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Numerical solution of integral equations system of the second kind by Block–Pulse functions

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Abstract

This paper endeavors to formulate Block–Pulse functions to propose solutions for the Fredholm integral equations system. To begin with we describe the characteristic of Block–Pulse functions and will go on to indicate that through this method a system of Fredholm integral equations can be reduced to an algebraic equation. Numerical examples presented to illustrate the accuracy of the method.

Keywords: Block–Pulse functions; Linear Fredholm integral equations system; Product operation; Operational matrix

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1. Introduction

In recent years, many different basic functions have been used to estimate the solution of integral equations. In this paper we use BPF (Block–Pulse functions) as a simple base for solving a system of integral equations. This set of functions was first introduced to electrical engineers by Harmuth. Then several researchers (Gopalsami and Deekshatulu, 1997 [8]; Chen and Tsay, 1977 [9]; Sannuti, 1977 [10]) discussed the Block–Pulse functions and their operational matrix [1,2].

2. Properties of BPF

2.1. Definition of BPF

An m-set of BPF is defined as follows:

$$\Phi_i(t) = \begin{cases} 1, & (i-1)\frac{T}{m} \leqslant t < i\frac{T}{m}; \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

with $t \in [0, T)$ and $i = 1, 2, \ldots, m, \frac{T}{m} = h$. Figs. 1–4 illustrate $\Phi_1(t)$ to $\Phi_4(t)$.

Now we explain the properties of BPF.

(i) *Disjointness:*
We clearly have

$$\Phi_i(t)\Phi_j(t) = \begin{cases} \Phi_i(t), & i = j; \\ 0, & i \neq j, \end{cases} \tag{2}$$

$t \in [0, T)$, $i, j = 1, 2, \ldots, m$. This property is obtained from definition of BPF.

(ii) *Orthogonality:*
We have

$$\int_0^T \Phi_i(t)\Phi_j(t)\mathrm{d}t = \begin{cases} h, & i = j; \\ 0, & i \neq j, \end{cases} \tag{3}$$

$t \in [0, T)$, $i, j = 1, 2, \ldots, m$. This property is obtained from the disjointness property.

(iii) *Completeness:*
For every $f \in L^2$, $\{\Phi\}$ is complete if $\int \Phi f = 0$ then $f = 0$ almost everywhere. Because of completeness of $\{\Phi\}$, we have

$$\int_0^T f^2(t)\mathrm{d}t = \sum_{i=1}^{\infty} f_i^2 \|\Phi_i(t)\|^2$$

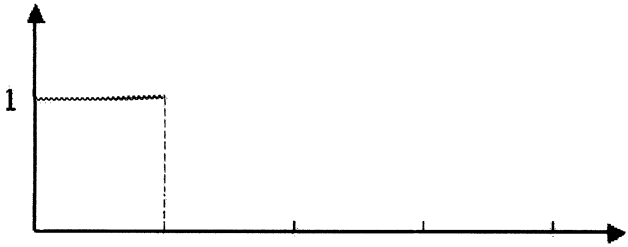

Fig. 1. $\Phi_1(t)$.

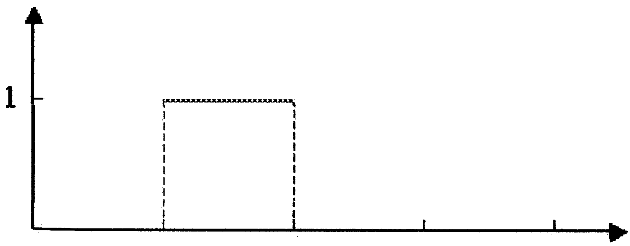

Fig. 2. $\Phi_2(t)$.

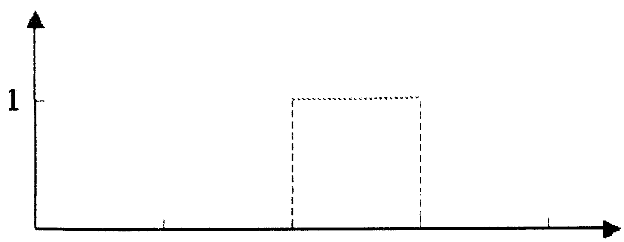

Fig. 3. $\Phi_3(t)$.

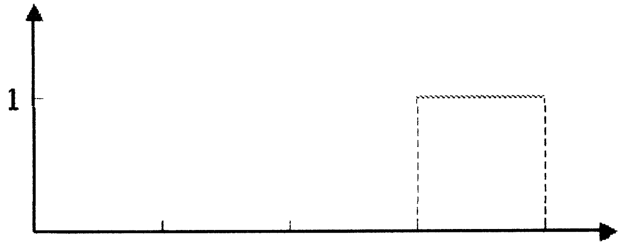

Fig. 4. $\Phi_4(t)$.

for every real bounded function $f(t)$ which is square integrable in the interval $t \in [0, T)$ and $f_i = \frac{1}{h} f(t) \Phi_i(t) \mathrm{d}t$.

