1.3 Induction and Other Proof Techniques

The purpose of this section is to study the proof technique known as **mathematical induction**. Before we do so, we will quickly review the other proof techniques used in mathematics.

1.3.1 Review of Proof techniques Other than Induction

Direct Proofs

We derive the result to prove by combining logically the given assumptions (if any), definitions, axioms and known theorems.

Example 64 *Prove that the sum of two odd integers is even.*
Recall that an integer n is even if $n = 2k$ and it is odd if $n = 2k + 1$ for some integer k. We start with two odd integers we call a and b. This means that there exist integers k_1 and k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Now,

$$\begin{aligned} a + b &= 2k_1 + 1 + 2k_2 + 1 \\ &= 2k_1 + 2k_2 + 2 \\ &= 2(k_1 + k_2 + 1) \end{aligned}$$

If k_1 and k_2 are integers, $k_1 + k_2 + 1$ is also an integer. Hence, $a + b$ is even.

Proof by Contrapositive

Suppose that P and Q are two statements. "If P then Q" is equivalent to its contrapositive "if not Q then not P ". Instead of proving one, the other can be proven.

Example 65 *Prove that if n^2 is even, so is n.*
Since a number is odd, the contrapositive of this statement is "if n is odd so is n^2. We prove that instead.

$$\begin{aligned} n \text{ odd} &\Longrightarrow n = 2k + 1 \text{ for some integer } k \\ &\Longrightarrow n^2 = (2k + 1)^2 \\ &\Longrightarrow n^2 = 4k^2 + 4k + 1 \\ &\Longrightarrow n^2 = 2\left(2k^2 + 2k\right) + 1 \\ &\Longrightarrow n^2 \text{ is odd since } 2k^2 + 2k \text{ is an integer} \end{aligned}$$

Proof by Contradiction

We prove that under the given assumptions, assuming a statement is true leads to some contradiction. Hence, the statement cannot be true. In general, what we assume to be true is the negation of what we have to prove. Since the negation of what we have to prove leads to a contradiction, hence cannot be true. It follows that the result to prove must be true. We show a classical example.

Example 66 *Show $\sqrt{2}$ is irrational.*
We do a proof by contradiction. We assume the opposite of what we want to prove, that is $\sqrt{2}$ is rational (there are only two possibilities, either a number is rational or it is irrational). We will show this leads to a contradiction. Thus, $\sqrt{2}$ cannot be rational, hence it must be irrational. So, suppose that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{m}{n}$ where m and n are integers with no common factors. This means that $m = n\sqrt{2}$ or $m^2 = 2n^2$. Thus m^2 is even, it follows that m is even (see above). If m is even, then $m = 2k$ for some integer k so that $m^2 = 4k^2$ but $m^2 = 2n^2$ hence $2n^2 = 4k^2$ or $n^2 = 2k^2$ thus n^2 is even,hence n is even. It follows that both m and n are even which is a contradiction since m and n were supposed to have no common factors.

Proof by Exhaustion

We divide the result to prove into cases and prove each one separately.

Proof by Construction

We prove an object having certain properties exists by constructing an example of an object with the required properties.

1.3.2 First Principle of Mathematical Induction

Proofs by induction are often used when one tries to prove a statement made about natural numbers or integers. Here are examples of statements where induction would be used.

- For every natural number n, $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$
- If $x > -1$, and n is a natural number, then $(1+x)^n \geq 1 + nx$

The principle of mathematical induction, states the following:

Theorem 67 (Induction) *Let $P(n)$ denote a statement about natural numbers with the following properties:*

1. *The statement is true when $n = 1$ i.e. $P(1)$ is true.*
2. *$P(k+1)$ is true whenever $P(k)$ is true for any positive integer k.*

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Remark 68 *The case $n = 1$ is called the **base case**.*

Remark 69 *The principle of mathematical induction is also true if instead of starting at 1, we start at any integer n_0. In other words, if we prove that $P(n_0)$ is true and $P(k+1)$ is true whenever $P(k)$ is true , $k \geq n_0$, then $P(n)$ will be true for all $n \in \mathbb{Z}$ such that $n \geq n_0$.*

Remark 70 *When doing a proof by induction, it is important to write explicitly what the statement $P(n)$ is so we know what we have to prove for a given n. Before proving $P(1)$, write clearly what $P(1)$ says. Similarly, when we assume $P(k)$ true and want to deduce $P(k+1)$, write clearly what both $P(k)$ and $P(k+1)$ say so we know what we are assuming and what we need to prove.*

The theorem can easily be proven if we assume an important result about $\mathbb{N}$. This result is called the well ordering principle, which we will take as an axiom.

