Extreme value theorem

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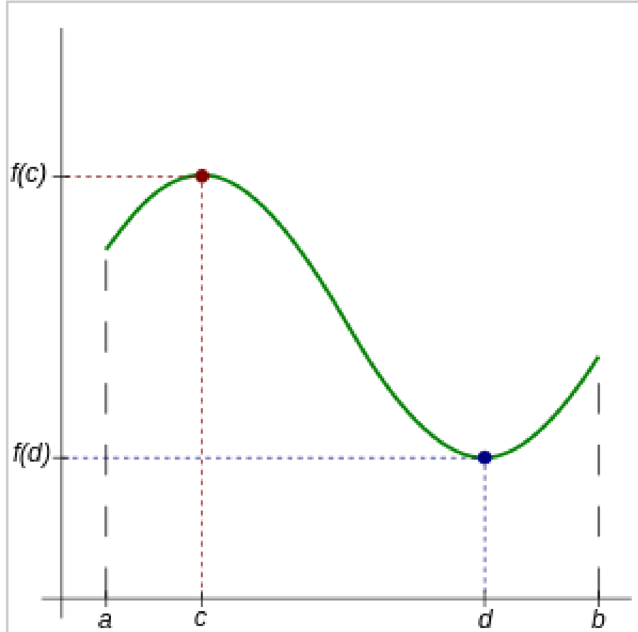
In calculus, the **extreme value theorem** states that if a real-valued function *f* is continuous in the closed and bounded interval [*a*,*b*], then *f* must attain a maximum and a minimum, each at least once. That is, there exist numbers *c* and *d* in [*a*,*b*] such that:

$$f(c) \geq f(x) \geq f(d) \quad \text{for all } x \in [a,b].$$

A related theorem is **the boundedness theorem** which states that a continuous function *f* in the closed interval [*a*,*b*] is bounded on that interval. That is, there exist real numbers *m* and *M* such that:

$$m \leq f(x) \leq M \quad \text{for all } x \in [a,b].$$

The extreme value theorem enriches the boundedness theorem by saying that not only is the function bounded, but it also attains its least upper bound as its maximum and its greatest lower bound as its minimum.



A continuous function $f(x)$ on the closed interval [*a*,*b*] showing the absolute max (red) and the absolute min (blue).

The extreme value theorem is used to prove Rolle's theorem. In a formulation due to Karl Weierstrass, this theorem states that a continuous function from a non-empty compact space to a subset of the real numbers attains a maximum and a minimum.

Contents

History

The extreme value theorem was originally proven by Bernard Bolzano in the 1830s in a work *Function Theory* but the work remained unpublished until 1930. Bolzano's proof consisted of showing that a continuous function on a closed interval was bounded, and then showing that the function attained a maximum and a minimum value. Both proofs involved what is known today as the Bolzano–Weierstrass theorem (Rusnock & Kerr-Lawson 2005). The result was also discovered later by Weierstrass in 1860.

Functions to which the theorem does not apply

The following examples show why the function domain must be closed and bounded in order for the theorem to apply. Each fails to attain a maximum on the given interval.

1. $f(x) = x$ defined over $[0, \infty)$ is not bounded from above.
2. $f(x) = x / (1 + x)$ defined over $[0, \infty)$ is bounded but does not attain its least upper bound 1.
3. $f(x) = 1 / x$ defined over $(0, 1]$ is not bounded from above.
4. $f(x) = 1 - x$ defined over $(0, 1]$ is bounded but never attains its least upper bound 1.

Defining $f(0) = 0$ in the last two examples shows that both theorems require continuity on $[a, b]$.

Generalization to arbitrary topological spaces

When moving from the real line to arbitrary topological spaces, the parallel of a closed bounded interval is a compact space.

It is known that compactness is preserved by continuous functions, i.e. the image of the compact space under a continuous mapping is also compact. A subset of the real line is compact if and only if it is both closed and bounded.

