multiplying everything by $f_h(t, \mathbf{y})$ and integrating over t, $\mathbf{x}$ and $\mathbf{y}$, we can use the same techniques as in the straight wire case to obtain

$$\int_{-L}^{L}\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} \frac{(\bar{m}_h \cdot n)(s, \mathbf{x})(\bar{m}_h \cdot n)(t, \mathbf{y})}{\sqrt{|\gamma(s) - \gamma(t)|^2 + h^2|\mathbf{x} - \mathbf{y}|^2}} \to -2\int_{-L}^{L}\int_{\partial\omega}\int_{\partial\omega} (m \cdot n)(t, \mathbf{x})(m \cdot n)(t, \mathbf{y}) \ln|\mathbf{x} - \mathbf{y}|, \quad (6.27)$$

where n is the normal to $\partial\Omega$. The lemma is proved.

Proof of Theorem 6.1. This can be proven in the same way as Theorem 2.1 using (6.12) and Lemma 6.1.

7. Conclusion

We have studied the micromagnetic energy for soft ferromagnetic wires with an arbitrary cross-section, curvature and torsion. Using variational methods and appropriate decomposition of the magnetostatic energy, we have shown that for ferromagnetic wires with a small diameter cross-section, the micromagnetic energy can be reduced to a simple 1D energy. An interesting feature of this reduced problem is that the non-local magnetostatic energy simplifies to a local 'shape anisotropy' that makes the magnetization prefer specific planes depending on the shape of the cross-section. This is different from the well-known magnetization behaviour inside wires with a circular cross-section. The magnetization inside the curved wires behaves as expected: it wants to align along the wire, and the optimal profile problem is analogous to the case of the straight wire. The methods used in this paper can also be extended to explain the optimal magnetization distribution inside nanowires with different geometries and in various regimes.

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