Continuity of convex functions in normed spaces

In this chapter, we consider continuity properties of real-valued convex functions defined on open convex sets in normed spaces. Recall that every infinite-dimensional normed space contains a discontinuous linear functional. Thus, in infinite-dimensional spaces, there exist discontinuous convex functions. One of the corollaries of the results of this chapter is that, in finite-dimensional spaces, this cannot happen.

Recall that $B(x,r)$ and $B^0(x,r)$ denote the open and closed ball of radius r, centered at x. A function $f\colon E\to\mathbb{R}$ is L-Lipschitz if $|f(x)-f(y)|\le L\|x-y\|$ whenever $x,y\in E$.

Proposition 0.1. *Let X be a normed space, $x_0\in X$, $r>0$, $\varepsilon\in(0,r)$, $m,M\in\mathbb{R}$. Let $f\colon B^0(x_0,r)\to\mathbb{R}$ be a convex function.*

(a) *If $f(x)\le m$ on $B^0(x_0,r)$, then $|f(x)|\le|m|+2|f(x_0)|$ on $B^0(x_0,r)$.*

(b) *If $|f(x)|\le M$ on $B^0(x_0,r)$, then f is $\left(\frac{2M}{\varepsilon}\right)$-Lipschitz on $B^0(x_0,r-\varepsilon)$.*

Proof. By translation, we can suppose that $x_0=0$. Denote $B=B^0(0,r)$ and $C=B^0(0,r-\varepsilon)$.

(a) Since $0=\frac{1}{2}x+\frac{1}{2}(-x)$ $(x\in B)$, we have $f(0)\le\frac{1}{2}f(x)+\frac{1}{2}f(-x)$. Consequently, $f(x)\ge 2f(0)-f(-x)\ge 2f(0)-m$, and hence

$$|f(x)|\le\max\{m,m-2f(0)\}\le|m|+2|f(0)|\qquad(x\in B)\,.$$

(b) Consider two distinct points $x,y\in C$. The point

$$z=y+\tfrac{\varepsilon}{\|y-x\|}(y-x)$$

belongs to B and $y\in(x,z)$. An easy calculation shows that

$$y=\tfrac{\varepsilon}{\varepsilon+\|y-x\|}x+\tfrac{\|y-x\|}{\varepsilon+\|y-x\|}z\qquad\text{(convex combination!)}.$$

Use convexity of f and multiply by the common denominator to get

$$(\varepsilon+\|y-x\|)f(y)\le\varepsilon f(x)+\|y-x\|f(z)\,.$$

Then $\varepsilon\big[f(y)-f(x)\big]\le\big[f(z)-f(y)\big]\|y-x\|\le 2M\|y-x\|$. Thus

$$f(y)-f(x)\le\tfrac{2M}{\varepsilon}\|y-x\|\,.$$

Interchanging the role of x and y, we obtain that f is $(\frac{2M}{\varepsilon})$-Lipschitz on C. □

Observation 0.2. *Let C be a convex set in a normed space X, $f\colon C\to\mathbb{R}$ a convex function, $B:=B^0(x_0,r)\subset C$. Let $x,y\in C$ be such that $x=(1-\lambda)x_0+\lambda y$ with $0<\lambda<1$. If $f\le m$ on B, then*

$$f\le\max\{m,f(y)\}\quad\textit{on}\ \operatorname{conv}\big[B\cup\{y\}\big]$$

(in particular, on $B^0(x,(1-\lambda)r)$).

Proof. Exercise. □

Theorem 0.3. *Let C be an open convex set in a normed space X, and $f\colon C\to\mathbb{R}$ a convex function. The following assertions are equivalent:*

(i) f *is locally Lipschitz on* C*;*
(ii) f *is continuous on* C*;*
(iii) f *is continuous at some point of* C*;*
(iv) f *is locally bounded on* C*;*
(v) f *is upper bounded on a nonempty open subset of* C*.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are obvious. It remains to show that (v) implies (i).

