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DECEMBER 23RD, 2012, 12:21 **#3**

> Originally Posted by **matqkks**
>
> I have been trying to prove the following result:
> If A is real symmetric matrix with an eigenvalue lambda of multiplicity m then lambda has m linearly independent e.vectors.
> Is there a simple proof of this result?

This is a slight variation of Deveno's argument. I will assume you already know that the eigenvalues of a real symmetric matrix are all real.

Let A be an $n\times n$ real symmetric matrix, and assume as an inductive hypothesis that all $(n-1)\times(n-1)$ real symmetric matrices are diagonalisable. Let λ be an eigenvalue of A, with a normalised eigenvector x_1. Using the Gram–Schmidt process, form an orthonormal basis $\{x_1,x_2,\ldots,x_n\}$ with that eigenvector as its first element.

Let P be the $n\times n$ matrix whose columns are $x_1,x_2,\ldots,x_n$, and denote by $T:\mathbb{R}^n\to\mathbb{R}^n$ the linear transformation whose matrix with respect to the standard basis is A. Then P is an orthogonal matrix ($P^{\mathrm{T}}=P^{-1}$), and the matrix of T with respect to the basis $\{x_1,x_2,\ldots,x_n\}$ is $P^{\mathrm{T}}AP$. The (i,j)-element of that matrix is $(P^{\mathrm{T}}AP)_{ij}=\langle Ax_j,x_i\rangle$. In particular, the elements in the first column are

$$(P^{\mathrm{T}}AP)_{i1}=\langle Ax_1,x_i\rangle=\langle\lambda x_1,x_i\rangle=\begin{cases}\lambda & (i=1)\\ 0 & (i>1)\end{cases}$$

(because the vectors x_i are orthonormal). Thus the first column of $P^{\mathrm{T}}AP$ has λ as its top element , and 0 for each of the other elements. Since $P^{\mathrm{T}}AP$ is symmetric, the top row also consists of a λ followed by all zeros. Hence the matrix $P^{\mathrm{T}}AP$ looks like this:

$$\begin{bmatrix}\lambda & 0 & \ldots & 0\\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{bmatrix},$$

where B is an $(n-1)\times(n-1)$ real symmetric matrix. By the inductive hypothesis, B is diagonalisable, so there is an orthogonal $(n-1)\times(n-1)$ matrix Q such that $Q^{\mathrm{T}}BQ$ is diagonal. Let

$$R = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{bmatrix}.$$

Then $R^{\mathrm{T}}P^{\mathrm{T}}APR$ is diagonal, as required.

Was sich überhaupt sagen lässt, lässt sich klar sagen; und wovon man nicht reden kann, darüber muss man schweigen.

(Anything that can be said at all, can be said clearly; and whereof one cannot speak, thereon one must be silent.)

– Ludwig Wittgenstein's good advice for forum contributors, in *Tractatus Logico-Philosophicus*.

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