A. The Schur complement of a symmetric matrix with a singular south-east block

Lemma A.1. *Consider a symmetric negative (positive) semi-definite matrix* $F = F^T$ *partitioned as*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

The definition of the Schur complement of F *with respect to invertible* F_{22}*, given by*

$$\bar{F} = F_{11} - F_{12} F_{22}^{-1} F_{21},$$

can be continuously extended to singular F_{22}*.*

Proof. After a possible change of coordinates we can write the singular matrix F_{22} into the form

$$F_{22} = \begin{bmatrix} F_{22}^a & 0 \\ 0 & 0 \end{bmatrix}$$

with F_{22}^a invertible. Then the properties of a negative (positive) semi-definite symmetric matrix (A.2) shown in Lemma A.2 yield

$$F_{12} = \begin{bmatrix} F_{12}^a & 0 \\ F_{12}^c & 0 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} F_{21}^a & F_{21}^b \\ 0 & 0 \end{bmatrix}.$$

Therefore the Schur complement with a perturbed F_{22} reads

$$\begin{aligned} \bar{F}_\varepsilon &= F_{11} - \begin{bmatrix} F_{12}^a & 0 \\ F_{12}^c & 0 \end{bmatrix} \begin{bmatrix} (F_{22}^a)^{-1} & 0 \\ 0 & \varepsilon^{-1} I \end{bmatrix} \begin{bmatrix} F_{21}^a & F_{21}^b \\ 0 & 0 \end{bmatrix} \\ &= F_{11} - \begin{bmatrix} F_{12}^a (F_{22}^a)^{-1} F_{21}^a & F_{12}^a (F_{22}^a)^{-1} F_{21}^b \\ F_{12}^c (F_{22}^a)^{-1} F_{21}^a & F_{12}^c (F_{22}^a)^{-1} F_{21}^b \end{bmatrix}, \end{aligned} \tag{A.1}$$

which is independent of ε. Hence we can let $\varepsilon \to 0$. This shows that the Schur complement can be still defined even for singular F_{22}. $\square$

Lemma A.2. *Consider a negative semi-definite symmetric matrix $F = F^T \leqslant 0$ partitioned as*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} \ker\ F_{22} &\subseteq \ker\ F_{12}, \\ \mathrm{im}\ F_{21} &\subseteq \mathrm{im}\ F_{22}. \end{aligned} \tag{A.2}$$

Proof. First we prove that $\ker\ F_{22} \subset \ker\ F_{12}$. Since F is negative semi-definite it follows that $x^T F x \leqslant 0$ for all real vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Take x_2 which is in the kernel of F_{22} and $x_1 = F_{12} x_2$. Then for a small positive constant ε we have

$$\begin{aligned} [\varepsilon x_1^T\ x_2^T] \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \varepsilon x_1 \\ x_2 \end{bmatrix} &= \varepsilon^2 x_1^T F_{11} x_1 + \varepsilon x_2^T F_{21} x_1 + \\ &\quad\ \varepsilon x_1^T F_{12} x_2 + x_2^T F_{22} x_2 = \\ &= \varepsilon^2 x_1^T F_{11} x_1 + 2\varepsilon \|x_1\|^2. \end{aligned}$$

Since the term $2\varepsilon\|x_1\|^2$ is strictly positive we can choose ε such that $2\varepsilon\|x_1\|^2$ prevails over $\varepsilon^2 x_1^T F_{11} x_1$ and therefore the expression above is positive. Since F is negative semi-definite, this implies that necessarily $x_1 = 0,$ showing that $x_2 \in \ker F_{12}$.

Furthermore, using the fact that the image of a matrix is orthogonal to the kernel of the transpose of the same matrix we write for any z which is in the image of F_{21}

$$\begin{aligned} z \in \mathrm{im}\ F_{21} &\implies z \perp \ker F_{21}^T \\ &\implies z \perp \ker F_{12} \\ &\implies z \perp \ker F_{22} \\ &\implies z \in \mathrm{im}\ F_{22}^T = \mathrm{im}\ F_{22}. \end{aligned}$$

Therefore $\mathrm{im}\ F_{21} \subseteq \mathrm{im}\ F_{22}$ which proves the claim. ☐

Remark A.3. To prove the expressions (A.2) for a positive semi-definite symmetric matrix take $x = \begin{bmatrix} -\varepsilon x_1 \\ x_2 \end{bmatrix}$.

