Fundamentals of Aerospace Navigation and Guidance

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December 2, 2013

To our families, Joséphine, Monique, Orianne and Louis, Michael, Leo and Mia.
With a special commendation for Mia who, at age 3 months, listened to the formal read-through of the whole book.

Chapter 8

Optimization

The next two chapters treat optimization and optimal control for purpose of their application to guidance. Chapter 8 focuses on optimization as a stepping-stone towards optimal control, which is treated in Chapter 9. Optimization is concerned with finding the best option from among several to solve a problem.

Optimization is often necessary in aerospace engineering because of the merciless requirements that physics and chemistry impose on flying systems. For instance, it is well known that to lift 1 kg of payload from Earth surface to orbit, using chemical rocket propulsion, it is required to use at least 80 kg of rocket structure, engine, fuel and propellant [78]. This staggering 80/1 ratio is one of many stark reminders that, when it comes to flight, optimization is of the essence.

Throughout this chapter, we discuss necessary and sufficient conditions for optimality. Let us clarify what we mean by these. A necessary condition for optimality is a statement of the form: "If item x is optimal (i.e., is the best), then condition $NC(x)$ must be satisfied." Typically, condition $NC(x)$ provides enough information to determine x. A sufficient condition for optimality is a statement of the form: "If item x satisfies condition $SC(x)$, then item x must be optimal." Here also, typically condition $SC(x)$ provides enough information to determine x. In view of these statements, the following caution is in order: *necessary conditions guarantee neither optimality of x, nor even existence of a solution to the optimization problem.* Sufficient conditions, on the other hand, are more powerful: if item x satisfies the sufficient condition, then not only can we guarantee that the problem does have a solution, we can also guarantee that x is a solution. Note that if the necessary condition for optimality has no solution, then we can conclude that the optimization problem has no solution either. However, if the sufficient condition has no solution, we cannot necessarily conclude that the optimization problem has no solution.

Section 8.1 covers unconstrained optimization of real functions of a vector variable. Section 8.2 treats optimization under equality constraints, and Section 8.3 extends the results to the case of inequality constraints. Section 8.4 treats optimal control of discrete-time systems, which is to segue into optimal control in Chapter 9. Sections 8.5, 8.6 and 8.7 present a summary of the key results in the chapter, bibliographic notes for further reading, and homework problems, respectively.

Remark 8.1 *Before proceeding with the study of optimization, another word of caution is in order. Optimization in engineering should always be approached with prudence. This is because engineering systems must often meet many competing performance specifications. As a consequence, optimizing an engineering system with respect to one performance metric is typically detrimental to other performance metrics that matter - as the adage says: The best is often the enemy of the good. Hence, optimization should be viewed as a convenient method for engineering synthesis, i.e., to obtain a configuration (e.g., a trajectory) of the engineering system as a function of specified parameters (e.g., a cost function). In that context, it is often practical to adjust the cost function until optimization synthesizes a satisfactory configuration.*

8.1 Unconstrained Optimization on $\mathbb{R}^n$

This section presents a hierarchy of results on optimization of real functions of a vector variable. The hierarchy is based on the smoothness of the function, proceeding from continuous functions to functions whose first derivative is continuous and then to functions whose second derivative is continuous.

Consider the function f defined as:

$$f : \mathbb{R}^n \to \mathbb{R} : x \mapsto f(x), \tag{8.1}$$

and let $X \subseteq \mathbb{R}^n$.

Definition 8.1 (Global Minimum) *$x^* \in X$ is a* ***global minimum*** *of f on X if:*

$$\forall x \in X, f(x^*) \leq f(x). \tag{8.2}$$

Definition 8.2 (Local Minimum) *$x^* \in X$ is a* ***local minimum*** *of f on X if there exists an open neighborhood $\mathring{N}$ of x^* such that x^* is a global minimum of f on $\mathring{N} \cap X$.*

Remark 8.2 *Note that if x^* is a global minimum of f on X, then x^* is also a local minimum of f on X.*

Example 8.1 *Examples of global and local minima are shown in Figure 8.1 for $f : \mathbb{R} \to \mathbb{R}$.*

The following basic result guarantees existence and achievement of minimum for continuous functions.

Proposition 8.1 *If X is closed and bounded, and if $f : X \to \mathbb{R}$ is continuous, then f achieves a minimum on X.*

Example 8.2 *In the example shown in Figure 8.2, $X = (a, b]$ and the problem has no minimum. This illustrates the importance of the set X.*

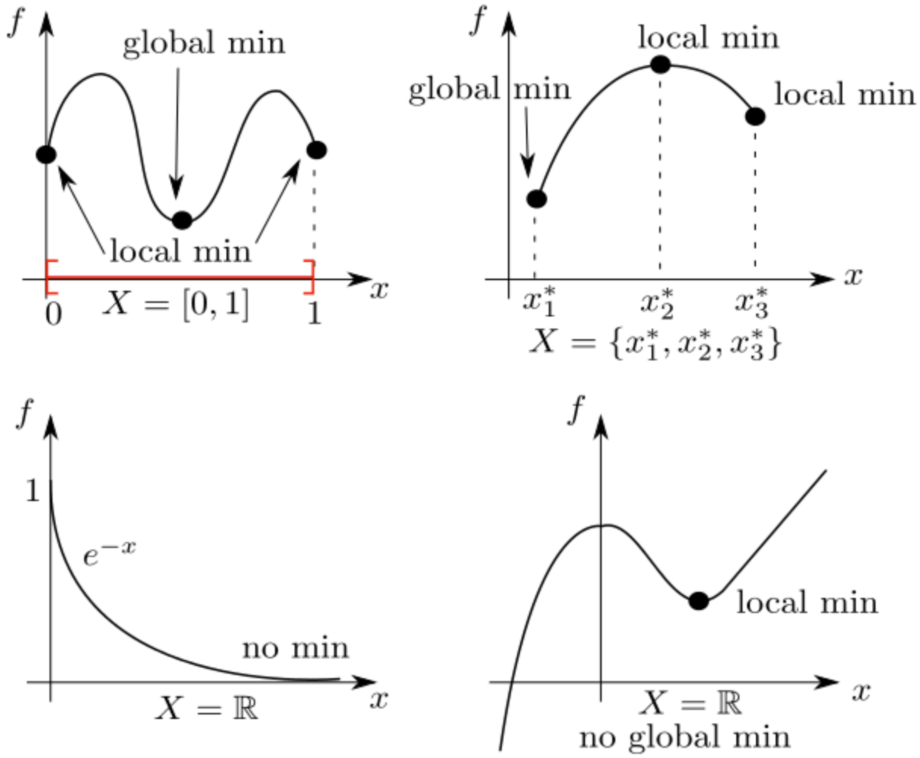

Figure 8.1: Examples of global and local minima. For all examples, $f : \mathbb{R} \to \mathbb{R}$.

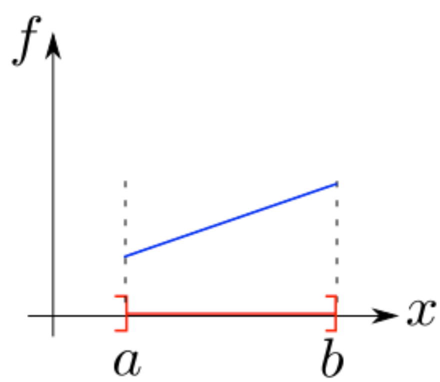

Figure 8.2: Importance of X for optimization problems: Example 8.2.

Definition 8.3 *We say that f is of **class** $\mathcal{C}^k$, written $f \in \mathcal{C}^k$, if and only if all partial derivatives of f exist and are continuous up to order k, i.e., the*

$$\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_l}} \tag{8.3}$$

exist and are continuous for all l such that $0 \leq l \leq k$.

Definition 8.4 *We say that f is of **class** $\mathcal{C}^{*k}$, written $f \in \mathcal{C}^{*k}$, if and only if $f \in \mathcal{C}^{k-1}$ and all partial derivatives of f of order k exist (but are not necessarily continuous).*

The following result is the basis for necessary and sufficient conditions for local optimization of differentiable functions. It is stated as A.1 in Appendix A.3 and repeated here for clarity.

Proposition 8.2 (Taylor's Theorem) *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^{*n+1}$ on $[a, x]$. Then,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x,a), \tag{8.4}$$

where

$$R_n(x,a) = \int_a^x \frac{(x-\tau)^n}{n!} f^{(n+1)}(\tau)\, \mathrm{d}\tau. \tag{8.5}$$

Moreover, if $f \in \mathcal{C}^{n+1}$ on $[a, x]$, then there exists $c \in [a, x]$ such that:

$$R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{8.6}$$

Remark 8.3 (Interpretation of Taylor's Theorem) *Taylor's theorem states that:*

$$\Delta f = f(x) - f(a) = \delta^1 f(x,a) + \delta^2 f(x,a) + \delta^3 f(x,a) + ... \tag{8.7}$$

where

$$\begin{aligned} \delta^1 f(x,a) &= f'(a)(x-a) \text{ is of order 1 in (x-a)}, \\ \delta^2 f(x,a) &= f''(a)(x-a)^2 \text{ is of order 2 in (x-a)}, \\ \delta^3 f(x,a) &= f'''(a)(x-a)^3 \text{ is of order 3 in (x-a)}, \\ &..., \end{aligned} \tag{8.8}$$

that is, the following limit is finite:

$$\lim_{|x-a|\to 0} \frac{\delta^k f(x,a)}{|x-a|^k}. \tag{8.9}$$

Hence, for $|x-a|$ small, $\delta^1 f$ dominates, followed by $\delta^2 f$, followed by $\delta^3 f$, etc. In other words, Δf is decomposed into contributions that can be ranked.

