Hybrid Method for Computing Demagnetizing Fields

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Abstract

We propose a hybrid finite element—boundary integral method for computing demagnetizing fields. For a three dimensional mesh with N nodes in the magnetic region, the method requires storage $O(N^{4/3})$ and no nodes outside of the magnetic region need to be considered. The method is suitable for nonlinear calculations, can be adapted to handle curved boundaries, and is especially convenient when the magnetic region consists of several parts whose relative positions are variable.

1 Introduction

Nonlinear magnetic computations require the calculation of the magnetic field due to the magnetization of the material (the demagnetizing field). The field calculation is done many times because the computational scheme is iterative, the governing equations being nonlinear. It is therefore necessary to make the field calculation time and space efficient.

Let us introduce a magnetic scalar potential, ϕ, so that $\vec{H} = -\nabla\phi$. The potential can be computed from the explicit formula

$$\phi(\vec{x}) = \int_{\mathcal{R}_m} \vec{M}(\vec{y}) \cdot \nabla_y \gamma(\vec{x},\vec{y})\, d^3y, \tag{1}$$

where $\vec{M}$ is the magnetization density, $\mathcal{R}_m$ is the region containing magnetic material, and $\gamma(\vec{x},\vec{y})$ is the Green function

$$\gamma(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}. \tag{2}$$

If we discretize with N nodes, (1) requires computation of $O(N^2)$ matrix elements for the discretized version of the Green function, or, for some methods, $O(N \times E)$ matrix elements, where E is the number of elements. Calculation of such a large number of matrix elements every time the field is needed is prohibitively slow, while storage of these elements is prohibitively expensive in storage if the mesh is irregular, as is likely to be the case when $\mathcal{R}_m$ has a general shape and the finite element method is used.

Solution of Poisson's equation for ϕ by the finite element method is a viable alternative to (1). Instead of storing the discretized Green function, which is a full matrix, one stores the discretized Laplacian, the "stiffness matrix," which is sparse. The storage requirement is $O(N)$. The difficulty is that ϕ must be found in all of space because one of the boundary conditions is at infinity. Even with special techniques, such as ballooning, elements outside of the magnetic material must be used. In three dimensions, the number of external elements grows rapidly as the accuracy demanded in the magnetic region is increased.

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In section 2 we present a method for calculating the magnetic scalar potential, ϕ, which has the advantages of the finite element method but which does not require introduction of elements outside of the magnetic region. We discuss some numerical aspects of the method and its application to regions with curved boundaries and give results for an illustrative example in section 3 and, in section 4 we discuss the additional advantages of the new method over a conventional finite element calculation when the magnetic system consists of several spatially separated components.

2 The Method

The object is calculation of the magnetic scalar potential $\phi(\vec{x})$ inside the magnetic region, $\vec{x} \in \mathcal{R}_m$. We start by considering $\phi(\vec{x})$ for all $\vec{x}$. We let $\mathcal{R}_e$ denote the non-magnetic region complementary to $\mathcal{R}_m$, and $\partial\mathcal{R}$ denote the boundary of $\mathcal{R}_m$. We require ϕ to satisfy the partial differential equation

$$\nabla^2\phi(\vec{x}) = \begin{cases} 4\pi\nabla\cdot\vec{M}(\vec{x}) & \text{for } \vec{x} \in \mathcal{R}_m \\ 0 & \text{for } \vec{x} \in \mathcal{R}_e \end{cases} \tag{3}$$

with the boundary condition on $\partial\mathcal{R}$

$$[\phi(\vec{x})] = 0 \tag{4}$$

$$\left[\frac{\partial\phi(\vec{x})}{\partial\hat{\nu}}\right] = -4\pi\hat{\nu}\cdot\vec{M}(\vec{x}), \tag{5}$$

where $[\ldots]$ denotes a discontinuity across $\partial\mathcal{R}$ and $\hat{\nu}(\vec{x})$ is the unit normal at $\vec{x}$ directed out of $\mathcal{R}_m$. We also require that $\phi(\vec{x}) \to 0$ for $|\vec{x}| \to \infty$. We make the split $\phi = \phi_1 + \phi_2$, where, for $\vec{x} \in \mathcal{R}_m$, $\phi_1(\vec{x})$ is the solution of the inhomogenous Neuman problem

$$\nabla^2\phi_1(\vec{x}) = 4\pi\nabla\cdot\vec{M}(\vec{x}) \tag{6}$$

with the boundary condition

$$\frac{\partial\phi_1(\vec{x})}{\partial\hat{\nu}} = 4\pi\hat{\nu}\cdot\vec{M}(\vec{x}), \tag{7}$$