B. Derivation of the effort- and flow-constraint reduced order models

B.1. Effort-constraint reduction

Consider the full order port-Hamiltonian system (3.20) with a splitting according to the dimension r chosen for the reduced order model:

$$\left\{\begin{aligned} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}G_{R_1}\\ G_{R_2}\end{bmatrix} f_R + \begin{bmatrix}G_1\\ G_2\end{bmatrix} u,\\ y &= \begin{bmatrix}G_1^T & G_2^T\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},\\ e_R &= \begin{bmatrix}G_{R_1}^T\, G_{R_2}^T\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix} f_R, \qquad f_R = -\bar{R}e_R. \end{aligned}\right. \tag{B.1}$$

The full order Dirac structure corresponding to the model (B.1) is given by the explicit equation in the DAE form (3.10)

$$F_x\dot{x} = E_x\frac{\partial H}{\partial x}(x) + F_R f_R + E_R e_R + F_P f_P + E_P e_P, \tag{B.2}$$

or

$$\begin{aligned} \begin{bmatrix}I_n\\ 0_{m\times n}\\ 0_{m_R\times n}\end{bmatrix}\dot{x} \quad = \quad & \begin{bmatrix}J\\ -G^T\\ -G_R^T\end{bmatrix}\tfrac{\partial H}{\partial x}(x) + \begin{bmatrix}G_R\\ 0_{m\times m_R}\\ 0_{m_R}\end{bmatrix} f_R + \begin{bmatrix}0_{n\times m_R}\\ 0_{m\times m_R}\\ I_{m_R}\end{bmatrix} e_R +\\ & \begin{bmatrix}G\\ 0_{m\times m}\\ 0_{m_R\times m}\end{bmatrix} f_P + \begin{bmatrix}0_{n\times m}\\ I_m\\ 0_{m_R\times m}\end{bmatrix} e_P, \end{aligned}$$

where m_R is the dimension of the resistive variables $f_R,\ e_R$, and m is that of the input-output variables $f_P = u,\ e_P = y$.

After the splitting, using the usual notation $e_x = \frac{\partial H}{\partial x}(x)$, the above equation reads

$$\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \\ \multicolumn{2}{c}{0_{m\times n}} \\ \multicolumn{2}{c}{0_{m_R\times n}} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ -G_1^T & -G_2^T \\ -G_{R_1}^T & -G_{R_2}^T \end{bmatrix} \begin{bmatrix} e_x^1 \\ e_x^2 \end{bmatrix} + \begin{bmatrix} G_{R_1} \\ G_{R_2} \\ 0_{m\times m_R} \\ 0_{m_R} \end{bmatrix} f_R + \begin{bmatrix} 0_{n\times m_R} \\ 0_{m\times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G_1 \\ G_2 \\ 0_{m\times m} \\ 0_{m_R\times m} \end{bmatrix} u + \begin{bmatrix} 0_{n\times m} \\ I_m \\ 0_{m_R\times m} \end{bmatrix} y. \tag{B.3}$$

Recall from Section 3.3 that the effort-constraint method assumes finding a (non-unique) maximal rank matrix L^{ec} satisfying

$$L^{\mathrm{ec}} F_x^2 = 0,$$

as well as setting $e_x^2 = 0$. We propose the following matrix L^{ec}

$$L^{\mathrm{ec}} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_{m_R} \end{bmatrix}. \tag{B.4}$$

One can readily verify that the matrix L^{ec} is the left annihilator for F_x^2. Indeed

$$L^{\mathrm{ec}} F_x^2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_{m_R} \end{bmatrix} \begin{bmatrix} 0_{r\times(n-r)} \\ I_{n-r} \\ 0_{m\times(n-r)} \\ 0_{m_R\times(n-r)} \end{bmatrix} = 0.$$

