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# Discrete homotopy analysis method for the nonlinear Fredholm integral equations

**T. Allahviranloo [a], M. Ghanbari [b,*]**

[a] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[b] *Department of Mathematics, Islamic Azad University, Aliabad Katoul Branch, Aliabad Katoul, Iran*

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**Abstract** Recently, Behiry et al. (in press) [8] have introduced a discretized version of the Adomian decomposition method, namely "Discrete Adomian Decomposition Method (DADM)", for solving nonlinear Fredholm integral equations. In this paper, we extent Behiry et al.'s idea on the well-known homotopy analysis method, and introduce "Discrete homotopy analysis method (DHAM)". The obtained numerical solutions by the present method are compared with the obtained results by DADM. Also, we present some advantages of DHAM which DADM has not them. Comparison of the DHAM with the DADM reveals that former is more powerful than the later and also DADM is only special case of the DHAM.



## 1. Introduction

The concepts of integral equations have motivated a large amount of research work in recent years. Nonlinear phenomena, which appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in biology, quantum mechanics, mathematical economics and queuing theory can be modeled by integral equations [1]. So, obtaining the solution with high accuracy for the such equations is very worth-while. As witnessed by literature, the Fredholm integral equation of the second kind is an important class of integral equations.

Up to now, several analytical and numerical methods were used such as the Adomian decomposition method (ADM), the variational iteration method (VIM), the direct computation method, the series solution method, the successive approximation method, the successive substitution method and the conversion to equivalent differential equations [2–7]. However, these analytical solution methods are not easy to use and require tedious calculation. Also, when applying the above methods to solve linear and nonlinear Fredholm integral

* Corresponding author. Tel.: +98 9113222331.
E-mail addresses: tofigh@allahviranloo.com (T. Allahviranloo), Mojtaba.Ghanbari@gmail.com (M. Ghanbari).



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equations many definite integrals need to be computed. For cases that evaluation of integrals analytically is impossible or complicated the above methods can not be applied.

Recently, in order to overcome this obstacle, Behiry et al. [8] have been introduced a discretized version of the ADM, namely "Discrete Adomian Decomposition Method (DADM)". In this paper, we extend Behiry et al.'s idea on an effective and reliable method which has shown great promise over the past few years, namely "homotopy analysis method (HAM)" .

The HAM proposed by Liao [9–14] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-difference equation. This method is unique among other similar methods as it allows us to effectively control the region of convergence and rate of convergence of a series solution to a nonlinear problem, via control of an initial approximation, an auxiliary linear operator, an auxiliary function and a convergence-control parameter [14]. Recently, Van Gorder and Vajravelu [15] have discussed about the selection of the initial approximation, auxiliary linear operator, auxiliary function and convergence-control parameter in the application of the HAM. They presented methods by which one may select the mentioned items when attempting to solve a nonlinear differential equation by using the HAM. Also, they presented necessary and sufficient conditions for the convergence of series solutions obtained via the HAM. In 2010, Abbasbandy and Shivanian [1] showed that the HAM can be applied to solve linear and nonlinear Fredholm integral equations with high accuracy. They disclosed that the ADM, which is well-known in solving integral equations , is only special case of the HAM. Recently, Jafari and Firoozjaee [16] have presented an efficient modification of the HAM, namely "multistage homotopy analysis method (MHAM)", for solving nonlinear integral equations.

Similar to the above mentioned methods, when applying the HAM to solve linear and nonlinear Fredholm integral equations many definite integrals need to be computed. For cases that evaluation of integrals analytically is impossible or complicated the HAM can not be applied. Due to such obstacle, we pursue the work of Behiry et al. [8] and introduce a discretized version of the HAM namely " discrete homotopy analysis method (DHAM)" for solving linear and nonlinear Fredholm integral equations. As a matter of the fact, the DHAM arises when the quadrature rules are used to approximate the definite integrals which can not be computed analytically. This method gives the numerical solution at nodes used in the quadrature rules. Comparison of the DHAM with the DADM reveals that former is more powerful than the later and DADM is only special case of the DHAM. Also, we present some advantages of DHAM which DADM has not them.

In this paper, we consider the nonlinear Fredholm integral equation (NFIE)

$$x(t) = y(t) + \lambda \int_a^b k(t,s)\ F[x(s)]\ ds, \quad \lambda \neq 0,\ a \leqslant t \leqslant b, \tag{1}$$

where $y(t)$ is known continuous function on $[a,b]$, $F[x(s)]$ is known nonlinear function, $k(t,s)$ is the kernel function which is known, continuous and bounded on the square $D = \{(t,s)|\ a \leqslant t \leqslant b,\ a \leqslant s \leqslant b\}$ and $x(t)$ is the unknown function which must be determined.