2.2. Function approximation

The orthogonality property of BPF is the basis of expanding functions into their Block–Pulse series. For every $f(t) \in L^2$

$$f(t) = \sum_{i=1} m f_i \Phi_i(t),$$

where f_i is the coefficient of Block–Pulse function, with respect to ith Block–Pulse function $\Phi_i(t)$.

The criterion of this approximation is that mean square error between $f(t)$ and its expansion be minimum

$$\epsilon = \frac{1}{T} \int_0^T \left(f(t) - \sum_{j=1}^{m} f_j \Phi_j(t) \right)^2 \mathrm{d}t \tag{4}$$

so we can evaluate BP coefficients.

$$\frac{\partial \epsilon}{\partial f_i} = -\frac{2}{T} \int_0^T \left(f(t) - \sum_{j=1}^{m} f_j \Phi_j(t) \right) \Phi_i(t) \mathrm{d}t = 0, \tag{5}$$

$$f_i = \frac{1}{h} \int_0^T f(t) \Phi_i(t) \mathrm{d}t \tag{6}$$

for further information see [1–3].

As an example we use BPF to approximate $f(t) = t^2$, $t \in [0, 1)$ with $m = 4$.

$$f_1 = 4 \int_0^1 t^2 \Phi_1(t) \mathrm{d}t = \frac{1}{48},$$

$$f_2 = 4 \int_0^1 t^2 \Phi_2(t) \mathrm{d}t = \frac{7}{48},$$

$$f_3 = 4 \int_0^1 t^2 \Phi_3(t) \mathrm{d}t = \frac{19}{48},$$

$$f_4 = 4 \int_0^1 t^2 \Phi_4(t) \mathrm{d}t = \frac{37}{48},$$

$$f(t) = \frac{1}{48} \Phi_1(t) + \frac{7}{48} \Phi_2(t) + \frac{19}{48} \Phi_3(t) + \frac{37}{48} \Phi_4(t).$$

In vector form we have

$$f(t) = \sum_{i=1}^{m} f_i \Phi_i(t) = F^T \Phi(t) = \Phi^{\mathrm{T}} F,$$

where $F = [f_1, f_2, \ldots, f_m]^{\mathrm{T}}$, $f_i = \frac{1}{h} \int_0^T f(t)\Phi_i(t)\mathrm{d}t$,

$$\Phi(t) = [\Phi_1(t), \Phi_2(t), \ldots, \Phi_m(t)]^{\mathrm{T}}.$$

Now let $k(t,s)$ be a two variable function defined on $t \in [0, T)$ and $s \in [0, 1)$, then $k(s,t)$ can be expand to BPF as

$$k(t,s) = \Phi^{\mathrm{T}}(t) K \Psi(s),$$

where $\Phi(t)$ and $\Psi(s)$ are m_1 and m_2 dimensional Block–Pulse function vectors, and k is an $m_1 \times m_2$ Block–Pulse coefficient matrix.

2.3. Multiplication of two BPF

There are two different cases of this multiplication (see [1,4]). The first case is

$$\Phi(t)\Phi^{\mathrm{T}}(t) = \begin{pmatrix} \Phi_1(t) & 0 & \vdots & 0 \\ 0 & \Phi_2(t) & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \Phi_m(t) \end{pmatrix}. \tag{7}$$

It is same as disjointness property of BPF. As you see it is a diagonal matrix with m Block–Pulse function.

The second case is

$$\Phi^{\mathrm{T}}(t)\Phi(t) = 1$$

because, $\sum_{i=1}^{m} (\Phi_i(t))^2 = \sum_{i=1}^{m} \Phi_i(t) = 1$.

2.4. Operational matrix of integration

BPF integration property expressed by an operational equation as

$$\int_0^T \Phi(t)\mathrm{d}t = P\Phi(t),$$

where

$$\Phi(t) = [\Phi_1(t), \Phi_2(t), \ldots, \Phi_m(t)]^{\mathrm{T}}$$

and P is operational matrix as you see in Figs. 5–8 for $m = 4$. A general formula for $P_{m\times m}$ can be written as

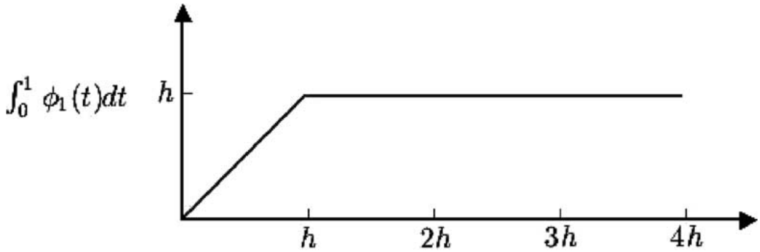

Fig. 5. $\int_0^t \Phi_1(t)\mathrm{d}t$.

