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To prove $P \to Q$, you can do the following:

1. Prove directly, that is assume P and show Q;
2. Prove by contradiction, that is assume P and $\neg Q$ and derive a contradiction; or
3. Prove the contrapositive, that is assume $\neg Q$ and show $\neg P$.

Sometimes the contradiction one arrives at in (2) is merely contradicting the assumed premise P, and hence, as you note, is essentially a proof by contrapositive (3). However, note that (3) allows us to assume *only* $\neg Q$; if we can then derive $\neg P$, we have a *clean* proof by contrapositive.

However, in (2), the aim is to derive a *contradiction*: the contradiction might *not* be arriving at $\neg P$, if one has assumed (P *and* $\neg Q$). Arriving at *any contradiction* counts in a proof by contradiction: say we assume P *and* $\neg Q$ and derive, say, Q. Since $Q \land \neg Q$ is a contradiction (can never be true), we are forced then to conclude *it **cannot** be that **both*** $(P \land \neg Q)$.

But note that $\neg(P \land \neg Q) \equiv \neg P \lor Q \equiv P \to Q$.

So a proof by contradiction usually looks something like this (R is often Q, or $\neg P$ or any other contradiction):

- $P \land \neg Q$ Premise
  - P
  - $\neg Q$
  - $\vdots$
  - R
  - $\vdots$
  - $\neg R$
  - $\neg R \land R$ Contradiction

$\therefore \neg(\neg P \land Q) \equiv P \to Q$