What is the proof that covariance matrices are always semi-definite?

Covariance matrix $\mathbf{C}$ is calculated by the formula,

$$\mathbf{C} \triangleq E\{(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\}.$$

For an arbitrary real vector $\mathbf{u}$, we can write,

$$\begin{aligned}\mathbf{u}^T\mathbf{C}\mathbf{u} &= \mathbf{u}^T E\{(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\}\mathbf{u} \\ &= E\{\mathbf{u}^T(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\mathbf{u}\} \\ &= E\{s^2\} \\ &= \sigma_s^2.\end{aligned}$$

Where σ_s is the variance of the zero-mean scalar random variable s, and it is a scalar real number whose value equals to,

$$\sigma_s = \mathbf{u}^T(\mathbf{x}-\bar{\mathbf{x}}) = (\mathbf{x}-\bar{\mathbf{x}})^T\mathbf{u}.$$

Square of any real number is equal to or greater than zero. That is,

$$\sigma_s^2 \geq 0.$$

Thus,

$$\mathbf{u}^T\mathbf{C}\mathbf{u} = \sigma_s^2 \geq 0.$$

Which implies that covariance matrix of any real random vector is always semi-definite.