Schur Complement Lemma

Lemma: Schur Complement

Let S be a symmetric matrix partitioned into blocks:

$$S = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where both A, C are symmetric and square. Assume that C is positive definite. Then the following properties are equivalent:

- S is positive semi-definite.
- The *Schur complement* of C in S, defined as the matrix $A - BC^{-1}B^T$, is positive semi-definite.

Proof: Recall that the matrix S is positive semi-definite if and only if $x^T Sx \geq 0$ for any vector x. Partitioning the vector x similarly to S, as $x = (y, z)$, we obtain that S is positive semi-definite if and only if

$$\forall\, (z, y) \;:\; g(y, z) := \begin{pmatrix} y \\ z \end{pmatrix}^T \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq 0.$$

This is equivalent to: for every y,

$$0 \geq f(y) := \min_z \; g(y, z).$$

Since S is positive semi-definite, the corresponding quadratic function g is convex, jointly in its two arguments. Due to the partial minimization result, we obtain that the partial minimum $f(y)$ is convex as well.

It is easy to obtain a closed-form expression for f. We simply have to minimize the convex quadratic function g with respect to its second argument. Since the problem of minimizing g is not constrained, we just set the gradient of g with respect to z to zero (see here):

$$\nabla_z g(y, z) = 2(Cz + B^T y) = 0,$$

which leads to the (unique) optimizer $z^*(y) := -C^{-1}B^T y$. Plugging this value we obtain:

$$f(y) = g(y, z^*(y)) = y^T(A - BC^{-1}B^T)y.$$

Since f is convex, its Hessian must be positive semi-definite. Hence $A - BC^{-1}B^T \succeq 0$, as claimed.