## 2. Homotopy analysis method (HAM)

Let us consider the following general nonlinear equation

$N[x(t)] = 0,$

where $N$ is a nonlinear operator, $t$ denotes independent variable, $x(t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [9–14] constructs the so-called zero-order deformation equation

$$(1-p)\ L[\varphi(t;p) - x_0(t)] = p\ \hbar\ H(t)\ N[\varphi(t;p)], \tag{2}$$

where $p \in [0,1]$ is called homotopy-parameter [14], $\hbar$ is a non-zero auxiliary parameter which is called convergence-control parameter [14], $H(t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $x_0(t)$ is an initial guess of $x(t)$, $\varphi(t;p)$ is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$\varphi(t;0) = x_0(t), \quad \varphi(t;1) = x(t),$

respectively. Thus, as $p$ increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $x_0(t)$ to the solution $x(t)$. Expanding $\varphi(t;p)$ in Maclaurin series with respect to $p$, we have

$$\varphi(t;p) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)\ p^m, \tag{3}$$

where

$$x_m(t) = \frac{1}{m!} \left.\frac{\partial^m \varphi(t;p)}{\partial p^m}\right|_{p=0}. \tag{4}$$

Here, the series (3) is called homotopy-series and Eq. (4) is called the $m$th-order homotopy-derivative of $\varphi$ [14]. If the auxiliary linear operator, the initial guess, the convergence-control parameter $\hbar$, and the auxiliary function are so properly chosen, the homotopy-series (3) converges at $p = 1$, then using the relationship $\varphi(t;1) = x(t)$, one has the so-called homotopy-series solution [14]

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t), \tag{5}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [9].

Based on Eq. (4), the governing equation can be deducted from the zero-order deformation equation (2). Define the vector

$\vec{x}_n = \{x_0(t), x_1(t), \ldots, x_n(t)\}.$

Differentiating Eq. (2) $m$ times with respect to the homotopy-parameter $p$ and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$L[x_m(t) - \chi_m x_{m-1}(t)] = \hbar\ H(t)\ R_m(\vec{x}_{m-1}), \tag{6}$$

where

$$R_m(\vec{x}_{m-1}) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}}\right|_{p=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{7}$$

It should be emphasized that $x_m(t)$ for $m \geqslant 1$ is governed by the linear equation (6) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as MAPLE, MATHEMATICA and MATLAB.

Finally an $n$th-order approximate solution is given by

$$\tilde{x}_n(t) = x_0(t) + \sum_{m=1}^{n} x_m(t), \tag{8}$$

and the exact solution is $x(t) = \lim_{n\to\infty}\tilde{x}_n(t)$.

## 3. HAM for NFIE (1)

In this section, we apply the HAM for the discussed problem (1). We choose the initial approximation

$$x_0(t) = y(t),$$

and the auxiliary linear operator

$$L[\varphi(t;p)] = \varphi(t;p).$$

Furthermore, Eq. (1) suggests to define the nonlinear operator

$$N[\varphi(t;p)] = \varphi(t;p) - y(t) - \lambda \int_a^b k(t,s)\ F[\varphi(s;p)]\ ds. \tag{9}$$

Using the above definitions, with assumption $H(t) = 1$, we construct the zero-order deformation equation

$$(1-p)\ [\varphi(t;p) - y(t)] = \hbar\ p\ N[\varphi(t;p)]. \tag{10}$$

Obviously, when $p = 0$ and $p = 1$, we have

$$\varphi(t;0) = y(t), \quad \varphi(t;1) = x(t),$$

respectively. Using Eq. (4) and differentiating the zero-order deformation equation (10) $m$ ($m \geqslant 1$) times with respect to $p$, and finally dividing by $m!$, we obtain the $m$th-order deformation equation

$$x_m(t) = \chi_m x_{m-1}(t) + \hbar\ R_m(\vec{x}_{m-1}), \quad m \geqslant 1, \tag{11}$$

where

$$R_m(\vec{x}_{m-1}) = \chi_m x_{m-1}(t) - \lambda \int_a^b k(t,s) \left\{ \frac{1}{(m-1)!} \frac{\partial^{m-1} F[\varphi(s;p)]}{\partial p^{m-1}}\bigg|_{p=0} \right\} ds, \tag{12}$$

and

$$\chi_m = \begin{cases} 0, & m = 1, \\ 1, & m > 1. \end{cases} \tag{13}$$

On the other hand, it is clear that since $\varphi(s;p) = \sum_{i=0}^{\infty} x_i(s)\ p^i$, then we have