and we define $\phi_1(\vec{x}) = 0$ for $\vec{x} \in \mathcal{R}_e$. From (3) and (6), we see that $\phi_2(\vec{x})$ satisfies Laplace's equation

$$\nabla^2\phi_2(\vec{x}) = 0, \tag{8}$$

and from (4), (5) and (7) we have, for $\vec{x} \in \partial\mathcal{R}$

$$[\phi_2(\vec{x})] = \phi_1(\vec{x}) \tag{9}$$

$$\left[\frac{\partial\phi_2(\vec{x})}{\partial\hat{\nu}}\right] = 0. \tag{10}$$

We must also require that $\phi_2(\vec{x}) \to 0$ for $|\vec{x}| \to \infty$. From potential theory, we have the expression

$$\phi_2(\vec{x}) = \frac{1}{4\pi} \int_{\partial\mathcal{R}} \phi_1(\vec{y}) \frac{\partial\gamma(\vec{x},\vec{y})}{\partial\hat{\nu}(\vec{y})}\, d^2y \tag{11}$$

In practice, we determine ϕ_1 by a finite element calculation, using the weak (Galerkin) form of (6) and (7)

$$\int_{\mathcal{R}_m} \nabla\psi(\vec{x}) \cdot \nabla\phi_1(\vec{x})\, d^3x = \int_{\mathcal{R}_m} \nabla\psi(\vec{x}) \cdot \vec{M}(\vec{x})\, d^3x, \tag{12}$$

for all $\psi \in \mathrm{H}^1(\mathcal{R}_m)$, where $\mathrm{H}^1(\mathcal{R}_m)$ is the usual Sobolev space. The variational condition $\delta I = 0$, with

$$I = \int_{\mathcal{R}_m} \frac{1}{2} |\nabla\phi_1(\vec{x})|^2\, d^3x \tag{13}$$

is equivalent to (12). Note that the boundary condition (7) is natural.

We do not regard (11) as computationally feasible because it still requires excessive storage ($O(N^{5/3})$) if the discretized form of $\partial\gamma(\vec{x},\vec{y})/\partial\hat{\nu}(\vec{y})$ is not to be repeatedly reevaluated. Instead, we use (11) to evaluate ϕ_2 on the boundary ($\partial\mathcal{R}$) and then we solve (8) inside $\mathcal{R}_m$ with the now known boundary values. For $\vec{x} \to \partial\mathcal{R}$, with the limit taken from within $\mathcal{R}_m$, (11) takes the form

$$\phi_2(\vec{x}) = \frac{1}{4\pi} \int_{\partial\mathcal{R}} \phi_1(\vec{y}) \frac{\partial\gamma(\vec{x},\vec{y})}{\partial\hat{\nu}(\vec{y})}\, d^2y + \left(\frac{\Omega(\vec{x})}{4\pi} - 1\right) \phi_1(\vec{x}), \tag{14}$$

where $\Omega(\vec{x})$ is the solid angle subtended by $\partial\mathcal{R}$ at $\vec{x}$[1]. If $\partial\mathcal{R}$ is smooth at $\vec{x}$, $\Omega(\vec{x}) = 2\pi$. Storage for the kernel in (14) is $O(N^{4/3})$, which we find acceptable. The integral in (14) is numerically well behaved and can be evaluated exactly if ϕ_1 is linear in each element[2]. Solution of the Dirichlet problem for ϕ_2 inside $\mathcal{R}_m$ using the finite element method uses exactly the same stiffness matrix as was used for the determination of ϕ_1, except for the suppression of the rows and columns pertaining to the boundary nodes.

Note that (6) and (7) only determine ϕ_1 up to an additive constant[3]. It is easy to see that (11) guarantees that, when a constant is added to ϕ_1, the same constant is subtracted from ϕ_2[4].

3 Numerical Considerations

The discretized version of (14) will have the form

$$\Phi_2 = B\Phi_1 \tag{15}$$

where Φ_1 and Φ_2 are vectors defined at the surface nodes and B is a matrix. Consider a subdivision of $\mathcal{R}_m$ into tetrahedral elements. If $\mathcal{R}_m$ is a polyhedron, each face will be subdivided into triangular elements. When the potential is linear, the elements of B can be obtained by applying the formulas of [1] or [2] to the surface triangles. Alternatively, B can be obtained by numerical integration. For example, for an n^{th} order integration formula, each triangle could be subdivided into n^2 identical subtriangles and an approximate integration scheme used in each subtriangle. The ϕ_1 values used in the subintegrals would be obtained by linear interpolation in the triangle using the values at the original nodes so that the contributions to the total integral from each original node can be ascertained and accumulated to find the elements of B. There are a various possibilities; we merely require convergence in the limit of large n. Numerical efficiency is not important, since B need only be computed once.

