Volume Average Demagnetizing Tensor of Rectangular Prisms

Hiroshi Fukushima, Yoshinobu Nakatani, and Nobuo Hayashi, *Member, IEEE*

***Abstract*— An exact analytic formula is presented for the magnetostatic field produced by a three-dimensional array of uniformly magnetized rectangular prisms. It is derived from the volume integration of a point-function demagnetizing tensor. The values of the point-function demagnetizing tensor at the centers of rectangular prisms of different aspect-ratios are compared with the volume averaged ones. This difference for the zz-component of demagnetizing tensors reaches 35% at its maximum when the aspect-ratio is 2.**

***Index Terms*—Computer simulation, demagnetizing factor, micromagnetics, rectangular prism.**

I. Introduction

RECENTLY, computer simulations in micromagnetics have become a significant method for examining the behavior of magnetic materials. In these simulations, a specimen is divided into cells, and the magnetization of each cell is assumed to be uniform. The cell can be a cubic, rectangular, or hexagonal prism. The distribution of magnetization in the specimen is obtained by solving the Landau–Lifshitz–Gilbert equation or the energy minimization. In the modeling, exchange, anisotropy, external, and demagnetizing fields are considered as the effective fields. By demagnetizing field we mean the magnetostatic field whose source is the magnetization of the specimen itself. It can be calculated from the demagnetizing tensor between each pair of the cells; at each cell the demagnetizing field is the product of this tensor and the corresponding magnetization [1]. For obtaining the averaged demagnetizing field at an observing cell, it is necessary to average the point-function demagnetizing tensor over the volume of the cell; the point-function demagnetizing tensor is defined at an observing point. The averaged demagnetizing factors (namely the components of the tensor) over the entire volume of the specimen are called the magnetometric demagnetizing factors [2], [3]. We use, however, the name of volume average demagnetizing tensor, because this tensor does not relate to the magnetometer methods for measuring magnetization, but is utilized for calculating the demagnetizing field in the specimen. The volume integration can be computed numerically with sufficient accuracy, but it is a time-consuming process. For the purpose of saving time, in some cases, the values at the center of an observing cell were used instead of the volume average values over the cell [4].

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H. Fukushima, retired, is at 3-73 Honda, Midori-ku, Chiba-shi, Chiba 266, Japan.

Y. Nakatani and N. Hayashi are with the University of Electro-Communications, Chofugaoka, Chofu-shi, Tokyo 182, Japan.

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Schabes and Aharoni derived an exact analytic formula for the magnetostatic field in a three-dimensional array of ferromagnetic cubes [5]. Their method provides an accurate value, but this is only applicable to cubic cell cases.

In the case of noncubic cells, such as thin or elongated rectangular prisms, the use of the averaged demagnetizing field, which is obtained by dividing a rectangular prism cell into cubic ones, produces improved results in the calculation of the switching field [6].

The present paper provides a method that computes analytically the demagnetizing field in a rectangular prism cell array.

In general, it is expected that the difference between the volume averaged demagnetizing tensor of a cubic cell and the central one becomes a maximum at the nearest neighbor cell. However, it is shown in Section III-A that the difference can reach a maximum at another cell.

Effects of the aspect-ratio of the rectangular prism cell on the difference are also described.

II. Formulas

When a magnetic specimen is discretized into an array of rectangular prism cells, in which the magnetization is assumed uniform, the magnetic field at a point r_j arising from the magnetization of a source cell i is described as follows:

$$\boldsymbol{H}(\boldsymbol{r}_j) = -\nabla_j \int_{V_i} \boldsymbol{M}_i \cdot \nabla_i \left(\frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_i|} \right) dv_i \tag{1}$$

where $\boldsymbol{H}(\boldsymbol{r}_j)$ is the magnetic field at an observing point $\boldsymbol{r}_j$, $\boldsymbol{r}_i$ is a point in the cell i, and $\boldsymbol{M}_i$ and V_i are the magnetization vector and the volume of the cell i, respectively. Equation (1) can be rewritten as

$$\boldsymbol{H}(\boldsymbol{r}_j) = -\boldsymbol{K}(\boldsymbol{r}_j) \cdot \boldsymbol{M}_i \tag{2}$$

where $\boldsymbol{K}(\boldsymbol{r}_j)$ is the point-function demagnetizing tensor [7].

