Real Analysis

<u>Motivation</u>: Many interesting problems do not admit closed form solutions. Instead, one hopes to "iterate" to a solution; that is, one forms a <u>sequence</u> of approximate solutions (x_k), and hopes/tries to prove that as k gets large, a "good enough" answer is obtained. We are now going to look at some of the underlying mathematics.

Let $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ be a <u>normed space</u>. <u>Recall</u> $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is a norm if

i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0 \Leftrightarrow x = 0$.

ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$, $x \in \mathcal{X}$

iii) $\|x+y\| \leq \|x\| + \|y\|$ all $x, y \in \mathcal{X}$.

<u>Recall</u>

<u>Def.</u> (a) For two points $x, y \in \mathcal{X}$,

$d(x,y) := \|x-y\|$ = distance from x to y.

(b) Let $x \in \mathcal{X}$ and $S \subset \mathcal{X}$ a subset.

Then $d(x,S) := \inf_{y \in S} \|x-y\|$

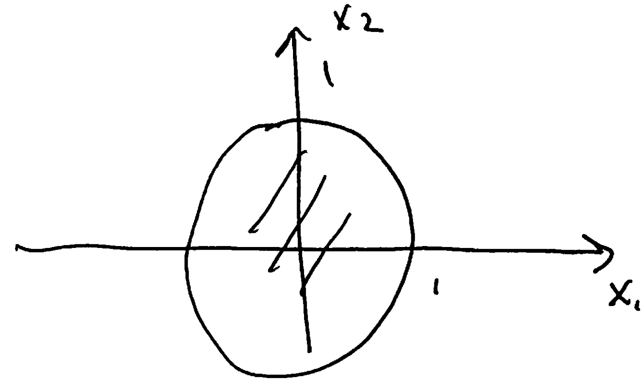
<u>Def.</u> <u>Open Ball</u>: Let $x_0 \in \mathcal{X}$ and $a \in \mathbb{R}$, $a > 0$. Then the <u>open ball of radius a centered at x_0</u> is

$$B_a(x_0) := \{ x \in \mathcal{X} \mid \|x - x_0\| < a \}$$

<u>Examples</u>

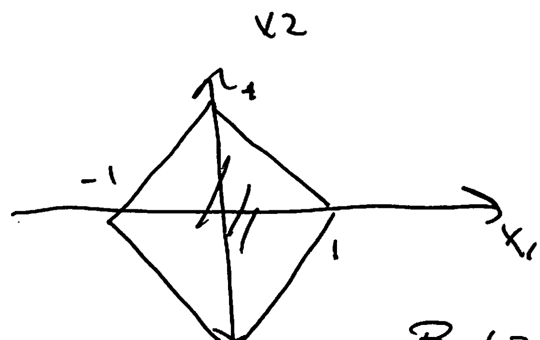
$(\mathbb{R}^2, \|\cdot\|_2)$

$B_1(0)$



$(\mathbb{R}^2, \|\cdot\|_1)$

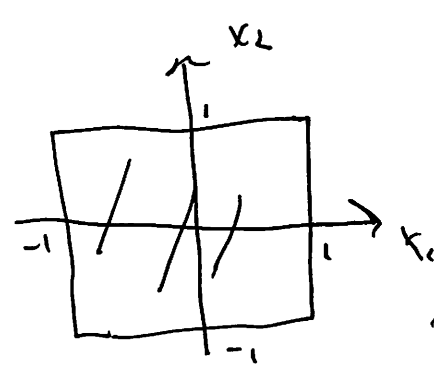
$\|(x_1, x_2)\|_1 = |x_1| + |x_2|$



$B_1(0)$

$(\mathbb{R}^2, \|\cdot\|_\infty)$

$\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|)$



$B_1(0)$

Exercise

Lemma $x \in X$ and $S \subset X$ a subset.

Then $d(x,S) = 0 \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap S \neq \emptyset$

$\iff \forall \varepsilon > 0,\ \exists\, y \in S,\ \|x-y\| < \varepsilon$

Corollary $d(x,S) > 0 \iff \exists\, \varepsilon > 0$ s.t.

$B_\varepsilon(x) \cap S = \emptyset$

In the following, $(X, \|\cdot\|)$ is a normed space.