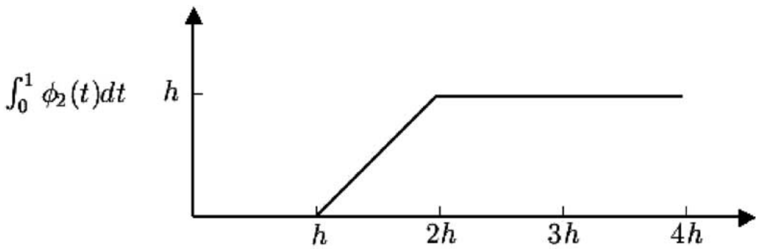

Fig. 6. $\int_0^t \Phi_2(t)\mathrm{d}t$.

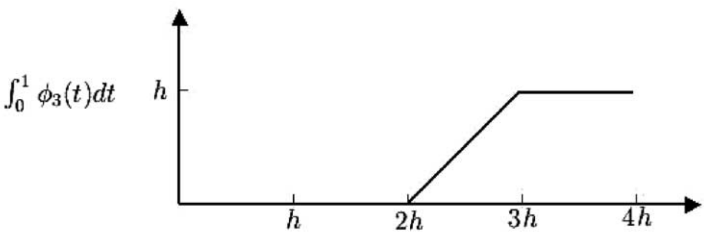

Fig. 7. $\int_0^t \Phi_3(t)\mathrm{d}t$.

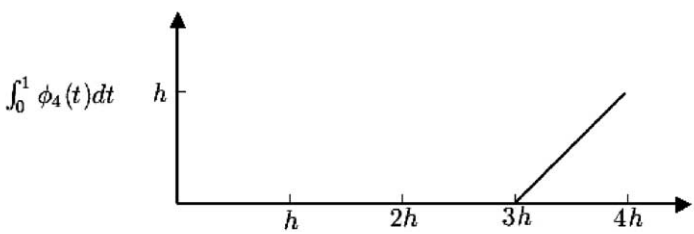

Fig. 8. $\int_0^t \Phi_4(t)\mathrm{d}t$.

$$P = \frac{1}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{8}$$

This matrix has a regular form i.e. it is an upper triangular matrix and its kth row can be obtained by shifting the first row $(k-1)$ positions to the right.

We can also verify that all the m eigenvalue of this upper triangular matrix are $\frac{h}{2}$.

By using this matrix we can express the integral of a function $f(t)$ into its Block–Pulse series

$$\int_0^t f(t)\mathrm{d}t = \int_0^t F^T\Phi(t)\mathrm{d}t = F^T P\Phi(t).$$

3. Linear integral equations system

Consider following integral equations system (see [5–7]):

$$\sum_{j=1}^{n} y_j(x) = f_i(x) + \lambda\sum_{j=1}^{n}\int_\alpha^\beta k_{ij}(x,t)y_j(t)\mathrm{d}t, \quad i = 1,2,\ldots,n. \tag{9}$$

Our problem is to determine Black–Pulse coefficient of $y_j(x)$, $j = 1,2,\ldots,n$ in the interval $x \in [\alpha,\beta)$ from the known functions $f_i(x)$, $i = 1,2,\ldots,n$ and the kernels $k_{ij}(x,t)$ $i,j = 1,2,\ldots,n$. Usually we set $\alpha = 0$ to facilities the use of Black–Pulse functions. In case $\alpha \neq 0$ we set $X = \frac{x-\alpha}{\beta-\alpha}T$ where $T = mh$.

We approximate f_i, y_j, k_{ij} by its BPF as follows:

$$f_i(x) \simeq F_i^T\Phi(x),$$

$$y_i(x) \simeq Y_j^T\Phi(x),$$

$$k_{ij}(x,t) \simeq \Phi^{\mathrm{T}}(x)k_{ij}\Psi(t),$$

where F_i, Y_i and k_{ij} are defined in section [3]. With substituting in Eq. (9) we have

$$\sum_{j=1}^{n} Y_j^T\Phi(x) = F_i^T\Phi(x) + \lambda Y_j^T\sum_{j=1}^{n}\int_0^{mh}\Psi(t)\Psi^T(t)\mathrm{d}t K_{ij}^T\Phi(x) \tag{10}$$