When the magnetostatic interaction between the source cell i and the observing cell j is considered, it is necessary to calculate the magnetic field which is averaged over the volume of the cell j. By taking volume average, (2) is transformed into

$$\langle \boldsymbol{H} \rangle_v = -\langle \boldsymbol{K} \rangle_v \cdot \boldsymbol{M}_i \tag{3}$$

where $\langle \boldsymbol{H} \rangle_v$ is the volume average magnetic field, $\langle \boldsymbol{K} \rangle_v$ is the volume average demagnetizing tensor [8].

The formulas for the xx- and xy-components of the point-function demagnetizing tensor of rectangular prism cells are described in CGS system of units as follows:

$$K_{xx} = \sum_{i,j,k=1}^{2} (-1)^{i+j+k-1} \arctan\left[\frac{(y-y_j)(z-z_k)}{(x-x_i)D(i,j,k)}\right] \quad (4)$$

$$K_{xy} = -\sum_{i,j,k=1}^{2} (-1)^{i+j+k-1} \ln|D(i,j,k)+(z-z_k)| \quad (5)$$

where (x,y,z) denote the observing point, (x_i,y_j,z_k) are the vertices of a source cell, $D(i,j,k) = \sqrt{(x-x_i)^2+(y-y_j)^2+(z-z_k)^2}$, here $i,j,k = 1,2$ [4]. The other components can be derived from (4) and (5) by permutations of coordinates.

By the triple integration of (4) and (5), the exact analytic expressions for the xx- and xy-components of the volume average demagnetizing tensor, $\langle K_{xx}\rangle_v$ and $\langle K_{xy}\rangle_v$, are obtained, respectively, as follows:

$$\langle K_{xx}\rangle_v = \frac{1}{v}\sum_{i,j,k=1}^{2}(-1)^{i+j+k-1}\int_{z+z_1}^{z+z_2}\int_{y+y_1}^{y+y_2} \cdot\int_{x+x_1}^{x+x_2}\arctan\left(\frac{(y'-y_j)(z'-z_k)}{(x'-x_i)D(i,j,k)}\right)dx'\,dy'\,dz' \quad (6)$$

$$\langle K_{xy}\rangle_v = -\frac{1}{v}\sum_{i,j,k=1}^{2}(-1)^{i+j+k-1}\int_{y+y_1}^{y+y_2}\int_{x+x_1}^{x+x_2} \cdot\int_{z+z_1}^{z+z_2}\ln|D(i,j,k)+(z'-z_k)|\,dz'\,dx'\,dy' \quad (7)$$

where v is the volume of the cell, the origin of coordinates is located at the center of the source cell, $(x,\ y,\ z)$ is the center of the observing cell, $(x_i,\ y_j,\ z_k)$, here $i,\ j,\ k = 1,\ 2$ are the vertices of the source cell, $D(i,\ j,\ k) = \sqrt{(x'-x_i)^2+(y'-y_j)^2+(z'-z_k)^2}$.

The integration procedure is very complicated. For avoiding entangled calculations, the indefinite integrations are carried out three times consecutively first, then the differences between the upper and lower bounds are calculated.

By introducing new variables $X = x - x_i$, $Y = y - y_j$, and $Z = z - z_k$, the terms in (4) and (5) can be written as (8) and (9), respectively,

$$\arctan\left(\frac{YZ}{XD}\right) \quad (8)$$

$$-\ln|D+Z| \quad (9)$$

where $D = \sqrt{X^2+Y^2+Z^2}$. The indefinite integrals of (8) and (9) with respect to X, Y, and Z are represented as (10) and (11), respectively,