Axiom 71 (Well Ordering Principle) *Every nonempty subset of $\mathbb{N}$ has a smallest element. In other words, if $A \subseteq \mathbb{N}$ and $A \neq \varnothing$ then there exists $n \in A$ such that $n \leq k$ for all $k \in A$.*

Example 72 *$S = \{1, 3, 5, 7, 9\}$ is a subset of $\mathbb{N}$. Its smallest element is 1.*

Remark 73 *Can we say the same about subsets of positive real numbers?*

We can now prove theorem 67 using a proof by contradiction.
Proof of the Principle of Mathematical Induction. Suppose the hypotheses of theorem 67 are true but the conclusion is false. That is, for some k, $P(k)$ is false. let $A = \{k \in \mathbb{N} : P(k) \text{ is false}\}$. Then $A \subseteq \mathbb{N}$ and $A \neq \varnothing$. So, it has a smallest element, call it k_0. In particular, $P(k_0 - 1)$ is true since k_0 is the smallest number for which $P(k)$ is false. But by the hypotheses of the theorem, if $P(k_0 - 1)$ is true, so should $P(k_0)$. Which means $k_0 \notin A$, which is a contradiction. So, $P(n)$ must be true for all $n \in \mathbb{N}$. ■

We illustrate this principle with some examples which we state as theorems.

Theorem 74 *If n is a natural number, then $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$*

Proof. *We do a proof by induction (though a nice direct proof also exists). Let $P(n)$ denote the statement that $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$. We would like to show that $P(n)$ is true for all n. $P(1)$ states that $1 = \dfrac{1(1+1)}{2}$ which is true. This establishes that $P(1)$ is true. Next, we assume that $P(k)$ holds for some natural number k. We wish to prove that $P(k+1)$ also holds. We begin by writing what $P(k)$ and $P(k+1)$ represent so that we know what we are assuming and what we have to prove. $P(k)$ says that $1+2+3+...+k = \dfrac{k(k+1)}{2}$. $P(k+1)$ says that $1 + 2 + 3 + ... + k + (k+1) = \dfrac{(k+1)(k+2)}{2}$. Now,*

$$
\begin{aligned}
1 + 2 + 3 + ... + k + (k+1) &= (1 + 2 + 3 + ... + k) + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \text{ by assumption} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

So, we see that $P(k+1)$ holds. Therefore, by induction, $P(n)$ holds for all n.
■

Remark 75 *Proving that $P(1)$ is true is essential. Consider the statement $n+1=n$ for all $n \geq 0$. This is obviously false. However, if we do not bother to check whether $P(1)$ is true and we assume that $P(k)$ is true then we can prove that $P(k+1)$ is also true. $P(k+1)$ says that $n+2=n+1$.*

$$\begin{aligned} n+2 &= n+1+1 \\ &= n+1 \text{ by assumption since } n+1=n \end{aligned}$$

Thus we would have proven that $n+1=n$.

Theorem 76 (Bernoulli's inequality) *If $x > -1$, and n is a natural number, then $(1+x)^n \geq 1+nx$*
Proof. *We do a proof by induction. Let $P(n)$ be the statement that $x \geq -1$, and n is a natural number, then $(1+x)^n \geq 1+nx$.*

- *$P(1)$ would be the statement $1+x \geq 1+x$, which is obviously true.*
- *Assume $P(k)$ is true, that is $(1+x)^k \geq 1+kx$. we wish to prove that $P(k+1)$ is also true, that is $(1+x)^{k+1} \geq 1+(k+1)x$.*

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) \text{ by assumption and since } x > -1 \\ &\geq 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x \end{aligned}$$

Thus, $P(k+1)$ holds. It follows by induction that $P(n)$ holds for every n.

■

Sometimes, it is not easy to deduce that $P(k+1)$ is true knowing that $P(k)$ is true, especially if we do not have a relationship between $P(k)$ and $P(k+1)$. In such cases, another form of mathematical induction can be used.

1.3.3 Second Principle of Mathematical Induction

Theorem 77 (Second Principle of Mathematical Induction) *Let $P(n)$ denote a statement about natural numbers with the following properties:*

1. *The statement is true when $n=1$ i.e. $P(1)$ is true.*
2. *$P(k)$ is true whenever $P(j)$ is true for all positive integers $1 \leq j < k$.*

Then, $P(n)$ is true for every natural number.