These contributions, the $\delta^k f(x,a)$, are called the ***variations of** f **at** a. They lead to the necessary or sufficient conditions for optimality. Their computation in functional spaces leads to the* ***calculus of variations****.*

Definition 8.5 (Convex Set) *A set $X \subset \mathbb{R}^n$ is* ***convex*** *if:*

$$\forall x_1, x_2 \in X, \forall \alpha \in [0,1], x = \alpha x_2 + (1-\alpha)x_1 \in X. \tag{8.10}$$

Example 8.3 (Convex and Non-convex Sets) *Examples of convex and non-convex sets are shown in Figure 8.3.*

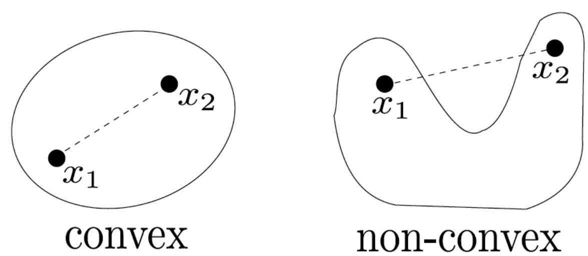

Figure 8.3: Examples of convex and non-convex sets.

Definition 8.6 (Convex (Strictly Convex) Function) *A function $f : X \to \mathbb{R}$ is* ***convex*** *if:*

$$\forall x_1, x_2 \in X, \forall \alpha \in [0,1], f(\alpha x_2 + (1-\alpha)x_1) \leq \alpha f(x_2) + (1-\alpha) f(x_1). \tag{8.11}$$

The function is called ***strictly convex*** *if:*

$$\forall x_1, x_2 \in X, x_1 \neq x_2, \forall \alpha, 0 < \alpha < 1, f(\alpha x_2 + (1-\alpha)x_1) < \alpha f(x_2) + (1-\alpha) f(x_1). \tag{8.12}$$

Example 8.4 (Convex Function) *An example of a convex function is shown in Figure 8.4.*

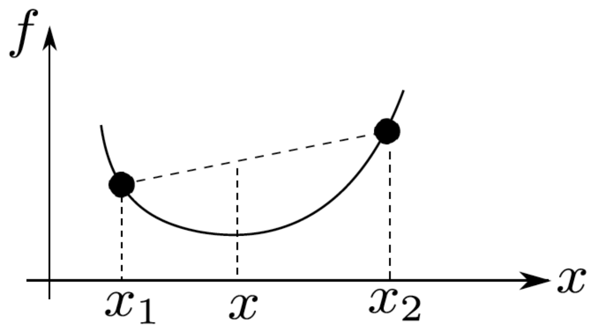

Figure 8.4: Convex function.

Taylor's theorem is generalized as follows. Let $X \in \mathbb{R}^n$ be convex, closed and bounded. Let $f : X \to \mathbb{R}$, $f \in \mathcal{C}^{m+1}$ on X. Assume that we want to calculate the quantity $f(x+h) - f(x)$. Let $g(\alpha) = f(x + \alpha h)$, where $\alpha \in [0, 1]$ is a scalar. We apply the scalar version of Taylor's theorem to g to yield:

$$g(1) = g(0) + g'(0) + \frac{g''(0)}{2!} + ... + R_m(1, 0). \tag{8.13}$$

Developing (8.13) yields:

$$\begin{aligned} f(x+h) - f(x) &= \left(\frac{\partial f}{\partial x}\right)_x^T h + \frac{1}{2!} h^T \left(\frac{\partial^2 f}{\partial x^2}\right)_x h \\ &+ \sum_{i_1}\sum_{i_2}\sum_{i_3} \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\right)_x h_{i_1}h_{i_2}h_{i_3} \\ &+ ... \\ &+ \sum_{i_1}\sum_{i_2}...\sum_{i_m} \frac{1}{m!}\left(\frac{\partial^m f}{\partial x_{i_1}\partial x_{i_2}...\partial x_{i_m}}\right)_x h_{i_1}h_{i_2}...h_{i_m} \\ &+ \sum_{i_1}\sum_{i_2}...\sum_{i_{m+1}} \frac{1}{(m+1)!}\left(\frac{\partial^{m+1} f}{\partial x_{i_1}\partial x_{i_2}...\partial x_{i_{m+1}}}\right)_{x+\alpha_0 h, \alpha_0 \in [0,1]} h_{i_1}h_{i_2}...h_{i_{m+1}}, \end{aligned} \tag{8.14}$$

where

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ ... \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{8.15}$$

is the gradient of f, and

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \cdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{8.16}$$

is the (symmetric) Hessian matrix of f. Here also, defining:

$$\Delta f(x; h) = f(x + h) - f(x), \tag{8.17}$$

we have:

$$\Delta f(x; h) = \delta f(x; h) + \delta^2 f(x; h) + \delta^3 f(x; h) + ... + \delta^m f(x; h) + R_m(x; h), \tag{8.18}$$

where

$$\delta^k f(x; h) \tag{8.19}$$

is called the k^{th} differential of f or the k^{th} variation of f, and is homogeneous in h with degree k, i.e.,

$$\delta^k f(x; \alpha h) = \alpha^k \delta^k f(x; h). \tag{8.20}$$

Hence, the difference of a smooth function is a sum of variations and remainder. Each variation is a homogeneous function of the increment, the degree of homogeneity being the index of the variation.

Based on Taylor's theorem and its generalization, we can now state the following:

Proposition 8.3 (Necessary Conditions for Optimality of Twice-Differentiable Functions on Open Sets) *If $X \in \mathbb{R}^n$, $X = \mathring{X}$ (X is open), and $x^* \in \mathring{X}$, then:*

- *$f \in \mathcal{C}^1$ has a local minimum at x^* implies that:*

$$\forall h, \delta^1 f(x^*; h) = 0. \tag{8.21}$$

- *$f \in \mathcal{C}^2$ has a local minimum at x^* implies that:*

$$\begin{aligned} \forall h, \delta^1 f(x^*; h) &= 0, \quad (8.22) \\ \forall h, \delta^2 f(x^*; h) &\geq 0. \quad (8.23) \end{aligned}$$

Remark 8.4 *Note that (8.21) is equivalent to requiring:*

$$\frac{\partial f}{\partial x} = 0, \tag{8.24}$$

that is, to solving n equations for x^*. The solution yields what are called stationary or singular points. Requirement (8.23) is equivalent to checking that:

$$\frac{\partial^2 f}{\partial x^2} \geq 0, \tag{8.25}$$

that is, that the Hessian matrix is positive semi-definite.

Remark 8.5 *Note that the first order necessary conditions for optimality have the form:*

$$\frac{\partial f}{\partial x} = 0, \tag{8.26}$$

which is that of a system of n equations in n unknowns. Such systems can be solved systematically by using Newton's method as discussed in Appendix A.4.

Remark 8.6 *To apply Newton's method to solve $\frac{\partial f}{\partial x} = 0$, the iteration becomes:*

$$x^{k+1} = x^k - \left(\frac{\partial^2 f}{\partial x^2}\right)^{-1}_{x^k} \left(\frac{\partial f}{\partial x}\right)_{x^k}, \tag{8.27}$$

*which can be interpreted as follows: approximate f locally by a **quadratic** function and optimize it to get the next iterate.*

Proposition 8.4 (Sufficient Conditions for Optimality of Twice-Differentiable Functions on Open Sets) *If $X \in \mathbb{R}^n$, $X = \mathring{X}$ (X is open), $x^* \in \mathring{X}$, $f \in \mathcal{C}^2$ on X and*

$$\left(\frac{\partial f}{\partial x}\right)_{x^*} = 0, \tag{8.28}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{x^*} > 0, \tag{8.29}$$

then f has a local minimum at x^*.

Proposition 8.5 (Uniqueness of (Global) Minimum) *If X is convex, closed and bounded, if $f \in \mathcal{C}^2$ on X, and if f is strictly convex on X, then there exists a unique $x^* \in X$ such that f has a global minimum at x^*.*

Remark 8.7 *If $f \in \mathcal{C}^2$ on X and if f is strictly convex on X, then:*

$$\frac{\partial^2 f}{\partial x^2} > 0 \text{ on } X. \tag{8.30}$$

8.2 Constrained Optimization on $\mathbb{R}^n$

Constrained optimization problems on $\mathbb{R}^n$ are of the form: given $f : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^2$, $g : \mathbb{R}^n \to \mathbb{R}^m \in \mathcal{C}^2$ with $m < n$, and given $X = \{x \in \mathbb{R}^n | g(x) = 0\}$, find:

$$\min_{x \in X} J = f(x), \tag{8.31}$$

or

$$\begin{aligned} \min_{x \in X} J &= f(x) \\ \text{subject to } g(x) &= 0. \end{aligned} \tag{8.32}$$

The key idea is to "eliminate" part of x using the constraint, and then apply the results of Section 8.1.

Definition 8.7 *$x^* \in X$ is a* ***regular point*** *if $\left(\frac{\partial g}{\partial x}\right)_{x^*}$ has full rank. Recall that:*

$$\begin{aligned} \left(\frac{\partial g}{\partial x}\right)_{ij} &= \frac{\partial g_j}{\partial x_i} \in \mathbb{R}^{n \times m}, \\ \frac{\partial g}{\partial x} &= \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} & \ldots & \frac{\partial g_m}{\partial x} \end{bmatrix}. \end{aligned} \tag{8.33}$$

Remark 8.8 *If x is a regular point, then it is always possible to re-order the entries of x so that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1 \in \mathbb{R}^m$, $x_2 \in \mathbb{R}^{n-m}$, and the Jacobian $\frac{\partial g}{\partial x_1}$ is nonsingular. Using the Implicit Function Theorem of Appendix A.5, x_1 can be eliminated from the implicit constraint $g(x_1, x_2) = 0$ using an explicit function $x_1 = \phi(x_2)$. Moreover, to evaluate the Jacobian of the explicitation, we do not need a closed form expression; all we need are the Jacobians of the implicit function.*

8.2.1 Lagrange Multipliers

We can now state necessary conditions for the optimization problem (8.32) when f and g are of class $\mathcal{C}^1$.