<u>Def.</u> (a) Let $P \subset X$ be a subset. A point $p \in P$ is an <u>interior point</u> of P if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(p) \subset P$

Remark:

(b) $\mathring{P} := \{p \in P \mid p \text{ is an interior point}\}$

$= \{p \in P \mid \exists\, \varepsilon > 0,\ B_\varepsilon(p) \subset P\}$

is the <u>interior</u> of P (Remark, $p \in \mathring{P}$
$\Leftrightarrow \exists\, \varepsilon > 0\ B_\varepsilon(p) \subset P$
$\Leftrightarrow \exists\, \varepsilon > 0\ B_\varepsilon(p) \cap \sim P = \emptyset$
$\Leftrightarrow d(x, \sim P) > 0$)

(c) P is <u>open</u> if $P = \mathring{P}$ [$\Leftrightarrow P \subset \mathring{P}$ because $\mathring{P} \subset P$ by definition.]

Let $\sim P = \{x \in X \mid x \notin P\}$ = complement of P. (Sometimes denoted P^c).

<u>Proposition</u> $x \in \mathring{P} \Leftrightarrow d(x, \sim P) > 0$.

(Easier Proof on Previous Page)

<u>Proof:</u> Suppose $x \in \mathring{P}$. Then $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset P$. That is, if $y \in X$, and $d(x,y) = \|x-y\| < \varepsilon$, then $y \in P$.

Hence, if $y \in \sim P$, $d(x,y) \geq \varepsilon$.

Therefore,

$$d(x, \sim P) = \inf_{y \in \sim P} \|x-y\| \geq \varepsilon > 0.$$

Now, suppose $x \notin \mathring{P}$. Then, $\forall\, \varepsilon > 0$, $B_\varepsilon(x) \cap (\sim P) \neq \phi$. Hence, $\forall\, \varepsilon > 0$,

$$d(x, \sim P) \leq \varepsilon \quad \Rightarrow \quad d(x, \sim P) = 0.$$

$\square$

<u>Remark:</u> P is open $\Leftrightarrow P = P^\circ$

$\Leftrightarrow P = \{ x \in X \mid d(x, \sim P) > 0. \}$

skip

Examples

- $P = (0,1) \subset (\mathbb{R}, |\cdot|)$ is open.
- $P = [0,1) \subset (\mathbb{R}, |\cdot|)$ is not open

because $0 \in P$, but $d(0, \sim P) = 0$

$\Leftrightarrow 0 \in P$, but $\nexists \, \varepsilon > 0$ s.t. $B_\varepsilon(0) \subset P$

$\Leftrightarrow 0 \in P$, but $\forall \varepsilon > 0$, $B_\varepsilon(0) \cap (\sim P) \neq \phi$.

Def.

(a) A point $x \in X$ is a closure point of P if $\forall \varepsilon > 0$, $\exists p \in P$ s.t. $\|x-p\| < \varepsilon$ (i.e, $d(x,P)=0$)

(b) Closure of $P = \overline{P} := \{x \in X \mid x$ is a closure point$\}$. (Note: $P \subset \overline{P}$ because $x \in P \Rightarrow d(x,P)=0$).

(c) P is closed if $P = \overline{P}$ [$\Leftrightarrow \overline{P} \subset P$ (because we noted above that $P \subset \overline{P}$ is automatic)]

Remark

Proposition

(a) P open $\Rightarrow \sim P$ is closed

(b) P closed $\Rightarrow \sim P$ is open

(i.e. P open $\Leftrightarrow \sim P$ closed)

Proof. (a) $\Rightarrow$ (b) : Suppose that P is open. Then $P = \mathring{P}$. Hence,

$$x \in P \Leftrightarrow x \in \mathring{P} \Leftrightarrow d(x, \sim P) > 0.$$

~~Hence, $x \notin P \Leftrightarrow$~~

Hence, $x \in \sim P \Leftrightarrow x \notin P \Leftrightarrow x \notin \mathring{P} \Leftrightarrow d(x, \sim P) = 0$

$\Leftrightarrow x \in \overline{\sim P}$

$$\therefore \sim P = \overline{\sim P}$$

and thus $\sim P$ is closed.

<u>(b) $\Rightarrow$ (a)</u>

Before we start, note that $A \subset B \Leftrightarrow A \cap \sim B = \emptyset$.

We assume that $P = \overline{P}$ and $x \in \sim P$.

~~To show: $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset \sim P$~~

~~$\Leftrightarrow B_\varepsilon(x) \cap P = \emptyset$.~~

$x \in \sim P \Leftrightarrow x \notin P \Leftrightarrow x \notin \overline{P}$ because $P = \overline{P}$.

