Exam Number:________

ROB 501 Exam-I

Tuesday, October 25, 2016, 7:10 PM–9:00 PM
Rooms (First letter of last name): (A-K) in Chrysler 133 and (L-T) in Chrysler 165
(U-Z) in Duderstadt Center 1180 Teleconference Rooms C & D

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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(Sign **after** the exam is completed)

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FILL IN YOUR NAME NOW. COPY THE HONOR CODE NOW. DO NOT COUNT PAGES.
DO NOT OPEN THE EXAM BOOKLET UNTIL TOLD TO DO SO.

RULES:

1. CLOSED TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. 2 SHEETS OF NOTE PAPER (Front and Back), US Letter Size.
6. NO CALCULATORS, CELL PHONES, HEADSETS, nor DIGITAL DEVICES of any KIND.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Enter Multiple Choice Answers Here

Record Answers Here	
	Your Answer
Problem 1	(a) (b) (c) (d)
Problem 2	(a) (b) (c) (d)
Problem 3	(a) (b) (c) (d)
Problem 4	(a) (b) (c) (d)
Problem 5	(a) (b) (c) (d)

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Tuesday, October 25, 2016, 7:10 PM–9:00 PM

____________________, ________________

LAST NAME (PRINTED) *FIRST NAME*

FILL IN YOUR NAME

Scores (Filled in by Instructor)		
	Your Score	Max Score
Problems 1-5		30
Problem 6		20
Problem 7		15
Problem 8		15
Total		**80**
Problem 9		A^+ Points (5)
Answer Prob. 9	$A = \left[\quad \right]$	

Problems 1 - 5 (30 points: 5 $\times$ 6)

Instructions. For each problem, select all of the answers that are correct and enter them in the table on page 2. For each problem, there is at least one answer that is correct (i.e., true) and one answer that is incorrect (i.e., false). *You will receive no credit for your response if you either circle all of the answers or none of the answers.*

1. (Questions on logic) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table".

(a) $\neg q \implies \neg p$ is the converse of $p \implies q$.

(b) $p \implies q$ if, and only if, $\neg(p \wedge \neg q)$.

(c) You seek to show $p \implies q$ by employing the method of *Proof by Contradiction.* This means that you assume that p is FALSE and q is TRUE, and then seek to deduce a logical statement R that is both TRUE and FALSE.

(d) The truth table given below is correct for p implies NOT q:

p	q	$p \implies \neg q$
1	1	0
1	0	1
0	1	1
0	0	1

2. (Select the correct negations.) Note that two of the statements are the SAME[1] so that the problem is quicker to work.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$ a point. `Statement:` For all $\epsilon > 0$, there exists $\delta > 0$ such that $y \in \mathbb{R}$ and $|y - x_0| < \delta$ together imply that $|f(y) - f(x_0)| < \epsilon$. `Negation:` There exists an $\epsilon > 0$, such that for all $\delta > 0$, there exists $y \in \mathbb{R}$ such that $|y - x_0| < \delta$ and $|f(y) - f(x_0)| \geq \epsilon$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$ a point. `Statement: (same as above)` For all $\epsilon > 0$, there exists $\delta > 0$ such that $y \in \mathbb{R}$ and $|y - x_0| < \delta$ together imply that $|f(y) - f(x_0)| < \epsilon$. `Negation:` There exist an $\epsilon > 0$, a $\delta > 0$, and a $y \in \mathbb{R}$ such that $|y - x_0| < \delta$ and $|f(y) - f(x_0)| \geq \epsilon$.

(c) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and let S and M be subsets of $\mathcal{X}$. `Statement:` The set $S + M := \{x + y \mid x \in S, y \in M\}$ is a subspace. `Negation:` $\exists\, x \in S,\ y \in M$ and $\alpha \in \mathcal{F}$ such that $x + \alpha y \notin S + M$.