Proposition 8.6 (Lagrange's Theorem) *If x^* is a solution to the minimization under constraints problem (8.31) or (8.32), where f and g are of class $\mathcal{C}^1$, and if x^* is regular, then there exist real numbers p_1, p_2, ..., p_m such that:*

$$\frac{\partial f}{\partial x} + \sum_{i=1}^{m} p_i \frac{\partial g_i}{\partial x} = 0. \tag{8.34}$$

Remark 8.9 *The p_i are called **Lagrange multipliers**. There are as many of them as there are constraints. The necessary conditions for optimality of dynamic optimization problems, presented in Chapter 9, feature similar quantities called the co-states.*

Remark 8.10 (Sketch of the Derivation) *Recall that x can be decomposed as $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^{n-m}$. We then have:*

$$\begin{aligned} \delta J &= \left(\frac{\partial f}{\partial x_1}\right)^T \delta x_1 + \left(\frac{\partial f}{\partial x_2}\right)^T \delta x_2, \\ \delta g &= \left(\frac{\partial g}{\partial x_1}\right)^T \delta x_1 + \left(\frac{\partial g}{\partial x_2}\right)^T \delta x_2 = 0, \end{aligned} \tag{8.35}$$

where $\left(\frac{\partial g}{\partial x_1}\right)^T \in \mathbb{R}^{m\times m}$ is assumed nonsingular, without loss of generality. Then,

$$\delta x_1 = -\left(\frac{\partial g}{\partial x_1}\right)^{-T} \left(\frac{\partial g}{\partial x_2}\right)^T \delta x_2, \tag{8.36}$$

hence,

$$\delta J = \left[-\left(\frac{\partial g}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right)^{-1}\left(\frac{\partial f}{\partial x_1}\right) + \left(\frac{\partial f}{\partial x_2}\right)\right]^T \delta x_2, \tag{8.37}$$

where J is now a function of x_2 alone, by use of the Implicit Function Theorem. Define:

$$p = -\left(\frac{\partial g}{\partial x_1}\right)^{-1}\left(\frac{\partial f}{\partial x_1}\right). \tag{8.38}$$

Then,

$$\left(\frac{\partial g}{\partial x_1}\right) p + \left(\frac{\partial f}{\partial x_1}\right) = 0, \tag{8.39}$$

or equivalently,

$$\frac{\partial}{\partial x_1}(f + p^T g) = 0. \tag{8.40}$$

Setting $\delta J = 0$ for all δx_2 in (8.37) yields:

$$\frac{\partial}{\partial x_2}(f + p^T g) = 0. \tag{8.41}$$

Combining (8.40) *and* (8.41) *yields the desired result:*

$$\frac{\partial}{\partial x}(f + p^T g) = 0. \tag{8.42}$$

Remark 8.11 (Interpretation of p) *In* (8.35), *let $\delta x_2 = 0$, to obtain:*

$$\delta x_1 = \left(\frac{\partial g}{\partial x_1}\right)^{-T} \delta g, \tag{8.43}$$

which implies that:

$$\delta J = \left[\left(\frac{\partial g}{\partial x_1}\right)^{-1}\left(\frac{\partial f}{\partial x_1}\right)\right]^T \delta g = -p^T \delta g. \tag{8.44}$$

Thus, a possible interpretation of p is given by:

$$p = -\left(\frac{\partial J}{\partial g}\right)_{x_2 \text{ fixed}}, \tag{8.45}$$

that is, p can be interpreted as the sensitivity of the cost with respect to the constraint.

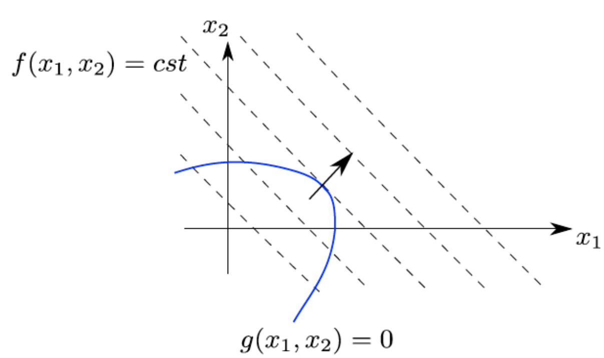

Figure 8.5: Geometric interpretation of minimization under constraints.

Remark 8.12 (Geometric Interpretation of p) *A geometric interpretation is as shown in Figure 8.5. At optimum, the gradients of f and g are collinear. Recalling that the gradient is a vector that is perpendicular to the local level curve, we obtain that the level curves of f and g have the same perpendicular, i.e., they are tangent.*

Remark 8.13 (First Order Necessary Conditions for Optimality) *For the problem of minimization under constraints as stated in (8.31) or (8.32) , define:*

$$L(x,p) = p^T g(x) + f(x) \tag{8.46}$$

*to be the **Lagrangian**. Then the first order necessary conditions for optimality can be expressed as:*

$$\begin{aligned} \frac{\partial L}{\partial p} &= g(x) = 0, \\ \frac{\partial L}{\partial x} &= \frac{\partial f}{\partial x} + \left(\frac{\partial g}{\partial x}\right) p = 0, \end{aligned} \tag{8.47}$$

which is a system of $n+m$ equations in $n+m$ unknowns.

Remark 8.14 *We can generalize the contents of Remarks 8.5 and 8.6 to optimization under constraints as follows. Assume we use Newton's iteration to solve the system of $n+m$ equations with $n+m$ unknowns (8.47). Then, each iteration amounts to locally approximating the objective function f by a **quadratic** function, the constraint g by a **linear** function, and choosing as the next iterate the solution of this subsidiary optimization problem.*

8.2.2 Second Order Conditions

Definition 8.8 (Tangent Plane) *For the constraints $g(x) = 0$, the **tangent plane** $\mathcal{T}$ is defined as:*

$$\mathcal{T} = \left\{ \delta x | \left(\frac{\partial g}{\partial x}\right)^T \delta x = 0 \right\}. \tag{8.48}$$

Proposition 8.7 *For the problem:*

$$\begin{aligned} \min_x J &= f(x) \\ \text{subject to } g(x) &= 0, \end{aligned} \tag{8.49}$$

where f and g are of class $\mathcal{C}^2$, if x^* is regular, a necessary condition for optimality of x^* is:

$$\frac{\partial^2 L}{\partial x^2} \geq 0 \tag{8.50}$$

in the tangent plane $\mathcal{T}$ at x^*, where $L = p^T g(x) + f(x)$.

Proposition 8.8 *For the problem:*

$$\begin{aligned} \min_x J &= f(x) \\ \text{subject to } g(x) &= 0, \end{aligned} \tag{8.51}$$

where f and g are of class $\mathcal{C}^2$, if x^* is regular, a sufficient condition for optimality of x^* is:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0, \\ \frac{\partial L}{\partial p} &= 0, \\ \frac{\partial^2 L}{\partial x^2} &> 0 \text{ in } \mathcal{T} \text{ at } x^*. \end{aligned} \tag{8.52}$$

Remark 8.15 *Propositions 8.7 and 8.8 require ascertaining the sign of the Hessian of the Lagrangian in the tangent plane. Note that if a symmetric matrix is of definite sign (i.e., positive definite or negative definite), restricting its quadratic form to a plane preserves the sign of the quadratic form. Hence, if the Hessian of the Lagrangian is positive definite, we can conclude that the candidate optimum is a minimum without scrutinizing the tangent plane.*

Example 8.5 (Least Energetic Ballistic Shot Revisited) *To illustrate the use of Lagrange multipliers, we revisit the optimization problem of Example 6.1: from among all ballistic trajectories that originate at polar coordinates $(r, \theta) = (r_0, \pi/2)$ and hit the target located at $(r, \theta) = (r_0, 0)$, we seek the one that does so with minimum energy. Here, r_0 is the radius of the planet. Formally, the optimization problem is:*

$$\min_{v_{r_1}, v_{\theta_1}} T(v_{r_1}, v_{\theta_1}) = \frac{1}{2}\left(v_{r_1}^2 + v_{\theta_1}^2\right) \tag{8.53}$$

subject to:

$$v_{r_1} - v_{\theta_1} + \frac{k}{r_0 v_{\theta_1}} = 0, \tag{8.54}$$

where v_{r_1} and v_{θ_1} are the radial and tangential components of velocity at cutoff, respectively, k is the gravitational constant of the planet, the objective function is the specific kinetic energy at cutoff and the constraint is the hit equation.

The Lagrangian is:

$$L(v_{r_1}, v_{\theta_1}, p) = \frac{1}{2}(v_{r_1}^2 + v_{\theta_1}^2) + p\left(v_{r_1} - v_{\theta_1} + \frac{k}{r_0 v_{\theta_1}}\right), \tag{8.55}$$

where p is the Lagrange multiplier. The first order necessary conditions for optimality are:

$$\begin{aligned}
\frac{\partial L}{\partial v_{r_1}} &= v_{r_1} + p = 0, && (8.56)\\
\frac{\partial L}{\partial v_{\theta_1}} &= v_{\theta_1} - p - \frac{pk}{r_0 v_{\theta_1}^2} = 0, && (8.57)\\
\frac{\partial L}{\partial p} &= v_{r_1} - v_{\theta_1} + \frac{k}{r_0 v_{\theta_1}} = 0, && (8.58)
\end{aligned}$$

which constitute a system of three equations for the three unknowns v_{r_1}, v_{θ_1} and p. This system is easily solved through a sequence of eliminations as follows. Equation (8.56) yields:

$$p = -v_{r_1}. \tag{8.59}$$

Equations (8.57) and (8.59) yield:

$$v_{\theta_1} + v_{r_1} + \frac{v_{r_1} k}{r_0 v_{\theta_1}^2} = 0. \tag{8.60}$$

