1.2.2. Converse statements and "if and only if" statements. Given a statement P implies Q, the reverse statement Q implies P is called the **converse** statement. For example, back to set theory, the converse of the statement

$$\text{If } x \in A, \text{ then } x \in B; \text{ that is, } A \subseteq B,$$

is just the statement that

$$\text{If } x \in B, \text{ then } x \in A; \text{ that is, } B \subseteq A.$$

These set theory statements make it clear that the converse of a true statement may not be true, for $\{e, \pi\} \subseteq \{e, \pi, i\}$, but $\{e, \pi, i\} \not\subseteq \{e, \pi\}$. Let us consider examples with real numbers.

Example 1.16. The statement "If $x^2 = 2$, then x is irrational" is true, but its converse statement, "If x is irrational, then $x^2 = 2$," is false.

Statements for which the converse is equivalent to the original statement are called "if and only if" statements.

Example 1.17. Consider the statement "If $x = -5$, then $2x + 10 = 0$." This statement is true. Its converse statement is "If $2x + 10 = 0$, then $x = -5$." By solving the equation $2x + 10 = 0$, we see that the converse statement is also true.

The implication $x = -5 \Longrightarrow 2x + 10 = 0$ can be written

$$2x + 10 = 0 \text{ if } x = -5, \tag{1.5}$$

while the implication $2x + 10 = 0 \Longrightarrow x = -5$ can be written

$$2x + 10 = 0 \text{ only if } x = -5. \tag{1.6}$$

Combining the two statements (1.5) and (1.6) into one statement, we get

$$2x + 10 = 0 \textit{ if } \text{and } \textit{only if } x = -5,$$

which is often denoted by a double arrow

$$2x + 10 = 0 \Longleftrightarrow x = -5,$$

or in more common terms, $2x + 10 = 0$ is equivalent to $x = -5$. We regard the statements $2x + 10 = 0$ and $x = -5$ as equivalent because if one statement is true, then so is the other one; hence the wording "is equivalent to". In summary, if both statements

$$Q \text{ if } P \text{ (that is, } P \Longrightarrow Q) \quad \text{and} \quad Q \text{ only if } P \text{ (that is, } Q \Longrightarrow P)$$

hold, then we write

$$Q \text{ if and only if } P \qquad \text{or} \qquad Q \Longleftrightarrow P.$$

Also, if you are asked to prove a statement "Q if and only if P", then you have to prove both the "if" statement "Q if P" (that is, $P \Longrightarrow Q$) and the "only if" statement "Q only if P" (that is, $Q \Longrightarrow P$).

The if and only if notation $\Longleftrightarrow$ comes in quite handy in proofs whenever we want to move from one statement to an equivalent one.

Example 1.18. Recall that in the proof of Theorem 1.3, we wanted to show that $A \cap \bigcup_\alpha A_\alpha = \bigcup_\alpha (A \cap A_\alpha)$, which means that $A \cap \bigcup_\alpha A_\alpha \subseteq \bigcup_\alpha (A \cap A_\alpha)$ and $\bigcup_\alpha (A \cap A_\alpha) \subseteq A \cap \bigcup_\alpha A_\alpha$; that is,

$$x \in A \cap \bigcup_\alpha A_\alpha \implies x \in \bigcup_\alpha (A \cap A_\alpha) \quad \text{and} \quad x \in \bigcup_\alpha (A \cap A_\alpha) \implies x \in A \cap \bigcup_\alpha A_\alpha,$$

which is to say, we wanted to prove that

$$x \in A \cap \bigcup_\alpha A_\alpha \iff x \in \bigcup_\alpha (A \cap A_\alpha).$$

We can prove this quick and simple using $\iff$:

$$\begin{aligned} x \in A \cap \bigcup_\alpha A_\alpha \iff x \in A \text{ and } x \in \bigcup_\alpha A_\alpha &\iff x \in A \text{ and } x \in A_\alpha \text{ for some } \alpha \\ &\iff x \in A \cap A_\alpha \text{ for some } \alpha \\ &\iff x \in \bigcup_\alpha (A \cap A_\alpha). \end{aligned}$$

Just make sure that if you use $\iff$, the expression to the immediate left and right of $\iff$ are indeed equivalent.