$x \notin \overline{P} \Leftrightarrow d(x, P) > 0 \Leftrightarrow \exists\, \varepsilon > 0$ s.t.

$d(x, P) \geq \varepsilon \Rightarrow B_\varepsilon(x) \cap P = \emptyset$

$\Rightarrow B_\varepsilon(x) \subset \sim P \Rightarrow \sim P$ is open.

~~**Def.** S is *closed* if $\sim S$ is open.~~

~~**Example** $S = [0,1] \subset \mathbb{R}$ is closed~~

~~because $\sim S = (-\infty, 0) \cup (1, \infty)$ is open.~~

Exercise

- Arbitrary unions of open sets are open and finite intersections of open sets are open

- Arbitrary intersections of closed sets are closed and finite unions of closed sets are closed.

Recall DeMorgan's Laws $\sim\left[\bigcup_{\alpha \in I} A_\alpha\right] = \bigcap_{\alpha \in I} [\sim A_\alpha]$

and $\sim \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} \sim A_\alpha$

Summary

- $d(x,S) = \inf_{y \in S} \|x-y\|$

$d(x,S)=0 \Leftrightarrow \forall \varepsilon>0\ \exists y \in S$ s.t. $\|x-y\|<\varepsilon$.

- $\mathring{P} = \{p \in P \mid \exists \varepsilon>0,\ B_\varepsilon(p) \subset P\}$

$= \{p \in P \mid d(p, \sim P) > 0\}$

$= \{x \in X \mid d(x, \sim P) > 0\}$

- Def P open $\Leftrightarrow$ $P = \mathring{P}$

- $\overline{P} = \{x \in X \mid d(x,P) = 0\}$

- Def P closed $\Leftrightarrow$ $P = \overline{P}$

- Thm P open $\Leftrightarrow$ $\sim P$ closed

Because

$$\underbrace{\sim P = \sim(\mathring{P})}_{P\text{ open}} = \{x \in X \mid d(x, \sim P) = 0\} = \underbrace{\overline{\sim P} = \sim P}_{\sim P\text{ closed}}$$

$\therefore$ P open $\Leftrightarrow$ $\sim P$ closed

Sequences

$(X, \mathbb{R}, \|\cdot\|)$ a normed space

Def. A set of vectors indexed by the non-negative integers is called a sequence, (x_n), or $\{x_n\}$.

Let (x_n) be a sequence and let $n_1 < n_2 < n_3 < \cdots$ be an infinite set of strictly increasing integers. Then (x_{n_i}) is called a subsequence of (x_n)

Example: $n_i = 2^i$, or $n_i = 2i+1$

<u>Def.</u> A sequence of vectors (x_n) <u>converges</u> to $x \in X$ if, $\forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$ s.t. $n \geq N \Rightarrow \|x - x_n\| < \varepsilon$.

I.E., $n \geq N \Rightarrow x_n \in B_\varepsilon(x)$. One writes

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x$$

<u>Proposition</u> Suppose $x_n \to x$. Then

a) $\|x_n\| \to \|x\|$

b) $\sup_n \|x_n\| < \infty$ (the sequence is bdd)

c) If $x_n \to y$, then $y = x$ (limits are unique)

<u>Remark</u> For $\bar{x}, \bar{y} \in X$

$$\|\bar{x}\| = \|\bar{x} + \bar{y} - \bar{y}\| \le \|\bar{x}+\bar{y}\| + \|-\bar{y}\|$$
$$\le \|\bar{x}+\bar{y}\| + \|\bar{y}\|$$

$$\therefore \big| \|\bar{x}\| - \|\bar{y}\| \big| \le \|\bar{x}+\bar{y}\|$$

(a) $\big| \|x_n\| - \|x\| \big| \le \|x_n - x\| \longrightarrow 0$

$\therefore \|x_n\| \longrightarrow \|x\|$

(b) Set $\varepsilon = 1$. Then $\exists N(\varepsilon)$ s.t. $n \ge N$

$\Rightarrow \|x_n - x\| \le 1$. $\forall n \ge N$,

$$\therefore \|x_n\| = \|x_n - x + x\| \le \|x_n - x\| + \|x\|$$
$$\le 1 + \|x\|$$

$$\therefore \sup_n \|x_n\| \le \max\{\underbrace{\|x_1\|, \|x_2\|, \ldots, \|x_{N-1}\|, 1+\|x\|}_{\text{finite}}\}$$
$$< \infty$$

(c) $\|x-y\| = \|x - x_n + x_n - y\|$

$$\leq \|x - x_n\| + \|x_n - y\| \xrightarrow[n\to\infty]{} 0$$

$\square$

$\therefore \|x-y\| = 0$ and thus $x = y$.