Now, (8.58) can be rewritten as:

$$v_{r_1} = v_{\theta_1} - \frac{k}{r_0 v_{\theta_1}}. \tag{8.61}$$

Equations (8.60) and (8.61) yield, after grouping terms:

$$2v_{\theta_1} - \frac{k^2}{r_0^2 v_{\theta_1}^3} = 0, \tag{8.62}$$

which can be solved for v_{θ_1} as:

$$v_{\theta_1} = -\sqrt{\frac{k}{r_0\sqrt{2}}}, \tag{8.63}$$

where we have chosen the negative sign to ensure that $v_{r_1} > 0$*, i.e., that the ballistic launch clears the surface of the planet. Then,* (8.61) *and* (8.63) *yield:*

$$v_{r_1} = (\sqrt{2} - 1)\sqrt{\frac{k}{r_0\sqrt{2}}}, \tag{8.64}$$

and, finally, (8.64) *and* (8.59) *yield:*

$$p = (1 - \sqrt{2})\sqrt{\frac{k}{r_0\sqrt{2}}}. \tag{8.65}$$

Equations (8.63)*,* (8.64) *and* (8.65) *are the solution of the first order necessary conditions for optimality.*

For the second order analysis, we evaluate the Hessian of the Lagrangian as follows:

$$\frac{\partial^2 L}{\partial(v_{r_1}, v_{\theta_1})^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{2pk}{r_0 v_{\theta_1}^3} \end{bmatrix}. \tag{8.66}$$

At the candidate optimum, (8.63) - (8.65) *imply that this Hessian is:*

$$\frac{\partial^2 L}{\partial(v_{r_1}, v_{\theta_1})^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + 2\sqrt{2}(\sqrt{2} - 1) \end{bmatrix}. \tag{8.67}$$

At this point, the attentive reader may recognize that this matrix is positive definite and hence we could expediently conclude, as per Remark 8.15, that (8.63) - (8.65) *provide a minimum. However, for the sake of instruction, we carry out the second order analysis to a dutiful end as follows. The tangent plane* (8.48) *is defined by the linear homogenous equation:*

$$\begin{bmatrix} 1 & -1 - \frac{k}{r_0 v_{\theta_1}^2} \end{bmatrix} \begin{bmatrix} \delta v_{r_1} \\ \delta v_{\theta_1} \end{bmatrix} = 0, \tag{8.68}$$

for which (8.63) - (8.65) *yield:*

$$\begin{bmatrix} 1 & -(1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} \delta v_{r_1} \\ \delta v_{\theta_1} \end{bmatrix} = 0. \tag{8.69}$$

A basis for this tangent plane is:

$$\begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix}. \tag{8.70}$$

Hence, we evaluate the sign of the 1×1 matrix:

$$\begin{bmatrix} 1+\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1+2\sqrt{2}(\sqrt{2}-1) \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} = (1+\sqrt{2})^2 + 1 + 2\sqrt{2}(\sqrt{2}-1). \quad (8.71)$$

Since this last expression is positive, we conclude, as per Proposition 8.8, that (8.63) - (8.65) provide the least energetic ballistic shot, which confirms the results of Example 6.1.

Example 8.6 (Maximum Lift-to-Drag Ratio) *Consider the computation of the maximum lift-to-drag ratio achievable by a conventional aircraft in steady flight, assuming standard models for lift and drag. Mathematically, the problem is:*

$$\max_{C_L, C_D} \frac{C_L}{C_D}, \quad (8.72)$$

subject to:

$$C_D = C_{D0} + K C_L^2, \quad (8.73)$$

where (8.73) is the ***drag polar equation*** *which describes the aerodynamics of the aircraft, C_L and C_D are the non-dimensional lift and drag coefficients, respectively, and C_{D0} and K are constants.*

The Lagrangian L is:

$$L(C_L, C_D, p) = \frac{C_L}{C_D} + p\left(C_{D0} + K C_L^2 - C_D\right), \quad (8.74)$$

where p is the Lagrange multiplier. The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial C_L} &= \frac{1}{C_D} + 2pKC_L = 0, \\ \frac{\partial L}{\partial C_D} &= -\frac{C_L}{C_D^2} - p = 0, \\ \frac{\partial L}{\partial p} &= C_{D0} + KC_L^2 - C_D = 0. \end{aligned} \quad (8.75)$$

Solving this system of algebraic equations yields:

$$C_L = \sqrt{\frac{C_{D0}}{K}} \quad (8.76)$$

and

$$C_D = 2C_{D0}, \tag{8.77}$$

or equivalently:

$$\left(\frac{C_L}{C_D}\right)_{max} = \frac{1}{2\sqrt{KC_{D0}}}. \tag{8.78}$$

The second order analysis is left as an exercise (see Problem 8.4).

Example 8.7 (Rocket Staging) *As mentioned in the introduction to this chapter, it takes 80 kg of rocket structure, engine, fuel and propellant to lift 1 kg of payload from Earth's surface into orbit. Just before burnout, rockets use propellant to lift an empty propellant tank. This is clearly wasteful. The idea behind* ***rocket staging*** *is to jettison part of the tank "as we go."*

Let us start by consider the ***single-stage case****. Initially, the mass of the rocket is* $m_0 = m_p + m_s + m_l$*, where* m_p *is the mass of propellant,* m_s *is the mass of the structure and engine, and* m_l *is the payload mass. The final mass is* $m_s + m_l$*.*

The ***Tsiolkovsky rocket equation****, or* ***ideal rocket equation*** *describes the motion of rockets. It relates the* Δv *(the maximum change of speed of the rocket if no other external forces act) with the effective exhaust velocity and the initial and final mass of the rocket:*

$$\begin{aligned} ||\Delta\vec{v}|| &= c\log\frac{m_0}{m_s + m_l}, \\ &= c\log\left(1 + \frac{m_p}{m_s + m_l}\right), \end{aligned} \tag{8.79}$$

where c *is the effective exhaust velocity, and is dictated by the chemistry. Let:*

$$\epsilon = \frac{m_s}{m_p + m_s} \tag{8.80}$$

be the structural coefficient, and:

$$m = m_p + m_s \tag{8.81}$$

be the mass of the single stage. Then,

$$\Delta v = c\log\frac{m + m_l}{\epsilon m + m_l}. \tag{8.82}$$

*In the **two-stage case**, let the initial mass be $m_0 = m_{s1} + m_{p1} + m_{l1} + m_{s2} + m_{p2} + m_{l2}$. The mass right after the first burnout is $m_{s1} + m_{s2} + m_{p2} + m_l$, and Δv_1 is given by:*

$$\Delta v_1 = c_1 \log \frac{m_{s1} + m_{p1} + m_{s2} + m_{p2} + m_l}{m_{s1} + m_{s2} + m_{p2} + m_l}. \tag{8.83}$$

The mass just before the second stage kicks in is $m_{s2} + m_{p2} + m_l$, and the mass after the second burnout is $m_{s2} + m_l$. The second velocity increment, Δv_2, is given by:

$$\Delta v_2 = c_2 \log \frac{m_{s2} + m_{p2} + m_l}{m_{s2} + m_l}. \tag{8.84}$$

Overall,

$$\Delta v = \Delta v_1 + \Delta v_2. \tag{8.85}$$

*Define $m_1 = m_{s1} + m_{p1}$ and $m_2 = m_{s2} + m_{p2}$ to be the mass of the first and second stages, respectively. Let $\epsilon_1 = \frac{m_{s1}}{m_1}$ and $\epsilon_2 = \frac{m_{s2}}{m_2}$ be the first and second structural parameters, respectively. Then, the **two-stage rocket equation** is:*

$$\Delta v = c_1 \log \frac{m_1 + m_2 + m_l}{\epsilon_1 m_1 + m_2 + m_l} + c_2 \log \frac{m_2 + m_l}{\epsilon_2 m_2 + m_l}. \tag{8.86}$$

*We can extend this to the **n-stage case**, where we have:*

$$\Delta v = \sum_{i=1}^{n} \Delta v_i, \tag{8.87}$$

where:

$$\Delta v_i = c_i \log \frac{m_i + m_{i+1} + ... + m_n + m_l}{\epsilon_i m_i + m_{i+1} + ... + m_n + m_l}. \tag{8.88}$$

Note that typically ϵ_i is known, and depends on materials used in rockets. For example, for the Ariane IV rocket, $\epsilon_1 = 0.7$, $\epsilon_2 = 0.01$ and $\epsilon_3 = 0.1$.

Optimal staging can be used to decide how to distribute mass to obtain the largest Δv, or alternatively to minimize mass for a given Δv. Mathematically, the problem we wish to solve is:

$$\min f(m_1, ..., m_n) = m_1 + m_2 + ... + m_n \tag{8.89}$$

subject to:

$$\sum_{i=1}^{n} c_i \log \frac{m_i + m_{i+1} + ... + m_n + m_l}{\epsilon_i m_i + m_{i+1} + ... + m_n + m_l} - \Delta v_d = g(m_1, ..., m_n) = 0, \tag{8.90}$$

where Δv_d *is the desired* Δv.

Let the Lagrangian be:

$$L(m_1, m_2, .., m_n, p) = f(m_1, ..., m_n) + pg(m_1, ..., m_n), \tag{8.91}$$

where p *is a scalar. The first order conditions are:*

$$\begin{aligned} \frac{\partial L}{\partial m_1} &= 0, \\ &\cdots, \\ \frac{\partial L}{\partial m_n} &= 0, \\ \frac{\partial L}{\partial p} &= 0, \end{aligned} \tag{8.92}$$

yielding $n+1$ *algebraic equations in the unknowns* m_1, ..., m_n *and* p.