Numerical integration is advantageous because it permits easy extension to higher order elements and because, with some modification, it can be extended to account for curved surfaces. Suppose that $\mathcal{R}_m$ has a curved boundary and that it has been approximated by a polyhedron. When computing B by numerical integration, we place all of the added nodes of the subtriangles on the true surface. In the limit of large n, this scheme approximates the true surface as closely as desired. It is necessary to know ϕ_1 at the true surface. No matter what order elements are used, interpolated values of ϕ_1 on the surface, accurate to that order, can be obtained using only values at all of the original nodes of the tetrahedral element containing the surface face under consideration. The resulting B matrix now has elements for each node which is connected to a surface face, however, its order is still that of the square of the number of surface nodes. It is also necessary to modify some elements of the stiffness matrix to account for the true surface. Space considerations prohibit an adequate discussion of this point.

We have tested the method by applying it to a uniformly magnetized sphere. The underlying polyhedron is a soccer ball configuration with an extra node placed on the surface of the sphere at the center of each five and six sided surface panel. About 6% of the total volume of the sphere is missing in the polyhedron. Further details of the structure are given in [5]. The magnetization is taken to be unity, and cgs Gaussian units are used, so that the magnetic field inside the sphere is uniform and equal to $4\pi/3 = 4.188790205$. The results are shown in table 1. The convergence of the numerical integration is demonstrated in table 2, where the average magnetic field inside the polyhedron just described is computed. The exact value, using the formula from [1], is 3.989434453. Note that this differs from the value for the sphere by 5%.

Order		Computed Field	Error (ppm)
1	1	4.0420325	35036
2	4	4.1539700	8313
4	16	4.1804095	2001
8	64	4.1867390	490
16	256	4.1882830	121
32	1024	4.1886645	30
64	4096	4.1887650	6

Table 1: Results for a uniformly magnetized sphere. The first column is the number of segments into which each surface edge is divided. The second column is the number of triangles into which each surface element is divided. For each subdivision, the exact formula from [1] was used.

4 Discussion

The accurate computation of the demagnetizing field of the sphere establishes the ability of this hybrid method both to account for the influence on ϕ of the external region and to treat a curved surface as accurately as desired.

The conceptually simple treatment of the curved surface is a true hybrid result. It is only possible because ϕ can be extrapolated away from a conventional boundary element surface mesh using using internal values which are known from the finite element part of the formulation.

Order		Nodal Error (ppm)	Face-Centered Error (ppm)	Edge-Centered Error (ppm)
1	1	36489	105514	6589
2	4	10376	31321	329
4	16	2688	8077	24
8	64	679	2037	2
16	256	170	510	0
32	1024	43	128	0

Table 2: Results for a uniformly magnetized polyhedron. The first column is the number of segments into which each surface edge is divided. The second column is the number of triangles into which each surface element is divided. The nodal integration formula evaluates the integral over a surface element by its area times the average of the integrand at the corners of the element. The face-centered formula uses the area times the integrand at the element center. The edge-centered formula, which is accurate to second order, uses the area times the average of the integrand at the centers of the edges of the surface element. Errors less than 1ppm are shown as zero.

The hybrid method is especially advantageous when the magnetic region consists of several disjoint components, $\mathcal{R}_m = \cup_i \mathcal{R}_m^{(i)}$. Since a mesh is only required for $\mathcal{R}_m$, it is only necessary to establish a mesh in each subregion $\mathcal{R}_m^{(i)}$. If it is desired to vary the geometry by moving the subregions with respect to one another, no remeshing is required. Further, the stiffness matrix for the finite element calculations has no elements connecting nodes in different subregions, so it, too, need not be recomputed when the subregions are moved. Only the matrix B, introduced in section 3, has to be partially recomputed.

References

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[2] C. J. Collie, "Magnetic Fields and Potentials of Linearly Varying Current or Magnetization in a Plane-Bounded Region," *Proc. COMPUMAG*, pp 86–95, (Oxford, England 1976).

[3] This indeterminacy is easily handled numerically. We have found that no special attention is required when solving the discretized equation by either SOR or ICCG.

[4] We are grateful to Rui-Wei Wang for this observation.

[5] D. R. Fredkin and T. R. Koehler, "Numerical Micromagnetics of Small Particles," *IEEE Transactions on Magnetics*, **24**, 2362 (1988). Invited paper at the 4th Joint MMM-Intermag Conference.