<u>Def.</u> $x \in \mathcal{X}$, $P \subset \mathcal{X}$ a subset. x is a <u>limit point</u> of P if $\exists$ a sequence of elements of P that converges to x; i.e, $\exists\,(x_n)$, s.t. $x_n \in P$ and $x_n \to x$.

<u>Prop.</u> x is a limit point of $P \Leftrightarrow x \in \overline{P}$.

<u>Pf.</u>
(a) Suppose x is a limit point. Then $\exists\,(x_n)$ s.t. $x_n \in P$ and $x_n \to x$. Because $x_n \to x$, $\forall\, \varepsilon > 0$, $\exists\, x_n \in P$ s.t. $\|x_n - x\| < \varepsilon$

$\Rightarrow d(x, P) = 0 \Rightarrow x \in \overline{P}$.

(b) Suppose $x \in \overline{P}$. Then $\forall \varepsilon > 0$, $\exists$ $y \in P$ s.t. $\|x-y\| < \varepsilon$. Let $\varepsilon = \frac{1}{n}$. Then $\exists$ $x_n \in P$ s.t. $\|x - x_n\| < 1/n$

$\Rightarrow$ $x_n \to x$. $\therefore$ x is a limit point.

<u>Corollary</u> P is closed $\Leftrightarrow$ it contains its limit points.

Complete Spaces (Banach Spaces)

A sequence (x_n) in $(\mathcal{X}, \|\cdot\|)$ is a <u>Cauchy sequence</u> if $\forall \varepsilon > 0$, $\exists$ $N(\varepsilon) < \infty$ s.t. $n, m \geq N \Rightarrow$ $\|x_n - x_m\| < \varepsilon$.

Notation $\|x_n - x_m\| \xrightarrow[n,m \to \infty]{} 0$

<u>Proposition</u> If $x_n \to x$, then (x_n) is Cauchy.

<u>Proof:</u> Let $\varepsilon > 0$ and choose $N < \infty$ s.t. $n \geq N \Rightarrow \|x_n - x\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\|x_n - x_m\| &= \|x_n - x + x - x_m\| \\ &\leq \|x_n - x\| + \|x - x_m\| \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &\leq \varepsilon \qquad \text{for all } n, m \geq N.\end{aligned}$$

$\square$

Unfortunately, not all Cauchy sequences are convergent.

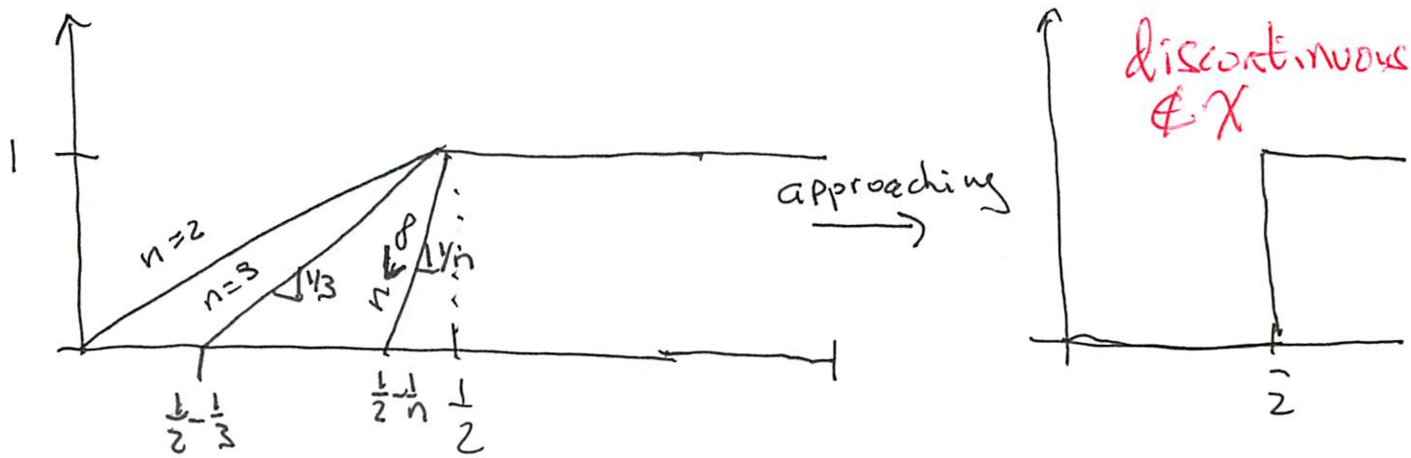
For a reason we will understand shortly, all counter examples are infinite dimensional.