Premultiplying (B.3) with L^{ec} while setting $e_x^2 = 0$ leads to

$$\begin{bmatrix} I_r \\ 0_{m\times r} \\ 0_{m_R\times r} \end{bmatrix} \dot{x}_1 = \begin{bmatrix} J_{11} \\ -G_1^T \\ -G_{R_1}^T \end{bmatrix} e_x^1 + \begin{bmatrix} G_{R_1} \\ 0_{m\times m_R} \\ 0_{m_R} \end{bmatrix} f_R + \begin{bmatrix} 0_{r\times m_R} \\ 0_{m\times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G_1 \\ 0_{m\times m} \\ 0_{m_R\times m} \end{bmatrix} u + \begin{bmatrix} 0_{r\times m} \\ I_m \\ 0_{m_R\times m} \end{bmatrix} \hat{y}, \tag{B.5}$$

which is the equational representation (3.12)

$$L^{\mathrm{ec}} F_x^1 f_x^1 + L^{\mathrm{ec}} E_x^1 e_x^1 + L^{\mathrm{ec}} F_R f_R + L^{\mathrm{ec}} E_R e_R + L^{\mathrm{ec}} F_P f_P + L^{\mathrm{ec}} E_P e_P = 0,$$

of the reduced order Dirac structure (note that $f_x^1 = -\dot{x}_1$).

Recall from Section 2.6.1 that setting $e_x^2 = 0$ implies that $e_x^1 = Q_s x_1$, where $Q_s = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$ is the Schur complement of the energy matrix Q. The equational representation (B.5) is then equivalent to

$$\begin{cases} \dot{x}_1 &= J_{11}Q_s x_1 + G_{R_1} f_R + G_1 u, \\ \hat{y} &= G_1^T Q_s x_1, \\ e_R &= G_{R_1}^T Q_s x_1. \end{cases} \tag{B.6}$$

This is the reduced order port-Hamiltonian model by the effort-constraint method with the open resistive port. Termination of the resistive port employing the original linear relation $f_R = -\bar{R}e_R$ (while using $R_{11} = G_{R_1}\bar{R}G_{R_1}^T$ from (3.24)) leads to the reduced order port-Hamiltonian model by the effort-constraint method (3.21)

$$\begin{cases} \dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + G_1 u, \\ y_{ec} &= G_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1, \end{cases} \tag{B.7}$$

where $F_{11} = J_{11} - R_{11}$.

B.2. Flow-constraint reduction

We start with the equational representation of the full order Dirac structure (B.3). A maximal rank matrix L^{fc} satisfying

$$L^{\text{fc}}E_x^2 = 0$$

is proposed to be

$$L^{\text{fc}} = \begin{bmatrix} I_r & -J_{12}J_{22}^{-1} & 0 & 0 \\ 0 & G_2^T J_{22}^{-1} & I_m & 0 \\ 0 & G_{R_2}^T J_{22}^{-1} & 0 & I_{m_R} \end{bmatrix}, \tag{B.8}$$

assuming that J_{22} is invertible (J_{22} is invertible for even dimensions). Indeed

$$L^{\text{fc}}E_x^2 = \begin{bmatrix} I_r & -J_{12}J_{22}^{-1} & 0 & 0 \\ 0 & G_2^T J_{22}^{-1} & I_m & 0 \\ 0 & G_{R_2}^T J_{22}^{-1} & 0 & I_{m_R} \end{bmatrix} \begin{bmatrix} J_{12} \\ J_{22} \\ -G_2^T \\ -G_{R_2}^T \end{bmatrix} = 0.$$