$$\frac{1}{(m-1)!} \frac{\partial^{m-1} F[\varphi(s;p)]}{\partial p^{m-1}}\bigg|_{p=0} = \frac{1}{(m-1)!} \frac{\partial^{m-1} F\left[\sum_{i=0}^{\infty} x_i(s)\ p^i\right]}{\partial p^{m-1}}\bigg|_{p=0} = A_{m-1}[x_0, x_1, \ldots, x_{m-1}] : = A_{m-1}(s),$$

where $A_{m-1}(s)$, $m \geqslant 1$ are the so-called Adomian polynomials. Consequently, Eq. (12) can be written as:

$$R_m(\vec{x}_{m-1}) = \chi_m\ x_{m-1}(t) - \lambda \int_a^b k(t,s)\ A_{m-1}(s)\ ds, \quad m \geqslant 1, \tag{14}$$

where $A_i(s)$, $i \geqslant 0$ are Adomian polynomials for the nonlinear term $F[x(s)]$ in Eq. (1).

**Remark 3.1.** It may be noted that based on the Molabahrami and Khani's Theorem [17], if $F[x(s)] = [x(s)]^{\alpha}$ then we have

$$A_i = \sum_{r_1=0}^{i} x_{j-r_1} \sum_{r_2=0}^{r_1} x_{r_1-r_2} \sum_{r_3=0}^{r_2} x_{r_2-r_3} \cdots \sum_{r_{\alpha-2}=0}^{r_{\alpha-3}} x_{r_{\alpha-3}-r_{\alpha-2}} \times \sum_{r_{\alpha-1}=0}^{r_{\alpha-2}} x_{r_{\alpha-2}-r_{\alpha-1}} x_{r_{\alpha-1}}.$$

According to HAM, the components $x_m(t)$, $m \geqslant 1$ are to be computed using the recursive relation (11) together with Eq. (14), then the homotopy-series solution of Eq. (1) is given by

$$x(t) = y(t) + \sum_{m=1}^{\infty} x_m(t). \tag{15}$$

Also, in this paper we present the $n$th-order approximate solution by

$$\tilde{x}_n(t) = y(t) + \sum_{m=1}^{n} x_m(t), \quad n \geqslant 1. \tag{16}$$

**Theorem 3.2.** *Eq. (15) ia an exact solution of Eq. (1) as long as the series $\sum_{m=1}^{\infty} x_m(t)$ is convergent.*

**Proof.** Since $\sum_{m=1}^{\infty} x_m(t)$ is convergent, we must have

$$\lim_{m\to\infty} x_m(t) = 0.$$

Due to the recursive relation (11) together with Eq. (14), it holds

$$\hbar\left[\sum_{i=1}^{\infty} x_i(t) - \lambda \int_a^b k(t,s) \sum_{i=0}^{\infty} A_i(s)\ ds\right] = \lim_{m\to\infty} x_m(t) = 0.$$

On the other hand, since

$$F[x(s)] = \sum_{i=0}^{\infty} A_i(s), \quad \sum_{i=1}^{\infty} x_i(t) = x(t) - y(t), \quad \hbar \neq 0,$$

then

$$x(t) = y(t) + \lambda \int_a^b k(t,s)\ F[x(s)]\ ds.$$

This ends the proof. □

According to the above Theorem 3.2, for obtaining exact solution, it is important to ensure that the homotopy-series solution (15) is convergent. Note that the series solution (15) contains the convergence-control parameter $\hbar$, which provides us with a simple way to adjust and control the convergence of the homotopy-series solution. Liao [9] suggested to choose a proper value of $\hbar$ by plotting the so-called $\hbar$-curve. Also, we can by the residual error present an other way to determine a region of $\hbar$ [18,19,14]. Let $R_n(t,\hbar)$ denote the residual error of the $n$th-order approximate solution, and $V_n(\hbar) = \int_a^b R_n^2(t,\hbar)\ dt$ denote the integral of the residual error. Plotting the curves of $V_n(\hbar) \sim \hbar$, it is straightforward to find a region of $\hbar$ in which $V_n(\hbar)$ decreases to zero as the order of approximation increases. Then, a convergent series solution is obtained by choosing a value in this region.

It is clear that the computation of each component $x_m(t)$, $m \geqslant 1$ requires the computation of an integral in Eq. (14). If evaluation of integrals analytically is possible, the HAM can be applied in a simple manner. Otherwise, if evaluation of integrals analytically is impossible, the HAM can not be applied. To overcome this possible problem, following [8], we introduce a discretized modified version of the HAM which will be called later the DHAM.

## 4. Discrete homotopy analysis method (DHAM)

Calculating the definite integral of a given real function $f(t)$ on interval $[a,b]$ is a classic problem [20]. For some simple cases, computation of integral is possible, but in many cases the integral is very complicated and consequently its computation analytically is impossible. Therefore, we have to apply the numerical integration methods for the evaluation of the such definite integrals.