Example

$$X = \{ f : [0,1] \to \mathbb{R} \mid f \text{ continuous} \}$$

$$\|f\|_1 = \int_0^1 f(\tau)\, d\tau$$

Define a sequence as follows



$$f_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + n(t - \frac{1}{2}) & \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & t \ge \frac{1}{2} \end{cases}$$

$$\|f_n - f_m\|_1 = \frac{1}{2}\left| \frac{1}{n} - \frac{1}{m} \right| \xrightarrow[n,m \to \infty]{} 0$$

$\therefore$ Cauchy, but $\nexists$ f a cont. function such that $f_n \to f$.

$\square$

<u>Def.</u> A normed space $(X, \mathbb{R}, \|\cdot\|)$ is <u>complete</u> if every Cauchy sequence in X has a limit in X. Such spaces are called Banach spaces.

There are many useful and known Banach spaces.

In EECS 562, you will use $(C[0,T], \|\cdot\|_\infty)$

<u>Def.</u> A subset P of a normed space is complete if every Cauchy sequence in P has a limit in P.

<u>Theorem</u> (a) In a normed linear space, any finite dimensional subspace is complete.

(b) Any closed subset of a complete set is also complete

(c) $C[a,b]$, $\|\cdot\|_\infty$ is complete
where $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$.

Note: $a < b$, both finite.

Newton's Algorithm

Only Read
Pages 1 to 3

Pages 4 to 7 used
in EECS 562

Fixed Points and Contraction Mappings

Let $T: S \to S$, for some $S \subset X$.

Questions

① When does there exist x such that $T(x) = x$? (fixed point)

(1b) $F(x) = 0 \leftrightarrow F(x) - x = x$, $T(x) = F(x) - x$

② If a fixed point exists, is it unique?

③ When can it be obtained as the limit of $x_{n+1} = T(x_n)$, the Method of Successive Approximations?

Give Newton's Algorithm

Def. Let S be a subset of $(X, \|\cdot\|)$ and let $T: S \to S$. Then T is a contraction mapping if $\exists\ 0 \le \alpha < 1$ such that $\forall x, y \in S$, $\|T(x) - T(y)\| \le \alpha \|x - y\|$.

Contraction Mapping Theorem

If T is a contraction mapping on a complete subset S of a normed linear space, then there is a unique vector $x^* \in S$ such that $T(x^*) = x^*$.

Moreover, $\forall\, x_0 \in S$, the sequence $(x_{n+1} := T(x_n),\ n \geq 0)$ is Cauchy and $x_n \longrightarrow x^*$.

Proof.

For all $n \geq 1$

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leq \alpha \|x_n - x_{n-1}\|$$

By induction,

$$\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\|.$$

Consider $\|x_m - x_n\|$, and w.l.o.g., suppose that $m = n+p$, $p>0$. Then

$$\|x_m - x_n\| = \|x_{n+p} - x_n\|$$

$$= \|x_{n+p} - x_{n+p-1} + x_{n+p-1} - \dots + x_{n+1} - x_n\|$$

$$\le \|x_{n+p} - x_{n+p-1}\| + \dots + \|x_{n+1} - x_n\|$$

$$\le (\alpha^{n+p-1} + \dots + \alpha^n)\, \|x_1 - x_0\|$$

$$\le \alpha^n \sum_{i=0}^{p-1} \alpha^i \, \|x_1 - x_0\|$$

$$\le \alpha^n \sum_{i=0}^{\infty} \alpha^i \, \|x_1 - x_0\|$$

$$\le \frac{\alpha^n}{1-\alpha} \|x_1 - x_0\| \xrightarrow[m,n \to \infty]{} 0$$

$\therefore$ (x_n) is Cauchy.

$\therefore$ By completeness, $\exists\, x^* \in S$ s.t.

$x_n \to x^*$.