$$\begin{aligned}
&\iiint \arctan\left(\frac{YZ}{XD}\right)dX\,dY\,dZ \\
&= XYZ\arctan\left(\frac{YZ}{XD}\right)+\frac{1}{2}Y(Z^2-X^2)\ln|D-Y| \\
&+\frac{1}{2}Z(Y^2-X^2)\ln|D-Z|+\frac{1}{6}(Y^2+Z^2-2X^2)D \\
&-\frac{1}{4}Y(Z^2-X^2)\ln|X^2+Z^2| \\
&-\frac{1}{4}Z(Y^2-X^2)\ln|X^2+Y^2| \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&-\iiint \ln|D+Z|\,dZ\,dX\,dY \\
&= -XYZ\ln|D+Z|+\frac{1}{6}Y(Y^2-3Z^2)\ln|D+X| \\
&+\frac{1}{6}X(X^2-3Z^2)\ln|D+Y|+\frac{1}{2}X^2Z\arctan\left(\frac{YZ}{XD}\right) \\
&+\frac{1}{2}Y^2Z\arctan\left(\frac{XZ}{YD}\right)+\frac{1}{6}Z^3\arctan\left(\frac{XY}{ZD}\right) \\
&+\frac{1}{3}XYD-\frac{1}{2}ZX^2\arctan\left(\frac{Y}{X}\right) \\
&-\frac{1}{2}ZY^2\arctan\left(\frac{X}{Y}\right)+\frac{3}{2}XYZ-\frac{2}{3}Z^3\arctan\left(\frac{Y}{Z}\right) \\
&+\frac{2}{3}YZ^2-\frac{1}{18}Y^3 \qquad (11)
\end{aligned}$$

where integral constants, which can be functions of one or two variables, are omitted. Several useful formulas for obtaining the above integrals are presented in the Appendix.

The definite integral is obtained from the indefinite integral as follows. Let $g(X,\ Y,\ Z)$ be an integrand. We define the three indefinite integrals: $G^X(X,\ Y,\ Z)$, $G^{XY}(X,\ Y,\ Z)$, and $G^{XYZ}(X,\ Y,\ Z)$, as

$$\begin{aligned}
G^X(XY,Z) &= \int g(X,Y,Z)\,dX, \\
G^{XY}(X,Y,Z) &= \int G^X(X,Y,Z)\,dY \\
&= \iint g(X,\ Y,\ Z)\,dX\,dY \\
G^{XYZ}(X,\ Y,\ Z) &= \int G^{XY}(X,\ Y,\ Z)\,dZ \\
&= \iint G^X(X,Y,Z)\,dY\,dZ \\
&= \iiint g(X,Y,Z)\,dX\,dY\,dZ. \qquad (12)
\end{aligned}$$

Then the definite integral of $g(X,\ Y,\ Z)$ with respect to X, Y, and Z is expressed as follows:

$$\begin{aligned}
&\int_{Z_a}^{Z_b}\int_{Y_a}^{Y_b}\int_{X_a}^{X_b} g(X,\ Y,\ Z)\,dX\,dY\,dZ \\
&= \int_{Z_a}^{Z_b}\int_{Y_a}^{Y_b}[G^X(X_b,\ Y,\ Z)-G^X(X_a,\ Y,\ Z)]\,dY\,dZ \\
&= \int_{Z_a}^{Z_b}[G^{XY}(X_b,\ Y_b,\ Z)-G^{XY}(X_a,\ Y_b,\ Z) \\
&\qquad -G^{XY}(X_b,\ Y_a,\ Z)+G^{XY}(X_a,\ Y_b,\ Z)]\,dZ \\
&= G^{XYZ}(X_b,\ Y_b,\ Z_b)-G^{XYZ}(X_a,\ Y_b,\ Z_b) \\
&\quad -G^{XYZ}(X_b,\ Y_a,\ Z_b)+G^{XYZ}(X_a,\ Y_a,\ Z_b) \\
&\quad -G^{XYZ}(X_b,\ Y_b,\ Z_a)+G^{XYZ}(X_a,\ Y_b,\ Z_a) \\
&\quad +G^{XYZ}(X_b,\ Y_a,\ Z_a)-G^{XYZ}(X_a,\ Y_a,\ Z_a) \qquad (13)
\end{aligned}$$

where X_a, X_b, Y_a, Y_b, Z_a, and Z_b are the lower and upper bounds of X, Y, and Z, respectively. By means of this procedure, the definite integrations in (6) and (7) are carried out.

The term in (11) that has less than three variables is canceled out in the calculation of definite integral, because it can be considered as integral constant; for example, the term $-\frac{2}{3}Z^3\arctan(Y/Z)$ vanishes in the definite integral with respect to X.