In discrete homotopy analysis method (DHAM), the definite integral in Eq. (14) is computed using discretization methods which approximate the integral by finite sum corresponding to some partition of the interval of integration $[a,b]$.

In this paper, we consider the integration formulas of Newton and Cotes. This formulas are obtained if the integrand is replaced by a suitable interpolating polynomial. For any natural number $n$, the Newton–Cotes formulas are given by

$$\int_a^b f(t)\ dt \approx \sum_{j=0}^{n} w_j\, f(t_j), \tag{17}$$

where $f(t)$ is continuous function on $[a,b]$, $P = \{t_0, t_1, \ldots, t_n\}$ is an uniform partition of the closed interval $[a,b]$ given by

$$t_j = a + jh, \quad j = 0, 1, \ldots, n,$$

of the step length $h = (b-a)/n$ and $w_j$, $j = 0, 1, \ldots, n$ are the weight functions and determined by

$$w_j = \int_{t_0}^{t_n} L_j(t)\ dt, \quad j = 0, 1, \ldots, n, \tag{18}$$

where $L_j(t)$, $j = 0, 1, \ldots, n$ are Lagrangian polynomials. Obviously, the weights $w_j$ depend solely on n, in particular, they do not depend on the function $f$ to be integrated, or on the boundaries $a$, $b$ of the integral. We present the weights $w_j$ in Table 1 for values of $n$ from 1 to 8. The value of $w_j$ in (17) is given by $hA\alpha_j$ where $h$ is the step length [21].

For high-order Newton–Cotes formulas, some of the values $w_j$ become negative and the corresponding formulas are unsuitable for numerical purposes. In fact, it can be shown [21] that only for $n \leqslant 7$ and $n = 9$ all the weights are positive. Since the sum of the weights is always the length of the interval $[a,b]$, then if some of the weights are negative, this adversely affects roundoff error [21].

A typical representative of Newton–Cotes formulas is Simpson's rule ($n = 2$), which is still the best-known and most widely used integration method. In this paper, for the numerical implementation of the DHAM, we will apply Simpson's rule to approximate the definite integral in Eq. (14).

In general, approximating the definite integral in Eq. (14) by applying formula (17) and substituting it in Eq. (11) we obtain

$$x_m(t) \approx (1+\hbar)\chi_m\ x_{m-1}(t) - \hbar\lambda \sum_{j=0}^{n} w_j k(t, t_j) A_{m-1}(t_j), \quad m \geqslant 1, \tag{19}$$

where $x_0(t) = y(t)$. Therefore, from Eq. (15) we conclude

$$x(t) \approx y(t) + \sum_{m=1}^{\infty} x_m(t), \tag{20}$$

as the approximate homotopy-series solution for Eq. (1). For numerical purposes, we approximate the exact solution of Eq. (1) by

$$\tilde{x}_n(t) = y(t) + \sum_{m=1}^{n} x_m(t), \tag{21}$$

which is called $n$th order approximate solution, where $x_m(t)$, $m = 1, 2, \ldots, n$ are obtained approximately via the recursive relation (19).

Obviously, the approximate solution $\tilde{x}_n(t)$ contains the convergence-control parameter $\hbar$. In order to determine the valid region of $\hbar$, we can plot the $\hbar$-curve and choose a proper value of $\hbar$ in this region. Also, by residual error we able to present an other way to determine the optimal value $\hbar$. To this end, let $R_n(t,\hbar)$ is the residual error of $\tilde{x}_n(t)$, then we have

$$R_n(t,\hbar) = \tilde{x}_n(t) - y(t) - \lambda \int_a^b k(t,s)\ F[\tilde{x}_n(s)]\ ds. \tag{22}$$

By approximating the above definite integral via formula (17), we estimate the residual error as

$$R_n(t,\hbar) \approx R_n^*(t,\hbar) = \tilde{x}_n(t) - y(t) - \lambda \sum_{j=0}^{2n} w_j k(t, t_j)\ F[\tilde{x}_n(t_j)]. \tag{23}$$

**Table 1** Weights for Newton–Cotes formulas (17).

| $n$ | A | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | $\alpha_8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | 1 | 1 | – | – | – | – | – | – | – |
| 2 | $\frac{1}{3}$ | 1 | 4 | 1 | – | – | – | – | – | – |
| 3 | $\frac{3}{8}$ | 1 | 3 | 3 | 1 | – | – | – | – | – |
| 4 | $\frac{2}{45}$ | 7 | 32 | 12 | 32 | 7 | – | – | – | – |
| 5 | $\frac{5}{288}$ | 19 | 75 | 50 | 50 | 75 | 19 | – | – | – |
| 6 | $\frac{1}{140}$ | 41 | 216 | 27 | 272 | 27 | 216 | 41 | – | – |
| 7 | $\frac{7}{17280}$ | 751 | 3577 | 1323 | 2989 | 2989 | 1323 | 3577 | 751 | – |
| 8 | $\frac{4}{14175}$ | 989 | 5888 | −928 | 10496 | −4540 | 10496 | −928 | 5888 | 989 |