<u>Claim</u> $x^* = T(x^*)$

<u>Pf.</u>

$$\|x^* - T(x^*)\| = \|x^* - x_n + x_n - T(x^*)\|$$
$$= \|x^* - x_n + T(x_{n-1}) - T(x^*)\|$$
$$\leq \|x^* - x_n\| + \|T(x_{n-1}) - T(x^*)\|$$
$$\leq \|x^* - x_n\| + \alpha\|x_{n-1} - x^*\| \xrightarrow[n\to\infty]{} 0$$

□

<u>Claim</u> x^* is unique

<u>Pf.</u> Suppose $y^* = T(y^*)$. Then

$$\|x^* - y^*\| = \|T(x^*) - T(y^*)\|$$
$$\leq \alpha\|x^* - y^*\|$$

$0 \leq \alpha < 1 \Rightarrow \|x^* - y^*\| = 0.$

$\therefore x^* = y^*$

□

(Skip)

<u>Corollary</u> If S is complete, $T: S \to S$ is continuous, and $\exists$ an inter $b > 0$ such that $T^b := \underbrace{T \circ \cdots \circ T}_{b\text{-times}}$ is a contraction mapping, then T has a unique fixed point and it can be found by the method of successive approximations.

<u>Pf.</u> See Luenberger or EECS 600.

Continuous Functions & Compact Sets

Def. Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be two normed spaces.

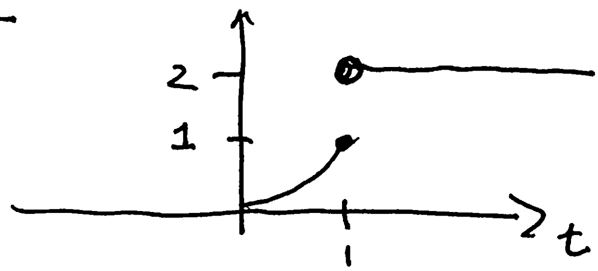
$f: X \to Y$ is continuous at $x_0 \in X$ if, $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon, x_0) > 0$ s.t. $\|x_0 - x\| < \delta \Rightarrow |||f(x) - f(x_0)||| < \varepsilon$.

(i.e. $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$.)

f is continuous if it is continuous at x_0 for all $x_0 \in X$.

Non-example

$f : \mathbb{R} \to \mathbb{R}$



f is not continuous at $t_0 = 1$ because

for $\varepsilon = \frac{1}{2}$ and $\forall \delta > 0 \ \exists t$ s.t.

$|t - t_0| < \delta$ but $|f(t) - f(t_0)| \geq \frac{1}{2}$.

Theorem Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$

be normed spaces and $f : X \to Y$

a function.

a) If f is continuous at x_0

and (x_n) is any sequence such

that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

(b) If f is not continuous at x_0, then $\exists$ a sequence (x_n) in X such that $x_n \to x_0$ and $f(x_n) \not\to f(x_0)$, i.e., the sequence $f(x_n)$ does <u>not</u> converge to $f(x_0)$.

<u>Corollary</u> f is continuous at x_0 $\Leftrightarrow$ for every sequence (x_n) in X that converges to x_0, the ~~conve~~ sequence $(f(x_n))$ in Y converges to $f(x_0)$.

Is $f(x^*) = f^*$? Let's see:

$$
\begin{aligned}
|f^* - f(x^*)| &\le |f^* - f(x_{n_i}) + f(x_{n_i}) - f(x^*)| \\
&\le |f^* - f(x_{n_i})| + |f(x_{n_i}) - f(x^*)| \\
&\le \frac{1}{i} + |f(x_{n_i}) - f(x^*)| . \\
&\le \xrightarrow[i \to \infty]{} 0
\end{aligned}
$$

because $f(x_{n_i}) \to f(x^*)$.

$\square$

<u>Bolzano-Weierstrass Thm</u> In a finite dimensional normed space $(X, \|\cdot\|)$ the following two properties are equivalent for a set $C \subset X$:

(a) C is closed and bounded[1]

(b) For every sequence (x_n) with elements in C, there exist $x_0 \in C$ and a subsequence ~~of~~ (x_{n_i}) such that

$x_{n_i} \longrightarrow x_0$

$\square$

[1] <u>Remark:</u> C is bounded if $\exists\, r < \infty$ s.t. $C \subset B_r(0)$. $\square$

(Weierstrass) Theorem: If C is compact and $f: C \to \mathbb{R}$ is continuous at each point of C, then f achieves its extreme values. That is, $\exists\ x^* \in C$ s.t.

$$f(x^*) = \sup_{x \in C} f(x)$$

and $\exists\ x_* \in C$ s.t.

$$f(x_*) = \inf_{x \in C} f(x)$$

$\square$

<u>Remark:</u> Powerful method to show existence of solutions to finite dimensional optimization problems, but <u>not</u> how to solve them.