For example, in the two-stage case, the problem under consideration is:

$$\min f(m_1, m_2) = m_1 + m_2 \tag{8.93}$$

subject to:

$$c_1 \log \frac{m_1 + m_2 + m_l}{\epsilon_1 m_1 + m_2 + m_l} + c_2 \log \frac{m_2 + m_l}{\epsilon_2 m_2 + m_l} - \Delta v = 0. \tag{8.94}$$

The Lagrangian is:

$$L(m_1, m_2, p) = m_1 + m_2 + p \left(c_1 \log \frac{m_1 + m_2 + m_l}{\epsilon_1 m_1 + m_2 + m_l} + c_2 \log \frac{m_2 + m_l}{\epsilon_2 m_2 + m_l} - \Delta v \right). \tag{8.95}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial m_1} &= 1 + pc_1 \frac{\epsilon_1 m_1 + m_2 + m_l}{m_1 + m_2 + m_l} \frac{\epsilon_1 m_1 + m_2 + m_l - \epsilon_1 (m_1 + m_2 + m_l)}{(\epsilon_1 m_1 + m_2 + m_l)^2} = 0, \\ \frac{\partial L}{\partial m_2} &= 1 + pc_1 \frac{\epsilon_1 m_1 + m_2 + m_l}{m_1 + m_2 + m_l} \frac{\epsilon_1 m_1 + m_2 + m_l - \epsilon_1 (m_1 + m_2 + m_l)}{(\epsilon_1 m_1 + m_2 + m_l)^2} \\ & \quad pc_2 \frac{\epsilon_2 m_2 + m_l}{m_2 + m_l} \frac{\epsilon_2 m_2 + m_l - \epsilon_2 (m_2 + m_l)}{(\epsilon_2 m_2 + m_l)^2} = 0, \\ \frac{\partial L}{\partial p} &= c_1 \log \frac{m_1 + m_2 + m_l}{\epsilon_1 m_1 + m_2 + m_l} + c_2 \log \frac{m_2 + m_l}{\epsilon_2 m_2 + m_l} - \Delta v = 0, \end{aligned} \tag{8.96}$$

yielding three equations that can be solved for m_1, m_2 and p.

8.3 Inequality Constraints on $\mathbb{R}^n$

We use the following notation. Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Then, $g(x) \leq 0$ if and only if $g_i(x) \leq 0, 1 \leq i \leq m$.

Let $X = \{x \in \mathbb{R}^n | g(x) \leq 0\}$. We seek:

$$\min_{x \in X} f(x). \tag{8.97}$$

Note that this generalizes the results of the previous section dealing with equality constraints, since equality constraints of the form $g(x) = 0$ can be rewritten as inequality constraints of the form $g(x) \leq 0 \wedge -g(x) \leq 0$.

Proposition 8.9 (Karush-Kuhn-Tucker (KKT) Conditions) *Assume that f and g are functions of class $\mathcal{C}^2$, that x^* is an optimum, and that the active constraints (i.e., those for which $g_i(x^*) = 0$) are linearly independent at x^*. Then, there exist p_1, p_2, ..., $p_m \in \mathbb{R}$ such that:*

$$\begin{aligned}
\left(\frac{\partial f}{\partial x}\right)_{x^*} &+ \sum_{i=1}^{m} p_i \left(\frac{\partial g}{\partial x_i}\right)_{x^*} = 0, \\
g_i(x^*) &\leq 0, \ 1 \leq i \leq m, \\
p_i g_i(x^*) &= 0, \ 1 \leq i \leq m, \\
p_i &\geq 0, \ 1 \leq i \leq m.
\end{aligned} \tag{8.98}$$

Remark 8.16 *Note that (8.98) is a system of $n + m$ equations in $n + m$ unknowns.*

Example 8.8 *Let us consider the following example:*

$$\begin{aligned}
\min_{x,y} J &= (5 - x - y)^2 \\
\textit{subject to } x^2 + y^2 &= 1,
\end{aligned} \tag{8.99}$$

i.e., the problem of finding the point on the unit disk that is closest to the line $x + y = 5$, as shown in Figure 8.6.

The Lagrangian is given by:

$$L = (5 - x - y)^2 + p(x^2 + y^2 - 1). \tag{8.100}$$

The necessary conditions for optimality are:

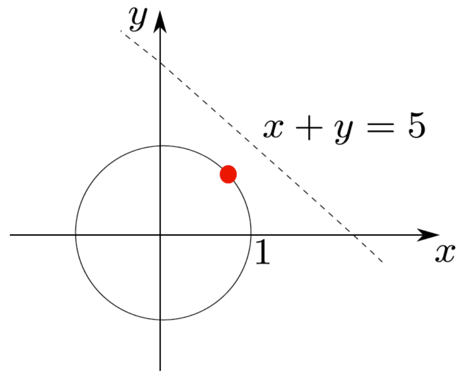

Figure 8.6: Setup for example illustrating use of the KKT conditions.

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -2(5-x-y)+2px=0, && (8.101)\\
\frac{\partial L}{\partial y} &= -2(5-x-y)+2py=0,\\
p(x^2+y^2-1) &= 0,\\
p &\geq 0,\\
x^2-y^2 &\leq 1.
\end{aligned}$$

We consider first the case of $p=0$, and second the case of $p>0$.

Case 1: $p=0$*: If $p=0$, then, $x+y=5$, and $y=5-x$. Then,*

$$x^2+(5-x)^2 \leq 1, \quad (8.102)$$

or, equivalently,

$$2x^2-10x+24 \leq 0. \quad (8.103)$$

The left hand side achieves a minimum for $4x-10=0$, that is, for $x=5/2$. For that value of x, the minimum is $23/2>0$, which is impossible.

Case 2: $p>0$*: If $p>0$, then $x=y=\frac{5}{p+2}$. In that case,*

$$x^2+y^2-1=0, \quad (8.104)$$

so

$$p=-2+\sqrt{50}, \quad (8.105)$$

and

$$x = y = \frac{\sqrt{2}}{2}, \tag{8.106}$$

which is illustrated in Figure 8.6.

8.4 Optimal Control of Discrete-Time Systems

Let us consider the following dynamic system, where the state $x(k) \in \mathbb{R}^n$, the control $u(k) \in \mathbb{R}^m$, and the dynamics are given by:

$$\begin{aligned} x(k+1) &= x(k) + f(x(k), u(k), k), \;\; k_0 \leq k \leq k_f, \\ x(k_0) &= x_0 \text{ given}, \\ x(k_f+1) & \quad \text{free}, \end{aligned} \tag{8.107}$$

and where the unknowns are $u(k_0), u(k_1), ..., u(k_f)$. We seek to solve the minimization problem:

$$\min_{u(k_0),...,u(k_f)} J = K(x(k_f+1)) + \sum L(x(k), u(k), k). \tag{8.108}$$

Let us define x as:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x(k_1) \\ x(k_2) \\ ... \\ x(k_f+1) \\ u(k_0) \\ ... \\ u(k_f) \end{bmatrix}, \tag{8.109}$$

where x_1 is of order $(k_f - k_0 + 1)n$ and x_2 is of order $(k_f - k_0 + 1)m$. Then, we seek:

$$\begin{gathered} \min_x J(x) \\ \text{subject to } F(x) = 0, \end{gathered} \tag{8.110}$$

where $F(x)$ has components:

$$F(x) = \begin{bmatrix} x(k_1) - x(k_0) - f(x(k_0), u(k_0), k_0) \\ x(k_2) - x(k_1) - f(x(k_1), u(k_1), k_1) \\ ... \\ x(k_f+1) - x(k_f) - f(x(k_f), u(k_f), k_f) \end{bmatrix}, \tag{8.111}$$

and $F(x)$ is of order $(k_f - k_0 + 1)n$. In terms of regularity, we can show that $\frac{\partial F}{\partial x_1}$ is nonsingular as follows:

$$\frac{\partial F}{\partial x_1} = \begin{bmatrix} I & -I - \frac{\partial f(x(k_1),u(k_1),k_1)}{\partial x(k_1)} & 0 & ... & ... & 0 \\ 0 & I & -I - \frac{\partial f(x(k_2),u(k_2),k_2)}{\partial x(k_2)} & 0 & ... & 0 \\ 0 & 0 & I & ... & ... & 0 \\ 0 & 0 & ... & ... & 0 & I \end{bmatrix}. \tag{8.112}$$

This matrix is square, of order $(k_f - k_0 + 1)n$, upper triangular, and with ones on the diagonal. Therefore, it is nonsingular. Note that this non-singularity guarantees uniqueness of the solution to the difference equation (8.107).

Consider the sequence of Lagrange multipliers $p(k) \in \mathbb{R}^n$, $p(k_1)$, $p(k_2)$, ..., $p(k_f + 1)$. Then, the Lagrangian is:

$$\begin{aligned} \mathcal{L} &= K(x(k_f+1)) + \sum_{k=k_0}^{k_f} L(x(k_f), u(k_f), k) \qquad (8.113) \\ &+ \sum_{k=k_0}^{k_f} p^T(k+1)\left(x(k+1) - x(k) - f(x(k), u(k), k)\right). \end{aligned}$$

The necessary conditions for optimality are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x(k)} &= p(k) - p(k+1) - \frac{\partial}{\partial x(k)} f(x(k), u(k), k) p(k+1) \qquad (8.114) \\ &+ \frac{\partial}{\partial x(k)} L(x(k), u(k), k) = 0,\ k_1 \le k \le k_f, \\ \frac{\partial \mathcal{L}}{\partial x(k_f+1)} &= \frac{\partial K(x(k_f+1))}{\partial x(k_f+1)} + p(k_f+1) = 0, \\ \frac{\partial \mathcal{L}}{\partial u(k)} &= \frac{\partial}{\partial u(k)} L(x(k), u(k), k) - \frac{\partial}{\partial u(k)} f(x(k), u(k), k) p(k+1) = 0, k_1 \le k \le k_f. \end{aligned}$$

Summarizing our progress, we have four types of equations so far: the **state dynamics**, which are propagated forward in time, are given by (8.107) and repeated here for convenience:

$$x(k+1) = x(k) + f(x(k), u(k), k), \; k_0 \leq k \leq k_f. \tag{8.115}$$

The **co-state dynamics**, propagated backwards in time, are obtained from (8.114) and also repeated for convenience:

$$\begin{aligned} p(k) &= p(k+1) + \frac{\partial}{\partial x(k)} f(x(k), u(k), k) p(k+1) \\ &- \frac{\partial}{\partial x(k)} L(x(k), u(k), k), \; k_1 \leq k \leq k_f. \end{aligned} \tag{8.116}$$

The **optimality condition**, given by (8.114) (line 3):

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u(k)} &= \frac{\partial}{\partial u(k)} L(x(k), u(k), k) - \frac{\partial}{\partial u(k)} f(x(k), u(k), k) p(k+1) \\ &= 0, k_1 \leq k \leq k_f. \end{aligned} \tag{8.117}$$

The **boundary conditions**, given by (8.107) (line 2)and (8.114) (line 2):

$$\begin{aligned} x(k_0) &= x_0, \\ p(k_f+1) &= -\frac{\partial K(x(k_f+1))}{\partial x(k_f+1)}. \end{aligned} \tag{8.118}$$

8.5 Summary of Key Results

The key results in Chapter 8 are as follows:

1. Proposition 8.1, which guarantees existence of a minimum.
2. Proposition 8.2, which provides the basis for all necessary or sufficient conditions for local optimization.
3. Proposition 8.3, which provides necessary conditions for unconstrained optimization.
4. Proposition 8.4, which provides sufficient conditions for unconstrained optimization.
5. Proposition 8.5, which guarantees existence and uniqueness of an optimizer.
6. Proposition 8.6, which provides first order necessary conditions for constrained optimization.
7. Proposition 8.7, which provides second order necessary conditions for constrained optimization.