Furthermore, the term in (10) or (11) that is linear with respect to X, Y, or Z, for example, $-\frac{1}{4}Y(Z^2-X^2)\ln|X^2+Z^2|$, cancels the other in the definite integral as the following explanation shows [5]. A term t in (11) is assumed to be linear with respect to X; namely $t = cX$. Besides, X is represented as $X = x + x_i - x_j$, where $i,\ j = 1,\ 2$ from (6) and (7). Therefore, X has four values corresponding to $i,\ j = 1,\ 2$. The values of X and t in the definite integral are shown as

$$\begin{aligned} &X_1 = x + x_1 - x_1, \qquad X_2 = x + x_1 - x_2\\ &X_3 = x + x_2 - x_1, \qquad X_4 = x + x_2 - x_2\\ &t_1 = cx, \qquad t_2 = -c(x + x_1 - x_2)\\ &t_3 = z - c(x + x_2 - x_1), \qquad t_4 = cx \end{aligned} \tag{14}$$

where t_1, t_2, t_3, and t_4 are values of t, c is a function of Y and Z. The sum of t_1, t_2, t_3, and t_4 becomes zero.

By omitting the ineffective terms mentioned above, we transform (10) and (11) into (15) and (16), respectively, as

$$\begin{aligned} F1(X;Y,Z) = {} & XYZ\ \arctan\left(\frac{YZ}{XD}\right)\\ &+\frac{1}{2}Y\left(Z^2-X^2\right)\ln|D-Y|\\ &+\frac{1}{2}Z\left(Y^2-X^2\right)\ln|D-Z|\\ &+\frac{1}{6}\left(Y^2+Z^2-2X^2\right)D \end{aligned} \tag{15}$$

$$\begin{aligned} F2(Z;X,Y) = {} & -XYZ\ln|D+Z|\\ &+\frac{1}{6}Y\left(Y^2-3Z^2\right)\ln|D+X|\\ &+\frac{1}{6}X\left(X^2-3Z^2\right)\ln|D+Y|\\ &+\frac{1}{2}X^2Z\ \arctan\left(\frac{YZ}{XD}\right)\\ &+\frac{1}{2}Y^2Z\ \arctan\left(\frac{XZ}{YD}\right)\\ &+\frac{1}{6}Z^3\ \arctan\left(\frac{XY}{ZD}\right)+\frac{1}{3}XYD \end{aligned} \tag{16}$$

where $D = \sqrt{X^2+Y^2+Z^2}$. In (15) and (16), the second and third arguments are symmetric with each other. These functions $F1$ and $F2$ are substantially the same as the functions F_2 and G_2 in [5].

By using $F1$ and $F2$, (6) and (7) are expressed as (17) and (18), respectively,

$$\begin{aligned} \langle K_{xx}\rangle_v = \frac{1}{v}&\sum_{i,\,j,\,k,\,l,\,m,\,n=1}^{2}(-1)^{i+j+k+l+m+n-1}\\ &\times F1(x+x_l-x_i;\ y+y_m-y_j,\ z+z_n-z_k) \end{aligned} \tag{17}$$

$$\begin{aligned} \langle K_{xy}\rangle_v = \frac{1}{v}&\sum_{i,\,j,\,k,\,l,\,m,\,n=1}^{2}(-1)^{i+j+k+l+m+n-1}\\ &\times F2(z+z_n-z_k;\ x+x_l-x_i,\ y+y_m-y_j) \end{aligned} \tag{18}$$

where the origin of coordinates is located at the center of the source cell, $(x,\ y,\ z)$ is the center of the observing cell, $(x_i,\ y_j,\ z_k)$, here $i,\ j,\ k = 1,\ 2$ are the vertices of the source cell.

