Proof that Newton Raphson method has quadratic convergence

I've googled this and I've seen different types of proofs but they all use notations that I don't understand.

But first of all, I need to understand what quadratic convergence means, I read that it has to do with the speed of an algorithm. Is this correct?

Ok, so I know that this is the Newton-Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

How do I proof that it's convergence?

Thanks.

(calculus) (computer-science) (numerical-linear-algebra)

edited May 8 at 14:11

asked May 8 at 13:57
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1 Answer

You look at the size of the next function value. For simple roots and close to the root, the function value is a measure for the distance to the root.

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x+\theta h)h^2$$

Denote $L = \max_{x \in I} |f''(x)|$ and set $f(x) + f'(x)h = 0$, then

$$|f(x+h)| \le \frac{L}{2}h^2 = \frac{L}{2}\frac{f(x)^2}{f'(x)^2}$$

Now put the first derivatives into the constant and return to the iteration sequence (x_n) to get

$$|f(x_{n+1})| \le C\,|f(x_n)|^2 \iff |C\,f(x_{n+1}| \le |C\,f(x_n)|^2$$

where $C = \frac{L}{2m^2}$ with

$$0 < m \le |f'(x)| \le M < \infty$$

Repeated squaring leads to a dyadic power in the exponent, so that

$$|C\,f(x_n)| \le |C\,f(x_0)|^{2^n}$$

This is what is meant with quadratic convergence, that the exponent is 2^n instead of n as in linear convergence.

The condition to guarantee convergence is then $|C\,f(x_0)| < 1$.

For the distance to the root x_* use

$$f(x) = f(x) - f(x_*) \le f'(x_* + \theta(x - x_*))\,(x - x_*)$$

so that

$$m\,|x - x_*| \le |f(x)| \le M\,|x - x_*| \iff \frac{|f(x)|}{M} \le |x - x_*| \le \frac{|f(x)|}{m}.$$

answered May 8 at 22:42

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