8. Proposition 8.8, which provides sufficient conditions for constrained optimization.

9. Proposition 8.9, which provides necessary conditions for optimization under inequality constraints.

8.6 Bibliographic Notes for Further Reading

The material in Chapter 8 is standard and is well covered in many texts, including [12] and [63].

One of the earliest formal applications of optimization in aeronautics was the optimization of range for a conventional airplane [2]. Since then, the theory has been applied in many aerospace instances, including optimization of wings [2], airline cruise altitude [75], rocket staging [36], ballistic re-entry [75] and many more.

8.7 Homework Problems

Problem 8.1 *Consider the constraint given by:*

$$x_1^2 + x_2^2 - 1 = 0. \tag{8.119}$$

When can x_1 be expressed as a function of x_2? Evaluate $\frac{\mathrm{d}x_1}{\mathrm{d}x_2}$ using the Implicit Function Theorem.

Problem 8.2 *Prove that:*

$$\underset{|u|\leq 1}{\operatorname{argmax}}(au) = \operatorname{sign}(a), \tag{8.120}$$

where the function sign(.) is defined as follows:

$$\operatorname{sign}[\alpha] = \begin{cases} 1 & \text{if } \alpha > 0, \\ -1 & \text{if } \alpha < 0, \\ \text{undefined} & \text{if } \alpha = 0. \end{cases} \tag{8.121}$$

Problem 8.3 *Let $(x, y) \in \mathbb{R}^2$. Use the method of Lagrange multipliers to optimize $J(x, y) = x^2 - y^2$ subject to the constraint $2x - y + 1 = 0$. Carry out the analysis to second order to ascertain whether the optimum is a minimum or a maximum.*

Problem 8.4 *Carry out the second order analysis in Example 8.6 to ascertain that (8.76) and (8.77) do indeed provide the* ***maximum*** *lift-to-drag ratio.*

Problem 8.5 *Perform numerically the optimization of Example 8.7 for a two-stage rocket, using the numerical values* $\epsilon_1 = 0.7$, $\epsilon_2 = 0.01$, $m_l = 1$ *and* Δv *appropriate for low-Earth orbit. Assume an exhaust speed of 3,000 m/s. Carry out the analysis to second order to ascertain the nature of the optimum.*

Problem 8.6 *Let* $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ *be symmetric positive definite,* $b \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$.

1. *Obtain explicitly the solution of the unconstrained optimization problem:*

$$\min_x f(x) = \frac{1}{2} x^T A x + x^T b + c.$$

2. *Let* $m < n$, $D \in \mathbb{R}^{m \times n}$ *have full rank, and* $e \in \mathbb{R}^m$. *Obtain explicitly the solution of the constrained optimization problem:*

$$\min_x f(x) = \frac{1}{2} x^T A x + x^T b + c,$$

subject to:

$$D^T x = e.$$

Problem 8.7 *Let* M *and* K *be real* $n \times n$ *symmetric positive definite matrices. Obtain explicitly the solution of the constrained optimization problem:*

$$\min_x f(x) = \frac{1}{2} x^T K x,$$

subject to:

$$\frac{1}{2} x^T M x = 1.$$

Give a geometric interpretation to the results.

Problem 8.8 *Answer both parts.*

1. *For a fixed perimeter, what is the rectangle with maximum area?*

2. *For a fixed area, what is the rectangular parallelepiped with maximum volume?*

Problem 8.9 *In designing a cylindrical canister, we want to maximize the volume for a fixed area. What is the optimal shape (e.g., radius to height ratio)?*

Problem 8.10 *What is the parallelepiped with maximum volume inscribed in the ellipsoid:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Assume that the edges are parallel to the axes.

Problem 8.11 *Let $p > 1$ and $q > 1$ be such that:*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Show that for all $a > 0$ and $b > 0$, we have:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Hint: Solve the constrained minimization problem:

$$\min_{x,y} f(x,y) = \frac{x^p}{p} + \frac{y^q}{q},$$

subject to:

$$xy = 1.$$

Problem 8.12 *Let m and n be integers satisfying $m < n$, $Q \in \mathbb{R}^{n\times n}$ be symmetric, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^m$. Show that if $x \in \mathbb{R}^n$ is a regular local optimum for the problem:*

$$\min_x \frac{1}{2} x^T Q x - b^T x,$$

subject to:

$$A^T x = c,$$

then x is a global optimum.

Problem 8.13 *Evaluate the sign of the matrix:*

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix},$$

restricted to the subspace:

$$T = \left\{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}.$$

Problem 8.14 (Least Squares Polynomial Interpolation) *An experiment has produced n measurements $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ where, for $1 \leq i, j \leq n$, we have $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$, and $x_i \neq x_j$. We want to model the variable y as a polynomial function of the variable x with degree $m < n - 1$. In other words, given the polynomial:*

$$p(\xi) = \alpha_m \xi^m + \alpha_{m-1}\xi^{m-1} + ... + \alpha_1 \xi + \alpha_0,$$

and defining:

$$v_i = p(x_i) - y_i,$$

we want to find the values of the coefficients $\alpha_0, \alpha_1, ..., \alpha_m$ *that minimize the cost function:*

$$f(\alpha) = \frac{1}{2}\sum_{i=1}^{n} v_i^2.$$

1. *What is the optimal value of the coefficients?*
2. *Show that this is indeed a minimum.*

Problem 8.15 *Answer both parts.*

1. *Find, if possible, a minimizer for the function:*

$$f(x, y, z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9.$$

2. *For the same function as in (a.), find, if possible, a minimizer under the constraint:*

$$x + y + z = 1.$$

Problem 8.16 *Let* $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Solve the constrained minimization problem:*

$$\min_{x,y} f(x, y) = x^T x + x^T y + y^T y - 2b^T x - 2a^T y,$$

subject to:

$$a^T x + b^T y = \alpha.$$

Show that you obtain a minimum indeed.

Problem 8.17 *Solve the minimization problem:*

$$\min_{x,y} f(x, y) = -xy,$$

subject to:

$$g(x, y) = (x - 3)^2 + y^2 - 5 = 0.$$

Problem 8.18 *For the problem:*

$$\max_{x,y} xy,$$

subject to:

$$(x + y - 2)^2 = 0,$$

an (almost) obvious solution is:

$$(x, y) = (1, 1).$$

However, writing the first order necessary conditions using a Lagrange multiplier leads to a difficulty. Explain why.

Appendix A

Useful Definitions and Mathematical Results

A.1 Results from Topology

Definition A.1 *Let $x^* \in \mathbb{R}^n$ and $\delta > 0$. The open ball around x^* with radius δ is the set:*

$$\mathring{B}(x^*, \delta) = \{x \in \mathbb{R}^n : ||x - x^*|| < \delta\}. \tag{A.1}$$

The closed ball around x^* with radius δ is the set:

$$\bar{B}(x^*, \delta) = \{x \in \mathbb{R}^n : ||x - x^*|| \leq \delta\}. \tag{A.2}$$

Definition A.2 *Let $x^* \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. We say that x^* is in the interior of X, or equivalently that X is a neighborhood of x^*, if there exists an open ball around x^* that is completely contained in X, i.e.,*

$$\exists \delta > 0 : \mathring{B}(x^*, \delta) \subset X. \tag{A.3}$$

Definition A.3 *Let $X \subseteq \mathbb{R}^n$. The interior of X, notated $\mathring{X}$, is the set of points of which X is a neighborhood, or equivalently, the set of points that are in the interior of X. In other words,*

$$\mathring{X} = \left\{x \in \mathbb{R}^n : \exists \delta > 0 : \mathring{B}(x, \delta) \subset X\right\}. \tag{A.4}$$

Note that the interior of a set is always a subset of the set, i.e.,

$$\forall X \subseteq \mathbb{R}^n, \mathring{X} \subseteq X. \tag{A.5}$$

Definition A.4 *Let $X \subseteq \mathbb{R}^n$. We say that X is open if it is equal to its interior, or equivalently if it is a neighborhood of each of its points, or equivalently if $X = \mathring{X}$.*

Definition A.5 *Let $X \subset \mathbb{R}^n$. We say that X has measure zero if its interior is the empty set, i.e., $\mathring{X} = \emptyset$.*

Note that, in $\mathbb{R}$, the interval $X = [0, 1]$ does not have measure zero. However, in $\mathbb{R}^2$, the line segment $X = [0, 1] \times \{0\} = \{(x, 0) : x \in [0, 1]\}$ does have measure zero.

