1. The answers are (a), (c), (d).

(a) True. From lecture, $\hat{x} = Ky$ where

$K = (C^T Q^{-1} C)^{-1} C^T Q^{-1}$

(d) True. When BLUE was developed, the "unbiased" condition arose from $KC = I$.

(b) False. The MVE reduces to BLUE when $P = \text{cov}(x, x) \rightarrow \infty I$, not $0 \cdot I$.

When $P = 0$, the MVE gain is $K = 0$. We already know that $x = 0$ with certainty, and we therefore ignore the measurements.

(c) True This a ~~standard~~ weighted least squares problem.

$\hat{x} = (C^T Q^{-1} C)^{-1} C^T Q^{-1} y$, which equals Ky.

2. The answers are (a) and (d) $\frac{2}{a}$

(a) True. ~~The quick~~ one answer is that R gives the representation of the columns of A in the basis defined by the columns of Q, and thus R has the same rank as A.

$\therefore R = 2\times 2$ and has rank 2 $\Rightarrow$ invertible.

A second answer is, we know R is upper triangular. ~~$R_1 = \|A_1\|$, where $A = [A_1 | A_2]$. If $R_{22} = 0$, then A_2 is~~ ~~Hence, $R_{11} \neq 0$.~~

Let write $Q = [v^1, v^2]$, and $A = [A_1, A_2]$.

Then $A_1 = R_{11} v^1$, and we have $|R_{11}| = \|A_1\| \neq 0$.

Also, $A_2 = R_{12} v^1 + R_{22} v^2$, and if $R_{22} = 0$, then A_2 is linearly dependent on A_1, which it is not. Hence $R_{22} \neq 0$. $\therefore \det(R) = R_{11} R_{22} \neq 0$.

2/b

(b) False.
R^TR is square, while S is 4×2, the same size as A.

(c) False. The columns of U are e-vectors of AA^T.

(d) True ~~Obv~~ This is very similar to HW 9, Prob 5. See HW 2, Prob 7 (b).

3/a

3. The answer ~~are~~ is (b)

(a) False. It's variance is simply 6

(b) True   In the handout on Gaussian Random Vectors, ~~[illegible]~~ we let $X_1 = X$ and $X_2 = [Y, Z]^T$. We then compute

$$\cancel{\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\left(\begin{bmatrix}4\\z\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}\right)}$$

$$E\{X \mid Y=4, Z=z\} = 1 + [0, 1]\begin{bmatrix}4 & 2\\ 2 & 6\end{bmatrix}^{-1}\left(\begin{bmatrix}4\\z\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= 1 + [0, 1]\left(\frac{1}{20}\right)\begin{bmatrix}6 & -2\\ -2 & 4\end{bmatrix}\left(\begin{bmatrix}4\\z\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= 1 - \frac{1}{10}(4-2) + \frac{1}{5}(z-3)$$

(c) and (d) are both False. The random vector $\begin{bmatrix} X \\ Y \end{bmatrix} \mid Z = z$ has covariance

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\tfrac{1}{6}\right) \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & 3\frac{1}{3} \end{bmatrix}$$

This does NOT match the answer in (d), and because the matrix is not diagonal, the conditional random variables $X \mid Z = z$ and $Y \mid Z = z$ are not independent. $\square$

4. Using Schur complements,

$P > 0$ if, and only if,

(i) $a\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} > 0$ and (ii) $4 - \begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} a & 2a \\ 2a & 5a \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} > 0$

From (i) we have $a > 0$.

From (ii) we have

$$4 - \frac{1}{a}\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} > 0$$

$$\Updownarrow$$

$$4 - \frac{1}{a}\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} > 0$$

$$\Updownarrow$$

$$4 - \frac{25}{a} > 0$$

$$\Updownarrow$$

$$a > \frac{25}{4}.$$

$\therefore$ $\boxed{P > 0 \Leftrightarrow a > \frac{25}{4}}$

5/1

5.

$$\hat{x} = A^T (A \cdot A^T)^{-1} b$$

Doing the calculations,

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \Rightarrow (A \cdot A^T)^{-1} = \frac{1}{50} \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 15 \\ 25 \\ 20 \end{bmatrix}$$

6. <u>Solution 1</u> $\quad\quad\quad\quad \frac{6}{a}$

(a) The problem is set up for RLS.

For $k \geq 3$

$$P_{k+1} = P_k - P_k C_{k+1}^T \left[I + C_{k+1} P_k C_{k+1}^T\right]^{-1} C_{k+1} P_k$$

$$K_{k+1} = P_{k+1} C_{k+1}^T$$

$$\hat{x}_{k+1} = \hat{x}_k + K_{k+1}\left(y_{k+1} - C_{k+1}\hat{x}_k\right)$$

(b) We are given that

$$Q_3 = A_3^T A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_3 = Q_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_4 = P_3 - P_3 C_4^T \left[1 + C_4 P_3 C_4^T\right]^{-1} C_4 P_3$$

$$= I - I \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(1 + \begin{bmatrix} 2 & 1 \end{bmatrix} I \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 2 & 1 \end{bmatrix} I$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 5/6 \end{bmatrix}$$

$$K_4 = P_4 C_4^T$$

$$= \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$$

$$\hat{x}_4 = \hat{x}_3 + K_4 (y_4 - C_4 \hat{x}_3)$$

$$= \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} \left(\overset{5}{\cancel{4}} - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} (6)$$

$$= \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Solution 2

(a) The problem is set up for RLS

For $k \geq 3$

$$Q_{k+1} = Q_k + C_{k+1}^T C_{k+1}$$
$$K_{k+1} = (Q_{k+1})^{-1} C_{k+1}^T$$
$$\hat{x}_{k+1} = \hat{x}_k + K_{k+1}(y_{k+1} - C_{k+1}\hat{x}_k)$$

(b) We are given that

$$Q_3 = A_3^T A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q_4 = Q_3 + C_4^T C_4$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

6/2

$$Q_4^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$$K_4 = Q_4^{-1} C_4^T$$

$$= \frac{1}{6}\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$$

$\hat{x}_4$ same as before.

$\square$