The flow-constraint method assumes applying such a matrix L^{fc} to the equational representation of the full order Dirac structure (B.3) along with setting $f_x^2 = -\dot{x}_2 = 0$. For details see again Section 3.3. Then the equational representation of the reduced order Dirac structure reads

$$\begin{bmatrix} I_r & -J_{12}J_{22}^{-1} \\ 0_{m\times r} & G_2^T J_{22}^{-1} \\ 0_{m_R\times r} & G_{R_2}^T J_{22}^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} J_s & 0 \\ G_2^T J_{22}^{-1} J_{21} - G_1^T & 0 \\ G_{R_2}^T J_{22}^{-1} J_{21} - G_{R_1}^T & 0 \end{bmatrix} \begin{bmatrix} e_x^1 \\ e_x^2 \end{bmatrix} +$$

$$\begin{bmatrix} G_{R_1} - J_{12}J_{22}^{-1}G_{R_2} \\ G_2^T J_{22}^{-1} G_{R_2} \\ G_{R_2}^T J_{22}^{-1} G_{R_2} \end{bmatrix} f_R + \begin{bmatrix} 0_{r\times m_R} \\ 0_{m\times m_R} \\ I_{m_R} \end{bmatrix} e_R +$$

$$\begin{bmatrix} G_1 - J_{12}J_{22}^{-1}G_2 \\ G_2^T J_{22}^{-1} G_2 \\ G_{R_2}^T J_{22}^{-1} G_2 \end{bmatrix} u + \begin{bmatrix} 0_{r\times m} \\ I_m \\ 0_{m_R\times m} \end{bmatrix} \hat{y}.$$

After using the notation as in (3.23)

$$\begin{aligned} &\alpha := G_2^T J_{22}^{-1} J_{21} - G_1^T, \quad && \beta := G_{R_2}^T J_{22}^{-1} J_{21} - G_{R_1}^T, \\ &\gamma := G_2^T J_{22}^{-1} G_{R_2}, && \delta := G_{R_2}^T J_{22}^{-1} G_{R_2}, \\ &\eta := G_2^T J_{22}^{-1} G_2, && \end{aligned}$$

the above equation transforms to

$$\begin{aligned} \begin{bmatrix} I_r \\ 0_{m\times r} \\ 0_{m_R\times r} \end{bmatrix} \dot{x}_1 \; = \; & \begin{bmatrix} J_s \\ \alpha \\ \beta \end{bmatrix} e_x^1 + \begin{bmatrix} -\beta^T \\ \gamma \\ \delta \end{bmatrix} f_R + \begin{bmatrix} 0_{r\times m_R} \\ 0_{m\times m_R} \\ I_{m_R} \end{bmatrix} e_R + \\ & \begin{bmatrix} -\alpha^T \\ \eta \\ -\gamma^T \end{bmatrix} u + \begin{bmatrix} 0_{r\times m} \\ I_m \\ 0_{m_R\times m} \end{bmatrix} \hat{y}, \end{aligned} \tag{B.9}$$

which is of the form (3.14).

The equational representation (B.9) of the reduced order Dirac structure implies the reduced order port-Hamiltonian model

$$\begin{cases} \dot{x}_1 = J_s e_x^1 - \beta^T f_R - \alpha^T u, \\ \hat{y} = -\alpha e_x^1 - \gamma f_R - \eta u, \\ 0 = \beta e_x^1 + \delta f_R + e_R - \gamma^T u. \end{cases} \tag{B.10}$$

The resistive relation $f_R = -\bar{R} e_R$ allows to solve the third equation for e_R, which, after substituting in the other equations and using the fact that e_x^1 is such that $e_x^1 = Q_{11}x_1$ ($\dot{x}_2 = 0$ implies $x_2 = constant$ taken to be zero), results in the reduced order port-Hamiltonian model by the flow-constraint method (3.22)

$$\begin{cases} \dot{x}_1 = (J_s - \beta^T Z \beta) Q_{11} x_1 + (-\alpha^T + \beta^T Z \gamma^T) u, \\ y_{fc} = (-\alpha - \gamma Z \beta) Q_{11} x_1 + (-\eta + \gamma Z \gamma^T) u, \end{cases} \tag{B.11}$$

where $Z = \bar{R}(I - \delta\bar{R})^{-1}$.