Definition A.6 *Let $x^* \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. We say that x^* adheres to X if every open ball around x^* has a non-empty intersection with X, i.e.,*

$$\forall \delta > 0, \mathring{B}(x^*, \delta) \cap X \neq \emptyset. \tag{A.6}$$

Definition A.7 *Let $X \subseteq \mathbb{R}^n$. The closure of X, or adherence of X, notated $\bar{X}$, is the set of points that adhere to X. In other words,*

$$\bar{X} = \left\{ x \in \mathbb{R}^n : \forall \delta > 0, \mathring{B}(x, \delta) \cap X \neq \emptyset \right\}. \tag{A.7}$$

Note that the closure of a set is always a superset of the set, i.e.,

$$\forall X \subset \mathbb{R}^n, \bar{X} \supseteq X. \tag{A.8}$$

Definition A.8 *Let $X \subseteq \mathbb{R}^n$. We say that X is closed if it is equal to its closure, i.e., if all the points that adhere to X belong to X also, or in other words, $X = \bar{X}$.*

Definition A.9 *Let $X \subset \mathbb{R}^n$. The boundary of X, notated ∂X, is the set of points that adhere to X without being in the interior of X. In other words,*

$$\partial X = \bar{X} \setminus \mathring{X}. \tag{A.9}$$

Definition A.10 *Let $X \subseteq \mathbb{R}^n$. We say that X is bounded if there exists a closed ball around the origin that contains X, i.e.,*

$$\exists \delta > 0 : \bar{B}(0, \delta) \supset X. \tag{A.10}$$

Definition A.11 *Let $X \subseteq \mathbb{R}^n$. We say that X is compact if it is both closed and bounded.*

Definition A.12 *Let $f : \mathbb{R}^n \to \mathbb{R}^m : x \mapsto f(x)$, and $x^* \in \mathbb{R}^n$. We say that the function f is continuous at x^* if its value at x^* can be approximated arbitrarily closely by approximation of x^*, i.e.,*

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}^n, ||x - x^*|| < \delta \implies ||f(x) - f(x^*)|| < \epsilon. \tag{A.11}$$

Proposition A.1 *If $X \subset \mathbb{R}^n$ is compact and $f : X \to \mathbb{R}$ is continuous on X, then f achieves both its minimum and maximum on X.*

Definition A.13 *Let $X \subset \mathbb{R}$ and $b \in \mathbb{R}$. We say that b in an upper bound (respectively, lower bound) for X if $\forall x \in X, x \leq b$ (respectively, $\forall x \in X, x \geq b$). We denote as $X^{\uparrow}$ the set of all the upper bounds of X, and as $X^{\downarrow}$ the set of all its lower bounds. In other words:*

$$\begin{aligned} X^{\uparrow} &= \{b \in \mathbb{R} : \forall x \in X, x \leq b\}, \\ X^{\downarrow} &= \{b \in \mathbb{R} : \forall x \in X, x \geq b\}. \end{aligned} \tag{A.12}$$

Definition A.14 *Let $X \subset \mathbb{R}$. The supremum (resp. infimum) of X is, if it exists, its least upper bound (resp. greatest lower bound). Let f be a real function or functional. The supremum (resp. infimum) of f is the supremum (resp. infimum) of its image - in other words, the supremum (resp. infimum) of the set of real values that f achieves. We use the notation* $\sup X$, $\sup f$ *for supremum, and* $\inf X$, $\inf f$ *for infimum.*

Note that both the supremum and infimum of a set adhere to the set. The existence of a supremum or infimum is secured by the following foundational result.

Proposition A.2 (Fundamental Axiom of Analysis) *If $X \subset \mathbb{R}$ has an upper (resp. lower) bound, then it has a supremum (resp. infimum). In other words,*

$$\begin{aligned} \left(X \subset \mathbb{R} \wedge X^{\uparrow} \neq \emptyset\right) &\implies \left(\exists x^* \in X^{\uparrow} : \forall x \in X^{\uparrow}, x^* \leq x\right), \\ \left(X \subset \mathbb{R} \wedge X^{\downarrow} \neq \emptyset\right) &\implies \left(\exists x^* \in X^{\downarrow} : \forall x \in X^{\downarrow}, x^* \geq x\right). \end{aligned} \tag{A.13}$$

A.2 Results from Linear Algebra

Definition A.15 *A real vector space $\mathcal{V}$ is a set of elements called "vectors", equipped with two operations: addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and scaling: $\mathbb{R} \times \mathcal{V} \to \mathcal{V}$ that satisfy the following axioms:*

$$\begin{aligned} \forall u, v \in \mathcal{V}, \exists! u + v \in \mathcal{V}, \\ \forall u, v \in \mathcal{V}, u + v = v + u, \\ \forall u, v, w \in \mathcal{V}, (u + v) + w = u + (v + w), \\ \exists 0 \in \mathcal{V} : \forall v \in \mathcal{V}, v + 0 = v, \\ \forall v \in \mathcal{V}, \exists (-v) \in \mathcal{V} : v + (-v) = 0, \\ \forall x \in \mathbb{R}, \forall v \in \mathcal{V}, \exists! xv \in \mathcal{V}, \\ \forall x, y \in \mathbb{R}, \forall v \in \mathcal{V}, (xy)v = x(yv), \\ \forall x \in \mathbb{R}, \forall u, v \in \mathcal{V}, x(u + v) = xu + xv, \\ \forall x, y \in \mathbb{R}, \forall v \in \mathcal{V}, (x + y)v = xv + yv, \\ \forall v \in \mathcal{V}, 1v = v. \end{aligned} \tag{A.14}$$

Note that the zero vector defined in (A.14) (line 4) and the additive inverse in (A.14) (line 5) are unique.

Definition A.16 *A subspace of a real vector space is a subset that is itself a real vector space.*

Definition A.17 *Let $\mathcal{V}$ be a real vector space and $V = \{v_1, v_2, ..., v_n\} \subset \mathcal{V}$. We say that V is linearly independent if:*

$$\forall x_1, x_2, ..., x_n \in \mathbb{R}, x_1v_1 + x_2v_2 + ... + x_nv_n = 0 \implies x_1 = x_2 = ... = x_n = 0. \quad \text{(A.15)}$$

An infinite set is linearly independent if all its finite subsets are linearly independent.

Definition A.18 *The dimension of a real vector space is the maximum number of elements in a linearly independent subset.*

Definition A.19 *Let $\mathcal{V}$ be a real vector space and $V = \{v_1, v_2, ..., v_n\} \subset \mathcal{V}$. We say that V is generating if:*

$$\forall v \in \mathcal{V}, \exists x_1, x_2, ..., x_n \in \mathbb{R} : v = x_1v_1 + x_2v_2 + ... + x_nv_n. \quad \text{(A.16)}$$

Definition A.20 *Let $\mathcal{V}$ be a real vector space and $V \subset \mathcal{V}$. We say that V is a basis for $\mathcal{V}$ if V is both linearly independent and generating.*

Note that, in a real vector space, all bases have a number of elements equal to the dimension. Also, the dimension is the minimum number of elements in a generating subset. Finally, every vector is expressed uniquely as a linear combination of the basis vectors, as in (A.16) - the coefficients of that linear combination are called the coordinates of the vector with respect to the basis.

Definition A.21 *Let $\mathcal{V}$ and $\mathcal{U}$ be real vector spaces, and $\mathcal{A} : \mathcal{V} \to \mathcal{U}$. We say that the mapping $\mathcal{A}$ is linear if:*

$$\forall v_1, v_2 \in \mathcal{V}, \forall x_1, x_2 \in \mathbb{R}, \mathcal{A}(x_1v_1 + x_2v_2) = x_1\mathcal{A}(v_1) + x_2\mathcal{A}(v_2). \quad \text{(A.17)}$$

Definition A.22 *Let $\mathcal{V}$ and $\mathcal{U}$ be real vector spaces, and $\mathcal{A} : \mathcal{V} \to \mathcal{U}$ be linear. The range or range space of $\mathcal{A}$ is the set:*

$$\mathcal{R}(\mathcal{A}) = \{u \in \mathcal{U} : \exists v \in \mathcal{V} : u = \mathcal{A}(v)\}. \quad \text{(A.18)}$$

The null space or nullspace or kernel of $\mathcal{A}$ is the set:

$$\mathcal{N}(\mathcal{A}) = \{v \in \mathcal{V} : \mathcal{A}(v) = 0\}. \quad \text{(A.19)}$$

Note that if $\mathcal{A} : \mathcal{V} \to \mathcal{U}$ is linear, then its range space is a subspace of $\mathcal{U}$ and its null space is a subspace of $\mathcal{V}$. This allows the following definition:

Definition A.23 *Let $\mathcal{V}$ and $\mathcal{U}$ be real vector spaces, and $\mathcal{A} : \mathcal{V} \to \mathcal{U}$ be linear. The rank of $\mathcal{A}$ is the dimension of its range space. The nullity of $\mathcal{A}$ is the dimension of its null space.*

Proposition A.3 (Fundamental Theorem of Linear Algebra) *Let $\mathcal{V}$ and $\mathcal{U}$ be real vector spaces, and $\mathcal{A} : \mathcal{V} \to \mathcal{U}$ be linear. Then the sum of its rank and nullity equals the dimension of $\mathcal{V}$.*

Definition A.24 *A matrix is a rectangular array of numbers denoted by:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}. \tag{A.20}$$

If A has m rows and n columns, we say that A is an m by n matrix, written $m \times n$. If $m = n$, we say that A is a square matrix of order n. For a square matrix A of order n, the diagonal elements are the entries a_{ii}, $1 \leq i \leq n$. A square matrix is called diagonal if its only non-zero entries are on its diagonal.

The connection between linear mappings and matrices is as follows. Let $\mathcal{V}$ and $\mathcal{U}$ be real vector spaces of dimensions n and m, respectively, with known ordered bases. Let $\mathcal{A} : \mathcal{V} \to \mathcal{U}$ be linear. Then, to $\mathcal{A}$ we associate a matrix such as (A.20) as follows: the j^{th} column of A contains the coordinates, with respect to the basis of $\mathcal{U}$, of the image, by $\mathcal{A}$, of the j^{th} vector of the basis of $\mathcal{V}$. If $v \in \mathcal{V}$ has coordinates $x \in \mathbb{R}^n$ and $u = \mathcal{A}(v)$, then u has coordinates $y \in \mathbb{R}^m$ satisfying $y = Ax$.