This implies the following generalization of the extreme value theorem: a continuous real-valued function on a nonempty compact space is bounded above and attains its supremum. Slightly more generally, this is true for an upper semicontinuous function. (see compact space#Functions and compact spaces).

Proving the theorems

We look at the proof for the upper bound and the maximum of f. By applying these results to the function $-f$, the existence of the lower bound and the result for the minimum of f follows. Also note that everything in the proof is done within the context of the real numbers.

We first prove the boundedness theorem, which is a step in the proof of the extreme value theorem. The basic steps involved in the proof of the extreme value theorem are:

1. Prove the boundedness theorem.
2. Find a sequence so that its image converges to the supremum of f.

3. Show that there exists a subsequence that converges to a point in the domain.
4. Use continuity to show that the image of the subsequence converges to the supremum.

Proof of the boundedness theorem

Suppose the function f is not bounded above on the interval $[a,b]$. Then, for every natural number n, there exists an x_n in $[a,b]$ such that $f(x_n) > n$. This defines a sequence $\{x_n\}$. Because $[a,b]$ is bounded, the Bolzano–Weierstrass theorem implies that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Denote its limit by x. As $[a,b]$ is closed, it contains x. Because f is continuous at x, we know that $\{f(x_{n_k})\}$ converges to the real number $f(x)$ (as f is sequentially continuous at x.) But $f(x_{n_k}) > n_k \geq k$ for every k, which implies that $\{f(x_{n_k})\}$ diverges to $+\infty$, a contradiction. Therefore, f is bounded above on $[a,b]$. ■

Proof of the extreme value theorem

By the boundedness theorem, f is bounded from above, hence, by the Dedekind-completeness of the real numbers, the least upper bound (supremum) M of f exists. It is necessary to find a d in $[a,b]$ such that $M = f(d)$. Let n be a natural number. As M is the *least* upper bound, $M – 1/n$ is not an upper bound for f. Therefore, there exists d_n in $[a,b]$ so that $M – 1/n < f(d_n)$. This defines a sequence $\{d_n\}$. Since M is an upper bound for f, we have $M – 1/n < f(d_n) \leq M$ for all n. Therefore, the sequence $\{f(d_n)\}$ converges to M.

The Bolzano–Weierstrass theorem tells us that there exists a subsequence $\{d_{n_k}\}$, which converges to some d and, as $[a,b]$ is closed, d is in $[a,b]$. Since f is continuous at d, the sequence $\{f(d_{n_k})\}$ converges to $f(d)$. But $\{f(d_{n_k})\}$ is a subsequence of $\{f(d_n)\}$ that converges to M, so $M = f(d)$. Therefore, f attains its supremum M at d. ■

Alternative proof of the extreme value theorem

The set $\{y \in \mathbf{R} : y = \mathrm{f}(x) \text{ for some } x \in [a,b]\}$ is a bounded set. Hence, its least upper bound exists by least upper bound property of the real numbers. Let $M = \sup(f(x))$ on $[a, b]$. If there is no point x on $[a, b]$ so that $f(x) = M$ then $f(x) < M$ on $[a, b]$. Therefore $1/(M - f(x))$ is continuous on $[a, b]$.

However, to every positive number ε, there is always some x in $[a, b]$ such that $M - f(x) < \varepsilon$ because M is the least upper bound. Hence, $1/(M - f(x)) > 1/\varepsilon$, which means that $1/(M - f(x))$ is not bounded. Since every continuous function on a $[a, b]$ is bounded, this contradicts the conclusion that $1/(M - f(x))$ was continuous on $[a, b]$. Therefore there must be a point x in $[a, b]$ such that $f(x) = M$. ■