For computer implementation, (17) and (18) can be further simplified by introducing the following four arrays:

$$\begin{aligned} &ax(1) = -ddx, \quad && ax(2) = 0, \quad && ax(3) = ddx\\ &ay(1) = -ddy, && ay(2) = 0, && ay(3) = ddy\\ &az(1) = -ddz, && az(2) = 0, && az(3) = ddz\\ &sn(1) = 1, && sn(2) = 2, && sn(3) = 1 \end{aligned} \tag{19}$$

where ddx, ddy, and ddz are the length of each side of the cell directing to x-, y-, and z-axis, respectively. By using these arrays, (17) and (18) are transformed into (20) and (21), respectively, as follows:

$$\begin{aligned} \langle K_{xx}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F1[x+ax(i);y+ay(j),z+az(k)] \end{aligned} \tag{20}$$

$$\begin{aligned} \langle K_{xy}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F2[z+az(k);x+ax(i),y+ay(j)]. \end{aligned} \tag{21}$$

The other components are derived from the above formulas by permutations of coordinates. Namely

$$\begin{aligned} \langle K_{yy}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F1[y+ay(j);z+az(k),x+ax(i)] \end{aligned} \tag{22}$$

$$\begin{aligned} \langle K_{zz}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F1[z+az(k);x+ax(i),y+ay(j)] \end{aligned} \tag{23}$$

$$\begin{aligned} \langle K_{xz}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F2[y+ay(j);z+az(k),x+ax(i)] \end{aligned} \tag{24}$$

$$\begin{aligned} \langle K_{yz}\rangle_v = \frac{1}{v}&\sum_{i,j,k=1}^{3}(-1)^{i+j+k-1}sn(i)sn(j)sn(k)\\ &\times F2[x+ax(i);y+ay(j),z+az(k)]. \end{aligned} \tag{25}$$

III. Discussion

A. Differences Between K_{xx} and $\langle K_{xx}\rangle_v$, and Between K_{xy} and $\langle K_{xy}\rangle_v$ for Cubic Cells

It was reported that the demagnetizing field at the center of a nearest-neighbor cell differs 0.17% from the volume-averaged one [4]. At the second nearest-neighbor cell, however, it differs 1.1%. The relative differences in the case of cubic cells which are located side by side in the direction of the x-axis are listed in Table I. The tables for K_{yy} and K_{zz} are not shown here, but similar distributions are obtained.

For an array of cubic cells located on the diagonal line, in the [110] direction, the relative differences between K_{xy} and

TABLE I
DISTRIBUTION OF RELATIVE DIFFERENCE BETWEEN K_{xx} AND $\langle K_{xx}\rangle_v$ FOR CUBIC CELLS. HERE *RD* STANDS FOR $(K_{xx} - \langle K_{xx}\rangle_v)/\langle K_{xx}\rangle_v$ IN %, d IS THE DISTANCE OF THE OBSERVING POINT FROM THE CENTER OF THE SOURCE CELL, AND l IS THE SIZE OF THE CELL. OBSERVING POINTS ARE LOCATED ALONG THE [100] AXIS

d/l	*RD* (%)
0	0
1	−0.17
2	1.12
3	0.26
4	0.08

TABLE II
DISTRIBUTION OF RELATIVE DIFFERENCE BETWEEN K_{xy} AND $\langle K_{xy}\rangle_v$ FOR CUBIC CELLS. HERE *RD* DENOTES $(K_{xy} - \langle K_{xy}\rangle_v)/\langle K_{xy}\rangle_v$ IN %. IN THIS TABLE, OBSERVING POINTS ARE LOCATED ALONG THE [110] AXIS. WHEN $d/l = 0$, K_{xy} AND $\langle K_{xy}\rangle_v$ ARE BOTH ZERO

d/l	*RD* (%)
0	—
$\sqrt{2}$	-5.8
$2\sqrt{2}$	-0.14
$3\sqrt{2}$	-0.03

TABLE III
DISTRIBUTION OF RELATIVE DIFFERENCE BETWEEN K_{xz} AND $\langle K_{xz}\rangle_v$ FOR CUBIC CELLS. HERE *RD* DENOTES $(K_{xz} - \langle K_{xz}\rangle_v)/\langle K_{xz}\rangle_v$ IN %. IN THIS TABLE, OBSERVING POINTS ARE LOCATED ALONG THE [111] AXIS. WHEN $d/l = 0$, K_{xz} AND $\langle K_{xz}\rangle_v$ ARE BOTH ZERO

