# Supplemental material to “Control Barrier Function Based Quadratic Programs with Application to Automotive Safety Systems”

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## I. Proof of Validity of CBFs In Appendix

First, we prove that the RCBF candidate $B_F^c := 1/h_F^c$ satisfies

$$\inf_{u\in U}\left[L_f B_F^c + L_g B_F^c u - \frac{1}{B_F^c}\right] \le 0, \tag{*}$$

where $h_F^c$ is given in (54)-(57) and $u$ satisfying (FC):

Case (i). $v_l \ge v_f, T_l \ge T_f$:

$$h_F^c(x) = D - \tau_d v_f.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f)}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + (v_l - v_f)}{(h_F^c)^2} < 0,$$

which means that (*) holds.

Case (ii). $v_l \ge v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(a_l v_f - a_f v_l)^2}{a_l a_f (a_l - a_f) g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(a_l v_f - a_f v_l)(u a_l/M - a_f a_L)}{(a_l a_f (a_l - a_f) g)}}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + (v_l - v_f) + \frac{(a_l v_f - a_f v_l)(a_l a_f g + a_f a_L)}{a_l a_f (a_l - a_f) g}}{(h_F^c)^2} < 0,$$

because $a_l > a_f$ and $a_l v_f > a_f v_l$ in this case. Then it follows that inequality (*) holds.

Case (iii). $v_l < v_f, T_l \ge T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(v_f - v_l)^2}{(a_f - a_l) g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(v_f - v_l)(u/M - a_L)}{(a_f - a_l) g}}{(h_F^c)^2}.$$

If $u = -a_f Mg$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + \frac{(v_f - v_l)(a_l g + a_L)}{(a_f - a_l)g}}{(h_F^c)^2} < 0,$$

because $v_f > v_l$ and $a_f > a_l$ in this case. It follows that inequality (*) holds.

Case (iv). $v_l < v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{v_f^2 a_l - v_l^2 a_f}{a_f a_l g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(a_l v_f u/M - a_f v_l a_L)}{a_f a_l g}}{(h_F^c)^2}.$$

If $u = -a_f Mg$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + v_l + v_l a_L / a_l g}{(h_F^c)^2} < 0,$$

which means that inequality (*) holds.

On the other hand, if $B_F^c$ is taken as

$$B_F^c = \log(\frac{h_F^c}{1 + h_F^c})$$

then because $\dot{B}_F^c = -\dot{h}_F^c/(h_F^c(1 + h_F^c))$, it is easy to check that argument above is still valid for each case after minor modifications. Similarly, the optimal RCBF candidate $B_F^o$ associated with $h_F^o$ can also be shown to be a valid RCBF for the ACC problem.

## II. Optimal CBF for ACC

Recall that the *optimal RCBF* is $B_F^o = \frac{1}{h_F^o}$ or $B_F^o = -log(\frac{h_F^o}{1+h_F^o})$ with $h_F^o(x) = D - \Delta^*$ where $\Delta^*$ is given as follows:
(i) if $T_f > T_l$, then

$$\Delta^* = \max_{t\in[0,T_f]} (\Delta_1(t) + \tau_d(v_f - a_f gt)).$$

with

$$\Delta_1(t) = \begin{cases} (v_f t - \frac{1}{2}a_f gt^2) - (v_l t - \frac{1}{2}a_l gt^2), t \in [0, T_l), \\ (v_f t - \frac{1}{2}a_f gt^2) - \frac{v_l^2}{2a_l g}, t \in [T_l, T_f], \end{cases}$$

(ii) if $T_f \le T_l$, then

$$\Delta^* = \max_{t\in[0,T_f]} (\Delta_2(t) + \tau_d(v_f - a_f gt)),$$

with

$$\Delta_2(t) = (v_f t - \frac{1}{2}a_f gt^2) - (v_l t - \frac{1}{2}a_l gt^2),\ t \in [0, T_f],$$

The explicit form of $\Delta^*$ can be derived by solving the optimization problem above. The results are as follows:

**Case(I)** $a_f = a_l$
**(Ia)**$v_f \le v_l$:

$$\Delta^* = 1.8v_f$$

**(Ib)**$v_f > v_l$

**(Ib-1)**$v_l < v_f \le v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

**(Ib-2)**$v_f > v_l + 1.8a_f$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

**Case(II)** $a_f > a_l$
**(IIa)** $v_f < \frac{a_f}{a_l} v_l$

**(IIa-1)** $v_l \ge v_f$:

$$\Delta^* = 1.8v_f$$

**(IIa-2)** $v_l < v_f$

**(IIa-2-1)** $v_f \le v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

**(IIa-2-2)** $v_f > v_l + 1.8a_f g$:

$$\Delta^* = \frac{1}{2}\frac{(v_l + 1.8a_f g - v_f)^2}{(a_f - a_l)g} + 1.8v_f$$

**(IIb)** $v_f \geq \frac{a_f}{a_l}v_l$

**(IIb-1)** $v_f \leq v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

**(IIb-2)** $v_f > v_l + 1.8a_f g$

**(IIb-2-1)** $\frac{v_f - 1.8a_f g}{a_f g} \leq \frac{v_l}{a_l g}$:

$$\Delta^* = \frac{1}{2}\frac{(v_l + 1.8a_f g - v_f)^2}{(a_f - a_l)g} + 1.8v_f$$

**(IIb-2-2)** $\frac{v_f - 1.8a_f g}{a_f g} > \frac{v_l}{a_l g}$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

**Case(III)** $a_f < a_l$

**(IIIa)** $v_f \leq \frac{a_f}{a_l}v_l$:

$$\Delta^* = 1.8v_f$$

**(IIIb)** $v_f > \frac{a_f}{a_l}v_l$

**(IIIb-1)** $v_f \leq v_l$

**(IIIb-1-1)** $v_f \leq 1.8a_f g + \frac{a_f}{a_l}v_l$:

$$\Delta^* = 1.8v_f$$

**(IIIb-1-2)** $v_f > 1.8a_f g + \frac{a_f}{a_l}v_l$:

If $(v_l - v_f + 1.8a_f g)^2 \geq \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = 1.8v_f$$

If $(v_l - v_f + 1.8a_f g)^2 < \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

**(IIIb-2)** $v_f > v_l$

**(IIIb-2-1)** $v_f \geq 1.8a_f g + v_l$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

**(IIIb-2-2)** $v_f < 1.8a_f g + v_l$

**(IIIb-2-2-1)** $v_f \leq 1.8a_f g + \frac{a_f}{a_l}v_l$:

$$\Delta^* = 1.8v_f$$

**(IIIb-2-2-2)** $v_f > 1.8a_f g + \frac{a_f}{a_l}v_l$

If $(v_l - v_f + 1.8a_f g)^2 \geq \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = 1.8v_f$$

If $(v_l - v_f + 1.8a_f g)^2 < \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$