Definition A.25 *The range or range space of a rectangular matrix $A \in \mathbb{R}^{m \times n}$, $\mathcal{R}(A)$, is given by:*

$$\mathcal{R}(A) = \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n : y = Ax\}. \tag{A.21}$$

Definition A.26 *The null space or nullspace or kernel of a rectangular matrix $A \in \mathbb{R}^{m \times n}$, $\mathcal{N}(A)$, is given by:*

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}. \tag{A.22}$$

Note that if $A \in \mathbb{R}^{m \times n}$, then its range space is a subspace of $\mathbb{R}^m$ and its null space is a subspace of $\mathbb{R}^n$.

Definition A.27 *Let $A \in \mathbb{R}^{m \times n}$. The rank of A is the dimension of its range space. The nullity of A is the dimension of its null space.*

A remarkable corollary of the fundamental theorem of linear algebra is that the rank of a matrix is equal to both the number of its linearly independent columns and the number of its linearly independent rows.

Definition A.28 *If $A = [a_{ij}]$ is an $m \times n$ real matrix, then the transpose of A, denoted A^T, is the $n \times m$ real matrix defined by $A^T = [a_{ji}]$.*

Definition A.29 *A real square matrix A is called symmetric if $A = A^T$.*

Proposition A.4 *If A is a real symmetric matrix, then all its eigenvalues and eigenvectors are real (see Definition 2.10). Also, none of its eigenvalues is defective (see Definitions 2.11 and 2.12). Moreover, its eigenvectors associated with different eigenvalues are orthogonal in the following sense: if λ_i and λ_j, $\lambda_i \neq \lambda_j$, are eigenvalues of A associated with the eigenvectors x_i and x_j, respectively, then:*

$$x_i^T x_j = 0. \tag{A.23}$$

As a consequence of Proposition A.4, if A is a real symmetric matrix of order n, then it can be factored as:

$$A = X \Lambda X^T, \tag{A.24}$$

where the square matrix X is orthogonal, i.e., satisfies:

$$X^T X = I, \tag{A.25}$$

and the square matrix Λ is diagonal. Specifically, the matrix X contains, columnwise, the eigenvectors of A, normalized to magnitude 1, and the matrix Λ has on its diagonal the corresponding eigenvalues of A.

Definition A.30 *If A is a symmetric matrix, then the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by:*

$$g(x) = x^T A x, \tag{A.26}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \tag{A.27}$$

is called a real quadratic form in the n variables x_1, x_2, ..., x_n.

Definition A.31 (Sign of a symmetric matrix) *The sign of a symmetric matrix (positive definite (> 0), positive semidefinite (≥ 0), negative definite (< 0), negative semidefinite (≤ 0), or indefinite ($\gtreqless$)) is defined by the sign of its quadratic form. For example, $A = A^T$ is positive definite if:*

$$\forall x \neq 0, x^T A x > 0. \tag{A.28}$$

The sign of a symmetric $n \times n$ matrix A in a linear subspace $\mathcal{S} = \{x \in \mathbb{R}^n : Cx = 0\}$, where matrix C is $m \times n$ and full rank, is defined as follows. Let S be an $n \times (n-m)$ matrix which contains a columnwise basis of $\mathcal{S}$, i.e., $CS = 0$. Then, the sign of A in $\mathcal{S}$ is, by definition, the sign of $S^T AS$.

Note that the sign of a symmetric matrix is determined by the sign of its eigenvalues. For instance, a symmetric matrix is positive definite (resp. positive semidefinite) if and only if all its eigenvalues are strictly positive (resp. positive or zero).

A.3 Taylor's Theorem

Theorem A.1 (Taylor's Theorem) *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^{*n+1}$ on $[a, x]$. Then,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x,a), \tag{A.29}$$

where

$$R_n(x,a) = \int_a^x \frac{(x-\tau)^n}{n!} f^{(n+1)}(\tau)\,\mathrm{d}\tau. \tag{A.30}$$

Moreover, if $f \in \mathcal{C}^{n+1}$ on $[a, x]$, then there exists $c \in [a, x]$ such that:

$$R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{A.31}$$

A.4 Newton's Method

Consider the system of equations:

$$y = g(x,t), \tag{A.32}$$

where $t \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are known, and $x \in \mathbb{R}^n$ is unknown.

Assume that we proceed iteratively to find a solution for the system of equations (A.32), and let x^k be the k^{th} iterate, i.e., the candidate solution at the k^{th} iteration. Assume that x^k is not a solution of (A.32), i.e., $y \neq g(x^k, t)$. However, assume that x^{k+1} is a solution, i.e., $y = g(x^{k+1}, t)$. Performing a first order Taylor series expansion around x^k, and using the same notation for Jacobian as in Chapter 2, we obtain:

$$y = g(x^k, t) + \left(\frac{\partial g}{\partial x}\right)_{x^k}^T (x^{k+1} - x^k), \tag{A.33}$$

where we have neglected the higher-order terms. We recognize that (A.33) is a **linear** equation for x^{k+1}. If the Jacobian matrix is nonsingular, we can solve this equation as:

$$x^{k+1} = x^k + \left(\frac{\partial g}{\partial x}\right)_{x^k}^{-T} \left(y - g(x^k, t)\right). \tag{A.34}$$

Equation (A.34) defines **Newton's iteration**. The above discussion is quite informal and does not precisely answer the question of when this iteration is guaranteed to converge to a solution. This question is settled by the following result.

Proposition A.5 (Newton's Theorem) *Let g be a differentiable function of x in the system of equations (A.32). Assume that the system (A.32) has a solution x^* such that the Jacobian matrix, $\left(\frac{\partial g}{\partial x}\right)_{x^*}$, is nonsingular. Then, there exists an open neighborhood of x^* such that whenever one chooses an initial condition x^0 from that neighborhood (i.e., whenever $||x^0 - x^*||$ is small enough), iteration (A.34) is guaranteed to converge towards x^*.*

Example A.1 *Figure A.1 illustrates Newton's iteration.*

A.5 The Implicit Function Theorem

Proposition A.6 (Implicit Function Theorem) *Let F be a real m-vector function of n real variables, i.e.,*

$$F : \mathbb{R}^n \to \mathbb{R}^m, \;\; m < n. \tag{A.35}$$

Assume that $x^0 \in \mathbb{R}^n$ satisfies $F(x^0) = 0$ and that F is of class $\mathcal{C}^p$ in a neighborhood of x^0. Let:

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}, \tag{A.36}$$

where $x_1^0 \in \mathbb{R}^m$ and $x_2^0 \in \mathbb{R}^{n-m}$.

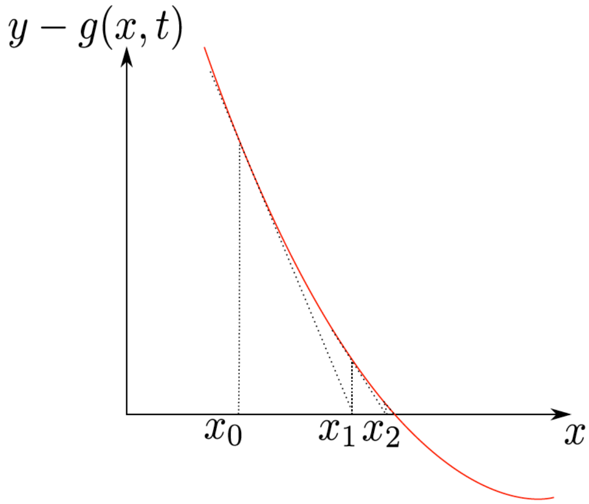

Figure A.1: Newton's iteration.

Assume that the $m \times m$ matrix of partial derivatives evaluated at x^0,

$$\left[\frac{\partial F}{\partial x_1}\right]_{x^0}, \tag{A.37}$$

is nonsingular. (Note that this is a crucial assumption.) Then, there exists an open ball around x_2^0, $B(x_2^0, \delta)$, and a function ϕ of class $\mathcal{C}^p$:

$$\phi : B(x_2^0, \delta) \to \mathbb{R}^m : x_2 \mapsto x_1 = \phi(x_2) \tag{A.38}$$

that makes the implicit relation $F(x) = 0$ explicit on that neighborhood, i.e., such that:

$$x_1^0 = \phi(x_2^0), \tag{A.39}$$

and

$$F(\phi(x_2), x_2) \equiv 0 \text{ on } B(x_2^0, \delta). \tag{A.40}$$

Moreover, this function $\phi(x_2)$ is at least once continuously differentiable at x_2^0, and its matrix of partial derivatives at x_2^0, $\left(\frac{\partial \phi}{\partial x_2}\right)_{x_2 = x_2^0}$, satisfies the linear algebraic equation:

$$\left(\frac{\partial F}{\partial x_1}\right)_0^T \left(\frac{\partial \phi}{\partial x_2}\right)_0^T + \left(\frac{\partial F}{\partial x_2}\right)_0^T = 0. \tag{A.41}$$

Remark A.1 *Note that formula (A.41) is easily remembered by setting the total differential on the left hand side of (A.40) to zero and using the chain rule. Also note that it is possible to evaluate the matrix of partial derivatives of ϕ at x_2^0 without solving for ϕ explicitly.*

Remark A.2 (Local Inverse) *The Implicit Function Theorem is sometimes called the* ***Local Inverse Theorem****. To see why, consider the function:*

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ x &\mapsto y = f(x). \end{aligned} \tag{A.42}$$

We want to know if f can be inverted, and particularly whether $x = \phi(y)$. Let $x_1 = x$, $x_2 = y$, and $g(x_1, x_2) = f(x) - y$. The Implicit Function Theorem states that if $\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x}$ is nonsingular, then the inverse exists locally. Moreover,

$$\frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial y} = -(-I)\left(\frac{\partial f}{\partial x}\right)^{-1} = \left(\frac{\partial f}{\partial x}\right)^{-1}. \tag{A.43}$$

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