Appendix B Matrix Inversion Lemma

The Matrix Inversion Lemma is often used and very powerful in the matrix analysis of signal processing. It comes in various forms, but they are all easily derived from the following basic form of the lemma. For any matrices A and B not necessarily square, as long as the products AB and BA exist and the relevant matrices are invertible,

$$(I + AB)^{-1} = I - A(I + BA)^{-1}B\,, \tag{B1}$$

where by "I" throughout this appendix we mean the identity matrix of size appropriate to its use.

The lemma can be proved by multiplying the inverse of the left-hand side of (B1) by its right-hand side and inspecting the result:

$$\begin{aligned} \text{LHS}^{-1} \,.\, \text{RHS} &= (I + AB)\big[I - A(I + BA)^{-1}B\big] \\ &= I + AB - A(I + BA)^{-1}B - ABA(I + BA)^{-1}B \\ &= I + A\big[I - (I + BA)^{-1} - BA(I + BA)^{-1}\big]B \\ &= I + A\big[I - (I + BA)(I + BA)^{-1}\big]B \\ &= I\,. \end{aligned} \tag{B2}$$

In that case, LHS = RHS, and the lemma is proved.

More complicated versions of the lemma make good use of the fact that $(PQ)^{-1} = Q^{-1}P^{-1}$ for any invertible matrices P, Q. For example, apply the lemma to $(A + BCD)^{-1}$:

$$\begin{aligned} (A + BCD)^{-1} &= \big[(I + BCDA^{-1})A\big]^{-1} \\ &= A^{-1}(I + BCDA^{-1})^{-1} \\ &\overset{\text{(B1)}}{=\!=}\; A^{-1}\big(I - BC[I + DA^{-1}BC]^{-1}DA^{-1}\big) \\ &= A^{-1}\big(I - B\big[(I + DA^{-1}BC)C^{-1}\big]^{-1}DA^{-1}\big) \\ &= A^{-1}\big(I - B\big[C^{-1} + DA^{-1}B\big]^{-1}DA^{-1}\big)\,. \end{aligned} \tag{B3}$$

Finally,

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}\,, \tag{B4}$$

which is a common form of the Matrix Inversion Lemma.