1.2.3. Negations and logical quantifiers. We already know that a statement and its contrapositive are always equivalent: "if P, then Q" is equivalent to "if not Q, then not P". Therefore, it is important to know how to "not" something, that is, find the **negation**. Sometimes the negation is obvious.

Example 1.19. The negation of the statement that $x > 5$ is $x \leq 5$, and the negation of the statement that x is irrational is that x is rational. (In both cases, we are working under the unstated assumptions that x represents a real number.)

But some statements are not so easy especially when there are **logical quantifiers**: "for every" = "for all" (sometimes denoted by $\forall$ in class, but not in this book), and "for some" = "there exists" = "there is" = "for at least one" (sometimes denoted by $\exists$ in class, but not in this book). The equal signs represent the fact that we mathematicians consider "for every" as another way of saying "for all", "for some" as another way of saying "there exists", and so forth. Working under the assumptions that all numbers we are dealing with are real, consider the statement

$$\text{For every } x,\ x^2 \geq 0. \tag{1.7}$$

What is the negation of this statement? One way to find out is to think of this in terms of set theory. Let $A = \{x \in \mathbb{R}\,;\, x^2 \geq 0\}$. Then the statement (1.7) is just that $A = \mathbb{R}$. It is obvious that the negation of the statement $A = \mathbb{R}$ is just $A \neq \mathbb{R}$. Now this means that there must exist some real number x such that $x \notin A$. In order for x to not be in A, it must be that $x^2 < 0$. Therefore, $A \neq \mathbb{R}$ just means that there is a real number x such that $x^2 < 0$. Hence, the negation of (1.7) is just

$$\text{For at least one } x,\ x^2 < 0.$$

Thus, the "for every" statement (1.7) becomes a "there is" statement. In general, the negation of a statement of the form

"For every x, P" is the statement "For at least one x, not P."

Similarly, the negation of a "there is" statement becomes a "for every" statement. Explicitly, the negation of

"For at least one x, Q" is the statement "For every x, not Q."

For instance, with the understanding that x represents a real number, the negation of "There is an x such that $x^2 = 2$" is "For every x, $x^2 \neq 2$".

EXERCISES 1.2.

1. In this problem all numbers are understood to be real. Write down the contrapositive and converse of the following statement:
$$\text{If } x^2 - 2x + 10 = 25, \text{ then } x = 5,$$
and determine which (if any) of the three statements are true.
2. Write the negation of the following statements, where x represents an integer.
   (a) For every x, $2x + 1$ is odd.
   (b) There is an x such that $2^x + 1$ is prime.[8]
3. Here are some more set theory proofs to brush up on.
   (a) Prove that $(A^c)^c = A$.
   (b) Prove that $A = A \cup B$ if and only if $B \subseteq A$.
   (c) Prove that $A = A \cap B$ if and only if $A \subseteq B$.

1.3. What are functions?

In high school we learned that a function is a "rule that assigns to each input exactly one output". In practice, what usually comes to mind is a formula, such as
$$p(x) = x^2 - 3x + 10.$$
In fact, Leibniz who in 1692 (or as early as 1673) introduced the word "function" [**221**, p. 272] and to all mathematicians of the eighteenth century, a function was always associated to some type of analytic expression "a formula". However, because of necessity to problems in mathematical physics, the notion of function was generalized throughout the years and in this section we present the modern view of what a function is; see [**118**] or [**137, 138**] for some history.

1.3.1. (Cartesian) product. If A and B are sets, their **(Cartesian) product**, denoted by $A \times B$, is the set of all 2-tuples (or ordered pairs) where the first element is in A and the second element is in B. Explicitly,
$$\boxed{A \times B := \{(a, b)\,;\, a \in A,\ b \in B\}.}$$
We use the adjective "ordered" because we distinguish between ordered pairs, e.g. $(e, \pi) \neq (\pi, e)$, but as sets we regard then as equal, $\{e, \pi\} = \{\pi, e\}$. Of course, one can also define the product of any finite number of sets
$$A_1 \times A_2 \times \cdots \times A_m$$
as the set of all m-tuples $(a_1, \ldots, a_m)$ where $a_k \in A_k$ for each $k = 1, \ldots, m$.

Example 1.20. Of particular interest is m-dimensional Euclidean space
$$\mathbb{R}^m := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{(m \text{ times})},$$
which is studied in Section 2.8.

[8] A number that is not prime is called composite.