Sketch of the Proof

Let $f^* := \sup_{x \in C} f(x)$. The sup always exists. In general, it may not be bounded, but in this case it is, though we do not prove it.

Because f^* is the supremum, $\forall \varepsilon > 0$ $\exists\, x_\varepsilon \in C$ such that $|f^* - f(x_\varepsilon)| < \varepsilon$.

We let $\varepsilon = 1/n$, and conclude that $\forall n \geq 1$, $\exists\, x_n \in C$ s.t. $|f^* - f(x_n)| < 1/n$.

Because C is compact, $\exists$ a subsequence (x_{n_i}) and a point $x^* \in C$ s.t. $x_{n_i} \xrightarrow[i \to \infty]{} x^*$.

By continuity of f, $f(x_{n_i}) \xrightarrow[i \to \infty]{} f(x^*)$.

(4) Consider $(\mathbb{R}^n, \mathbb{R})$, and

A a real $m \times n$ matrix

and $b \in \mathbb{R}^m$.

$$K = \{ x \in \mathbb{R}^n \mid A x \leq b \}$$

row wise

is convex.

$$K = \{ x \in \mathbb{R}^n \mid A x = b \}$$

is convex (Note: $\emptyset$ is convex)

$$\therefore \quad K = \{ x \in \mathbb{R}^n \mid A_{in} x \leq b_{in}, A_{eq} x = b_{eq} \}$$

is convex (by intersection property).

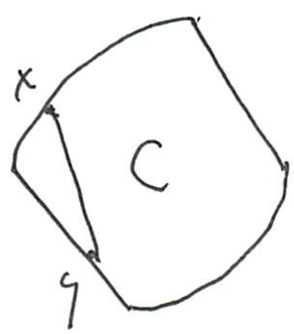
Convex Sets & Functions

Let $(V, \mathbb{R})$ be a vector space.

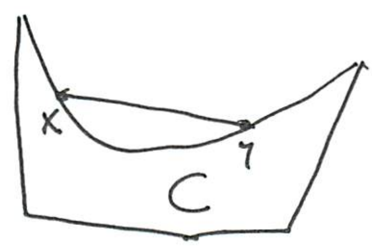
<u>Def.</u> $C \subset V$ is <u>convex</u> if

$\forall\ x, y \in C$, and $0 \le \lambda \le 1$,

$\lambda x + (1-\lambda) y \in C$.



Yes



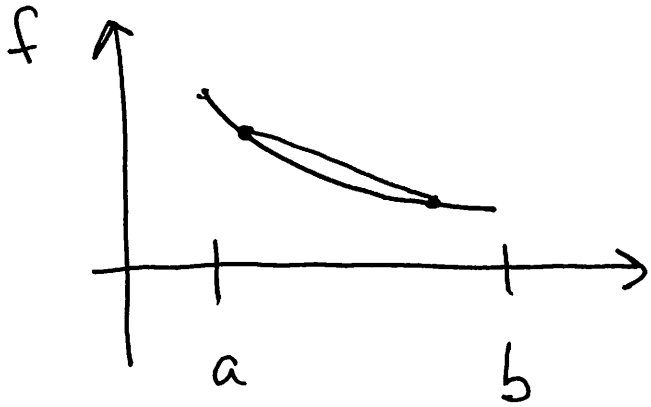
NO

<u>Def.</u> Suppose C is convex. Then

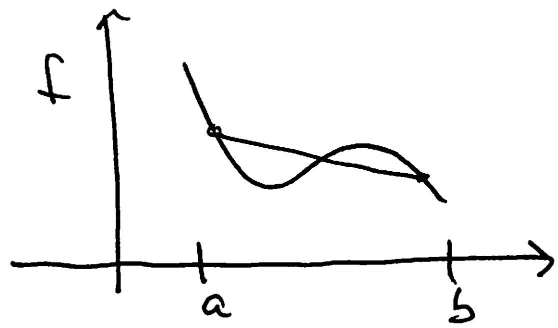
$f : C \to \mathbb{R}$ is <u>convex</u> if

$\forall\ x, y \in C$, $0 \le \lambda \le 1$

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y).$$



Yes



No

Suppose $(V, \mathbb{R}, \|\cdot\|)$ is a normed space, and $f : D \to \mathbb{R}$ a function.