d/l	*RD* (%)
0	—
$\sqrt{3}$	-2.81
$2\sqrt{3}$	-0.11
$3\sqrt{3}$	-0.02

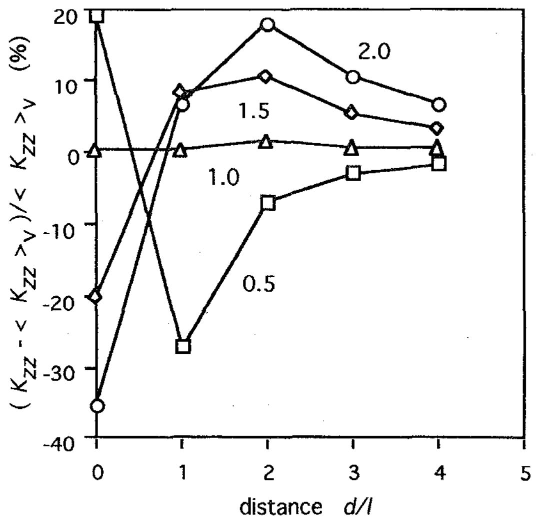

Fig. 1. Distribution of relative differences between K_{zz} and $\langle K_{zz}\rangle_v$ for various values of the aspect-ratio of rectangular prism cell. The figure by a curve is the aspect-ratio. d is the distance of the observing point from the center of the source cell, and l is the lateral size of the cell. Observing points are located along the x-axis.

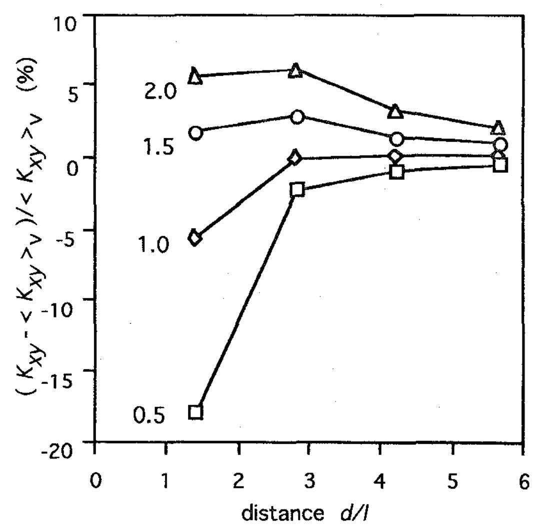

Fig. 2. Distribution of relative differences between K_{xy} and $\langle K_{xy}\rangle_v$ for various values of the aspect-ratio of rectangular prism cell. Observing points are located along the [110]-axis on the x–y plane.

$\langle K_{xy}\rangle_v$ are shown in Table II. Table III shows the relative differences between K_{xz} and $\langle K_{xz}\rangle_v$ for cubic cells on the diagonally upward line in the [111] direction.

In these tables, the difference reaches 5.8% at its maximum. The difference of this amount may not be ignored.

B. Effects of the Aspect-Ratio on the Difference

In the case of the computer simulation of magnetic films, it sometimes happens that the use of cells of rectangular prism, thin or elongated, is convenient to save the computing time. When the cells of rectangular prism are employed, attention should be paid to the effect of aspect-ratio on the calculation of the demagnetizing field.

In this paper, a rectangular prism cell has a square cross section. The aspect-ratio is defined as the ratio of the longitudinal length of the cell to its lateral one. The cell is so placed in the coordinate system that the cross section cut by the x–y plane is square. The longitudinal direction is parallel to the z-axis.

The distribution of relative differences between K_{zz} and $\langle K_{zz}\rangle_v$ for the cells with various aspect-ratios is shown in Fig. 1. In this figure, observing points are located along x-axis. It is seen that the difference changes largely between a source cell and the nearest-neighbor cell, and it reaches about 35% when the aspect-ratio is 2. In the case of the aspect-ratio which differs extremely from 1, when the demagnetizing field is calculated by using K_{zz} instead of $\langle K_{zz}\rangle_v$, large errors can occur. Fig. 2 shows a similar distribution for K_{xy} and $\langle K_{xy}\rangle_v$. In Fig. 2, observing points are located along the diagonal on the x–y plane. The change of differences is not so drastic as in Fig. 1.

IV. Conclusion

Analytical formulas for the volume average demagnetizing tensor of rectangular prisms have been presented. The use of the volume average formulas is probably important for the calculation of the demagnetizing field arising from the neighbors of an observing cell in the interacting array of rectangular prism cells, in particular, when the aspect-ratio of the cell is far different from one.