(d) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and let S and M be subsets of $\mathcal{X}$. `Statement (same as above):` The set $S + M := \{x + y \mid x \in S, y \in M\}$ is a subspace. `Negation:` $\exists\, x_1, x_2 \in S,\ y_1, y_2 \in M$ and $\alpha \in \mathcal{F}$ such that

$$(x_1 + y_1) + \alpha(x_2 + y_2) \notin S + M.$$

[1] Hence, you only have to negate two statements. To be clear, it could be the case that both of the `proposed negations` for a given statement are correct or neither one is correct.

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Anything written here will not be graded. You can use as scratch paper.

3. Let $(\mathcal{X}, \mathcal{F})$ be a finite dimensional vector space and let $\{v^1, v^2, v^3, v^4\}$ be a set of vectors in $\mathcal{X}$. Which of the following statements imply that the dimension of $\mathcal{X}$ is greater than or equal to 4?

(a) $\forall x \in \text{span}\{v^1, v^2, v^3, v^4\}$, there exist unique coefficients $\alpha_1, \ldots, \alpha_4 \in \mathcal{F}$ such that $x = \sum_{i=1}^4 \alpha_i v^i$

(b) $\mathcal{X} = \text{span}\{v^1, v^2, v^3, v^4\}$

(c) $\exists\, v^5 \in \mathcal{X}$ such that $v^5 \notin \text{span}\{v^1, v^2, v^3, v^4\}$

(d) $\mathcal{X} = \text{span}\{v^1, v^2\} \oplus \text{span}\{v^3, v^4\}$

4. Let $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ be a finite dimensional inner product space. Let $x, y \in \mathcal{X}$ be two vectors in $\mathcal{X}$, let $S \subset \mathcal{X}$ be a nonempty subset, and let $M = \text{span}\{S\}$. (underlined to make sure you did not misread S as a subspace). The norm on $\mathcal{X}$ is the standard one, namely, $||x|| :=< x, x >^{(1/2)}$

(a) If $x \in S^\perp$, then $d(x, M) = ||x||$.

(b) Let $\hat{x} = \arg\min d(x, M)$. Then $||x - \hat{x}||^2 + ||\hat{x}||^2 = ||x||^2$.

(c) If $S^\perp = M^\perp$, then $S = M$.

(d) If $m_1 \in M$ and $m_2 \in M$ satisfy $||x - m_1|| = d(x, M)$ and $||y - m_2|| = d(y, M)$, then

$$d(x + 3y, M) = ||x + 3y - m_1 - 3m_2||.$$

5. (Fields and vector spaces) Select all of the statements that are correct.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\mathbb{R}^3, \mathbb{R})$, but not a basis for $(\mathbb{C}^3, \mathbb{C})$.

(b) $(\mathbb{C}^2, \mathbb{R})$ is a 2-dimensional vector space.

(c) Consider the vector space $(\mathbb{R}^4, \mathbb{R})$ and let $\alpha_0 \in \mathbb{R}$ be fixed. Then $\mathcal{Y} = \{x \in \mathbb{R}^4 \mid [1,\ 2, -1,\ 2]x = \alpha_0\}$ is a subspace if, and only if, $\alpha_0 = 0$.

(d) Consider the set $F = \{a + b\sqrt{2} \mid a,\ b\ \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers. Then F satisfies: for each non-zero $\alpha \in F$, there exists $\gamma \in F$ such that $\alpha\gamma = 1$ (i.e., each non-zero element has a multiplicative inverse) [Yes, this problem is having you check ONE of the Axioms of a Field.]

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Anything written here will not be graded. You can use as scratch paper.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. *If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.* For example, we proved that the Gram Schmidt Process produces orthogonal vectors. So if you need this fact, simply state it and use it.

6. (20 points) (Place your answers in the **boxes** and show your work below.) Consider the real inner product space $\mathbb{R}^{2\times 3}$ consisting of real 2×3 matrices, with inner product $< A, B >= \text{trace}(A^\top B)$. Define $M = \text{span}\{y^1, y^2\}$ for

$$y^1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}, \; y^2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Recall that the `trace` of a square matrix is the sum of its diagonal terms.

(a) (5 points) Compute the norm of y^1.