By (v), there exists an open ball $B^0(x_0, r) \subset C$ on which f is upper bounded. Let $x \in C$. There exists $y \in C$ such that $x \in (x_0, y)$. By Observation 0.2, f is uper bounded on some ball $B^0(x, \varrho)$. By Proposition 0.1, f is Lipschitz on $B^0(x, \frac{\varrho}{2})$. □

As an easy corollary, we obtain the following result on automatic continuity of convex functions in finite-dimensional spaces.

Corollary 0.4. *Each convex function on an open convex subset of* $\mathbb{R}^d$ *is locally Lipschitz (hence continuous).*

Proof. Let $C \subset \mathbb{R}^d$ be open and convex, and $f \colon C \to \mathbb{R}$ a convex function. Fix $x_0 \in C$. There exist finitely many points $c_1, \ldots, c_n \in C_0$ such that $x_0 \in U := \mathrm{int}[\mathrm{conv}\{c_1, \ldots, c_n\}]$ (take, e.g., the vertices of a small d-dimensional cube centered at x_0). By convexity, $f \le \max\{f(c_1), \ldots, f(c_n)\}$ on U. By Theorem 0.3, f is locally Lipschitz on U. □

Corollary 0.5. *Let* C *be a finite-dimensional convex set in a normed space* X*. Then every convex function* $f \colon C \to \mathbb{R}$ *is continuous on* $\mathrm{ri}(C)$ *(the relative interior of* C*).*

Proof. Ecercise. (Hint: use Corollary 0.4.) □

Continuity of semicontinuous convex functions.

Let M be a topological space, $x_0 \in M$. Recall that a function $f \colon M \to \overline{\mathbb{R}}$ is:

- *lower semicontinuous* (*l.s.c.*) at x_0 if $\forall t \in (-\infty, f(x_0))\ \exists \delta > 0$: $f(x) > t$ whenever $d(x, x_0) < \delta$.
- *upper semicontinuous* (*u.s.c.*) at x_0 if $\forall t \in (f(x_0), +\infty)\ \exists \delta > 0$: $f(x) < t$ whenever $d(x, x_0) < \delta$.

Clearly, f is u.s.c. at x_0 if and only if $-f$ is l.s.c. at x_0.

Observation 0.6. *Let* M, f *be as above. Then the following assertions are equivalent:*

(i) f *is l.s.c.;*
(ii) *for each* $t \in \mathbb{R}$*, the set* $\{f > t\}$ *is open;*
(iii) *for each* $t \in \mathbb{R}$*, the set* $\{f \le t\}$ *is closed;*
(iv) *the epigraph* $\mathrm{epi}(f) := \{(x, t) \in M \times \mathbb{R} : f(x) \le t\}$ *is closed (in* $M \times \mathbb{R}$*).*

Proof. Exercise. □

Proposition 0.7. *Let* C *be an open convex set in a normed space* X*,* $f \colon C \to \mathbb{R}$ *a convex function.*

(a) *If* f *is u.s.c., then* f *is continuous on* C*.*

(b) *If X is a Banach space and f is l.s.c., then f is continuous on C.*

Proof. (a) Fix $x_0 \in C$ and $t > f(x_0)$. Then the set $\{x \in C : f(x) < t\}$ is a nonempty open subset of C, on which f is bounded above. Apply Theorem 0.3.
(b) If $C = X$, put $F_n = \{x \in C : f(x) \leq n\}$; otherwise define

$$F_n = \{x \in C : \ f(x) \leq n, \ \mathrm{dist}(x, X \setminus C) \geq \tfrac{1}{n}\}\,.$$

The sets F_n $(n \in \mathbb{N})$ are closed in C; but they are also closed in X since $\overline{F_n} \subset C$. By the Baire Category Theorem, there exists $k \in \mathbb{N}$ such that F_k has a nonempty interior. This implies that f is upper bounded on a nonempty open set. Apply Theorem 0.3. □

Families of convex functions. Let $\mathcal{F}$ be a family of fuctions on a set E. We say that $\mathcal{F}$ is *pointwise bounded* if, for each $x \in E$, the set $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is bounded (in $\mathbb{R}$).