Appendix

Several useful formulas for triple integrations of the demagnetizing tensor are presented.

$$\begin{aligned}\int \arctan\left(\frac{A\cdot u}{B\sqrt{u^2+A^2+B^2}}\right)du &= u\cdot\arctan\left(\frac{A\cdot u}{B\sqrt{u^2+A^2+B^2}}\right)\\ &\quad+\frac{1}{2}B\cdot\ln\left|\frac{\sqrt{u^2+A^2+B^2}+A}{\sqrt{u^2+A^2+B^2}-A}\right|\end{aligned}\tag{A1}$$

$$\begin{aligned}\int \arctan\left(\frac{A\cdot B}{u\sqrt{u^2+A^2+B^2}}\right)du &= u\cdot\arctan\left(\frac{A\cdot B}{u\sqrt{u^2+A^2+B^2}}\right)\\ &\quad+\frac{1}{2}A\cdot\ln\left|\frac{\sqrt{u^2+A^2+B^2}-B}{\sqrt{u^2+A^2+B^2}+B}\right|\\ &\quad+\frac{1}{2}B\cdot\ln\left|\frac{\sqrt{u^2+A^2+B^2}-A}{\sqrt{u^2+A^2+B^2}+A}\right|\end{aligned}\tag{A2}$$

$$\begin{aligned}\int u\cdot\arctan\left(\frac{A\cdot B}{u\sqrt{u^2+A^2+B^2}}\right)du &= \frac{1}{2}u^2\arctan\left(\frac{A\cdot B}{u\sqrt{u^2+A^2+B^2}}\right)\\ &\quad-\frac{1}{2}A^2\arctan\left(\frac{B\cdot u}{A\sqrt{u^2+A^2+B^2}}\right)\\ &\quad-\frac{1}{2}B^2\arctan\left(\frac{A\cdot u}{B\sqrt{u^2+A^2+B^2}}\right)\\ &\quad+A\cdot B\cdot\ln\left|\sqrt{u^2+A^2+B^2}+u\right|\end{aligned}\tag{A3}$$

$$\begin{aligned}\int u\cdot\arctan\left(\frac{A}{u}\right)du &= \frac{1}{2}u^2\arctan\left(\frac{A}{u}\right)-\frac{1}{2}A^2\arctan\left(\frac{u}{A}\right)+\frac{1}{2}A\cdot u\end{aligned}\tag{A4}$$

$$\begin{aligned}\int \ln\left|\sqrt{u^2+A^2+B^2}+u\right|du &= u\cdot\ln\left|\sqrt{u^2+A^2+B^2}+u\right|\\ &\quad-\sqrt{u^2+A^2+B^2}\end{aligned}\tag{A5}$$

$$\begin{aligned}\int \ln\left|\sqrt{u^2+A^2+B^2}+A\right|du &= u\cdot\ln\left|\sqrt{u^2+A^2+B^2}+A\right|\\ &\quad+A\cdot\ln\left|\sqrt{u^2+A^2+B^2}+u\right|\\ &\quad+B\left[\arctan\left(\frac{u}{B}\right)-\arctan\left(\frac{A\cdot u}{B\sqrt{u^2+A^2+B^2}}\right)\right]-u\end{aligned}\tag{A6}$$

$$\begin{aligned}\int u\cdot\ln\left|\sqrt{u^2+A^2+B^2}+A\right|du &= \frac{1}{2}(u^2+B^2)\ln\left|\sqrt{u^2+A^2+B^2}+A\right|\\ &\quad-\frac{1}{4}u^2+\tfrac{1}{2}A\sqrt{u^2+A^2+B^2}\end{aligned}\tag{A7}$$

$$\begin{aligned}\int u^2\ln\left|\sqrt{u^2+A^2+B^2}+A\right|du &= \frac{1}{3}u^3\ln\left|\sqrt{u^2+A^2+B^2}+A\right|\\ &\quad+\frac{1}{6}A\cdot u\sqrt{u^2+A^2+B^2}\\ &\quad-\frac{1}{3}B^3\left[\arctan\left(\frac{u}{B}\right)-\arctan\left(\frac{A\cdot u}{B\sqrt{u^2+A^2+B^2}}\right)\right]\\ &\quad-\frac{1}{9}u^3+\tfrac{1}{3}B^2u\\ &\quad-\frac{1}{6}A(A^2+3B^2)\ln\left|\sqrt{u^2+A^2+B^2}+u\right|.\end{aligned}\tag{A8}$$