It should be noted that we use $h' = h/2$ in the estimation of definite integral in (22). Because we want to obtain the residual error with more accuracy and precision. Then, we set

$$V_n(\hbar) = \int_a^b \left[R_n^*(t,\hbar)\right]^2 dt \approx \sum_{j=0}^{2n} w_j\left[R_n^*(t_j,\hbar)\right]^2 dt = V_n^*(\hbar), \quad (24)$$

and determine the value of $\hbar$ by minimizing $V_n(\hbar)$. To this end, we can plot $V_n^*(\hbar) \sim \hbar$ and choose $\hbar^*$ such that the value of $V_n^*(\hbar)$ be minimum.

Now, discretize the independent variable at the nodes used for the quadrature rule in Eq. (19). Thus, the discrete version of Eqs. (11) and (19) can be written as:

$$x_0(t_i) = y(t_i), \quad (25)$$

and

$$x_m(t_i) \approx (1+\hbar)\chi_m\, x_{m-1}(t_i) - \hbar\lambda \sum_{j=0}^{n} w_j k(t_i,t_j) A_{m-1}(t_j), \quad m \geqslant 1, \quad (26)$$

where $i = 0,1,\ldots,n$. Therefore, according to DHAM, the values of exact solution of Eq. (1) at the notes $t_i$, $i = 0,1,\ldots,n$ are approximated by summing the approximate values to the components $x_m(t)$, $m \geqslant 1$ represented by Eq. (26) at the nodes $t_i$, $i = 0,1,\ldots,n$, i.e.,

$$x(t_i) \approx y(t_i) + \sum_{m=1}^{\infty} x_m(t_i), \quad i = 0,1,\ldots,n. \quad (27)$$

Rewriting Eqs. (25)–(27) in matrix form, we have

$$\mathbf{X}_0 = \mathbf{Y}, \quad (28)$$

and

$$\mathbf{X}_m \approx (1+\hbar)\chi_m\, \mathbf{X}_{m-1} - \hbar\mathbf{B}\, \mathbf{A}_{m-1}, \quad m \geqslant 1, \quad (29)$$

and

$$\mathbf{X} \approx \mathbf{Y} + \sum_{m=1}^{\infty} \mathbf{X}_m, \quad (30)$$

where $\mathbf{Y}$, $\mathbf{X}_m$ and $\mathbf{A}_m$ are all vectors of dimension $(n+1)$ and $\mathbf{B}$ is $(n+1)\times(n+1)$ matrix as:

$$\mathbf{Y} = \begin{bmatrix} y(t_0) \\ y(t_1) \\ \vdots \\ y(t_n) \end{bmatrix}, \quad \mathbf{X}_m = \begin{bmatrix} x_m(t_0) \\ x_m(t_1) \\ \vdots \\ x_m(t_n) \end{bmatrix}, \quad \mathbf{A}_m = \begin{bmatrix} A_m(t_0) \\ A_m(t_1) \\ \vdots \\ A_m(t_n) \end{bmatrix},$$

and

$$\mathbf{B} = \lambda \begin{bmatrix} w_0k(t_0,t_0) & w_1k(t_0,t_1) & \ldots & w_nk(t_0,t_n) \\ w_0k(t_1,t_0) & w_1k(t_1,t_1) & \ldots & w_nk(t_1,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ w_0k(t_n,t_0) & w_1k(t_n,t_1) & \ldots & w_nk(t_n,t_n) \end{bmatrix}.$$

Recently Behiry et al. [8], have been introduced a discretized version of the ADM namely "Discrete Adomian Decomposition Method (DADM)". They mentioned several main advantages of applying DADM to solve Eq. (1). We can see that DHAM has all of the main advantages of DADM, which are:

(1) The matrix $\mathbf{B}$ is unchanged during the computation of components $x_m(t)$, $m \geqslant 1$.
(2) The computation of the solution need not to solve linear or nonlinear algebraic system of equations.
(3) The solution is simply obtained for arbitrary kernels.
(4) The computer program is very simple.In addition to the above mentioned advantages, DHAM has other advantages such that DADM has not them, which are:
(5) When $\hbar = -1$ the results obtained by DHAM are exactly the same as results obtained by DADM. To prove this claim, compare Eqs. (28)–(30) when $\hbar = -1$, with Eqs. (17)–(19) of [8]. Therefore, DADM is a special case of DHAM.
(6) The DHAM solution contains the convergence-control parameter $\hbar$, which we can choose properly by plotting the so-called $\hbar$-curves to ensure that the series solution is convergent, as suggested by Liao [9].