Note that the presented ways of constructing the reduced order port-Hamiltonian models (B.7), (B.11) are not unique, since the annihilator matrices L^{ec}, L^{fc} are not unique. For example, in case of the flow-constraint method, premultiplying the differential equation in (B.1) with J^{-1} (J is invertible for even dimensions) and proceeding with the maximal rank annihilator matrix L^{fc}, given as

$$L^{\mathrm{fc}} = \begin{bmatrix} 0 & G_2^T & I_m & 0 \\ 0 & G_{R_2}^T & 0 & I_{m_R} \\ I_r & 0 & 0 & 0 \end{bmatrix}$$

instead of that in (B.8), lead to the same reduced order model (B.11), (3.22).

Lemma B.1. *Consider the matrix Z from* (3.23) *given as*

$$Z := \bar{R}(I - \delta\bar{R})^{-1}$$

for a skew-symmetric matrix $\delta = -\delta^T = G_{R_2}^T J_{22}^{-1} G_{R_2}$, and a symmetric positive definite matrix $\bar{R} = \bar{R}^T > 0$. Then the matrix Z can be decomposed into its symmetric Z_{sym} and skew-symmetric Z_{sk} parts as follows:

$$Z_{sym} = (\bar{R}^{-1} - \delta\bar{R}\delta)^{-1}, \quad Z_{sk} = (\bar{R}^{-1}\delta^{-1}\bar{R}^{-1} - \delta)^{-1}.$$

Furthermore, the symmetric part of the matrix Z is positive definite:

$$Z_{sym} = (\bar{R}^{-1} - \delta\bar{R}\delta)^{-1} > 0.$$

Proof. The matrix Z can be rewritten as $Z = (\bar{R}^{-1} - \delta)^{-1}$. Then straightforward calculations show that

$$\begin{aligned} Z_{sym} &= \tfrac{1}{2}(Z + Z^T) \\ &= \tfrac{1}{2}[(\bar{R}^{-1} - \delta)^{-1} + (\bar{R}^{-1} + \delta)^{-1}] \\ &= \tfrac{1}{2}(\bar{R}^{-1} - \delta)^{-1}[(\bar{R}^{-1} + \delta) + (\bar{R}^{-1} - \delta)](\bar{R}^{-1} + \delta)^{-1} \\ &= (\bar{R}^{-1} - \delta)^{-1}\bar{R}^{-1}(\bar{R}^{-1} + \delta)^{-1} \\ &= (\bar{R}^{-1} - \delta)^{-1}(I + \delta\bar{R})^{-1} \\ &= [(I + \delta\bar{R})(\bar{R}^{-1} - \delta)]^{-1} \\ &= (\bar{R}^{-1} - \delta\bar{R}\delta)^{-1}. \end{aligned}$$

Similarly

$$Z_{sk} = \frac{1}{2}(Z - Z^T) = (\bar{R}^{-1}\delta^{-1}\bar{R}^{-1} - \delta)^{-1}.$$

Moreover, $Z = (\bar{R}^{-1} - \delta)^{-1}$ implies that $Z^{-1} = \bar{R}^{-1} - \delta$. Hence, the symmetric part of Z^{-1}, which is $\bar{R}^{-1}$, is necessarily positive definite.The matrix Z^{-1} is positive definite as well: $Z^{-1} > 0$.

Since any real vector w of an appropriate dimension can be written as $w = Z^{-1}v$ for a certain v, it follows that

$$w^T Z w = v^T Z^{-T} Z Z^{-1} v = v^T Z^{-T} v = v^T Z^{-1} v > 0.$$

This leads to

$$\begin{aligned} w^T Z w > 0 \quad &\Longrightarrow \quad w^T Z_{sym} w + w^T Z_{sk} w > 0 \\ &\Longrightarrow \quad w^T Z_{sym} w > 0, \end{aligned}$$

and therefore the symmetric part of Z is positive definite. □

Note that in case of a lossless full order port-Hamiltonian system $\bar{R} = 0$ and, consequently, $Z = 0$.