Proof using the hyperreals

In the setting of non-standard calculus, let N be an infinite hyperinteger. The interval $[0, 1]$ has a natural hyperreal extension. Consider its partition into N subintervals of equal infinitesimal length $1/N$, with partition points $x_i = i/N$ as i "runs" from 0 to N. The function f is also naturally extended to a function f^* defined on the hyperreals between 0 and 1. Note that in the standard setting (when N is finite), a point with the maximal value of f can always be chosen among the $N+1$ points x_i, by induction. Hence, by the transfer principle, there is a hyperinteger i_0 such that $0 \le i_0 \le N$ and $f^*(x_{i_0}) \ge f^*(x_i)$ for all $i = 0, \ldots, N$. Consider the real point

$$c = \mathbf{st}(x_{i_0})$$

where **st** is the standard part function. An arbitrary real point x lies in a suitable sub-interval of the partition, namely $x \in [x_i, x_{i+1}]$, so that $\mathbf{st}(x_i) = x$. Applying **st** to the inequality $f^*(x_{i_0}) \ge f^*(x_i)$, we obtain $\mathbf{st}(f^*(x_{i_0})) \ge \mathbf{st}(f^*(x_i))$. By continuity of f we have

$$\mathbf{st}(f^*(x_{i_0})) = f(\mathbf{st}(x_{i_0})) = f(c).$$

Hence $f(c) \ge f(x)$, for all real x, proving c to be a maximum of f. See Keisler (1986, p. 164).

Extension to semi-continuous functions

If the continuity of the function f is weakened to semi-continuity, then the corresponding half of the boundedness theorem and the extreme value theorem hold and the values $-\infty$ or $+\infty$, respectively, from the extended real number line can be allowed as possible values. More precisely:

Theorem: If a function $f : [a,b] \to [-\infty,\infty)$ is upper semi-continuous, meaning that

$$\limsup_{y\to x} f(y) \le f(x)$$

for all x in $[a,b]$, then f is bounded above and attains its supremum.

Proof: If $f(x) = -\infty$ for all x in $[a,b]$, then the supremum is also $-\infty$ and the theorem is true. In all other cases, the proof is a slight modification of the proofs given above. In the proof of the boundedness theorem, the upper semi-continuity of f at x only implies that the limit superior of the subsequence $\{f(x_{n_k})\}$ is bounded above by $f(x) < \infty$, but that is enough to obtain the contradiction. In the proof of the extreme value theorem, upper semi-continuity of f at d implies that the limit superior of the subsequence $\{f(d_{n_k})\}$ is bounded above by $f(d)$, but this suffices to conclude that $f(d) = M$. ■

Applying this result to $-f$ proves:

Theorem: If a function $f : [a,b] \to (-\infty,\infty]$ is lower semi-continuous, meaning that

$$\liminf_{y\to x} f(y) \ge f(x)$$

for all x in $[a,b]$, then f is bounded below and attains its infimum.

A real-valued function is upper as well as lower semi-continuous, if and only if it is continuous in the usual sense. Hence these two theorems imply the boundedness theorem and the extreme value theorem.

References

- Keisler, H. Jerome (1986). *Elementary calculus. An infinitesimal approach* (http://www.math.wisc.edu/~keisler/calc.html). Boston, Massachusetts: Prindle, Weber & Schmidt. ISBN 0-87150-911-3.

External links

- A Proof for extreme value theorem (http://www.cut-the-knot.org/fta/fta_note.shtml) at cut-the-knot
- Boundedness Theorem (http://planetmath.org/?op=getobj&from=objects&id=6022) at PlanetMath.org.
- Extreme Value Theorem (http://planetmath.org/?op=getobj&from=objects&id=6023) at PlanetMath.org.
- Extreme Value Theorem (http://demonstrations.wolfram.com/ExtremeValueTheorem/) by Jacqueline Wandzura with additional contributions by Stephen Wandzura, the Wolfram Demonstrations Project.
- Weisstein, Eric W., "Extreme Value Theorem" (http://mathworld.wolfram.com/ExtremeValueTheorem.html), *MathWorld*.

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