<u>Def</u>: (a) $x^* \in D$ is a <u>local</u> <u>minimum</u> of f ~~if~~ $\exists\ \delta > 0$ such that $\forall\ x \in B_\delta(x^*)$, $f(x^*) \le f(x)$.

(b) $x^* \in D$ is a <u>global minimum</u> of f if $\forall\ x \in D$, $f(x^*) \le f(x)$.

<u>Theorem</u> If D and f are convex, then any local minimum is also a global minimum.

<u>Proof.</u> We prove the contrapositive. Hence, we assume $x \in D$ is NOT a global minimum and prove that x is also not a local minimum.

x NOT a global minimum implies $\exists\, y \in D$ such that $f(y) < f(x)$.

Let $\delta > 0$ be arbitrary. Then $\exists$ $0 < \lambda < 1$ such that $(1-\lambda)x + \lambda y \in B_\delta(x)$

<u>Aside</u> Indeed, $\|(1-\lambda)x+\lambda y - x\| =$

$= \|\lambda(y-x)\| = \lambda\|y-x\|$. Hence,

$0 < \lambda < \dfrac{\delta}{\|y-x\|}$ works. $\square$

<u>Claim</u> $f((1-\lambda)x+\lambda y) < f(x)$, showing that x cannot be a local minimum.

<u>Pf.</u> By convexity

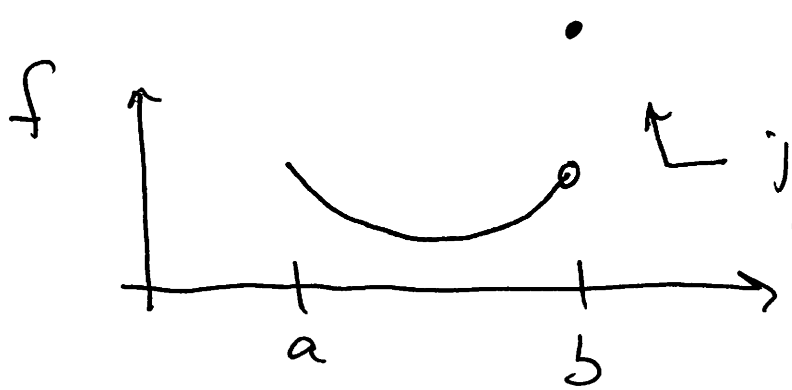
$$\begin{aligned} f((1-\lambda)x+\lambda y) &\le (1-\lambda)f(x)+\lambda f(y) \\ &< (1-\lambda)f(x)+\lambda f(x) \\ &\quad \text{(because } f(x)>f(y)\text{)} \\ &\le f(x) \end{aligned}$$

$\therefore f((1-\lambda)x+\lambda y) < f(x)$

$\square$

<u>Theorem (Harder)</u> Suppose that $(X, \mathbb{R}, \|\cdot\|)$ is a finite dimensional normed space, C is convex and $f : C \to \mathbb{R}$ is convex. Then f is continuous on $\mathring{C}$.

Proof is non trivial. To see that ~~the boundary of C can have jm~~ f can have jumps on the boundary of C, consider



jump does not destroy convexity.

Additional Facts
$\underbrace{}_{}$ (useful)

① All norms $\|\cdot\| : X \to [0,\infty)$ are convex (follows from the triangle inequality.)

② For all $1 \le \beta < \infty$, $\|\cdot\|^{\beta}$ is convex. Hence, on $\mathbb{R}^n$, for any $1 < p < \infty$, $\sum_{i=1}^{n} |x_i|^p$ is a convex function.

③ Suppose K_1 and K_2 are convex. Then $K_1 \cap K_2$ is also convex.

Quadratic Programs

$x \in \mathbb{R}^n, \quad Q \geq 0$

minimize $\quad \frac{1}{2} x^T Q x + f x$

subject to $\quad A_{in} x \leq b_{in}$

$\quad\quad A_{eq} x = b_{eq}$

Very powerful means to "distribute" torque in a robot, while assuring important bounds are met

- Gripping force
- Friction cone on foot
- Motor torque limits
- Stability
- See MABEL Video, where QP runs ≈ 50 μs in real time!