C. Sketch of the proof of the $\mathcal{H}_2$ error bound for model reduction of structured systems

In the proof of the error bound on p. 125 in [83] it is shown by straightforward calculations using (9.15) that

$$\begin{aligned} &\|\mathcal{E}\|^2_{\mathcal{H}_2} = \operatorname{trace}\{\hat{C}_2\Lambda_2\hat{C}_2^*\} \\ &+\tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} \operatorname{trace}\{(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)(\hat{C}_1 L(i\omega)\hat{F}_2(i\omega))^*\}d\omega, \end{aligned} \tag{C.1}$$

where Λ_2 comes from the eigenvalue decomposition of the reachability Gramian W, $\hat{C}_1 = CV_1$ as in (9.12), $\hat{C}_2 = CV_2$ as in (9.14), $L(s) = (\hat{K}_{11}(s))^{-1}\hat{K}_{12}(s)$ as in (9.13) and $\hat{F}_2(s)$ is given in (9.16). The second term in the above expression has the following upper bound [83]

$$\begin{aligned} &\tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} \operatorname{trace}\{(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)(\hat{C}_1 L(i\omega)\hat{F}_2(i\omega))^*\}d\omega \\ &\quad\leqslant \sup_{\omega} \|(\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\|_2 \operatorname{trace}\{\Lambda_2\}. \end{aligned}$$

Proof. We directly proceed with the second term

$$\begin{aligned} &\tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} \operatorname{trace}\{(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)(\hat{C}_1 L(i\omega)\hat{F}_2(i\omega))^*\}d\omega \\ &= \tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} \operatorname{trace}\{(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)\hat{F}_2(i\omega)^*(\hat{C}_1 L(i\omega))^*\}d\omega \\ &= /\text{property of a trace: } \operatorname{trace}\{YZ\} = \operatorname{trace}\{ZY\}/ \\ &= \tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} \operatorname{trace}\{(\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)\hat{F}_2(i\omega)^*\}d\omega. \end{aligned}$$

Using the notation $N(i\omega) := (\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)$, $M(i\omega) :=$

$\hat{F}_2(i\omega)\hat{F}_2(i\omega)^*$, we proceed with the above expression:

$$\begin{aligned}
&\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\{(\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)\hat{F}_2(i\omega)^*\}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\{N(i\omega)M(i\omega)\}d\omega \\
&\leqslant /\mathrm{trace}\{YZ\} \leqslant \sigma_{max}(Y)\mathrm{trace}\{Z\} \text{ see Lemma C.1 below; the trace is real}/ \\
&\leqslant \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_{max}(N(i\omega))\mathrm{trace}\{M(i\omega)\}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|N(i\omega)\|_2\mathrm{trace}\{M(i\omega)\}d\omega \\
&\leqslant \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \sup_{\omega}\|N(i\omega)\|_2\mathrm{trace}\{M(i\omega)\}d\omega \\
&= \sup_{\omega}\|N(i\omega)\|_2\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\{M(i\omega)\}d\omega \\
&= \sup_{\omega}\|N(i\omega)\|_2\mathrm{trace}\{\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_2(i\omega)\hat{F}_2(i\omega)^*d\omega\} \\
&= /\text{see (9.15)}/ \\
&= \sup_{\omega}\|N(i\omega)\|_2\mathrm{trace}\{\Lambda_2\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tfrac{1}{2\pi}\int_{-\infty}^{\infty} &\mathrm{trace}\{(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\hat{F}_2(i\omega)(\hat{C}_1 L(i\omega)\hat{F}_2(i\omega))^*\}d\omega \\
&\leqslant \sup_{\omega}\|(\hat{C}_1 L(i\omega))^*(\hat{C}_1 L(i\omega) - 2\hat{C}_2)\|_2\mathrm{trace}\{\Lambda_2\}.
\end{aligned}$$

□

Lemma C.1. *Let $Y,\ Z \in \mathbb{C}^{n\times n}$, and assume that Z is nonnegative semi-definite. Then*

$$|\mathrm{trace}\{YZ\}| \leqslant \sigma_{max}(Y)\mathrm{trace}\{Z\},$$

where $\sigma_{max}(Y)$ is the largest singular value of Y.

Proof. See [11], Fact 8.12.14. □