In the next section, we solve two nonlinear Fredholm integral equations considered in [8] by DHAM, while the HAM can not be applied for them. Also, we compare the obtained results by DHAM with the obtained results by DADM [8].

## 5. Numerical examples

**Example 5.1** [8]. Consider the nonlinear Fredholm integral equation

$$x(t) = t - \frac{\exp(1+t^4) - \exp(t^4)}{40} + \int_0^1 \frac{\exp(s^4+t^4)}{10}[x(s)]^3\, ds,$$

with the exact solution $x(t) = t$. For the same reason stated in [8] the HAM can not be applied, because the definite integral

$$\int_0^1 \exp(t^4)\, ds$$

is not analytically computable. To overcome this obstacle, we apply the DHAM to obtain an approximate solution for the above integral equation. To determine the components $x_m(t)$, $m \geqslant 1$ it is useful to list the first few Adomian polynomials. To this end, for the nonlinear term $F[x(s)] = [x(s)]^\alpha$, we find

$$A_0 = x_0^\alpha,$$
$$A_1 = \alpha\, x_0^{\alpha-1}x_1,$$
$$A_2 = \alpha\, x_0^{\alpha-1}x_2 + \frac{\alpha(\alpha-1)}{2!}x_1^2x_0^{\alpha-2},$$

and other polynomials can be derived (see Remark 3.1). By the Adomian polynomials derived above and using the recursive relation (19) together with considering the Simpson rule with step length $h = 1/8$, the components $x_m(t)$, $m \geqslant 1$ of the approximate series solution (20) are approximated as:

$$x_1(t) \approx [-(3.2009e-02)\,\hbar]\exp(t^4), x_2(t) \approx [-(3.2009e-02)\,\hbar - (2.4559e-02)\,\hbar^2]\exp(t^4), x_3(t) \approx [-(3.2009e-02)\,\hbar - (4.9119e-02)\,\hbar^2 - (1.9442e-02)\,\hbar^3]\exp(t^4), \quad \vdots$$

and so on. We approximate the exact solution by

$$\tilde{x}_3(t) = y(t) + \sum_{m=1}^{3} x_m(t)$$
$$\approx t - [(4.2957e-02) + (9.6027e-02)\ \hbar]\exp(t^4)$$
$$- [(7.3678e-02)\ \hbar^2 + (1.9442e-02)\ \hbar^3]\exp(t^4).$$

Our approximate solution contains the convergence-control parameter $\hbar$ which can be employed to adjust the convergence region of the approximate series solution. By means of so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the approximate series solution . As pointed out by Liao [9], the appropriate region for $\hbar$ is a horizontal line segment. In Fig. 1, we plot the $\hbar$-curve of $\tilde{x}_3(0.5)$. Thus, the valid region of $\hbar$ in this case is $-1.5 < \hbar < -0.9$. We can choose an appropriate value of $\hbar$ to ensure that the approximate series solution converge. To determine the best value of $\hbar$ ($\hbar^*$) we can plot $V_3^*(\hbar)$ and choose $\hbar^*$ such that the value of $V_3^*(\hbar)$ be minimum. Plotting $V_3^*(\hbar)$, it can be seen that $V_3^*(\hbar)$ is minimum about at $\hbar^* = -1.275$, see Fig. 2. In Table 2, we compare the obtained absolute error by DHAM with the obtained absolute error by DADM [8]. A clear conclusion can be draw from the numerical results that the DHAM algorithm provides highly accurate numerical solutions. It is also worth noting that the advantage of the DHM displays a fast convergence of the solutions by means of the convergence-control parameter $\hbar$. The illustration show the DHAM is numerically more accurate than the DADM.