by (10) and $\int_0^{mh} \Psi(t)\Psi^T(t)\mathrm{d}t = hI$ gives

$$\sum_{j=1}^{n} Y_j^T \Phi(x) = F_i^T \Phi(x) + \lambda Y_j^T \sum_{j=1}^{n} hIK_{ij}^T \Phi(x)$$

or

$$\sum_{j=1}^{n} (I - h\lambda K_{ij}^T) Y_j = F_i, \quad i = 1, 2, \ldots, n.$$

Set $A_{ij} = I - h\lambda K_{ij}^T$, then we have

$$\sum_{j=1}^{n} A_{ij} Y_j = F_i, \quad i = 1, 2, \ldots, n$$

which is a linear system

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}. \tag{11}$$

After solving the above system we can find

$$Y_j; \quad i = 1, 2, \ldots, n$$

and $y_j \simeq \Phi^T Y_j$.

4. Numerical examples

Example 1. Consider the integral equations system

$$y_1(t) + \int_0^1 (t+s) y_1(s)\mathrm{d}s + \int_0^1 (t + 2s^2) y_2(s)\mathrm{d}s = \frac{11}{6} t + \frac{11}{6},$$

$$y_2(t) + \int_0^1 ts^2 y_1(s)\mathrm{d}s + \int t^2 s y_2(s)\mathrm{d}s = \frac{5}{4} t^2 + \frac{1}{4} t$$

with exact solution $y_1(t) = t$, $y_2(t) = t^2$ results with $m = 16$ and $m = 32$ shown in Table 1.

Example 2. Consider the linear integral equations system

$$y_1(t) + \int_0^1 \mathrm{e}^{t-s} y_1(s)\mathrm{d}s \int_0^1 \mathrm{e}^{(t+2)s} y_2(s)\mathrm{d}s = 2\mathrm{e}^t + \frac{\mathrm{e}^{t+s} - 1}{t+1},$$

Table 1
Numerical results for polynomial kernel

t	$m = 16$	$m = 32$	Exact solution
0	(.02154, .06644)	(.01421, .04331)	(0,0)
.1	(.01302, .00944)	(.09802, .00971)	(.1, .01)
.2	(.21898, .04281)	(.20345, .04172)	(.2, .04)
.3	(.28145, .08177)	(.29146, .08866)	(.3, .09)
.4	(.40542, .17123)	(.38790, .16812)	(.4, .16)
.5	(.46877, .28136)	(.48614, .26409)	(.5, .25)
.6	(.51462, .39171)	(.60641, .38405)	(.6, .36)
.7	(.71876, .46299)	(.70914, .48991)	(.7, .49)
.8	(.78123, .66666)	(.81314, .65795)	(.8, .64)
.9	(.90615, .90855)	(.91512, .80551)	(.9, .81)
1	(.96863, .98843)	(.99315, .99971)	(1,1)

Table 2
Numerical results for exponential kernel

t	$m = 16$	$m = 32$	Exact solution
0	(1.03065, .96950)	(1.01047, .98470)	(1,1)
.1	(1.0976, .91079)	(1.11641, .89657)	(1.10517, .90483)
.2	(1.24352, .80392)	(1.22496, .81636)	(1.22140, .81873)
.3	(1.32380, .75520)	(1.34547, .74351)	(1.34986, .74081)
.4	(1.50014, .66666)	(1.47776, .67682)	(1.49182, .67032)
.5	(1.59687, .62642)	(1.6230, .61621)	(1.64872, .60653)
.6	(1.8099, .55283)	(1.83910, .54386)	(1.82211, .54881)
.7	(2.0512, .48790)	(2.01982, .49520)	(2. 01376, .49659)
.8	(2.1837, .45870)	(2.2190, .45010)	(2.22554, .44932)
.9	(2.47439, .40490)	(2.43651, .41070)	(2.45960, .40657)
1	(2.6352, .38087)	(2.67611, .37401)	(2.71828, .36788)

$$y_2(t) + \int_0^1 \mathrm{e}^{ts} y_1(s)\mathrm{d}s \int \mathrm{e}^{(t+s)} y_2(s)\mathrm{d}s = \mathrm{e}^t + \mathrm{e}^{-t} + \frac{\mathrm{e}^{t+s} - 1}{t+1}$$

with exact solution $y_1(t) = \mathrm{e}^t$ and $y_2(t) = \mathrm{e}^{-t}$. The computational results for $m = 16$ and $m = 32$ together with the exact solution are given in Table 2.

5. Conclusion

Block–Pulse functions which had previously been used in control theory have been developed in this paper to solve a linear Fredholm integral equations system of the second kind by means of references [6,7]. As examples indicate, in order to increase the accuracy of the numerical results, it is necessary to increase m. Other orthogonal functions can also be used for solving integral equations system of the second kind.

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