Example 78 *Consider $f:\mathbb{N}\to\mathbb{R}$ defined by $f(1)=0$, $f(2)=\frac{1}{3}$, and for $n>2$ by $f(n)=\frac{n-1}{n+1}f(n-2)$. By computing values of $f(n)$ for $n=3,4,5,6$, give a conjecture as to what a direct formula for f might be. Prove your conjecture by induction.*

- $f(3)=\frac{2}{4}f(1)=\frac{2}{4}0=0$
- $f(4)=\frac{3}{5}f(2)=\frac{3}{5}.\frac{1}{3}=\frac{3}{15}=\frac{1}{5}$
- $f(5)=\frac{4}{6}f(3)=0$
- $f(6)=\frac{5}{7}f(4)=\frac{5}{7}.\frac{1}{5}=\frac{1}{7}$
- *It seems that* $f(n)=\begin{cases}0 \text{ if } n \text{ is odd}\\ \frac{1}{n+1} \text{ if } n \text{ is even}\end{cases}$ *We need to prove this.*
- *Proof of the conjecture. We can see from the computations that the conjecture is true for $n=1,2$. Suppose that $n>2$. Suppose our conjecture holds for all $k<n$. We need to prove the conjecture also holds for n. If n is odd, then $f(n)=\frac{n-1}{n+1}f(n-2)$. Since n is odd, so is $n-2$. Because $n-2<n$, the conjecture is true for $n-2$, so $f(n-2)=0$ hence $f(n)=0$. If n is even, then $f(n)=\frac{n-1}{n+1}f(n-2)$. $n-2$ is also even and the conjecture holds for it. So, $f(n-2)=\frac{1}{n-1}$. Therefore*

$$\begin{aligned} f(n) &= \frac{n-1}{n+1}f(n-2) \\ &= \frac{n-1}{n+1}\frac{1}{n-1} \\ &= \frac{1}{n+1} \end{aligned}$$

  The conjecture is proven.

1.3.4 Exercises

1. Prove by induction that $(1+x)^n \geq 1+nx+\frac{n(n-1)}{2}x^2$ when $x\geq 0$.
2. Prove by induction that $(1-x)^n \leq 1-nx+\frac{n(n-1)}{2}x^2$ when $0\leq x<1$.
3. Prove by induction that if $a_1,a_2,...a_n$ are all non-negative, then $(1+a_1)(1+a_2)...(1+a_n)\geq 1+a_1+a_2+...+a_n$.

4. Prove by induction that $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+...+\dfrac{1}{n\,(n+1)}=\dfrac{n}{n+1}$.

5. Use mathematical induction to show that the identities below are valid for any $n\in\mathbb{N}$.

   (a) $1+3+5+...+(2n-1)=n^2$.

   (b) $1^2+2^2+3^2+...+n^2=\dfrac{n\,(n+1)\,(2n+1)}{6}$.

   (c) $1^3+2^3+3^3+...+n^3=\left[\dfrac{1}{2}n\,(n+1)\right]^2$.

   (d) $2+2^2+2^3+...+2^n=2\,(2^n-1)$.

   (e) $x^{n+1}-y^{n+1}=(x-y)\left(x^n+x^{n-1}y+x^{n-2}y^2+...+xy^{n-1}+y^n\right)$.

6. Use mathematical induction to establish the identities below for the given values of n. If no value is specified, you also need to find the smallest value of n that will work.

   (a) $2^n>n$ for all $n\in\mathbb{N}$.

   (b) $2^n>n^2$ for all $n\in\mathbb{N}$ such that $n\geq 5$.

   (c) $n!>2^n$ for all $n\in\mathbb{N}$ such that $n\geq 4$.

   (d) $n!>2^{n-1}$.

7. In the questions below, f is a function with domain $\mathbb{N}$. Use the given information to find a formula for $f(n)$ then use mathematical induction to prove your formula is correct.

   (a) $f(1)=\dfrac{1}{2}$, and for $n>1$, $f(n)=(n-1)\,f(n-1)-\dfrac{1}{n+1}$.

   (b) $f(1)=1$, $f(2)=4$, and for $n>2$, $f(n)=2f(n-1)-f(n-2)+2$.

8. Let $f:\mathbb{N}\to\mathbb{R}$ be defined recursively by:

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2 \\ f(n+2) &= \frac{1}{2}\left[f(n+1)+f(n)\right] \end{aligned}$$

   Use mathematical induction to prove that $1\leq f(n)\leq 2$ for every $n\in\mathbb{N}$.