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## What is the proof that covariance matrices are always semi-definite?

Covariance matrix  $\mathbf{C}$  is calculated by the formula,

$$\mathbf{C} \triangleq E\{(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T\}.$$

For an arbitrary real vector  $\mathbf{u}$ , we can write,

$$\begin{aligned}\mathbf{u}^T \mathbf{C} \mathbf{u} &= \mathbf{u}^T E\{(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T\} \mathbf{u} \\ &= E\{\mathbf{u}^T (\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{u}\} \\ &= E\{s^2\} \\ &= \sigma_s^2.\end{aligned}$$

Where  $\sigma_s$  is the variance of the zero-mean scalar random variable  $s$ , and it is a scalar real number whose value equals to,

$$\sigma_s = \mathbf{u}^T (\mathbf{x} - \bar{\mathbf{x}}) = (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{u}.$$

Square of any real number is equal to or greater than zero. That is,

$$\sigma_s^2 \geq 0.$$

Thus,

$$\mathbf{u}^T \mathbf{C} \mathbf{u} = \sigma_s^2 \geq 0.$$

Which implies that covariance matrix of any real random vector is always semi-definite.