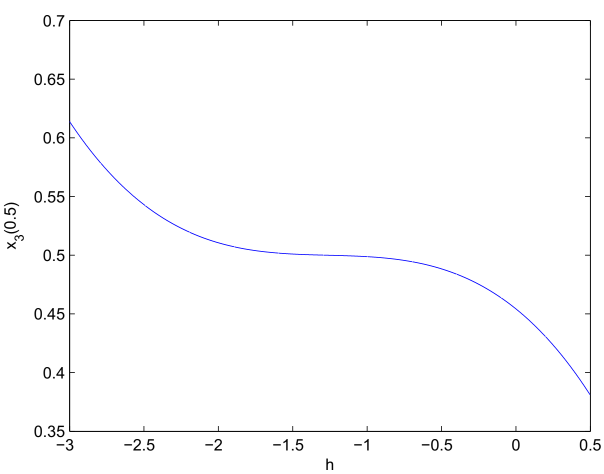


**Figure 1** The $\hbar$-curve of the 3rd-order approximation for Example 5.1.

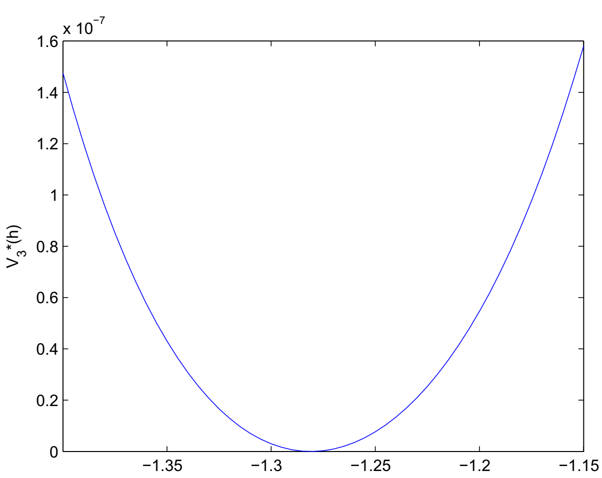


**Figure 2** Plot of $V_3^*(\hbar)$ for Example 5.1.

**Example 5.2** [8]. Consider the nonlinear Fredholm integral equation

$$x(t) = t + \frac{\cos(\exp(1)+t) - \cos(1+t)}{20} + \int_0^1 \frac{\sin(\exp(s)+t)}{20} \times \exp(x(s))\ ds,$$

with the exact solution $x(t) = t$. For the above equation the HAM can not be applied, because the definite integral

$$\int_0^1 \frac{\sin(\exp(s))}{20} \exp\left(\frac{\cos(\exp(1)+s) - \cos(1+s)}{20}\right) ds$$

is not available practically [8]. To overcome this difficulty, we suggest here to use DHAM. Using the recursive relation (19) together with considering the Simpson rule with step length $h = 1/8$, we obtain the components $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the approximate series solution (20) and present the approximate solution as

$$\tilde{x}_3(t) = y(t) + \sum_{m=1}^{3} x_m(t)$$
$$\approx t + \frac{\cos(\exp(1)+t) - \cos(1+t)}{20} - \sin(1+t)\ [(5.8123e-03)\hbar + (5.4106e-03)\hbar^2 + (1.6804e-03)\hbar^3]$$
$$- \sin(\exp(0.125)+t)\ [(2.6431e-02)\hbar + (2.4688e-02)\hbar^2 + (7.6949e-03)\hbar^3] - \sin(\exp(0.250)+t)\ [(1.5040e-02)\hbar + (1.4111e-02)\hbar^2 + (4.4188e-03)\hbar^3] - \sin(\exp(0.375)+t)\ [(3.4268e-02)\hbar + (3.2329e-02)\hbar^2 + (1.0181e-02)\hbar^3]$$
$$- \sin(\exp(0.500)+t)\ [(1.9538e-02)\hbar + (1.8550e-02)\hbar^2 + (5.8796e-03)\hbar^3] - \sin(\exp(0.625)+t)\ [(4.4594e-02)\hbar + (4.2645e-02)\hbar^2 + (1.3615e-02)\hbar^3] - \sin(\exp(0.750)+t)\ [(2.5465e-02)\hbar + (2.4543e-02)\hbar^2 + (7.8969e-03)\hbar^3]$$
$$- \sin(\exp(0.875)+t)\ [(5.8199e-02)\hbar + (5.6563e-02)\hbar^2 + (1.8351e-02)\hbar^3] - \sin(\exp(1.000)+t)\ [(1.1625e-02)\hbar + (1.6309e-02)\hbar^2 + (5.3364e-03)\hbar^3].$$

To find the convergence region of $\hbar$ we plot the $\hbar$-curve of $\tilde{x}_3(0.5)$, see Fig. 3. Therefore, the valid region of $\hbar$ in this case is $-1.3 < \hbar < -0.9$. Also, in Fig. 4, we plot $V_3^*(\hbar)$ to determine the best value of $\hbar(\hbar^*)$. Plotting $V_3^*(\hbar)$, it can be seen that $V_3^*(\hbar)$ is minimum about at $\hbar^* = -1.063$, see Fig. 4. In Table 3, we compare the obtained absolute error by DHAM with the obtained absolute error by DADM [8]. The illustration