Theorem 0.8. *Let C be an open convex set in a Banach space X. Let $\mathcal{F}$ be a family of continuous convex functions on C. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is locally equi-Lipschitz and locally equi-bounded.*

Proof. The (real-valued!) function

$$g(x) = \sup \mathcal{F}(x) = \sup_{f \in \mathcal{F}} f(x)$$

is easily seen to be l.s.c. (e.g., by Observation 0.6). By Proposition 0.7, g is continuous on C; in particular, g is locally bounded. Hence $\mathcal{F}$ is locally equi-bounded above on C. The rest follows from Proposition 0.1. □

Theorem 0.9. *Let C be an open convex set in a Banach space X. Let $\{f_n\}$ be a sequence of continuous convex functions on C that converges pointwise on C to a (convex) function $f\colon C \to \mathbb{R}$. Then f is continuous and the convergence is uniform on compact sets.*

Proof. The sequence $\{f_n\}$ is pointwise bounded, hence, by the previous theorem, locally equi-bounded and equi-Lipschitz. Consequently, f is locally bounded and hence continuous (Theorem 0.3). Moreover, on each compact set $K \subset C$, the restrictions $f_n|_K$ $(n \in \mathbb{N})$ are equi-bounded and equi-Lipschitz. An easy application of the Ascoli-Arzelà theorem (*exercise!*) gives that they converge uniformly on K. □

Let us recall the so-called *diagonal method*, a standard argument used in many areas of Mathematical Analysis. Given a sequence $\sigma = (\sigma(1), \sigma(2), \ldots)$ of elements of a set S, and $m \in \mathbb{N}$, we denote by $\sigma|_{[m,\infty)}$ the sequence $(\sigma(m), \sigma(m+1), \ldots)$.

Lemma 0.10 (Diagonal method)**.** *Let S be a set, and σ_n $(n \in \mathbb{N})$ countably many sequences of elements of S such that σ_{n+1} is a subsequence of σ_n for each $n \in \mathbb{N}$. Then there exists a sequence σ_∞ of elements of S, such that, for each $n \in \mathbb{N}$,*

$$\sigma_\infty|_{[n,\infty)} \text{ is a subsequence of } \sigma_n.$$

Proof. The "diagonal" sequence $\sigma_\infty = (\sigma_1(1), \sigma_2(2), \ldots)$ has the desired property. □

Theorem 0.11. *Let C be an open convex set in a separable Banach space X. Let $\{f_n\}$ be a pointwise bounded sequence of continuous convex functions on C. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges pointwise and uniformly on compact sets to a continuous convex function on C.*

Proof. Fix a countable dense set $D \subset C$. Since $\{f_n\}$ is pointwise bounded, an easy application of diagonal method produces a subsequence $\{f_{n_k}\}$ that converges at each point of D. Let us show that this subsequence converges pointwise on C. Given $x \in C$, there exists an open ball $U \subset C$, centered in x, on which $\{f_n\}$ is equi-Lipschitz with a certain Lipschitz constant $L > 0$. Choose $d \in U$ such that $\|d - x\| < \frac{1}{L}$. Given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|f_{n_k}(d) - f_{n_j}(d)| < \varepsilon$ whenever $k, j \geq k_0$. Then, for such indices, we have $|f_{n_k}(x) - f_{n_j}(x)| \leq |f_{n_k}(x) - f_{n_k}(d)| + |f_{n_k}(d) - f_{n_j}(d)| + |f_{n_j}(d) - f_{n_j}(x)| < L\|x - d\| + \varepsilon + L\|x - d\| < 3\varepsilon$. It follows that $\{f_{n_k}(x)\}$ is Cauchy and hence convergent. The rest follows from Theorem 0.9. □