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References

[1] E. Della Torre, "Magnetization calculation of fine particles," *IEEE Trans. Magn.*, vol. MAG-22, pp. 484–489, 1986.
[2] W. F. Brown Jr., *Magnetostatic Principles in Ferromagnetism.* Amsterdam, The Netherlands: North-Holland, 1962, p. 84.
[3] D.-X. Chen, J. A. Brug, and R. B. Goldfarb, "Demagnetizing factors for cylinders," *IEEE Trans. Magn.*, vol. 27, pp. 3601–3619, 1991.
[4] Y. Nakatani, Y. Uesaka, and N. Hayashi, "Direct solution of the Landau–Lifshitz–Gilbert equation for micromagnetics," *Jpn. J. Appl. Phys.*, vol. 28, pp. 2485–2507, 1989.
[5] M. E. Schabes and A. Aharoni, "Magnetostatic interaction fields for a three-dimensional array of ferromagnetic cubes," *IEEE Trans. Magn.*, vol. MAG-23, pp. 3882–3888, 1987.
[6] Y. Uesaka, Y. Nakatani, and N. Hayashi, "Computer simulation of magnetization reversal of cubic, thin and elongated particles; effect of method of calculating demagnetizing field," *J. Magn. Magn. Mater.*, vol. 145, pp. 205–210, 1995.
[7] E. Schlömann, "A sum rule concerning the inhomogeneous demagnetizing field in nonellipsoidal samples," *J. Appl. Phys.*, vol. 33, pp. 2825–2826, 1962.
[8] R. Moskowitz and E. Della Torre, "Theoretical aspects of demagnetization tensors," *IEEE Trans. Magn.*, vol. MAG-2, pp. 739–744, 1966.

Hiroshi Fukushima was born in Otaru, Japan, on November 5, 1930. He received the B.S. degree from the University of Tokyo, Tokyo, Japan, in 1954.

Since joining Hitachi, Ltd. in 1954, he worked mainly on the development of electron tubes in Mobara Works of the company. During the last 10 years before retiring in 1990, his interest was the computer simulation of physical phenomena in various products of the works. After retirement, he has been studying numerical micromagnetics in the capacity of a private person.

Mr. Fukushima is a member of the Physical Society of Japan, the Magnetics Society of Japan, and the Information Processing Society of Japan.

Yoshinobu Nakatani was born in Nara, Japan, on June 9, 1962. He received the B.E. and M.E. degrees from the University of Electro-Communications, Tokyo, Japan, in 1985 and 1987, respectively, and received the Dr.E. degree from the University of Nagoya, Japan, in 1993.

He is now Associate Professor in the Department of Computer Science and Information Mathematics at the University of Electro-Communications. His current research interests include numerical micromagnetics, computer simulation of magnetic recording media, computer simulation of magneto-optical disk, and parallel and distributed numerical calculation.

Dr. Nakatani is a member of the Magnetics Society of Japan and the Information Processing Society of Japan.

Nobuo Hayashi (M'81) was born in Tokyo, Japan, on May 7, 1939. He received the B.S., M.S., and Dr.S. degrees from the University of Tokyo, Tokyo, Japan, in 1962, 1964, and 1967, respectively.

From 1967 to 1972, he was a Research Assistant in the Department of Physics, the University of Tokyo, investigating exchange-coupled thin magnetic films. From 1970 to 1972, he was on leave from the University of Tokyo to the IBM Thomas J. Watson Research Center, Yorktown Heights, NY, as a Visiting Scientist, where he worked on bubble domain devices. He is now Professor in the Department of Computer Science and Information Mathematics, the University of Electro-Communications, Tokyo, Japan, being engaged in research concerning to numerical micromagnetics.

Dr. Hayashi is a member of the Physical Society of Japan, the Magnetics Society of Japan, and the Information Processing Society of Japan.