**Table 2** The absolute error of DHAM and DADM in 3rd-order approximation.

| $x$ | DHAM ($\hbar$ = −1.275) | DAD [8] |
|---|---|---|
| 0.000 | 6.4569e−07 | 1.1673e−03 |
| 0.125 | 6.4584e−07 | 1.1676e−03 |
| 0.250 | 6.4821e−07 | 1.1718e−03 |
| 0.375 | 6.5859e−07 | 1.1906e−03 |
| 0.500 | 6.8733e−07 | 1.2426e−03 |
| 0.625 | 7.5213e−07 | 1.3597e−03 |
| 0.750 | 8.8600e−07 | 1.6017e−03 |
| 0.875 | 1.1604e−06 | 2.0977e−03 |
| 1.000 | 1.7552e−06 | 3.1730e−03 |

**Table 3** The absolute error of DHAM and DADM in 3rd-order approximation.

| $x$ | DHAM ($\hbar$ = −1.063) | DADM [8] |
|---|---|---|
| 0.000 | 1.1660e−06 | 5.1932e−05 |
| 0.125 | 6.3597e−07 | 5.2954e−05 |
| 0.250 | 9.6002e−08 | 5.3149e−05 |
| 0.375 | 4.4546e−07 | 5.2515e−05 |
| 0.500 | 9.7998e−07 | 5.1061e−05 |
| 0.625 | 1.4992e−06 | 4.8810e−05 |
| 0.750 | 1.9950e−06 | 4.5798e−05 |
| 0.875 | 2.4597e−06 | 4.2071e−05 |
| 1.000 | 2.8860e−06 | 3.7688e−05 |

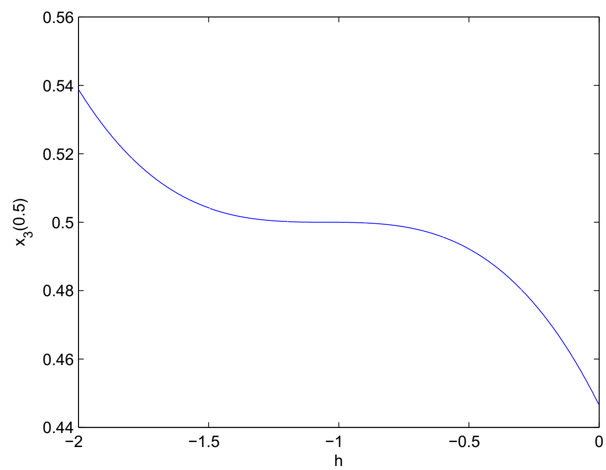


**Figure 3** The $\hbar$-curve of the 3rd-order approximation for Example 5.2.

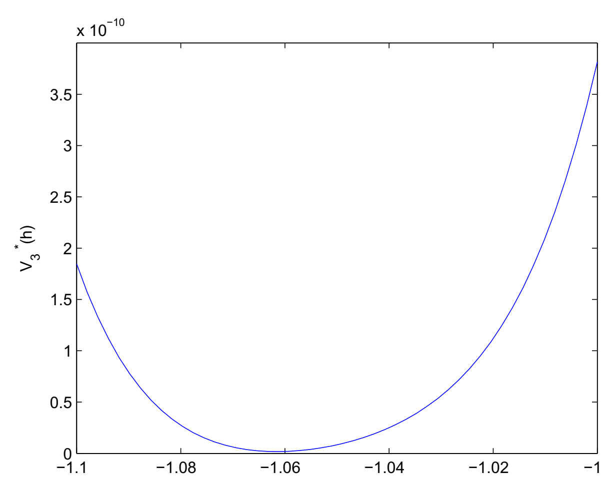


**Figure 4** Plot of $V_3^*(\hbar)$ for Example 5.2.

show the DHAM is numerically more accurate than the DADM.

## 6. Conclusion

In this paper, we have converted the non-numerical HAM to a numerical discretized version DHAM. The DHAM gives the numerical solution at the nodes used in the quadrature rules. In fact, for cases that evaluation of integrals analytically is impossible or complicated we apply DHAM instead of HAM. Also, in this paper we compared the DHAM with DADM. The results show that the DHAM is more powerful than the DADM and also the DADM is a special case of the DHAM.

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**T. Allahviranloo**, Department of Applied Mathematics, Science and Research Branch, Islamic Azad University, Post Code 14778, Tehran, Iran. T. Allahviranloo's interests include fuzzy linear system of equations, fuzzy differential equations, fuzzy integral equations and fuzzy partial differential equations.

**M. Ghanbari**, Department of Mathematics, Aliabad Katoul Branch, Islamic Azad University, Aliabad Katoul, Iran. M. Ghanbari's interests include numerical analysis, integral equations, homotopy analysis method and variational iteration method.