Direct applications to linear mappings and functionals. Let us show that some significant results of Functional Analysis are easy consequences of the above theorems for families of convex functions. While usual proofs of the Banach-Steinhaus theorem (Corollary 0.12) go in a similar way as Theorem 0.8, the "subsequence theorem" Corollary 0.14 is usually proved using compactness of B_{X^*} (the dual unit ball) in the w^*-topology (Alaoglu's theorem) and its metrizability.

Let X be a normed space, and $x \in X$. Recall the following well-known consequence of the Hahn-Banach theorem

$$\|x\| = \sup\{x^*(x) : \; x^* \in X^*, \; \|x^*\| = 1\}\,.$$

Corollary 0.12 (Banach-Steinhaus Uniform Boundedness Principle)**.** *Let X be a Banach space, Y a normed space. Let $\mathcal{T}$ be a family of coninuous linear mappings from X into Y. Suppose that $\mathcal{T}$ is pointwise bounded, that is, for each $x \in X$, the set $\{Tx : T \in \mathcal{T}\}$ is bounded in Y. Then the family $\mathcal{T}$ is bounded in the normed space $\mathcal{L}(X,Y)$ of all continuous linear mappings from X into Y.*

Proof. The family $\{y^* \circ T : \; T \in \mathcal{T}, \; y^* \in Y^*, \; \|y^*\| = 1\}$ is a pointwise bounded family of continuous linear functionals on X; indeed, $|(y^* \circ T)(x)| \leq \|y^*\| \, \|Tx\| = \|Tx\|$ $(x \in X)$. By Theorem 0.8, there exist $r > 0$ and $M \in \mathbb{R}$ such that

$$y^*(Tx) = (y^* \circ T)(x) \leq M \qquad (\, x \in B(0,r), \; \|y^*\| = 1, \; T \in \mathcal{T}\,).$$

Passing to supremum w.r.t. y^*, we obtain

$$\|Tx\| \leq M \qquad (\, x \in B(0,r), \; T \in \mathcal{T}\,)\,,$$

which easily implies that $\|T\| \leq \frac{M}{r}$ for each $T \in \mathcal{T}$. □

Corollary 0.13. *Let X be a Banach space, and $\{x_n^*\} \subset X^*$ a sequence that converge pointwise on X to a (linear) functional $\ell \colon X \to \mathbb{R}$. Then $\{x_n^*\}$ is bounded, $\ell \in X^*$, and the convergence is uniform on compact sets.*

Proof. By Corollary 0.12, $\{\|x_n^*\|\}$ is bounded, say, by a constant M. This easily implies that $\|\ell\| \leq M$. The last part follows by Theorem 0.9. □

Let us recall that a sequence $\{x_n^*\} \subset X^*$ is w^*-*convergent* if there exists $x^* \in X^*$ such that $x_n^* \to x^*$ pointwise on X.

Corollary 0.14. *Let X be a separable Banach space. Then every pointwise bounded sequence $\{x_n^*\} \subset X^*$ is bounded and admits a w^*-convergent subsequence.*

Proof. Apply Theorem 0.8 and Theorem 0.11. □

Remarks on boundary points. Let X be a normed space, C a convex set with $\mathrm{int}(C) \neq \emptyset$, and $f\colon C \to \mathbb{R}$ a convex function which is continuous on $\mathrm{int}(C)$. The following example shows that, even for finite-dimensional X, the behaviour of f at boundary points of C can be very bad.

Example 0.15. Let $|\cdot|_e$ denote the Euclidean norm on $X = \mathbb{R}^d$. Then each function $f\colon C \to [0, +\infty)$ such that $f|_{\mathrm{int}(C)} = 0$ is convex on C.