(b) (15 points) Solve $\hat{x} = \arg\ \min d(x, M)$ for $x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Note 1: It is fine to leave the answer as a linear combination of at most three matrices as long as you specify clearly all of the constants. All I am trying to do is avoid having you waste time adding up matrices. Zero, one, two and three are all less than or equal to three.

Note 2: You are given that $< y^1, y^2 >= 1$ and $\text{trace}(x^\top y^1) = 1$. **You do not have to verify these calculation.** If they are useful to you, just use them. If they are not useful, then ignore them.

(a) $\|\|y^1\|\| =$

(b) $\hat{x} = \begin{bmatrix} & \\ & \end{bmatrix}$

Show your calculations below

Please show your work for question 6.

7. (15 points) Consider the vector spaces $(\mathbb{R}^2, \mathbb{R})$ and $(\mathbb{R}^{2\times 2}, \mathbb{R})$. You are given standard bases for each of them, namely:

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ for } (\mathbb{R}^2, \mathbb{R}), \text{ and } V = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ for } (\mathbb{R}^{2\times 2}, \mathbb{R}).$$

Define the linear operator $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, by for $x \in \mathbb{R}^2$, $\mathcal{L}(x) = xw^\top$ where $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. As an example,

$$\text{if } x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathcal{L}(x) = \mathcal{L}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}.$$

(a) (9 points) **Find** the matrix representation A of $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ with respect to the (given) standard bases.

(b) (6 points) Define a new basis for $(\mathbb{R}^{2\times 2}, \mathbb{R})$ by $W = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \right\}$. Let P denote the change of basis matrix from V to W and let $\bar{P}$ denote the change of basis matrix from W to V. **You are asked to compute either** P **or** $\bar{P}$. **Only compute one of them!** You must clearly indicate which change of basis matrix you are computing. If you compute P and label it $\bar{P}$ or vice versa, you will lose half the points.

(a) $A = \begin{bmatrix} \quad \end{bmatrix}$

(b) $P = \begin{bmatrix} \quad \end{bmatrix}$ or $\bar{P} = \begin{bmatrix} \quad \end{bmatrix}$

Show your calculations below

Please show your work for question 7.

8. (15 points) (Proof Problem) Let $(\mathcal{X}, \mathcal{F})$ be an n-dimensional vector space, let v^i be vectors in $\mathcal{X}$, and let $1 \leq k \leq n$ be an integer. **Prove that**

- **IF** $v^1 \neq 0$, $v^2 \notin \text{span}\{v^1\}$, ..., and $v^k \notin \text{span}\{v^1, \cdots, v^{k-1}\}$,
- **THEN** the set $\{v^1, v^2, \cdots, v^k\}$ is linearly independent.

Show your work below. You can use as true anything we have established in ROB 501 lecture or HW. I cannot answer a question of the form: **"do I have to prove this?" or "can I assume this?" or "have I shown enough?"**.

Remark: If you can only prove it for $k = 3$, you will still earn seven (7) points.

Please show your work for question 8.

9. (5 points) A$^+$ Problem: Points earned here will go toward deciding who goes from an A to an A^+ at the end of the term. **You must place your answer on page 3 of the exam booklet.**

Given: Consider $(\mathbb{R}^3, \mathbb{R})$ with the standard inner product defined by $<x, y>= x^\top y$. Given a linearly independent set of vectors $\left\{ v^1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v^2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$, define $M = \mathrm{span}\{v^1, v^2\}$, and $M^\perp$ is the orthogonal complement of M.

Using the natural basis $\left\{ e^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ on $\mathbb{R}^3$, **find** the matrix representation A of the orthogonal projection operator $P^\perp : \mathbb{R}^3 \to \mathbb{R}^3$, which projects a vector $x \in \mathbb{R}^3$ to the subspace $M^\perp$. To be more clear, the definition of the projection operator is

$$P^\perp(x) = \arg \min_{y \in M^\perp} ||x - y||.$$

Show your work below:

(Scratch Paper)

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(If you write anything here, be sure to indicate to which problem it applies.)